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STRATONOVICH-WEYL CORRESPONDENCE FOR DISCRETE SERIES REPRESENTATIONS

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Abstract. Let $M = G/K$ be a Hermitian symmetric space of the noncompact type and let π be a discrete series representation of G holomorphically induced from a unitary character of K. Following an idea of Figueroa, Gracia-Bondìa and Vàrilly, we construct a Stratonovich-Weyl correspondence for the triple (G, π, M) by a suitable modification of the Berezin calculus on M. We extend the corresponding Berezin transform to a class of functions on M which contains the Berezin symbol of $d\pi(X)$ for X in the Lie algebra $\mathfrak{g}$ of G. This allows us to define and to study the Stratonovich-Weyl symbol of $d\pi(X)$ for $X \in \mathfrak{g}$.

1. Introduction

The notion of Stratonovich-Weyl correspondence was introduced in [35] as a generalization of the classical Weyl correspondence [1]. The systematic study of the Stratonovich-Weyl correspondences began with the work of J. M. Gracia-Bondìa, J. C. Vàrilly and their co-workers (see [22], [19], [17] and [21]).

Definition 1.1 ([21]). Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a (suitably normalized) G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G, π, M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a space of (generalized) functions on M satisfying the following properties:

(1) W maps the identity operator of $\mathcal{H}$ to the constant function 1;
(2) the function $W(A^*)$ is the complex-conjugate of $W(A)$;
(3) Covariance: we have $W(\pi(g)\, A\, \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x)$;
(4) Traciality: we have

$$\int_M W(A)(x) W(B)(x)\, d\mu(x) = \mathrm{Tr}(AB)\,.$$

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For example, if G is the $(2n+1)$-dimensional Heisenberg group H_n which acts on $\mathbb{R}^{2n}$ by translations and π is the Schrödinger representation of H_n on $L^2(\mathbb{R}^n)$ then the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple $(H_n, \pi, \mathbb{R}^{2n})$ [20], [21].

When G is a compact semisimple Lie group, π a unitary irreducible representation of G on a finite dimensional Hilbert space $\mathcal{H}$ and M the coadjoint orbit of G which is associated with π by the Kostant-Kirillov method of orbits [26], a Stratonovich-Weyl correspondence for (G, π, M) was constructed in [19] by a suitable modification of the Berezin calculus on M (see also [12] and [15]).

Let us also mention that, in [17], a Stratonovich-Weyl correspondence for the massive representations of the Poincaré group was constructed. Another examples of Stratonovich-Weyl correspondences can be found in [5] and [6]. A generalization of the notion of Stratonovich-Weyl correspondence was introduced in [9].

In the present paper, we consider a connected semisimple noncompact real Lie group G with finite center. Let K be a maximal compact subgroup of G. We assume that the center of K has positive dimension. Then $M = G/K$ is a Hermitian symmetric space of the noncompact type which is diffeomorphic to a bounded symmetric domain $\mathcal{D}$. Let π_χ be a discrete series representation of G holomorphically induced from a unitary character χ of K. The representation π_χ can be realized on a Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$. The domain $\mathcal{D}$ can be quantized by the general method of quantization introduced by Berezin [7], [8]. In [14], we gave explicit formulas for the Berezin symbols of $\pi_\chi(g)$ for $g \in G$ and $d\pi_\chi(X)$ for X in the Lie algebra $\mathfrak{g}$ of G (see also [13]). The Berezin symbol of $\pi_\chi(g)$ plays a central role in the Fourier theory for G [4], [38]. On the other hand, the Berezin symbol of $d\pi_\chi(X)$ is related to the coadjoint orbit of G associated with π_χ by the Kirillov-Kostant method of orbits (see [14, Proposition 5.5]; also, see [13, Proposition 3.3]). However, for the Fourier theory of G and for physical applications, it is convenient to use Stratonovich-Weyl symbols instead of Berezin symbols [19].

Berezin quantization on $\mathcal{D}$ gives an isomorphism S_χ from the space of Hilbert-Schmidt operators on $\mathcal{H}_\chi$ (endowed with the Hilbert-Schmidt norm) onto $L^2(\mathcal{D}, \mu)$ where μ is a G-invariant measure on $\mathcal{D}$. Here, we construct a Stratonovich-Weyl correspondence W_χ for the triple $(G, \pi_\chi, \mathcal{D})$ as in the compact case [19]. In fact, if we revisit [19] in the light of [3], [2], [30], [18] and [32], then we see that W_χ is the isometric part in the polar decomposition of S_χ, that is, $W_\chi = B_\chi^{-1/2} S_\chi$ where $B_\chi = S_\chi S_\chi^*$ is the so-called Berezin transform. Note that Berezin transforms for weighted Bergman spaces on bounded symmetric domains and their spectral decompositions have been intensively studied (see for instance [36], [32], [39] and [18]).

Here, in contrast to the compact case, the operator $d\pi_\chi(X)$ is generally not of the Hilbert-Schmidt type and then $W_\chi(d\pi_\chi(X))$ is not defined a priori. In this paper, we show how to extend B_χ to a class of functions on $\mathcal{D}$ which contains the Berezin symbols $S_\chi(d\pi_\chi(X))$ for $X \in \mathfrak{g}$. This allows us to define $W_\chi(d\pi_\chi(X))$. More precisely, we show that there exists a constant $a_\chi > 0$ such that $W_\chi(d\pi_\chi(X)) = a_\chi S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}$. This result is similar to that obtained in the compact case, see [15, Proposition 5.2], and it implies that W_χ is generally not an adapted Weyl correspondence in the sense of [11].

This paper is organized as follows. In Section 2, we introduce the representation π_χ, the Berezin calculus on $\mathcal{D}$ and we review some results from [14]. In Section 3, we construct a Stratonovich-Weyl correspondence W_χ for $(G, \pi_\chi, \mathcal{D})$ as mentioned above. In Section 4, we show how to extend the Berezin transform to functions of the form $S_\chi(d\pi_\chi(u))$ where $u \in \mathcal{U}(\mathfrak{g})$. As an application, we extend W_χ to the operators $d\pi_\chi(X)$ $(X \in \mathfrak{g})$ and we determine the form of $W_\chi(d\pi_\chi(X))$ (Section 5). Finally, in Section 6, we study the case of the holomorphic discrete series of $G = SU(1,1)$.

2. Berezin quantization for discrete series representations

In this section, we first review some well-known facts on Hermitian symmetric spaces of the noncompact type and on holomorphic discrete series representations. Our main references are [23, Chapter VIII], [27, Chapter 6], [29, Chapter XII] and [34, Chapter II].

Let G be a connected semisimple noncompact real Lie group with finite center and let K be a maximal compact subgroup of G. We assume that the center of the Lie algebra of K is non trival. Then the homogeneous space G/K is a Hermitian symmetric space of the noncompact type.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{g}^c$ and $\mathfrak{k}^c$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ and G^c, K^c the corresponding complex Lie groups containing G and K, respectively. We denote by β the Killing form of $\mathfrak{g}^c$, that is, $\beta(X,Y) = \mathrm{Tr}(\operatorname{ad} X \operatorname{ad} Y)$ for X, $Y \in \mathfrak{g}^c$. Let $\mathfrak{p}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to β. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}^c$ the complexification of $\mathfrak{h}$. Let H the connected subgroup of K with Lie algebra $\mathfrak{h}$. Let Δ be the root system of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and let $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Then we have the direct decompositions $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_c} \mathfrak{g}_\alpha$ and $\mathfrak{p}^c = \sum_{\alpha\in\Delta_n} \mathfrak{g}_\alpha$ where $\mathfrak{p}^c$ denotes the complexification of $\mathfrak{p}$ and Δ_c (resp. Δ_n) denotes the set of compact (resp. noncompact) roots. We choose an ordering on Δ as in [23, p. 384], and we denote by Δ^+, Δ_c^+ and Δ_n^+ the corresponding sets of positive roots, positive compact roots and positive noncompact roots, respectively. We set $\mathfrak{p}^+ = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_{-\alpha}$. Then we have $[\mathfrak{k}^c, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$ and $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subspaces [23, Proposition 7.2.]. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we also have $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}^c$. We denote by P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, respectively.

For each $\mu \in (\mathfrak{h}^c)^*$, we denote by H_μ the element of $\mathfrak{h}^c$ satisfying $\beta(H, H_\mu) = \mu(H)$ for all $H \in \mathfrak{h}^c$. Note that if μ is real-valued on $i\mathfrak{h}$ then $iH_\mu \in \mathfrak{g}$. For μ, $\nu \in (\mathfrak{h}^c)^*$, we set $(\mu, \nu) := \beta(H_\mu, H_\nu)$.

Let θ denotes conjugation over the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X \in \mathfrak{g}^c$, we set $X^* = -\theta(X)$. We denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$ to G^c. Recall that the multiplication map $(z, k, y) \to zky$ is a diffeomorphism from $P^+ \times K^c \times P^-$ onto an open submanifold of G^c containing G [23, Lemma 7.9]. Following [29], we introduce the projections

$\zeta\colon P^+K^cP^- \to P^+$, $\kappa\colon P^+K^cP^- \to K^c$ and $\eta\colon P^+K^cP^- \to P^-$. Then the map $gK \to \log\zeta(g)$ from G/K to $\mathfrak{p}^+$ induces a diffeomorphism from G/K onto a bounded domain $\mathcal{D} \subset \mathfrak{p}^+$ [23, p. 392]. The natural action of G on G/K corresponds to the action of G on $\mathcal{D}$ given by $g\cdot Z = \log\zeta(g\exp Z)$. The G-invariant measure on $\mathcal{D}$ is $d\mu(Z) = \chi_0(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where χ_0 is the character on K^c defined by $\chi_0(k) = \mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\,k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [29]. To simplify the notation, we set $k(Z) := \kappa(\exp Z^*\exp Z)$ for $Z \in \mathcal{D}$.

We introduce the holomorphic discrete series representations of scalar type of G as follows. Let χ be a unitary character of K. We also denote by χ the extension of χ to K^c. Let us introduce the Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|^2_\chi := \int_{\mathcal{D}} |f(Z)|^2\chi(k(Z))\,c_\chi d\mu(Z) < +\infty$$

where the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} (\chi\cdot\chi_0)\big(k(Z)\big)\,d\mu_L(Z)\,.$$

Note that $\chi(k(Z)) > 0$ for all $Z \in \mathcal{D}$. Indeed, for each $Z \in \mathcal{D}$ there exists $g_Z \in G$ such that $g_Z\cdot 0 = Z$. Writing $g_Z = \exp Zky$ with $k \in K^c$ and $y \in P^-$, we have $k(Z) = (k^*)^{-1}k^{-1}$ which gives $\chi(k(Z)) = \overline{\chi(k)}^{-1}\chi(k) = |\chi(g_Z^{-1}\exp Z)|^2 > 0$.

Proposition 2.1 ([31], [27])**.** *Let $\lambda := d\chi|_{\mathfrak{h}^c}$ and $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Then $\mathcal{H}_\chi$ is nonzero if and only if $(\lambda+\delta,\alpha) < 0$ for every noncompact positive root α. In that case, $\mathcal{H}_\chi$ contains all polynomials. Moreover, the action of G on $\mathcal{H}_\chi$ defined by*

$$\pi_\chi(g)f(Z) = \chi\big(\kappa(g^{-1}\exp Z)\big)^{-1} f(g^{-1}\cdot Z)$$

is a unitary representation of G which belongs to the holomorphic discrete series of G.

In the rest of the paper, we assume that χ satisfies the preceding condition. Note that $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, we have the reproducing property $f(Z) = \langle f, e_Z\rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$, where the coherent states $e_Z \in \mathcal{H}_\chi$ are defined by $e_Z(W) = \chi(\kappa(\exp Z^*\exp W))^{-1}$ (see [29], Chapter XII for instance). Here $\langle\cdot,\cdot\rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

Now we introduce the Berezin calculus on $\mathcal{D}$ as follows. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. The Berezin (covariant) symbol of A is the function defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) = \frac{\langle A\,e_Z\,,\,e_Z\rangle_\chi}{\langle e_Z\,,\,e_Z\rangle_\chi}\,.$$

From the equality

$$\pi_\chi(g)\,e_Z = \overline{\chi\big(\kappa(g\exp Z)\big)}^{-1}\,e_{g\cdot Z} \tag{2.1}$$

for $g \in G$ and $Z \in \mathcal{D}$ (see [14, Proposition 2.2]), we deduce that, for each $Z \in \mathcal{D}$, e_Z is a smooth vector for π_χ and hence the Berezin symbol of $d\pi_\chi(X)$ ($X \in \mathfrak{g}$) is well-defined.

Also, note that if A is an operator on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z ($Z \in \mathcal{D}$) then, for each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi(\pi_\chi(g^{-1})A\pi_\chi(g))(Z) = S(A)(g \cdot Z) \tag{2.2}$$

for each $g \in G$ and $Z \in \mathcal{D}$.

In [14], we gave explicit expressions for the derived representation $d\pi_\chi$, for the Berezin symbols of $\pi_\chi(g)$ and $d\pi_\chi(X)$. In the rest of this section, we recall some results from [14].

If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the right invariant vector field on L generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$.

Let $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ be the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$. By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

Lemma 2.1 ([14])**.** *Let $X \in \mathfrak{g}^c$ and $g = z\,k\,y$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. We have*

(1) $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z)$.

(2) $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\, X))^+(k)$.

(3) $d\eta_g(X^+(g)) = (\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X))^+(y)$.

From this lemma, we deduce the following propositions (see [14] and also [29, Proposition XII.2.1]).

Proposition 2.2. *For $X \in \mathfrak{g}^c$ and $f \in \mathcal{H}_\chi$, we have*

$$d\pi_\chi(X)f(Z) = d\chi(p_{\mathfrak{k}^c}(\mathrm{Ad}((\exp Z)^{-1})\, X))\, f(Z) - (df)_Z\big(p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z}\, X)\big)\,.$$

In particular, we have

(1) *If $X \in \mathfrak{p}^+$ then $d\pi_\chi(X)f(Z) = -(df)_Z(X)$.*

(2) *If $X \in \mathfrak{k}^c$ then $d\pi_\chi(X)f(Z) = d\chi(X)f(Z) + (df)_Z([Z, X])$.*

(3) *If $X \in \mathfrak{p}^-$ then $d\pi_\chi(X)f(Z) = -d\chi([Z, X])f(Z) - \frac{1}{2}(df)_Z([Z, [Z, X]])$.*

Proposition 2.3.

(1) *Let $g \in G$. We have*

$$S_\chi\big(\pi_\chi(g)\big)(Z) = \chi\big(\kappa(\exp Z^* g^{-1} \exp Z)^{-1} \kappa(\exp Z^* \exp Z)\big)\,.$$

(2) *Let $X \in \mathfrak{g}^c$. We have*

$$S_\chi\big(d\pi_\chi(X)\big)(Z) = d\chi\big(p_{\mathfrak{k}^c}(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)\, X\big)\,.$$

In particular, for $X \in \mathfrak{k}^c$, we have

$$S_\chi\big(d\pi_\chi(X)\big)(Z) = d\chi\big(X + [\log \eta((\exp Z^* \exp Z), [X, Z]]\big)\,.$$

(3) *We can write*

$$S\big(d\pi_\chi(X)\big)(Z)=i\beta\big(\psi_\chi(Z),X\big)$$

where the map ψ_χ *defined by*

$$\psi_\chi(Z):=\mathrm{Ad}\big(\exp(-Z^*)\,\zeta(\exp Z^*\exp Z)\big)\,(-iH_\lambda)$$

is a diffeomorphism from $\mathcal{D}$ *onto the orbit* $\mathcal{O}_\chi$ *of* $-iH_\lambda\in\mathfrak{g}$ *for the adjoint action of* G.

3. Berezin transform and Stratonovich-Weyl correspondence

We retain the notation from Section 2. Also, we denote by $\mathcal{L}_2(\mathcal{H}_\chi)$ the space of the Hilbert-Schmidt operators on $\mathcal{H}_\chi$ and by μ_χ the G-invariant measure on $\mathcal{D}$ defined by $d\mu_\chi(Z)=c_\chi d\mu_0(Z)=c_\chi\chi_0(k(Z))d\mu_L(Z)$. Then the map S_χ is a bounded operator on $\mathcal{L}_2(\mathcal{H}_\chi)$ into $L^2(\mathcal{D},\mu_\chi)$ which is one-to-one and has dense range [33], [36]. It is not hard to verify that the adjoint operator $S_\chi^*\colon L^2(\mathcal{D},\mu_\chi)\to\mathcal{L}_2(\mathcal{H}_\chi)$ is given by

$$S_\chi^*F=\int_{\mathcal{D}}F(Z)\,P_Z\,d\mu_\chi(Z) \tag{3.1}$$

where P_Z is the orthogonal projection operator of $\mathcal{H}_\chi$ on the line generated by e_Z. The Berezin transform $B_\chi=S_\chi S_\chi^*$ is then the operator on $L^2(\mathcal{D},\mu_\chi)$ given by

$$B_\chi F(Z)=\int_{\mathcal{D}}F(W)\,\frac{|\langle e_Z,e_W\rangle|_\chi^2}{\langle e_Z,e_Z\rangle_\chi\langle e_W,e_W\rangle_\chi}\,d\mu_\chi(W) \tag{3.2}$$

(see, for instance, [7], [36], [39]). Note that B_χ commute with $\rho(g)$ $(g\in G)$ where ρ denotes the left-regular representation of G on $L^2(\mathcal{D},\mu_\chi)$.

Now, we introduce the polar decomposition of S_χ: $S_\chi=(S_\chi S_\chi^*)^{1/2}W=B_\chi^{1/2}W_\chi$ where $W_\chi:=B_\chi^{-1/2}S_\chi$ is a unitary operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ onto $L^2(\mathcal{D},\mu_\chi)$. The following proposition is analogous to Theorem 3 of [19].

Proposition 3.1. *The map* $W_\chi\colon\mathcal{L}_2(\mathcal{H}_\chi)\to L^2(\mathcal{D},\mu_\chi)$ *is a Stratonovich-Weyl correspondence for the triple* $(G,\pi_\chi,\mathcal{D})$.

Proof. We have to verify that the properties (1), (2) and (3) of Definition 1.1 are satisfied. Property (1) follows from the fact that $B_\chi 1=1$. Since we have the properties $S_\chi(A^*)=\overline{S_\chi(A)}$ and $S_\chi^*(\overline{F})=(S_\chi^*F)^*$, we see that B_χ hence $B_\chi^{-1/2}$ commute with complex conjugation. This gives Property (2). Finally, Property (3) is a consequence of Equality (2.2). □

In the rest of this section, we show that the Stratonovich-Weyl correspondence W_χ is related to the operator Q introduced in [32] as a natural generalization of the Weyl transform.

Let $A \in \mathcal{L}_2(\mathcal{H}_\chi)$. For $Z \in \mathcal{D}$, we have

$$\begin{aligned} A\,f(Z) &= \langle A\,f\,,\,e_Z\rangle_\chi = \langle f\,,\,A^*\,e_Z\rangle_\chi \\ &= \int_{\mathcal{D}} f(W)\overline{A^*\,e_Z(W)}\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W) \\ &= \int_{\mathcal{D}} f(W)\overline{\langle A^*\,e_Z, e_W\rangle}_\chi\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W) \\ &= \int_{\mathcal{D}} f(W)\langle A\,e_W, e_Z\rangle_\chi\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W)\,. \end{aligned}$$

This shows that the kernel of A is the function

$$k_A(Z,W) = \langle A\,e_W, e_Z\rangle_\chi \tag{3.3}$$

which is holomorphic in the variable Z and anti-holomorphic in the variable W.

Now, let $\mathcal{H}_\chi^-$ be the Hilbert space conjugate to $\mathcal{H}_\chi$, that is, the elements of $\mathcal{H}_\chi^-$ are the functions $\overline{f}$ where $f \in \mathcal{H}_\chi$ and the Hilbert norm on $\mathcal{H}_\chi^-$ is defined by $\|\overline{f}\|_{\mathcal{H}_\chi^-} = \|f\|_\chi$. We form the Hilbert space tensor product $\mathcal{H}_\chi \otimes \mathcal{H}_\chi^-$ which can be identified with $\mathcal{L}_2(\mathcal{H}_\chi)$ endowed with the Hilbert-Schmidt norm by means of the map $\mathcal{K} : A \to k_A$. In [32], the authors introduced the restriction operator $D : \mathcal{H}_\chi \otimes \mathcal{H}_\chi^- \to L^2(\mathcal{D}, \mu_\chi)$

$$k(Z,W) \to k(Z,Z)\langle e_Z, e_Z\rangle_\chi^{-1}$$

and its polar decomposition $D = |D|Q$. Then, by using (3.3), we see immediately that $S_\chi = D \circ \mathcal{K}$. Hence we can conclude that $W_\chi = Q \circ \mathcal{K}$.

4. Extension of the Berezin transform

We introduce some additional notation. Let $(E_\alpha)_{\alpha\in\Delta_n^+}$ be a basis for $\mathfrak{p}^+$ as in [23, Chapter VIII, Corollary 7.6]. In particular, we have $E_\alpha \in \mathfrak{g}_\alpha$ and $[E_\alpha, E_{-\alpha}] = \frac{2}{\alpha(H_\alpha)}H_\alpha$ for each $\alpha \in \Delta_n^+$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an enumeration of Δ_n^+. We write $Z = \sum_{k=1}^n z_k E_{\alpha_k}$ for the decomposition of $Z \in \mathfrak{p}^+$ in the basis (E_{α_k}). If f is a holomorphic function on $\mathcal{D}$, then we denote by $\partial_k f$ the partial derivative of f with respect to z_k. We say that a function $f(Z)$ on $\mathcal{D}$ is a polynomial of degree q in the variable Z if $f(\sum_{k=1}^n z_k E_{\alpha_k})$ is a polynomial of degree q in the variables $z_1, z_2, \ldots, z_n$. For Z, $W \in \mathcal{D}$, we set $l_Z(W) := \log\eta(\exp Z^*\exp W) \in \mathfrak{p}^-$. We first establish some technical lemmas.

Lemma 4.1. (1) *For Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\,e_Z(W+tV)\Big|_{t=0} = -e_Z(W)\,d\chi([l_Z(W), V])\,.$$

(2) *For Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\,l_Z(W+tV)\Big|_{t=0} = \frac{1}{2}[l_Z(W), [l_Z(W), V]]\,.$$

(3) *The function* $(\partial_{k_1}\partial_{k_2}\ldots\partial_{k_q}\,e_Z)(W)$ *is of the form* $e_Z(W)P(l_Z(W))$ *where* P *is a polynomial of degree* $\leq q$.

(4) *For each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, *the operator* $d\pi_\chi(X_1X_2\ldots X_q)$ *is a sum of terms of the form* $P(Z)\partial_{k_1}\partial_{k_2}\ldots\partial_{k_q}$ *where* P *is a polynomial in* Z *of degree* $\leq 2q$.

Proof. By (2) of Lemma 2.1, we have

$$\begin{aligned}\frac{d}{dt}\,e_Z(W+tV)\big|_{t=0} &= \frac{d}{dt}\,\chi^{-1}\big(\kappa(\exp Z^*\exp W\exp tV)\big)\big|_{t=0}\\ &= d\chi^{-1}_{\kappa(\exp Z^*\exp W)}d\kappa_{\exp Z^*\exp W}\big(\big(\operatorname{Ad}(\exp Z^*\exp W)\,V\big)^+(\exp Z^*\exp W)\big)\\ &= -\chi^{-1}(\kappa(\exp Z^*\exp W))\,d\chi\big(p_{\mathfrak{k}^c}\big(\operatorname{Ad}\big(\kappa(\exp Z^*\exp W)\eta(\exp Z^*\exp W)\big)V\big)\big)\,.\end{aligned}$$

Since $d\chi(p_{\mathfrak{k}^c}(\operatorname{Ad}(k)X)) = d\chi(\operatorname{Ad}(k)p_{\mathfrak{k}^c}(X)) = d\chi(p_{\mathfrak{k}^c}(X))$ for each $k \in K^c$ and each $X \in \mathfrak{g}^c$, we obtain

$$\begin{aligned}\frac{d}{dt}\,e_Z(W+tV)\big|_{t=0} &= -e_Z(W)d\chi\left(p_{\mathfrak{k}^c}\big(\operatorname{Ad}\big(\eta(\exp Z^*\exp W)\big)V\big)\right)\\ &= -e_Z(W)d\chi\left([\log\eta(\exp Z^*\exp W), V]\right)\,.\end{aligned}$$

Then Statement (1) is proved. Similarly, by using (3) of Lemma 2.1, we have

$$\begin{aligned}&\frac{d}{dt}\,l_Z(W+tV)\big|_{t=0}\\ &\quad= d\log_{\eta(\exp Z^*\exp W)}\,d\eta_{\exp Z^*\exp W}\,\big((\operatorname{Ad}(\exp Z^*)V)^+(\exp Z^*\exp W)\big)\\ &\quad= \operatorname{Ad}\kappa(\exp Z^*\exp W)^{-1}\,p_{\mathfrak{p}^-}\,\big(\operatorname{Ad}\big(\zeta(\exp Z^*\exp W)^{-1}\exp Z^*\big)V\big)\\ &\quad= p_{\mathfrak{p}^-}\,\big(\operatorname{Ad}\big(\eta(\exp Z^*\exp W)\big)V\big)\\ &\quad= \frac{1}{2}\,[\log\eta(\exp Z^*\exp W), [\log\eta(\exp Z^*\exp W), V]]\end{aligned}$$

and hence we have proved (2). Now, by induction on q, we easily obtain (3). Finally, (4) is a consequence of Proposition 2.3. □

The following lemma is an immediate consequence of Lemma 4.1 (see also [16]).

Lemma 4.2. *Each holomorphic differential operator on* $\mathcal{D}$ *with polynomial coefficients has Berezin symbol. In particular, for each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))$ *is well-defined and is a sum of terms of the form* $P(Z)Q(l_Z(Z))$ *where* P *is a polynomial of degree* $\leq 2q$ *and* Q *is a polynomial of degree* $\leq q$.

Lemma 4.3. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_r$ *be a subset of* Δ_n^+ *consisting of strongly orthogonal roots.*

(1) *Let* $\tilde{\chi}$ *be a character (non necessarily unitary) on* K *and* $\tilde{\lambda} = d\tilde{\chi}|_{\mathfrak{h}^c}$. *Then* $(\tilde{\lambda}, \gamma_k)$ *does not depend on* $k = 1, 2, \ldots, r$.

(2) *In particular, let* $\lambda_0 := d\chi_0|_{\mathfrak{h}^c}$. *Then* $q_\chi = -2\frac{(\lambda_0+\lambda,\gamma_k)}{(\gamma_k,\gamma_k)}$ *does not depend on* $k = 1, 2, \ldots, r$.

Proof. (1) By [28, Lemma 2.1], each γ_r is of the form $\gamma_r=\mu_1+\sum_{i\geq 2}n_i\mu_i$ where μ_1 is the unique noncompact simple root and the μ_i $(i\geq 2)$ are the compact simple roots. Since $(\tilde{\lambda},\mu_i)=0$ for each $i\geq 2$, we have $(\tilde{\lambda},\gamma_k)=(\tilde{\lambda},\mu_1)$ for each k.
(2) By [28, Theorem 2], (γ_k,γ_k) does not depend on k. The result then follows from (1). □

We are now in position to extend the Berezin transform to a class of Berezin symbols of unbounded operators. Note that, by fixing an Iwasawa decomposition $G=NAK$, we get a smooth section $G/K\rightarrow NA\subset G$ and then we obtain a smooth section $\mathcal{D}\rightarrow G$, $Z\rightarrow g_Z$.

Proposition 4.1. *If $q\leq q_\chi$ then for each $X_1,X_2,\ldots,X_q\in\mathfrak{g}^c$, the Berezin transform of $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))$ is well-defined.*

Proof. First, note that if we change variables $W\rightarrow g_Z\cdot W$ in the integral (3.2) then by (2.1) we obtain

$$\begin{aligned}(B_\chi F)(Z)&=\int_{\mathcal{D}}F(g_Z\cdot W)\langle e_W,e_W\rangle_\chi^{-1}\,d\mu_\chi(W)\\ &=\int_{\mathcal{D}}F(g_Z\cdot W)\,c_\chi(\chi\cdot\chi_0)(k(W))\,d\mu_L(W)\,.\end{aligned}\tag{4.1}$$

In particular, if $F(W)=S_\chi(d\pi_\chi(X_1X_2\ldots X_q))(W)$ then, by (2.2), we have $F(g_Z\cdot W)=S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$ where $Y_k:=\mathrm{Ad}(g_Z^{-1})X_k$ for $k=1,2,\ldots,q$.

We will show that, under the condition that $q\leq q_\chi$, the function

$$W\rightarrow S_\chi\big(d\pi_\chi(Y_1Y_2\ldots Y_q)\big)(W)(\chi\cdot\chi_0)\big(k(W)\big)$$

is bounded hence integrable on $\mathcal{D}$. Recall that $S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$ is the sum of terms of the form $P(W)Q(\log\eta(\exp W^*\exp W))$ where P is a polynomial and Q is a polynomial of degree $\leq q$.

Let $\gamma_1,\gamma_2,\ldots,\gamma_r$ as in Lemma 4.3. Then each $W\in\mathcal{D}$ can be written as $W=\mathrm{Ad}(k)\big(\sum_{k=1}^r t_sE_{\gamma_s}\big)$ for $k\in K$ and $-1<t_s<1$, $1\leq s\leq r$ (see for instance [23, Chapter VIII]). From matrix calculations in the group $SL(2,\mathbb{C})$ and strongly orthogonality of the roots γ_s, we have

$$\log\eta(\exp W^*\exp W)=\mathrm{Ad}(k)\Big(-\sum_{s=1}^r\frac{t_s}{1-t_s^2}\,E_{-\gamma_s}\Big)\tag{4.2}$$

and

$$k(W)=\kappa(\exp W^*\exp W)=k\,\exp\Big(\sum_{s=1}^r\log\frac{1}{1-t_s^2}\,[E_{\gamma_s},E_{-\gamma_s}]\Big)\,k^{-1}\,.\tag{4.3}$$

Then

$$(\chi\cdot\chi_0)\big(k(W)\big)=\prod_{s=1}^r(1-t_s^2)^{-(\lambda+\lambda_0)([E_{\gamma_s},E_{-\gamma_s}])}$$

and, since we have

$$-(\lambda+\lambda_0)([E_{\gamma_s},E_{-\gamma_s}])=-2\,\frac{(\lambda+\lambda_0)(H_{\gamma_s})}{\gamma_s(H_{\gamma_s})}=-2\frac{(\lambda_0+\lambda,\gamma_s)}{(\gamma_s,\gamma_s)}=q_\chi\,,$$

we obtain

$$(\chi\cdot\chi_0)\big(k(W)\big)=\prod_{s=1}^{r}(1-t_s^2)^{q_\chi}\,. \tag{4.4}$$

Hence we see that the condition $q\le q_\chi$ guarantees that the functions

$$W\to P(W)\,Q\big(l_W(W)\big)(\chi\cdot\chi_0)\big(k(W)\big)$$

are bounded on $\mathcal{D}$. This finishes the proof. □

Remarks.

(1) Since we have

$$\int_{\mathcal{D}}(\chi\cdot\chi_0)\big(k(W)\big)\,d\mu_L(W)<+\infty\,,$$

we see immediately from (4.4) that $q_\chi\ge 0$.

(2) By [13, Lemma 5.2], we have $\chi(k(Z))\le 1$ for each $Z\in\mathcal{D}$ with equality if and only if $Z=0$. This implies that $-\lambda([E_{\gamma_s},E_{-\gamma_s}])>0$ for each $s=1,2,\ldots,r$.

(3) An extension of the Berezin transform to another class of functions on $\mathcal{D}$ is given in [36].

5. Stratonovich-Weyl symbols of derived representation operators

When $q_\chi\ge 1$, the Berezin transform of $S_\chi(d\pi_\chi(X))$ $(X\in\mathfrak{g}^c)$ is well-defined by Proposition 4.1. In this section, we determine the form of $B_\chi S_\chi(d\pi_\chi(X))$ and we show how to extend the Stratonovich-Weyl correspondence to the operators $d\pi_\chi(X)$ $(X\in\mathfrak{g}^c)$. To this aim, we first study the linear form b_χ defined on $\mathfrak{g}^c$ by

$$b_\chi(X):=B_\chi S_\chi\big(d\pi_\chi(X)\big)(0)=\int_{\mathcal{D}}S_\chi\big(d\pi_\chi(X)\big)(Z)\chi\big(k(Z)\big)\,d\mu_\chi(Z)\,. \tag{5.1}$$

Proposition 5.1. *There exists a real number $a_\chi\ge 1$ such that $b_\chi(X)=a_\chi\lambda(p_{\mathfrak{h}^c}(X))$ for each $X\in\mathfrak{g}^c$. Here $p_{\mathfrak{h}^c}$ denotes the projection operator from $\mathfrak{g}^c$ onto $\mathfrak{h}^c$ associated with the decomposition $\mathfrak{g}^c=\mathfrak{h}^c\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$.*

Proof. For each $k\in K$ and each $Z\in\mathcal{D}$, we have $\pi_\chi(k)e_Z=\chi(k)e_{k\cdot Z}$ and then $\langle e_{k\cdot Z},e_{k\cdot Z}\rangle_\chi=\langle e_Z,e_Z\rangle_\chi$. Thus, by changing variables $Z\to k^{-1}\cdot Z$ in the integral (5.1) and by using the fact that

$$S_\chi\big(d\pi_\chi(X)\big)(k^{-1}\cdot Z)=S_\chi\big(d\pi_\chi(\mathrm{Ad}(k)X)\big)(Z)\,,$$

we get

$$b_\chi(X)=b_\chi\big(\,\mathrm{Ad}(k)X\big) \tag{5.2}$$

for each $k\in K$ and each $X\in\mathfrak{g}^c$. Specializing to $X=E_\alpha$ $(\alpha\in\Delta)$ and $k=\exp Y$ where $Y\in\mathfrak{h}$ and noting that $\mathrm{Ad}(k)E_\alpha=e^{\alpha(Y)}E_\alpha$, we find that $b_\chi(E_\alpha)=0$ for each $\alpha\in\Delta$.

On the other hand, observe that, for each $X\in\mathfrak{g}$,

$$\overline{b_\chi(X)}=B_\chi S_\chi\big(d\pi_\chi(X^*)\big)(0)=b_\chi(X^*)=-b_\chi(X)$$

and then $b_\chi(X) \in i\mathbb{R}$. Now, introduce the element $H_{b_\chi} \in \mathfrak{k}^c$ satisfying $b_\chi(Y) = \beta(Y, H_{b_\chi})$ for each $Y \in \mathfrak{k}^c$. Then $H_{b_\chi} \in i\mathfrak{k}$. By (5.2), we have $\mathrm{Ad}(k)H_{b_\chi} = H_{b_\chi}$ for each $k \in K$. This implies that iH_{b_χ} lies in the center of $\mathfrak{k}$. Since the center of $\mathfrak{k}$ is one-dimensional (see for instance [24]) and contains iH_λ, there exists a real number a_χ such that $iH_{b_\chi} = a_\chi iH_\lambda$. Thus we have $b_\chi = a_\chi\lambda$ on $\mathfrak{h}^c$. Hence, we have obtained that $b_\chi(X) = a_\chi\lambda(p_{\mathfrak{h}^c}(X))$ for each $X \in \mathfrak{g}^c$. It remains to show that $a_\chi \geq 1$. To this goal, we consider the function φ_χ defined on $\mathcal{D}$ by $\varphi_\chi(Z) = S_\chi(d\pi_\chi(H_\lambda))(Z)$. By Proposition 2.3, we have

$$\varphi_\chi(Z) = \lambda(H_\lambda) + \lambda([\log\eta(\exp Z^*\exp Z), [H_\lambda, Z]])\,.$$

Moreover, since iH_λ is central in $\mathfrak{k}$, we have $\varphi_\chi(\mathrm{Ad}(k)Z) = \varphi_\chi(Z)$ for each $k \in K$ and $Z \in \mathcal{D}$.

As in the proof of Proposition 4.1 we write each $Z \in \mathcal{D}$ as $Z = \mathrm{Ad}(k)\left(\sum_{s=1}^r t_s E_{\gamma_s}\right)$ with $k \in K$ and $-1 < t_s < 1$ for $s = 1, 2, \ldots, r$. Then, for each $Z \in \mathcal{D}$, we have

$$\begin{aligned}\varphi_\chi(Z) &= \lambda(H_\lambda) + \lambda\Big(\Big[-\sum_{s=1}^r \frac{t_s}{1-t_s^2}E_{-\gamma_s}, [H_\lambda, \sum_{s=1}^r t_s E_{\gamma_s}]\Big]\Big)\\ &=\lambda(H_\lambda) + 2\sum_{s=1}^r \frac{t_s^2}{1-t_s^2}\frac{(\gamma_s,\lambda)^2}{(\gamma_s,\gamma_s)} \geq \lambda(H_\lambda)\end{aligned}$$

Thus

$$a_\chi\lambda(H_\lambda) = b_\chi(H_\lambda) = \int_{\mathcal{D}} \varphi_\chi(Z)\chi\big(k(Z)\big)\,d\mu_\chi(Z) \geq \lambda(H_\lambda)\,.$$

Hence $a_\chi \geq 1$. □

Proposition 5.2. *With the notation of Proposition 5.1, for each $X \in \mathfrak{g}^c$, we have $B_\chi S_\chi(d\pi_\chi(X)) = a_\chi S_\chi(d\pi_\chi(X))$.*

Proof. Applying successively Equality (4.1), Proposition 5.1, Proposition 2.3 and Equality (2.2), we have

$$\begin{aligned}B_\chi S_\chi(d\pi_\chi(X))(Z) &= B_\chi S_\chi\big(d\pi_\chi(\mathrm{Ad}(g_Z^{-1})X)\big)(0)\\ &= a_\chi\lambda\big(p_{\mathfrak{h}^c}(\mathrm{Ad}(g_Z^{-1})X)\big) = a_\chi S_\chi\big(d\pi_\chi(\mathrm{Ad}(g_Z^{-1})X)\big)(0)\\ &= a_\chi S_\chi\big(d\pi_\chi(X)\big)(g_Z\cdot 0) = a_\chi S_\chi\big(d\pi_\chi(X)\big)(Z)\end{aligned}$$

for each $Z \in \mathcal{D}$ and each $X \in \mathfrak{g}^c$. □

Consequently, we can define $B_\chi^{-1/2}$ on the space of functions of the form $S_\chi(d\pi_\chi(X))$ and W_χ on the space $\{d\pi_\chi(X) : X \in \mathfrak{g}^c\}$. Moreover, we have $W_\chi(d\pi_\chi(X)) = a_\chi^{-1/2}S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}^c$.

In [14], we showed that S_χ is adapted to π_χ in the sense that the linear form $X \to -iS_\chi(d\pi_\chi(X))$ lies in the coadjoint orbit of G associated with π_χ by the method of orbits (see also Proposition 2.3). In general, we have $a_\chi \neq 1$ (see for example Section 6) and then W_χ is not adapted to π_χ. However, the following proposition shows that W_χ is 'asymptotically adapted'.

Proposition 5.3. *We have* $\lim_{m\to+\infty} a_{\chi^m} = 1$.

Proof. Here we use the same notation as in the proofs of Proposition 4.1 and Proposition 5.1. We have

$$a_{\chi^m} = \frac{1}{(m\lambda, m\lambda)} \int_{\mathcal{D}} \varphi_{\chi^m}(Z)(\chi^m \cdot \chi_0)\big(k(Z)\big)\, c_{\chi^m} d\mu_L(Z)\,.$$

Then

$$a_{\chi^m} - 1 = \int_{\mathcal{D}} \frac{\varphi_{\chi^m}(Z) - (m\lambda, m\lambda)}{(m\lambda, m\lambda)}\, (\chi^m \cdot \chi_0)\big(k(Z)\big)\, c_{\chi^m} d\mu_L(Z)\,.$$

Changing variables $Z \to Z/\sqrt{m}$ in this integral, we get

$$a_{\chi^m} - 1 = m^{-n} c_{\chi^m} \int_{\sqrt{m}\mathcal{D}} I_m(Z)\, d\mu_L(Z)$$

where we have put

$$I_m(Z) := \frac{\varphi_{\chi^m}(Z/\sqrt{m}) - (m\lambda, m\lambda)}{(m\lambda, m\lambda)}\, (\chi^m \cdot \chi_0)\big(k(Z/\sqrt{m})\big)\,.$$

By [13, Lemma 5.3], we have $\lim_{m\to+\infty} m^{-n} c_{\chi^m} = \pi^{-n}$. On the other hand, we have

$$I_m(Z) = \Big(\sum_{s=1}^{r} 2\frac{(\gamma_s, \lambda)^2}{(\lambda, \lambda)(\gamma_s, \gamma_s)}\, \frac{(t_s/\sqrt{m})^2}{1 - (t_s/\sqrt{m})^2}\Big) \times \prod_{s=1}^{r} (1 - (t_s/\sqrt{m})^2)^{-(\lambda_0 + m\lambda)([E_{\gamma_s}, E_{-\gamma_s}])}$$

where $|t_s| < \sqrt{m}$ for each s and we see that $\lim_{m\to+\infty} I_m(Z) = 0$. In order to obtain the desired result, it suffices to verify that the Lebesgue dominated convergence theorem can be applied. This can be done as follows. Recall that we have $-\lambda([E_{\gamma_s}, E_{-\gamma_s}]) > 0$ for each $s = 1, 2, \ldots, r$. Then we fix m_0 so that we have

$$-m\lambda([E_{\gamma_s}, E_{-\gamma_s}]) - 1 \geq -\frac{m}{2}\lambda([E_{\gamma_s}, E_{-\gamma_s}])$$

for each $m \geq m_0$ and each $s = 1, 2, \ldots, r$. Thus for each $m \geq m_0$ and each $Z \in \sqrt{m}\mathcal{D}$, we have

$$\begin{aligned} I_m(Z) &\leq \sum_{s=1}^{r} 2\frac{(\gamma_s, \lambda)^2}{(\lambda, \lambda)(\gamma_s, \gamma_s)} \prod_{s=1}^{r} (1 - (t_s/\sqrt{m})^2)^{-(\lambda_0 + m\lambda)([E_{\gamma_s}, E_{-\gamma_s}]) - 1} \\ &\leq C \prod_{s=1}^{r} (1 - (t_s/\sqrt{m})^2)^{-\frac{m}{2}\lambda([E_{\gamma_s}, E_{-\gamma_s}])} \\ &\leq C \exp\Big(\sum_{s=1}^{r} \frac{1}{2}\lambda([E_{\gamma_s}, E_{-\gamma_s}]) t_s^2\Big) \end{aligned}$$

where $C > 0$ is a constant which does not depend on m. Hence we obtain the estimate

$$I_m(Z) \leq C e^{-D|Z|^2}$$

where $D > 0$ is a constant and $|\cdot|$ is an Euclidean norm on $\mathcal{P}^+$. This ends the proof. □

6. Example

In this section, we consider the case of the holomorphic discrete series of $SU(1,1)$ (see [10]). We take

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1, \quad a, b \in \mathbb{C} \right\}$$

and

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}$ of G has basis

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and its complexification $\mathfrak{g}^c$ is $sl(2,\mathbb{C})$. Then we have $G^c = SL(2,\mathbb{C})$ and

$$K^c = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad a \in \mathbb{C} \setminus (0) \right\}.$$

The conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ is given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\bar{a} & \bar{c} \\ \bar{b} & -\bar{d} \end{pmatrix}$$

and we have $X^* = -\theta(X)$ for $X \in \mathfrak{g}^c$.

The root system of $\mathfrak{g}^c = sl(2,\mathbb{C})$ relative to $\mathfrak{k}^c$ consists in the two noncompact roots α and $-\alpha$ where $\alpha(iu_3) = 1$. The corresponding root spaces are

$$\mathfrak{g}_\alpha = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-\alpha} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We say that a root is positive if it is positive on $iu_3 \in i\mathfrak{h}$. Then α is the positive root and $\mathfrak{p}^+ = \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \mathfrak{g}_{-\alpha}$. The corresponding groups are

$$P^+ = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}, \qquad P^- = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C} \right\}.$$

In the rest of this section, we identify $\mathfrak{p}^+$ to $\mathbb{C}$ by means of the map

$$z \to Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}.$$

Each element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ such that $d \neq 0$ has the following $P^+K^cP^-$-decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}.$$

In particular we have $G \subset P^+K^cP^-$.

The map $gK = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} K \in G/K \to \log \zeta(g) = b/\bar{a}$ is then a diffeomorphism from G/K onto the unit disk $D = \{z \in \mathbb{C} : |z| = 1\}$ and we can verify that the natural action of G on G/K corresponds to the action of G on D by fractional linear transformations defined by

$$g \cdot z = \frac{az+b}{\bar{b}z+\bar{a}}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ z \in D.$$

Note that the map

$$z \to g_z := \frac{1}{\sqrt{1-z\bar{z}}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$$

is a section for the action of G on D, that is, we have $g_z \cdot 0 = z$ for each $z \in D$. One can easily verify that a G-invariant measure on D is $d\mu(z) = (1-z\bar{z})^{-2}\, d\mu_L(z)$ where $d\mu_L(z) := dx\, dy$ denotes the Lebesgue measure on D ($z = x+iy$, $x, y \in \mathbb{R}$).

Now, we fix an integer m and we consider the unitary character χ_m of K defined by

$$\chi_m \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{-im\theta}\,.$$

We denote also by χ_m the extension of χ_m to K^c. We obtain immediately

$$\chi_m\big(\kappa(\exp Z^* \exp Z)\big) = (1-z\bar{z})^m\,.$$

The space $\mathcal{H}_{\chi_m}$ is the Hilbert space of holomorphic functions f such that

$$\|f\|_m^2 := \int_D |f(z)|^2\, (1-z\bar{z})^{m-2}\, \tfrac{m-1}{\pi}\, dxdy < +\infty\,. \tag{6.1}$$

Let $\lambda_m = d\chi_m$. By Proposition 2.1, $\mathcal{H}_{\chi_m}$ is nonzero if and only if the condition

$$(\lambda_m + \tfrac{1}{2}\alpha, \alpha) > 0$$

holds. Since $\lambda_m = -\frac{m}{2}\alpha$, this condition reads $\frac{1-m}{2}\,(\alpha,\alpha) < 0$ and, as the restriction of β to $i\mathfrak{k}$ is positive definite, it is equivalent to $m \geq 2$.

Also, note that the normalization of the measure in (6.1) is taken so that $\|1\|_m = 1$.

For each $m \geq 2$, the representation π_m of $G = SU(1,1)$ corresponding to m is realized in $\mathcal{H}_{\chi_m}$ as

$$\begin{aligned}\big(\pi_m(g)\big)f(z) &= \chi_m^{-1}\big(\kappa(g^{-1}\exp Z)\big)\, f(g^{-1}\cdot z)\\ &= (-\bar{b}z+a)^{-m}\, f(g^{-1}\cdot z)\end{aligned}$$

for $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$, $f \in \mathcal{H}_{\chi_m}$ and $z \in D$.

One can easily show that the family $f_p(z) := \binom{m+p-1}{p}^{1/2} z^p$ is an orthonormal basis for $\mathcal{H}_{\chi_m}$ (see [29, p. 11], for instance). From this, we see that the coherent states

$$e_z(w) = \chi_m(\kappa(\exp Z^* \exp W)^{-1}) = (1-\bar{z}w)^{-m} = \sum_{p\geq 0} \overline{f_p(z)} f_p(w)$$

satisfy the reproducing property $\langle f, e_z^m\rangle_m = f(z)$ for each $f \in \mathcal{H}_{\chi_m}$ and each $z \in D$.

Here we obtain the following formula for the Berezin symbol of $\pi_m(g)$ for $g \in G$

$$S_m\big(\pi_m(g)\big)(z) = \frac{(\pi_m(g)e_z)(z)}{e_z(z)} = \frac{(1-z\bar{z})^m}{(a-\bar{b}z+b\bar{z}-\bar{a}z\bar{z})^m}, \qquad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

Moreover, since $d\pi_m$ is given by

$$\begin{aligned} d\pi_m\,(u_1)f(z) &= \frac{m}{2}\,i\,z\,f(z) + \frac{1}{2}\,i\,(z^2+1)\,f'(z) \\ d\pi_m\,(u_2)f(z) &= \frac{m}{2}\,z\,f(z) + \frac{1}{2}\,(z^2-1)\,f'(z) \\ d\pi_m\,(u_3)f(z) &= \frac{m}{2}\,i\,f(z) + izf'(z) \end{aligned}$$

we get

$$\begin{aligned} S_m\big(d\pi_m\,(u_1)\big)\,(z) &= i\frac{m}{2}\,\frac{z+\bar{z}}{1-z\bar{z}} \\ S_m\big(d\pi_m\,(u_2)\big)\,(z) &= \frac{m}{2}\,\frac{z-\bar{z}}{1-z\bar{z}} \\ S_m\big(d\pi_m\,(u_3)\big)\,(z) &= i\frac{m}{2}\,\frac{1+z\bar{z}}{1-z\bar{z}}\,. \end{aligned}$$

From this we deduce that $S_m(d\pi_m(X))(z) = i\beta(X, \psi_m(z))$ where the map ψ_m is defined by

$$\psi_m(z) := \frac{m}{8}i\begin{pmatrix} \frac{1+z\bar{z}}{1-z\bar{z}} & -\frac{2z}{1-z\bar{z}} \\ \frac{2\bar{z}}{1-z\bar{z}} & -\frac{1+z\bar{z}}{1-z\bar{z}} \end{pmatrix}.$$

Note that $\psi_m(0) = -iH_m$ where H_m is the coroot vector of λ_m and that $\psi_m(z) = \mathrm{Ad}(g_z)\,(-iH_m)$. Then ψ_m is a diffeomorphism from D onto the orbit of $-iH_m$ under the adjoint action of G.

Now, we turn to the Berezin transform B_m. Here we have

$$B_m(f)(z) = \int_D F(w)\,\frac{|1-\bar{z}w|^4}{(1-z\bar{z})^2}\,(1-w\bar{w})^{m-2}\,\tfrac{m-1}{\pi}d\mu_L(w)\,. \tag{6.2}$$

Let us compute q_{χ_m} (see Section 4). We have

$$q_{\chi_m} = -2\,\frac{(d\chi_0+\lambda_m,\alpha)}{(\alpha,\alpha)} = -2(1-\frac{m}{2}) = m-2$$

and Proposition 4.1 asserts that if $q \leq q_{\chi_m}$ then for each $X_1, X_2, \ldots, X_q$ in $\mathfrak{g}^c$, the Berezin transform of $S_m(d\pi_m(X_1X_2\ldots X_q))$ is well-defined. Here, this can be directly verified as follows. By using the formulas for $d\pi_m$ given above, we immediately see that $d\pi_m(X_1X_2\ldots X_q)$ is a linear combination of the differential operators $D_{p,r} := z^p(\frac{d}{dz})^r$ where $r \leq q$. By differentiating $e_z(w) = (1-\bar{z}w)^{-m}$, we get

$$S_m(D_{p,r})(w) = m(m+1)\ldots(m+r-1)\,w^p\bar{w}^r(1-w\bar{w})^{-r}\,.$$

Taking formula (6.2) into account, we see that the Berezin transform of $S_m(D_{p,r})$ is well-defined. Hence the result.

Now, we want to compute the constant a_{χ_m} for $m > 2$ (see Section 5). To this aim, we apply the equality $(B_m F)(0) = a_{\chi_m} F(0)$ to the function

$$F(Z) := S_m\big(d\pi_m(iu_3)\big)(z) = -\frac{m}{2}\,\frac{1+z\bar{z}}{1-z\bar{z}}\,.$$

We then obtain $(B_m F)(0) = -m^2/2(m-2)$ and hence we find that $a_{\chi_m} = m/m-2$. In particular, we have $\lim_{m\to+\infty} a_{\chi^m} = 1$, in accordance with Proposition 5.3.

Finally, let us mention that the computation of a_{χ_m} can be performed similarly when $G = SU(p,q)$, $K = S(U(p)\times U(q))$ and χ_m is the unitary character of K defined by

$$\chi_m \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = (\text{Det } A)^{-m}\,.$$

In that case, by adapting some methods from [25], we find that $a_{\chi_m} = m/m-p-q$ for $m > p+q$.

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