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STRONG CONVERGENCE OF AN ITERATIVE METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a general iterative scheme to investigate the problem of finding a common element of the fixed point set of a strict pseudocontraction and the solution set of a variational inequality problem for a relaxed cocoercive mapping by viscosity approximate methods. Strong convergence theorems are established in a real Hilbert space.

1. Introduction and preliminaries

Variational inequalities introduced by Stampacchia [16] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, see [4]–[22] and references therein. In this paper, we consider the problem of approximation of solutions of the classical variational inequality problem by iterative methods.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, C a nonempty closed convex subset of H and $A\colon C\to H$ a nonlinear mapping.

Recall the following definitions.

(a) A is said to be monotone if

$$\langle Ax-Ay, x-y\rangle \geq 0\,,\quad \forall\, x,y\in C\,.$$

(b) A is said to be ν-strongly monotone if there exists a constant $\nu>0$ such that

$$\langle Ax-Ay, x-y\rangle \geq \nu\|x-y\|^2\,,\quad \forall\, x,y\in C\,.$$

(c) A is said to be μ-cocoercive if there exists a constant $\mu>0$ such that

$$\langle Ax-Ay, x-y\rangle \geq \mu\|Ax-Ay\|^2\,,\quad \forall\, x,y\in C\,.$$

Clearly, every μ-cocoercive mapping is $1/\mu$-Lipschitz continuous.

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(d) A is said to be relaxed μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2 , \quad \forall\, x, y \in C .$$

(e) A is said to be relaxed (μ, ν)-cocoercive if there exist two constants $\mu, \nu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)\|Ax - Ay\|^2 + \nu\|x - y\|^2 , \quad \forall\, x, y \in C .$$

The classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 , \quad \forall\, v \in C . \tag{1.1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of the problem (1.1).

It is easy to see that an element $u \in C$ is a solution to the problem (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where P_C denotes the metric projection from H onto C, λ is a positive constant and I is the identity mapping.

Let $T: C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of the mapping T. Recall the following definitions.

(1) T is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\| , \quad \forall\, x, y \in C .$$

(2) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| , \quad \forall\, x, y \in C .$$

(3) T is said to be strictly pseudo-contractive with the coefficient $k \in (0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 , \quad \forall\, x, y \in C .$$

For such a case, T is also said to be a k-strict pseudo-contraction.

(4) T is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 , \quad \forall\, x, y \in C .$$

Clearly, the class of strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of strict pseudo-contractions; see, for example [1, 24].

The class of strict pseudo-contractions which was introduced by Browder and Petryshyn [2] is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of fixed points for strict pseudo-contractions. Recently, Zhou [23] considered a convex combination method to study strict pseudo-contractions. More precisely, take $t \in (0, 1)$ and define a mapping S_t by

$$S_t x = tx + (1 - t)Tx , \quad \forall\, x \in C ,$$

where T is a strict pseudo-contraction. Under appropriate restrictions on t, it is proved that the mapping S_t is nonexpansive. Therefore, the techniques of studying nonexpansive mappings can be applied to study more general strict pseudo-contractions.

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoercive mapping, Takahashi and Toyoda [19] introduced the following iterative process:

$$x_0 \in C\,, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C (x_n - \lambda_n A x_n)\,, \quad \forall\, n \geq 0\,,$$

where A is α-cocoercive mapping and S is a nonexpansive mapping with a fixed point. They showed that if $F(S) \cap VI(C,A)$ is nonempty then the sequence $\{x_n\}$ converges weakly to some $z \in F(S) \cap VI(C,A)$ under some restrictions imposed on the sequence $\{\alpha_n\}$ and $\{\lambda_n\}$; see [18] for more details.

On the other hand, for solving the variational inequality problem in the finite--dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous and $VI(C,A)$ is nonempty, Korpelevich [9] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C\,, \\ y_n = P_C(x_n - \lambda A x_n)\,, \\ x_{n+1} = P_C(x_n - \lambda A y_n)\,, \quad \forall\, n \geq 0\,, \end{cases}$$

where $\lambda \in (0, 1/k)$. He proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(C,A)$.

To obtain strong convergence theorems, Iiduka and Takahashi [8] proposed the following iterative scheme:

$$x_0 \in C\,, \quad x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C (x_n - \lambda_n A x_n)\,, \quad \forall\, n \geq 0\,,$$

where A is α-cocoercive mapping and S is a nonexpansive mapping with a fixed point. They showed that if $F(S) \cap VI(C,A)$ is nonempty then the sequence $\{x_n\}$ converges strongly to some $z \in F(S) \cap VI(C,A)$ under some restrictions imposed on the sequence $\{\alpha_n\}$ and $\{\lambda_n\}$; see [8] for more details.

Recently, Yao and Yao [22], further improved Iiduka and Takahashi [9]' results by considering the following iterative process

$$\begin{cases} x_0 \in C\,, \\ y_n = P_C(x_n - \lambda_n A x_n)\,, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C (I - \lambda_n A) y_n\,, \quad \forall\, n \geq 0\,, \end{cases}$$

where A is α-cocoercive mapping and S is a nonexpansive mapping with a fixed point. A strong convergence theorem was also established in the framework of Hilbert space under some restrictions imposed on the sequence $\{\alpha_n\}$ and $\{\lambda_n\}$; see [22] for more details.

In this paper, motivated by the research working going on in this direction, we continue to study the variational inequality problem and the fixed point problem by the viscosity approximation method which was first considered by Moudafi [10]. To be more precise, we introduce a general iterative process to find a common element of the set of fixed points of a strict pseudocontraction and the set of solutions of the variational inequality problem (1.1) for a relaxed cocoercive mapping in a real Hilbert space. Strong convergence of the purposed iterative process is obtained.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([23]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T\colon C \to C$ a k-strict pseudo-contraction with a fixed point. Define $S\colon C \to C$ by $Sx = kx + (1-k)Tx$ for each $x \in C$. Then S is nonexpansive with $F(S) = F(T)$.*

The following lemma is a corollary of Bruck's result in [3].

Lemma 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1 and T_2 be two nonexpansive mappings from C into C with a common fixed point. Define a mapping $S\colon C \to C$ by*

$$Sx = \lambda T_1 x + (1-\lambda) T_2 x\,, \quad \forall\, x \in C\,,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$

Lemma 1.3 ([2]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero*

Lemma 1.4 ([17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1\,.$$

Suppose $x_{n+1} = (1-\beta_n) y_n + \beta_n x_n$ for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0,.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5 ([21]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n\,,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H and $A\colon C \to H$ a relaxed (μ,ν)-cocoercive and L-Lipschitz continuous mapping. Let $f\colon C \to C$ be a contraction with the coefficient $\alpha \in (0,1)$ and $T\colon C \to C$ a strict pseudocontraction with a fixed point. Define a mapping $S\colon C \to C$ by*

$Sx = kx + (1-k)Tx$ for each $x \in C$. Assume that $\mathcal{F} = F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm: $x_1 \in C$ and

$$\begin{cases} z_n = \omega_n x_n + (1-\omega_n) P_C(x_n - t_n A x_n)\,, \\ y_n = \delta_n S x_n + (1-\delta_n) z_n\,, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n\,, \quad \forall\, n \geq 1\,, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\omega_n\}$ are sequences in $(0,1)$ and $\{t_n\}$ is a positive sequence. Assume that the above control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(d) $0 < t \leq t_n \leq \frac{2(\nu - L^2\mu)}{L^2}$, *where t is some constant, for each* $n \geq 1$;
(e) $\lim_{n\to\infty} |t_n - t_{n+1}| = 1$
(f) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$, $\lim_{n\to\infty} \omega_n = \omega \in (0,1)$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = P_{\mathcal{F}} f(\bar{x})$, which solves the following variational inequality

$$\langle f(\bar{x}) - \bar{x}, \bar{x} - x \rangle \leq 0\,, \quad \forall\, x \in \mathcal{F}\,.$$

Proof. First, we show the mapping $I - t_n A$ is nonexpansive for each $n \geq 1$. Indeed, from the relaxed (μ, ν)-cocoercive and L-Lipschitz definition on A, we have

$$\begin{aligned} \|(I - t_n A)x - (I - t_n A)y\|^2 &= \|(x-y) - t_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2t_n\langle x - y, Ax - Ay\rangle + t_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - 2t_n[-\mu\|Ax - Ay\|^2 + \nu\|x-y\|^2] + t_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 + 2t_n\mu L^2\|x-y\|^2 - 2t_n\nu\|x-y\|^2 + L^2t_n^2\|x-y\|^2 \\ &= (1 + 2t_nL^2\mu - 2t_n\nu + L^2t_n^2)\|x-y\|^2 \\ &\leq \|x-y\|^2\,, \end{aligned}$$

which implies the mapping $I - t_n A$ is nonexpansive for each $n \geq 1$.

Next, we show that the sequence $\{x_n\}$ is bounded. Fix $p \in F(T) \cap VI(C, A)$. From Lemma 1.1, we see that $F(T) = F(S)$. It follows that

$$\begin{aligned} \|z_n - p\| =& \|\omega_n(x_n - p) + (1-\omega_n)(P_C(I - t_nA)x_n - p)\| \\ \leq& \omega_n\|x_n - p\| + (1-\omega_n)\|x_n - p\| \\ =& \|x_n - p\|\,. \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - p\| &= \|\delta_n(Sx_n - p) + (1-\delta_n)z_n - p)\| \\ &\leq \delta_n\|Sx_n - p\| + (1-\delta_n)\|z_n - p\| \\ &\leq \|x_n - p\|\,. \end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n y_n-p)\| \\ &\le \alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|y_n-p\| \\ &\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|p-f(p)\|+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\| \\ &\le \alpha\alpha_n\|x_n-p\|+\alpha_n\|p-f(p)\|+(1-\alpha_n)\|x_n-p\| \\ &= [1-\alpha_n(1-\alpha)]\|x_n-p\|+\alpha_n\|p-f(p)\| \\ &\le \max\left\{\|x_n-p\|, \frac{\|p-f(p)\|}{1-\alpha}\right\}.\end{aligned}$$

By simple inductions, we have

$$\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{\|p-f(p)\|}{1-\alpha}\right\}, \quad \forall\, n \ge 1,$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Put $\rho_n = P_C(I-t_nA)y_n$. It follows that

$$\begin{aligned}\|\rho_n-\rho_{n+1}\| &= \|P_C(I-t_nA)x_n-P_C(I-t_{n+1}A)x_{n+1}\| \\ &\le \|(I-t_nA)x_n-(I-t_{n+1}A)x_{n+1}\| \\ &= \|(I-t_nA)x_n-(I-t_nA)x_{n+1}+(t_{n+1}-t_n)Ax_{n+1}\| \\ &\le \|x_n-x_{n+1}\|+|t_{n+1}-t_n|\,\|Ax_{n+1}\|.\end{aligned} \tag{2.1}$$

Note that

$$\begin{cases} z_n = \omega_n x_n + (1-\omega_n)\rho_n, \\ z_{n+1} = \omega_{n+1}x_{n+1}+(1-\omega_{n+1})\rho_{n+1}. \end{cases}$$

It follows that

$$z_n - z_{n+1} = \omega_n(x_n-x_{n+1})+(1-\omega_n)(\rho_n-\rho_{n+1})+(\rho_{n+1}-x_{n+1})(\omega_{n+1}-\omega_n),$$

which yields that

$$\begin{aligned}\|z_n-z_{n+1}\| \le \omega_n\|x_n-x_{n+1}\|&+(1-\omega_n)\|\rho_n-\rho_{n+1}\| \\ &+\|\rho_{n+1}-x_{n+1}\|\,|\omega_{n+1}-\omega_n|.\end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), we see that

$$\|z_n-z_{n+1}\| \le \|x_n-x_{n+1}\|+(|t_{n+1}-t_n|+|\omega_{n+1}-\omega_n|)M_1, \tag{2.3}$$

where M_1 is an appropriate constant such that

$$M_1 = \max\left\{\sup_{n\ge1}\{\|Ax_n\|\}, \sup_{n\ge1}\{\|\rho_n-x_n\|\}\right\}.$$

On the other hand, we have

$$y_n-y_{n+1} = \delta_n(Sx_n-Sx_{n+1})+(1-\delta_n)(z_n-z_{n+1})+(z_{n+1}-Sx_{n+1})(\delta_{n+1}-\delta_n),$$

which yields that

$$\begin{aligned}\|y_n-y_{n+1}\| \le \delta_n\|x_n-x_{n+1}\|&+(1-\delta_n)\|z_n-z_{n+1}\| \\ &+\|z_{n+1}-Sx_{n+1}\||\delta_{n+1}-\delta_n|.\end{aligned} \tag{2.4}$$

Substituting (2.3) into (2.4), we see that

$$\|y_n - y_{n+1}\| \le \|x_n - x_{n+1}\| + (|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |\delta_{n+1} - \delta_n|)M_2\,, \tag{2.5}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\{M_1, \sup_{n\ge 1}\{\|z_n - Sx_n\|\}\}.$$

Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$ for each $n \ge 1$. That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for each $n \ge 1$. Now, we compute $\|l_{n+1} - l_n\|$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1}) y_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1-\alpha_n - \beta_n) y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(f(x_n) - y_n) + y_{n+1} - y_n \end{aligned}$$

we obtain that

$$\|l_{n+1} - l_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| - \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \|y_{n+1} - y_n\|\,,$$

which combines with (2.5) yields that

$$\begin{aligned} \|l_{n+1} - l_n\| - \|x_n - x_{n+1}\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| \\ &\quad + \|y_{n+1} - y_n\| + (|t_{n+1} - t_n| + |\omega_{n+1} - \omega_n| + |\delta_{n+1} - \delta_n|)M_2\,. \end{aligned}$$

It follows from the conditions (a), (b), (c), (e) and (f) that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0\,.$$

In view of Lemma 1.4, we obtain that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n - x_n\| = 0\,.$$

Observing that

$$x_{n+1} - x_n = \alpha_n\big(f(x_n) - x_n\big) + \gamma_n(y_n - x_n)\,,$$

we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0\,. \tag{2.6}$$

Note that $P_{\mathcal{F}}f$ is a contraction. Indeed, for all $x, y \in C$, we have

$$\|P_{\mathcal{F}}f(x) - P_{\mathcal{F}}f(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|\,.$$

Banach's contraction mapping principle guarantees that $P_{\mathcal{F}}f$ has a unique fixed point, say $\bar{x} \in C$. That is, $\bar{x} = P_{\mathcal{F}}f(\bar{x})$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle \le 0\,.$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x})-\bar{x}, x_{n_i}-\bar{x}\rangle\,.$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging weakly to $\widehat{x}$. We may without loss of generality assume that $x_{n_i} \rightharpoonup \widehat{x}$, where $\rightharpoonup$ denotes the weak convergence. From the condition (d), we see that there exits a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \to s \in \left[t, \frac{2(\nu-L^2\mu)}{L^2}\right]$. Next, we prove that $\widehat{x} \in \mathcal{F}$. Indeed, define a mapping $R_1\colon C \to C$ by

$$R_1 x = \omega x + (1-\omega)P_C(I-sA)x\,, \quad \forall\, x \in C\,.$$

From Lemma 1.2, we see that R_1 is nonexpansive with

$$F(R_1) = F(I)\cap F(P_C(I-sA)) = VI(C,A)\,.$$

Now, we define another mapping $R_2\colon C\to C$ by

$$R_2 x = \delta S x + (1-\delta)R_1 x\,, \quad \forall\, x\in C\,.$$

From Lemma 1.2, we also obtain that R_2 is nonexpansive with

$$F(R_2) = F(S)\cap F(R_1) = F(T)\cap VI(C,A) = \mathcal{F}\,.$$

Note that

$$\begin{aligned}
\|R_1x_{n_i}-z_{n_i}\| &= \|\omega x_{n_i} + (1-\omega)P_C(I-sA)x_{n_i} - z_{n_i}\|\\
&= \|\omega x_{n_i} + (1-\omega)P_C(I-sA)x_{n_i} - [\omega_{n_i}x_{n_i} + (1-\omega_{n_i})P_C(I-t_{n_i}A)x_{n_i}]\|\\
&\le |\omega-\omega_{n_i}|(\|x_{n_i}\| + \|P_C(I-t_{n_i}A)x_{n_i}\|)\\
&\quad + (1-\omega)\|P_C(I-sA)x_{n_i} - P_C(I-t_{n_i}A)x_{n_i}\|\\
&\le |\omega-\omega_{n_i}|(\|x_{n_i}\| + \|P_C(I-t_{n_i}A)x_{n_i}\|) + (1-\omega)|s-t_{n_i}|\|Ax_{n_i}\|\,. \qquad (2.7)
\end{aligned}$$

In view of the condition (f), we obtain that $\lim_{i\to\infty}\|R_1x_{n_i}-z_{n_i}\| = 0$. On the other hand, we have

$$\begin{aligned}
\|R_2x_{n_i}-y_{n_i}\| &= \|\delta Sx_{n_i} + (1-\delta)R_1x_{n_i} - y_{n_i}\|\\
&= \|\delta Sx_{n_i} + (1-\delta)R_1x_{n_i} - [\delta_{n_i}Sx_{n_i} + (1-\delta_{n_i})z_{n_i}]\|\\
&\le |\delta-\delta_{n_i}|(\|Sx_{n_i}\| + \|z_{n_i}\|) + (1-\delta)\|R_1x_{n_i}-z_{n_i}\|\,.
\end{aligned}$$

From the condition (f) and $\lim_{i\to\infty}\|R_1x_{n_i}-z_{n_i}\| = 0$, we obtain that

$$\lim_{i\to\infty}\|R_2x_{n_i}-y_{n_i}\| = 0\,. \qquad (2.8)$$

Note that

$$\|R_2x_{n_i}-x_{n_i}\| \le \|R_2x_{n_i}-y_{n_i}\| + \|y_{n_i}-x_{n_i}\|\,.$$

Combining (2.6) and (2.8), we see that

$$\lim_{i\to\infty}\|R_2x_{n_i}-x_{n_i}\| = 0\,.$$

Note that $x_{n_i} \rightharpoonup \widehat{x}$. From Lemma 1.3, we obtain that $\widehat{x} \in \mathcal{F}$. It follows that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, \widehat{x} - \bar{x}\rangle \le 0\,.$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&= \alpha_n\langle f(x_n) - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\quad + \gamma_n\langle y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \alpha_n\langle f(x_n) - f(\bar{x}), x_{n+1} - \bar{x}\rangle + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\quad + \beta_n\|x_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| + \gamma_n\|y_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| \\
&\le \alpha_n\alpha\|x_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\quad + \beta_n\|x_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| + \gamma_n\|x_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| \\
&\le \frac{1 - \alpha_n(1-\alpha)}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \frac{1 - \alpha_n(1-\alpha)}{2}\|x_n - \bar{x}\|^2 + \frac{1}{2}\|x_{n+1} - \bar{x}\|^2 + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle\,.
\end{aligned}$$

This implies that

$$\|x_{n+1} - \bar{x}\|^2 \le [1 - \alpha_n(1-\alpha)]\|x_n - \bar{x}\|^2 + 2\alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle\,.$$

In view of Lemma 1.5, we can conclude the desired conclusion easily. □

As corollaries of Theorem 2.1, we have the following results immediately.

Corollary 2.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H and $A\colon C \to H$ a relaxed (μ,ν)-cocoercive and L-Lipschitz continuous mapping. Let $f\colon C \to C$ be a contraction with the coefficient $\alpha \in (0,1)$ and $T\colon C \to C$ a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} = F(T) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm: $x_1 \in C$ and*

$$\begin{cases} z_n = \omega_n x_n + (1-\omega_n)P_C(x_n - t_n A x_n)\,, \\ y_n = \delta_n T x_n + (1-\delta_n) z_n\,, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall\, n \ge 1\,, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\omega_n\}$ are sequences in $(0,1)$ and $\{t_n\}$ is a positive sequence. Assume that the above control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \ge 1$;
(b) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(c) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(d) $0 < t \le t_n \le \frac{2(\nu - L^2\mu)}{L^2}$, *where t is some constant, for each* $n \ge 1$;
(e) $\lim_{n\to\infty}|t_n - t_{n+1}| = 1$;
(f) $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$, $\lim_{n\to\infty}\omega_n = \omega \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in \mathcal{F}$, *where* $\bar{x} = P_{\mathcal{F}} f(\bar{x})$, *which solves the following variational inequality*

$$\langle f(\bar{x}) - \bar{x}, \bar{x} - x \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Corollary 2.2. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H *and* $A : C \to H$ *a relaxed* (μ, ν)*-cocoercive and* L*-Lipschitz continuous mapping. Let* $f : C \to C$ *be a contraction with the coefficient* $\alpha \in (0,1)$. *Assume that* $VI(C, A) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:* $x_1 \in C$ *and*

$$\begin{cases} y_n = [\delta + (1-\delta)\omega]x_n + (1-\delta)(1-\omega)P_C(x_n - t_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$, δ *and* ω *are two constant in* $(0,1)$ *and* $\{t_n\}$ *is a positive sequence. Assume that the above control sequences satisfy the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(d) $0 < t \leq t_n \leq \frac{2(\nu - L^2\mu)}{L^2}$, *where* t *is some constant, for each* $n \geq 1$;
(e) $\lim_{n\to\infty} |t_n - t_{n+1}| = 1$;

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in VI(C, A)$, *where* $\bar{x} = P_{VI(C,A)} f(\bar{x})$, *which solves the following variational inequality*

$$\langle f(\bar{x}) - \bar{x}, \bar{x} - x \rangle \leq 0, \quad \forall x \in VI(C, A).$$

Corollary 2.3. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H *and* $A\colon C \to H$ *a relaxed* (μ, ν)*-cocoercive and* L*-Lipschitz continuous mapping. Let* $f\colon C \to C$ *be a contraction with the coefficient* $\alpha \in (0,1)$ *and* $T\colon C \to C$ *a strict pseudocontraction with a fixed point. Define a mapping* $S\colon C \to C$ *by* $Sx = kx + (1-k)Tx$ *for each* $x \in C$. *Assume that* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \delta_n S x_n + (1-\delta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are sequences in* $(0,1)$. *Assume that the above control sequences satisfy the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(d) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in F(T)$, *where* $\bar{x} = P_{F(T)} f(\bar{x})$, *which solves the following variational inequality*

$$\langle f(\bar{x}) - \bar{x}, \bar{x} - x \rangle \leq 0, \quad \forall\, x \in F(T).$$

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