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CLASSIFICATION OF PRINCIPAL CONNECTIONS NATURALLY INDUCED ON W^2PE

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Abstract. We consider a vector bundle $E \to M$ and the principal bundle PE of frames of E. Let K be a principal connection on PE and let Λ be a linear connection on M. We classify all principal connections on $W^2PE = P^2M \times_M J^2PE$ naturally given by K and Λ.

Introduction

Principal connections on principal prolongation W^rP of a principal bundle P naturally given by a principal connection Γ on P and a linear connection Λ on M (considered as a principal connection on the principal frame bundle P^1M) play an important role in gauge invariant field theory, [4]. This problem was studied by many authors, see for instance [1, 2, 8, 10]. Complete classification is known for the first prolongation W^1P but for general r the problem is still open. In this paper we give the full classification of principal connections for the linear gauge group $\mathrm{GL}(n)$ playing the role the structure group and for the order $r = 2$.

We use the terminology and methods of the theory of natural and gauge-natural bundles and natural operators, see [3, 9, 11, 12, 13].

We denote by $\mathcal{PB}_m(G)$ the category of principal G-bundles with m-dimensional bases and principal bundle morphisms over diffeomorphisms of bases and by $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases and fibered morphisms over diffeomorphisms of bases. Then a gauge-natural bundle functor is a functor from the category $\mathcal{PB}_m(G)$ to the category $\mathcal{FM}_m$ transforming the diagram for a morphism in $\mathcal{PB}_m(G)$ to a commutative diagram in $\mathcal{FM}_m$. Moreover, such functor has to satisfy some locality condition.

All manifolds and maps are assumed to be smooth.

1. Principal bundles and principal connections

1.1. **Principal bundle and its principal prolongation.** We consider a principal bundle $P = (P, M, \pi; G)$ with a structure group G. We denote by (x^λ, z^a) fibered coordinates on P, $\lambda = 1, \dots, \dim M$, $a = 1, \dots, \dim G$.

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By $\mathrm{ad}(P)$ we denote the vector bundle associated to P with respect to the adjoint action of G on its Lie algebra $\mathfrak{g}$. We denote by (B_a) a base of $\mathfrak{g}$ and we denote by (x^λ, u^a) the induced fiber linear local coordinates on $\mathrm{ad}(P)$. Let us denote by c^a_{bd} the related structure constants, i.e. $[B_b, B_d] = c^a_{bd} B_a$.

For the right action $r_g \colon P \to P$ given by an element $g \in G$ we consider the tangent mapping $Tr_g \colon TP \to TP$. Let Ξ be a vector field on P. We say that Ξ is right invariant if $\Xi(pg) = Tr_g \Xi(p)$ for all $p \in P$ and $g \in G$. In coordinates we have

$$\Xi = \xi^\lambda(x)\partial_\lambda + \Xi^a(x)\widetilde{B}_a \,, \tag{1.1}$$

where $(\widetilde{B}_a)$ is the base of vertical right invariant vector fields on P which are induced by (B_a). So Ξ are sections of the bundle $TP/G \to M$.

Example 1.1. Let as recall that the bundle of r-th order frames on a manifold is the principal bundle $P^rM = (P^rM, M, \pi^r; G^r_m)$, where $P^rM = invJ^r_0(\mathbb{R}^m, M)$ is the bundle of invertible r-jets with the source 0 and the structure group $G^r_m = invJ^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$, [9]. Local coordinates (x^λ) on M then induce the fibered coordinates $(x^\lambda, x^\lambda_\mu, \dots, x^\lambda_{\mu_1\dots\mu_r})$ on P^rM. The coordinates in the structure group G^r_m are denoted by $(a^\lambda_\mu, \dots, a^\lambda_{\mu_1\dots\mu_r})$.

The base of the induced right invariant vertical vector fields $(\widetilde{B}^{\mu_1\dots\mu_i}_\lambda)$, $i = 1, \dots, r$, on P^rM can be obtained by the following construction. Let us consider a vector field $\xi = \xi^\lambda(x)\partial_\lambda$ on M. Then we have the flow lift $\mathcal{P}^r(\xi)$ given by $P^r(Fl_t\xi) = Fl_t(\mathcal{P}^r(\xi))$ which is a vector field on P^rM projectable on ξ. Then, [4],

$$\mathcal{P}^r\xi = \xi^\lambda(x)\partial_\lambda + \sum_{i=1}^{r} \frac{\partial^i \xi^\lambda(x)}{\partial x^{\mu_1}\dots\partial x^{\mu_i}} \, \widetilde{B}^{\mu_1\dots\mu_i}_\lambda \tag{1.2}$$

and, if we compare it with the expression of $\mathcal{P}^r\xi$ in the canonical coordinates, we obtain the relations between the canonical base of vertical vector fields $(\partial^{\mu_1\dots\mu_i}_\lambda)$ and the base of right invariant vertical vector fields $(\widetilde{B}^{\mu_1\dots\mu_i}_\lambda)$. For instance for $r = 1$ we have

$$\mathcal{P}^1(\xi) = \xi^\lambda \, \partial_\lambda + \frac{\partial \xi^\lambda}{\partial x^\rho} \, x^\rho_\mu \, \partial^\mu_\lambda \,, \tag{1.3}$$

which implies that

$$\widetilde{B}^\rho_\lambda = x^\rho_\mu \, \partial^\mu_\lambda \,.$$

For $r = 2$ we have

$$\begin{aligned} \mathcal{P}^2(\xi) &= \xi^\lambda \, \partial_\lambda + \frac{\partial \xi^\lambda}{\partial x^\rho} \, x^\rho_\mu \, \partial^\mu_\lambda + \Big(\frac{\partial^2 \xi^\lambda}{\partial x^\rho \partial x^\sigma} \, x^\rho_\mu \, x^\sigma_\nu + \frac{\partial \xi^\lambda}{\partial x^\rho} \, x^\rho_{\mu\nu} \Big) \, \partial^{\mu\nu}_\lambda \\ &= \xi^\lambda \, \partial_\lambda + \frac{\partial \xi^\lambda}{\partial x^\rho} \Big(x^\rho_\mu \, \partial^\mu_\lambda + x^\rho_{\mu\nu} \, \partial^{\mu\nu}_\lambda \Big) + \frac{\partial^2 \xi^\lambda}{\partial x^\rho \partial x^\sigma} \, x^\rho_\mu \, x^\sigma_\nu \, \partial^{\mu\nu}_\lambda \,. \end{aligned} \tag{1.4}$$

Then we obtain the right invariant vector fields on P^2M given by

$$\widetilde{B}^\rho_\lambda = x^\rho_\mu \, \partial^\mu_\lambda + x^\rho_{\mu\nu} \, \partial^{\mu\nu}_\lambda \,, \qquad \widetilde{B}^{\rho\sigma}_\lambda = x^\rho_\mu \, x^\sigma_\nu \, \partial^{\mu\nu}_\lambda \,. \tag{1.5}$$

Definition 1.1. For each principal bundle $P = (P, M, \pi; G)$ and for each integer r we can define the principal bundle $W^rP = P^rM \times_M J^rP \equiv (W^rP, M, p; W^r_mG)$. The structure group is the semidirect product $W^r_mG = G^r_m \rtimes T^r_mG$. See [9].

The group $W^r_m G$ is the group of r-jets at $(0,e)$ of all automorphisms $\varphi\colon \mathbb{R}^m\times G \to \mathbb{R}^m \times G$ with $\underline{\varphi}(0)=0$, where the multiplication μ is defined by the composition of jets,

$$\mu\big(j^r\varphi(0,e), j^r\psi(0,e)\big) = j^r(\psi\circ\varphi)(0,e)\,. \tag{1.6}$$

If we replace holonomic jets by nonholonomic or semiholonomic ones, we obtain the nonholonomic or semiholonomic principal prolongations $\widetilde{W}^r P$ and $\overline{W}^r P$, respectively. Analogously we have

$$\widetilde{W}^r P = \tilde{P}^r M \times_M \tilde{J}^r P\,, \qquad \overline{W}^r P = \bar{P}^r M \times_M \bar{J}^r P$$

and

$$\widetilde{W}^r_m G = \tilde{G}^r_m \rtimes \tilde{T}^r_m G \qquad \overline{W}^r_m G = \bar{G}^r_m \rtimes \bar{T}^r_m G\,.$$

Remark 1.1. Moreover, $W^r\colon \mathcal{PB}_m(G) \to \mathcal{PB}_m(W^r_m G)$ defines a functor. Let φ be a principal morphism in the category $\mathcal{PB}_m(G)$ over a base diffeomorphism $\underline{\varphi}$ then $W^r\varphi = (P^r\underline{\varphi}, J^r\varphi)$. W^r is then a gauge-natural bundle functor of order r and plays a fundamental role in the theory of gauge-natural bundles. Really, any r-th order G-gauge-natural bundle is a fiber bundle associated with W^rP. See [3, 9].

Remark 1.2. On W^rP we have the induced natural local fiber coordinates $(x^\lambda, x^\lambda_\mu, \dots, x^\lambda_{\mu_1\dots\mu_r}, z^a, z^a_\nu, \dots, z^a_{\nu_1\dots\nu_r})$. The base of the induced right invariant vector fields $(\widetilde{B}^{\mu_1\dots\mu_i}_\lambda, \widetilde{B}^{\nu_1\dots\nu_j}_a)$, $i=1,\dots,r$, $j=0,\dots,r$, on W^rP can be obtained by the following construction, [4]. Let us consider a right invariant vector field Ξ on M given by (1.1). Then we have the flow lift $\mathcal{W}^r(\Xi)$ which is a vector field on W^rP obtained as

$$\mathcal{W}^r(\Xi) = \mathcal{P}^r(\xi) \times_\xi \mathcal{J}^r(\Xi)\,, \tag{1.7}$$

where $\mathcal{P}^r(\xi)$ is the flow lift of the vector field ξ on P^rM and $\mathcal{J}^r(\Xi)$ is the r-th order jet lift of Ξ on J^rPE. Then

$$\mathcal{W}^r(\Xi) = \xi^\lambda(x)\partial_\lambda + \sum_{i=1}^r \frac{\partial^i\xi^\lambda(x)}{\partial x^{\mu_1}\dots\partial x^{\mu_i}}\,\widetilde{B}^{\mu_1\dots\mu_i}_\lambda + \sum_{j=0}^r \frac{\partial^j\Xi^a(x)}{\partial x^{\nu_1}\dots\partial x^{\nu_j}}\,\widetilde{B}^{\nu_1\dots\nu_j}_a\,. \tag{1.8}$$

Comparing (1.8) with (1.7) expressed in the canonical coordinates on W^rP we obtain relations between the canonical base of vertical vector fields $(\partial^{\mu_1\dots\mu_i}_\lambda, \partial^{\nu_1\dots\nu_j}_a)$ and the base of right invariant vertical vector fields $(\widetilde{B}^{\mu_1\dots\mu_i}_\lambda, \widetilde{B}^{\nu_1\dots\nu_j}_a)$.

We have two canonical group homomorphisms

$$\pi^r_{r-1}\colon W^r_m G \to W^{r-1}_m G\,, \qquad p_1\colon W^r_m G \to G^r_m\,, \tag{1.9}$$

which induce homomorphisms of Lie algebras

$$\pi^r_{r-1}\colon \mathfrak{w}^{\mathfrak{r}}_{\mathfrak{m}}\mathfrak{g} \to \mathfrak{w}^{\mathfrak{r}-1}_{\mathfrak{m}}\mathfrak{g}\,, \qquad p_1\colon \mathfrak{w}^{\mathfrak{r}}_{\mathfrak{m}}\mathfrak{g} \to \mathfrak{g}^{\mathfrak{r}}_{\mathfrak{m}}\,, \tag{1.10}$$

principle bundle morphisms, over the identity of M,

$$\pi^r_{r-1}\colon W^rP \to W^{r-1}P\,, \qquad p_1\colon W^rP \to P^rM\,, \tag{1.11}$$

and homomorphisms of associated vector bundles

$$\pi^r_{r-1}\colon\ \mathrm{ad}(W^rP) \to \mathrm{ad}(W^{r-1}P)\,, \qquad p_1\colon\ \mathrm{ad}(W^rP) \to \mathrm{ad}(P^rM)\,. \tag{1.12}$$

1.2. **Principal connection on P and on W^rP.** A principal connection on P is defined as a lifting linear mapping

$$\Gamma\colon TM \to TP/G\,.$$

In coordinates

$$\Gamma = \mathrm{d}^\lambda \otimes \big(\partial_\lambda + \Gamma^a{}_\lambda(x)\widetilde{B}_a\big)\,, \tag{1.13}$$

where $\Gamma^a{}_\lambda(x)$ are functions on M. If we identify Γ with the functions $\Gamma^a{}_\lambda(x)$ then Γ can be considered as a section of the bundle $QP \to M$ of principal connections on P.

Moreover, we have [9].

Proposition 1.1. *Let $P \to M$ be a principal bundle and $QP \to M$ is the bundle of principal connections on P. Then QP is a 1-order G-gauge-natural affine bundle associated with the vector bundle* $\mathrm{ad}(P)\otimes T^*M \to M$*. This implies that the standard fiber of the functor Q is $\mathfrak{g}\otimes\mathbb{R}^{m*}$.*

Remark 1.3. Let us recall that every principal connection Γ on P induce the linear adjoint connection $\mathrm{Ad}(\Gamma)$ on $\mathrm{ad}(P)$ given in coordinates by

$$\mathrm{Ad}(\Gamma) = \mathrm{d}^\lambda \otimes (\partial_\lambda + c^a_{de}\Gamma^d{}_\lambda u^e \partial_a)\,.$$

Example 1.2. Let us consider the r-order frame bundle P^rM of M (see Example 1.1). A principal connection Λ_r on P^rM has a coordinate expression

$$\Lambda_r = \mathrm{d}^\lambda \otimes \Big(\partial_\lambda + \sum_{i=1}^{r} \Lambda^\nu_{\mu_1\dots\mu_i\lambda}\widetilde{B}^{\mu_1\dots\mu_i}_\nu\Big)\,. \tag{1.14}$$

We remark that principal connections on P^1M are in the bijection with classical connections on M (linear connections on TM).

Let Γ_r be a principal connection on W^rP given in coordinates by

$$\Gamma_r = \mathrm{d}^\lambda \otimes \Big(\partial_\lambda + \sum_{i=1}^{r} \Lambda^\nu_{\mu_1\dots\mu_i\lambda}\widetilde{B}^{\mu_1\dots\mu_i}_\nu + \sum_{j=0}^{r} \Gamma^a{}_{\kappa_1\dots\kappa_j\lambda}\widetilde{B}^{\kappa_1\dots\kappa_j}_a\Big)\,. \tag{1.15}$$

Remark 1.4. We remark that coefficients $\Lambda^\nu_{\mu_1\dots\mu_i\lambda}$, respective $\Gamma^a{}_{\kappa_1\dots\kappa_j\lambda}$, satisfy the symmetry condition with respect to indices $\mu_1\dots\mu_i$, respective $\kappa_1\dots\kappa_j$.

The projections (1.11) of W^rP and the functor Q induce the projections

$$\pi^r_{r-1}\colon QW^rP \to QW^{r-1}P\,, \qquad p_1\colon QW^rP \to QP^rM\,, \tag{1.16}$$

so any principal connection Γ_r on W^rP projects on the principal connection Λ_r on P^rM, see (1.14), and on the principal connection Γ_{r-1} on $W^{r-1}P$.

Remark 1.5. By Proposition 1.1 $QW^rP \to M$ is the affine bundle modeled over the vector bundle $\mathrm{ad}(W^rP)\otimes T^*M \to M$. Now, if we consider two principal connections Γ_r and $\bar{\Gamma}_r$ on W^rP such that they are over the same Λ_r on P^rM and Γ_{r-1} on $W^{r-1}P$, then the difference $\Gamma_r - \bar{\Gamma}_r$ is in the intersection of the kernels of projections $\pi^r_{r-1}\otimes\mathrm{id}_{T^*M}\colon\ \mathrm{ad}(W^rP)\otimes T^*M \to \mathrm{ad}(W^{r-1}P)\otimes T^*M$ and $p_1\otimes\mathrm{id}_{T^*M}\colon\ \mathrm{ad}(W^rP)\otimes T^*M \to \mathrm{ad}(P^rM)\otimes T^*M$.

Let ξ be a vector field on M, Γ be a principal connection on P and Λ be a principal connection on P^1M such that the horizontal lift $h^\Lambda(\xi) = \mathcal{P}^1(\xi)$. Let $h^\Gamma(\xi)$ denote the horizontal lift of ξ with respect to Γ. Let us denote by $\mathrm{Fl}_t(h^\Gamma(\xi))$ the flow of $h^\Gamma(\xi)$. Then the expression

$$W^r(\mathrm{Fl}_t(h^\Gamma(\xi))) = (P^r(\mathrm{Fl}_t(\xi)), J^r(\mathrm{Fl}_t(h^\Gamma(\xi))) = \mathrm{Fl}_t(h^{\mathcal{W}^r\Gamma}(\xi))$$

gives a principal connection $\mathcal{W}^r\Gamma$ on W^rP which depends on Γ in order r and on Λ in order $(r-1)$. So $\mathcal{W}^r\Gamma$ is a natural operator

$$\mathcal{W}^r\Gamma\colon J^{r-1}QP^1M \times_M J^rQP \to QW^rP$$

called the *flow prolongation* of Γ.

Remark 1.6. There is another construction of a prolongation of the pair (Λ, Γ) of principal connections to principal connections on W^rP. In literature this construction is usually denoted by $p(\Lambda, \Gamma)$ and in order one was described in [10] and in any order r in [1].

Consider general problem of the classification of all principal connections on W^rP which are naturally given by a principal connection Γ on P and by a classical connection Λ on the base M. In general this problem is still open. There are only some partial results. Namely, the classification of principal connections on W^1P for a torsion free Λ and any Γ is given in [10] and the full classification for the linear gauge group $\mathrm{GL}(n)$ and order one is given in [8].

Clearly, the solution of this problem in any order depends essentially on the structure group of principal bundles. In this paper we give the full classification of principal connections for the linear gauge group $\mathrm{GL}(n)$ and for the order $r = 2$.

1.3. **Connections on frame bundle PE.** Let $E \to M$ be a vector bundle with m-dimensional base and n-dimensional fibers. Let us denote by (x^λ, y^i) local linear fiber coordinate charts on E. Let $PE \to M$ be the frame bundle of E, i.e. PE is the principal bundle with the structure group $\mathrm{GL}(n)$ and the induced fiber coordinates (x^λ, x^i_j).

Principal connections on PE are in bijection with general linear connections on E. The coordinate expression of a linear connection K on E is of the type

$$K = \mathrm{d}^\lambda \otimes (\partial_\lambda + K^i_{j\lambda}(x) y^j \partial_i)$$

and, if we consider K as a principal connection on PE,

$$K = \mathrm{d}^\lambda \otimes (\partial_\lambda + K^i_{p\lambda}(x) x^p_j \partial^j_i) = \mathrm{d}^\lambda \otimes (\partial_\lambda + K^i_{j\lambda}(x) \widetilde{B}^j_i)\,.$$

Applying the functor W^r on PE we obtain the principal prolongation W^rPE with the structure group $W^r_m\,\mathrm{GL}(n) = G^r_m \rtimes T^r_m\,\mathrm{GL}(n)$. For example, in order 2 we have the structure group $W^2_m\,\mathrm{GL}(n)$ with coordinates $(a^\lambda_\mu, a^\lambda_{\mu\nu}, a^i_j, a^i_{j\mu}, a^i_{j\mu\nu})$. If we denote by $(X^\lambda_\mu, X^\lambda_{\mu\nu}, X^i_j, X^i_{j\mu}, X^i_{j\mu\nu})$ the induced coordinates on the Lie algebra $\mathfrak{w}^2_m\mathfrak{gl}(n)$ we can compute easily the adjoint action of the group $W^2_m\,\mathrm{GL}(n)$ on

$\mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n)$. In coordinates if $\bar{X} = \mathrm{Ad}(g)(X)$ we have

$$\bar{X}^\lambda_\mu = a^\lambda_\varrho X^\varrho_\tau \tilde{a}^\tau_\mu\,, \qquad \bar{X}^\lambda_{\mu\nu} = a^\lambda_\varrho X^\varrho_{\alpha\beta}\tilde{a}^\alpha_\mu \tilde{a}^\beta_\nu + \mathrm{pol}(a,X)\,,$$
$$\bar{X}^i_j = a^i_p X^p_q \tilde{a}^q_j\,, \qquad \bar{X}^i_{j\varrho} = a^i_p X^p_{q\alpha}\tilde{a}^\alpha_\varrho \tilde{a}^q_j + \mathrm{pol}(a,X)\,,$$
$$\bar{X}^i_{j\varrho\sigma} = a^i_{q\tau} X^\tau_{\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j + a^i_p X^p_{q\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j + \mathrm{pol}(a,X)\,,$$

where $\mathrm{pol}(a,X)$ is a polynomial on $W^2_m\,\mathrm{GL}(n) \times \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{gl}(n)$ such that any monomial contains exactly one coordinate of $\pi^2_1(X)$ of orders less then the leading terms.

Proposition 1.2. *1. The restriction of the adjoint action of the group $W^2_m\,\mathrm{Gl}(n)$ on the kernel of the projection $\pi^2_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{gl}(n)$ has the form*

$$\bar{X}^\lambda_{\mu\nu} = a^\lambda_\varrho X^\varrho_{\alpha\beta}\tilde{a}^\alpha_\mu \tilde{a}^\beta_\nu\,,$$
$$\bar{X}^i_{j\varrho\sigma} = a^i_{q\tau} X^\tau_{\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j + a^i_p X^p_{q\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j\,.$$

2. The restriction of the adjoint action of the group $W^2_m\,\mathrm{Gl}(n)$ on the kernel of the projection $p_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{g}^2_{\mathfrak{m}}$ has the form

$$\bar{X}^i_j = a^i_p X^p_q \tilde{a}^q_j\,, \qquad \bar{X}^i_{j\varrho} = a^i_p X^p_{q\alpha}\tilde{a}^\alpha_\varrho \tilde{a}^q_j + \mathrm{pol}(a,X)\,,$$
$$\bar{X}^i_{j\varrho\sigma} = a^i_p X^p_{q\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j + \mathrm{pol}(a,X)\,,$$

where $\mathrm{pol}(X,a)$ is a polynomial on $W^2_m\,\mathrm{GL}(n) \times \mathfrak{t}^1_{\mathfrak{m}}\mathfrak{gl}(n)$ such that any monomial contains exactly one coordinate of orders less then the leading terms.

Proof. 1. The kernel of the projection $\pi^2_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{gl}(n)$ is given by $X^\lambda_\mu = 0$, $X^i_j = 0$ and $X^i_{j\mu} = 0$, i.e. all $\mathrm{pol}(a,X) = 0$.

2. The kernel of the projection $p_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{g}^2_{\mathfrak{m}}$ is given by $X^\lambda_\mu = 0$ and $X^\lambda_{\mu_1\mu_2} = 0$. □

Now, as a direct consequence of Proposition 1.2, we have.

Theorem 1.1. *The restriction of the adjoint action of the group $W^2_m\,\mathrm{Gl}(n)$ on the intersection of kernels of the projections $\pi^2_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{w}^1_{\mathfrak{m}}\mathfrak{gl}(n)$ and $p_1\colon \mathfrak{w}^2_{\mathfrak{m}}\mathfrak{gl}(n) \to \mathfrak{g}^2_{\mathfrak{m}}$ is given by*

$$\bar{X}^i_{j\varrho\sigma} = a^i_p X^p_{q\alpha\beta}\tilde{a}^\alpha_\varrho \tilde{a}^\beta_\sigma \tilde{a}^q_j\,,$$

i.e. $\mathrm{Ker}\,\pi^2_1 \cap \mathrm{Ker}\,p_1$ *is* $\mathfrak{gl}(n) \otimes S^2\mathbb{R}^{m*}$ *with the action of the group* $G^1_m \times \mathrm{GL}(n)$ *given as the tensor product of the adjoint action of* $\mathrm{GL}(n)$ *on its Lie algebra* $\mathfrak{gl}(n)$ *and the tensor action of the group* G^1_m *on* $S^2\mathbb{R}^{m*}$*.*

Corollary 1.1. *The intersection of kernels of the projections* $\pi^2_1\colon\ \mathrm{ad}\,(W^2_m PE) \to \mathrm{ad}\,(W^1_m PE)$ *and* $p_1\colon\ \mathrm{ad}\,(W^2_m PE) \to \mathrm{ad}\,(P^2 M)$ *is the vector bundle*

$$\mathrm{ad}(PE) \otimes S^2 T^* M \to M\,.$$

Next we will describe the flow lift (see Section 1.2) of a principal connection K on PE and a classical connection Λ on M on the principal bundle $W^2 PE$. We have the induced fibered coordinates on $W^2 PE$ denoted by $(x^\lambda_\mu, x^\lambda_{\mu\nu}, x^i_j, x^i_{j\mu}, x^i_{j\mu\nu})$.

A right invariant vector field on PE has the coordinate expression

$$\Xi = \xi^\lambda(x)\partial_\lambda + \Xi^i_j(x)\widetilde{B}^j_i = \xi^\lambda(x)\partial_\lambda + \Xi^i_p(x) x^p_j \partial^j_i\,.$$

The flow lift of Ξ on W^2PE is then

$$\mathcal{W}^2(\Xi) = \mathcal{P}^2(\xi) \times_\xi \mathcal{J}^2(\Xi)\,,$$

where $\mathcal{P}^2(\xi)$ is the flow lift of the vector field ξ on P^2M given by (1.4) and $\mathcal{J}^2(\Xi)$ is the 2nd order jet lift of Ξ on J^2PE, see [9, 11]. We have

$$\begin{aligned}\mathcal{J}^2(\Xi) &= \xi^\lambda\partial_\lambda + \Xi^i_p g^p_j \partial^j_i + (\partial_\lambda \Xi^i_p g^p_j + \Xi^i_p g^p_{j\lambda} - \partial_\lambda \xi^\mu g^i_{j\mu})\partial^{j\lambda}_i \\ &\quad + (\partial_{\lambda\mu}\Xi^i_p g^p_j + \partial_\lambda \Xi^i_p g^p_{j\mu} + \partial_\mu \Xi^i_p g^p_{j\lambda} + \Xi^i_p g^p_{j\lambda\mu} \\ &\quad - \partial_{\lambda\mu}\xi^\varrho g^i_{j\varrho} - \partial_\lambda \xi^\varrho g^i_{j\varrho\mu} - \partial_\mu \xi^\varrho g^i_{j\varrho\lambda})\partial^{j\lambda\mu}_i\end{aligned} \tag{1.17}$$

and from (1.17) and (1.4) we obtain the relations between the right invariant vertical vector fields on W^2PE and the canonical vertical vector fields in the form

$$\begin{aligned}\widetilde{B}^\varrho_\lambda &= x^\varrho_\mu \partial^\mu_\lambda + x^\varrho_{\mu\nu}\partial^{\mu\nu}_\lambda - g^i_{j\lambda}\partial^{j\varrho}_i - g^i_{j\lambda\mu}\partial^{j\varrho\mu}_i\,,\\ \widetilde{B}^{\varrho\sigma}_\lambda &= x^\varrho_\mu x^\sigma_\nu \partial^{\mu\nu}_\lambda - g^i_{j\lambda}\partial^{j\varrho\sigma}_i\,,\\ \widetilde{B}^p_i &= g^p_j\partial^j_i + g^p_{j\lambda}\partial^{j\lambda}_i + g^p_{j\lambda\mu}\partial^{j\lambda\mu}_i\,,\\ \widetilde{B}^{p\lambda}_i &= g^p_j \partial^{j\lambda}_i + g^p_{j\mu}\partial^{j\lambda\mu}_i\,, \qquad \widetilde{B}^{p\lambda\mu}_i = g^p_j\partial^{j\lambda\mu}_i\,.\end{aligned} \tag{1.18}$$

Now, let us assume a principle connection on Γ_2 on W^2PE given in coordinates by (see (1.15))

$$\Gamma_2 = \mathrm{d}^\lambda \otimes (\partial_\lambda + \Lambda^\nu_{\mu_\lambda}\widetilde{B}^\mu_\nu + \Lambda^\nu_{\mu_1\mu_2\lambda}\widetilde{B}^{\mu_1\mu_2}_\nu + K^i_{p\lambda}\widetilde{B}^p_i + \Gamma^i_{p\mu\lambda}\widetilde{B}^{p\mu}_i + \Gamma^i_{p\mu_1\mu_2\lambda}\widetilde{B}^{p\mu_1\mu_2}_i)$$

and, in the canonical base, by using the relations (1.18),

$$\begin{aligned}\Gamma_2 = \mathrm{d}^\lambda \otimes\big(&\partial_\lambda + \Lambda^\nu_{\mu\lambda}x^\mu_\varrho\partial^\varrho_\nu + (\Lambda^\nu_{\mu_1\mu_2\lambda}x^{\mu_1}_\varrho x^{\mu_2}_\sigma + \Lambda^\nu_{\mu\lambda}x^\mu_{\varrho\sigma})\partial^{\varrho\sigma}_\nu \\ &+ K^i_{p\lambda}x^p_j\partial^j_i + (\Gamma^i_{p\mu\lambda}x^p_j + K^i_{p\lambda}x^p_{j\mu} - \Lambda^\nu_{\mu\lambda}x^i_{j\nu})\partial^{j\mu}_i \\ &+ (\Gamma^i_{p\mu_1\mu_2\lambda}x^p_j + \Gamma^i_{p\mu_1\lambda}x^p_{j\mu_2} - \Lambda^\nu_{\mu_1\mu_2\lambda}x^i_{j\nu} \\ &+ K^i_{p\lambda}x^p_{j\mu_1\mu_2} - \Lambda^\nu_{\mu_1\lambda}x^i_{j\nu\mu_2})\partial^{j\mu_1\mu_2}_i\big)\,.\end{aligned} \tag{1.19}$$

Now, if we assume Ξ to be the horizontal lift of ξ given by the connection K and $\mathcal{P}^1(\xi)$ to be the horizontal lift of ξ given by Λ, i.e. $\Xi^i_j = K_j{}^i{}_\lambda \xi^\lambda$ and $\partial_\mu \xi^\lambda = \Lambda^\lambda_{\mu\rho}\xi^\rho$, we have the following coordinates expression

$$\begin{aligned}\mathcal{W}^2(\Xi) = \xi^\lambda\big[&\partial_\lambda + \Lambda^\nu_{\mu\lambda}x^\mu_\varrho\partial^\varrho_\nu + \big((\partial_{\mu_1}\Lambda^\nu_{\mu_2\lambda} + \Lambda^\nu_{\mu_1\alpha}\Lambda^\alpha_{\mu_2\lambda})x^{\mu_1}_\varrho x^{\mu_2}_\sigma + \Lambda^\nu_{\alpha\lambda}x^\alpha_{\sigma\varrho}\big)\partial^{\varrho\sigma}_\nu \\ &+ K^i_{p\lambda}x^p_j\partial^j_i + \big((\partial_\mu K^i_{p\lambda} + K^i_{p\alpha}\Lambda^\alpha_{\mu\lambda})x^p_j + K^i_{p\lambda}x^p_{j\mu} - \Lambda^\varrho_{\mu\lambda}x^i_{j\varrho}\big)\partial^{j\mu}_i \\ &+ \big((\partial_{\mu_1\mu_2}K^i_{p\lambda} + \partial_{\mu_1}K^i_{p\varrho}\Lambda^\varrho_{\mu_2\lambda} + K^j_{p\varrho}\partial_{\mu_1}\Lambda^\varrho_{\mu_2\lambda} + K^j_{p\varrho}\Lambda^\varrho_{\mu_1\sigma}\Lambda^\sigma_{\mu_2\lambda})x^p_j \\ &+ (\partial_{\mu_1}K^i_{p\lambda} + K^i_{p\varrho}\Lambda^\varrho_{\mu_1\lambda})x^p_{j\mu_2} - (\partial_{\mu_1}\Lambda^\varrho_{\mu_2\lambda} + \Lambda^\varrho_{\mu_1\sigma}\Lambda^\sigma_{\mu_2\lambda})x^p_{j\varrho} \\ &+ K^i_{p\lambda}x^p_{j\mu_1\mu_2} - \Lambda^\varrho_{\mu_1\lambda}x^i_{j\varrho\mu_2}\big)\partial^{j\mu_1\mu_2}_i\big]\,.\end{aligned}$$

So, the vector field $\mathcal{W}^2(\Xi)$ can be considered as the horizontal lift of ξ with respect to the connection (1.19) on W^2PE where Γ_2 is given by connections Λ and K. Especially, $\Gamma_2 = \mathcal{W}^2K(\Lambda, K)$. From the symmetry of $\partial^{\varrho\sigma}_\nu$ in ϱ, σ and $\partial^{j\mu_1\mu_2}_i$ in μ_1, μ_2 we obtain the coordinate expression of $\mathcal{W}^2K$ given by

$$\Lambda^{\nu}_{\mu_1\mu_2\lambda} = \frac{1}{2}\big(\partial_{\mu_1}\Lambda^{\nu}_{\mu_2\lambda} + \partial_{\mu_2}\Lambda^{\nu}_{\mu_1\lambda} + \Lambda^{\nu}_{\mu_1\alpha}\Lambda^{\alpha}_{\mu_2\lambda} + \Lambda^{\nu}_{\mu_2\alpha}\Lambda^{\alpha}_{\mu_1\lambda}\big)\,, \tag{1.20}$$

$$\Gamma^{i}_{p\mu\lambda} = \partial_{\mu}K^{i}_{p\lambda} + K^{i}_{p\varrho}\Lambda^{\varrho}_{\mu\lambda}\,, \tag{1.21}$$

$$\begin{aligned}\Gamma^{i}_{p\mu_1\mu_2\lambda} = {} & \partial_{\mu_1\mu_2}K^{i}_{p\lambda} + \partial_{\mu_1}K^{i}_{p\varrho}\Lambda^{\varrho}_{\mu_2\lambda} + \partial_{\mu_2}K^{i}_{p\varrho}\Lambda^{\varrho}_{\mu_1\lambda} \\ & + \frac{1}{2}K^{j}_{p\varrho}\big(\partial_{\mu_1}\Lambda^{\varrho}_{\mu_2\lambda} + \partial_{\mu_2}\Lambda^{\varrho}_{\mu_1\lambda} + \Lambda^{\varrho}_{\mu_1\sigma}\Lambda^{\sigma}_{\mu_2\lambda} + \Lambda^{\varrho}_{\mu_2\sigma}\Lambda^{\sigma}_{\mu_1\lambda}\big)\,.\end{aligned} \tag{1.22}$$

Remark 1.7. The flow prolongation $\mathcal{W}^1K(\Lambda,K)$ we obtain by the projection π^2_1 of $\mathcal{W}^2K(\Lambda,K)$ on W^1PE, so it is given by (1.21) and projects on Λ and K.

Remark 1.8. The second prolongation $p(\Lambda,K)$ on W^1PE from Remark 1.6 is over the same connections Λ and K and has the coordinate expresion

$$\Gamma^{i}_{p\mu\lambda} = \partial_{\lambda}K^{i}_{p\mu} + K^{i}_{j\mu}K^{j}_{p\lambda} - K^{i}_{j\lambda}K^{j}_{p\mu} + K^{i}_{p\varrho}\Lambda^{\varrho}_{\lambda\mu}\,.$$

The difference $p(K,\Lambda) - \mathcal{W}^1\Gamma(K,\bar{\Lambda})$ is a section $M \to \mathrm{ad}(PE)\otimes\otimes^2T^*M$ given by

$$\partial_{\lambda}K^{i}_{p\mu} - \partial_{\mu}K^{i}_{p\lambda} + K^{i}_{j\mu}K^{j}_{p\lambda} - K^{i}_{j\lambda}K^{j}_{p\mu}\,, \tag{1.23}$$

where $\bar{\Lambda}$ denotes the classical conjugate connection to Λ (obtain by the exchange of indices). It is easy to see that (1.23) is exactly the curvature tensor of the connection K.

2. Classification of naturally induced connections on W^2PE

Let us recall that any principal connection Γ_2 on W^2PE projects by (1.16) on principal connections Λ_2 on P^2M and Γ_1 on W^1PE. So first we will classify principal connections on P^2M naturally given by a classical connection Λ and a general linear connection K (considered as a principal connection on the corresponding frame bundle).

2.1. **Natural connections on the 2nd order frame bundle.** The connection $\mathcal{W}^2K(\Lambda,K)$ described in Section 1.3 projects on the (natural) principal connection $\Lambda_2(\Lambda,K)$ on P^2M given by (1.20), i.e. it is independent of K. $\Lambda_2(\Lambda,K)$ can be constructed by an other construction. Let as remark that $P^2M \subset \bar{P}^2M \subset W^1P^1M$ are natural reductions with respect to the subgroups $G^2_m \subset \bar{G}^2_m \subset W^1_mG^1_m$, [9, p. 153]. Here $\bar{P}^2M = \mathrm{inv}\,\bar{J}^2_0(\mathbb{R}^m,M)$ and $\bar{G}^2_m = \mathrm{inv}\,\bar{J}^2_0(\mathbb{R}^m,\mathbb{R}^m)_0$, where $\bar{J}^2$ denotes the semiholonomic 2-jet functor. Then we can consider the flow lift of a principal connection Λ on P^1M with respect to the same principal connection Λ and obtain the principal connection $\mathcal{W}^1\Lambda = \mathcal{W}^1\Lambda(\Lambda,\Lambda)$ on W^1P^1M. This connection is reducible to the principal connection on $\bar{P}^2M$. Now, if Λ_2 is a principal connection on the semiholonomic 2-order frame bundle, then it can be considered as $\bar{G}^2_m$-invariant section $\Lambda_2\colon \bar{P}^2M \to J^1\bar{P}^2M$ and the composition $i(\Lambda_2) = J^1i\circ\Lambda_2$ is the principal connection on $\bar{P}^2M$ conjugated with Λ_2. Here $i\colon \bar{P}^2M \to \bar{P}^2M$ is the canonical natural involution. Let us remark, that if Λ_2 is given by coefficients $(\Lambda^{\nu}_{\mu\lambda}, \Lambda^{\nu}_{\mu_1\mu_2\lambda})$, then $i(\Lambda_2)$ is given by coefficients $(\Lambda^{\nu}_{\mu\lambda}, \Lambda^{\nu}_{\mu_2\mu_1\lambda})$. Then the connection $\frac{1}{2}(\mathcal{W}^1\Lambda + i(\mathcal{W}^1\Lambda))$ is reducible to the principal connection on

P^2M and we obtain the principal connection on P^2M which is exactly the above connection $\Lambda_2(\Lambda, K)$. In what follows we shall denote this connection by $\mathcal{P}^2\Lambda$ and call it the *flow prolongation* of Λ.

Any natural principal connection on P^2M given by Λ is projectable on a natural connection on P^1M given by Λ. So, we have to classify first all natural connections Λ_1 on P^1M given by Λ. This result is well known and we have, [9, p. 220].

Proposition 2.1. *All natural operators transforming a principal connection Λ on P^1M into principal connections $\Lambda_1(\Lambda)$ on P^1M is of zero order and form a 3-parameter family*

$$\Lambda_1(\Lambda) = \Lambda + \Phi_1(\Lambda)\,,$$

where $\Phi_1(\Lambda)$ is a natural $(1,2)$-tensor field of the form

$$\Phi_1 = a_1\, T + a_2\, \mathrm{Id}_{TM} \otimes \hat{T} + a_3\, \hat{T} \otimes \mathrm{Id}_{TM}\,, \qquad a_i \in \mathbb{R}\,,$$

where T denote the torsion tensor of Λ and $\hat{T}$ denote the contraction.

Corollary 2.1. *Any connection $\Lambda_2(\Lambda)$ on P^2M naturally given by Λ is projectable on a natural connection in the family $\Lambda_1(\Lambda)$.*

Theorem 2.1. *Let $\Lambda_2(\Lambda)$ on P^2M be a natural connection given by Λ projectable on $\Lambda_1(\Lambda)$ then the difference*

$$\Lambda_2(\Lambda) - \mathcal{P}^2\Lambda_1(\Lambda)\colon M \to TM \otimes T^*M \otimes S^2T^*M\,.$$

Proof. As a corollary of Theorem 1.1 we obtain that the kernel of the projection $\pi^2_1\colon \mathfrak{g}^2_m \to \mathfrak{g}^1_m$ is $\mathfrak{g}^1_m \otimes \mathbb{R}^{m*}$ which implies that the kernel of the projection $\mathrm{ad}(P^2M) \to \mathrm{ad}(P^1M)$ is the vector bundle $\mathrm{ad}(P^1M) \otimes T^*M$. The difference of two principle connections on P^2M over the same $\Lambda_1(\Lambda)$ on P^1M is then in the kernel $\ker(\pi^2_1 \otimes \mathrm{id}_{T^*M}) = \mathrm{ad}(P^1M) \otimes T^*M \otimes T^*M$. Moreover, $\mathrm{ad}(P^1M) = TM \otimes T^*M$ and from the symmetry property we obtain Theorem 2.1. □

Lemma 2.1. *Any natural tensor field $\Phi_2(\Lambda)\colon M \to TM \otimes T^*M \otimes S^2T^*M$ naturally given by Λ is of maximal order one and all $\Phi_2(\Lambda)$ form 16-parameter family.*

Proof. A non-symmetric connection Λ can be decomposed as the sum of the classical symmetric connection $\widetilde{\Lambda}$ (obtained by the symmetrization of Λ) and the torsion tensor T. From the reduction theorems for classical non-symmetric connections, [7], we obtain that $\Phi_2(\Lambda)$ is given as a zero order operator of the curvature tensor $R[\widetilde{\Lambda}]$ of $\widetilde{\Lambda}$ and its covariant differentials $\widetilde{\nabla} R[\widetilde{\Lambda}]$ (with respect to $\widetilde{\Lambda}$) and the torsion tensor T and its covariant differentials $\widetilde{\nabla} T$ (with respect to $\widetilde{\Lambda}$). From the homogeneous function theorem, [9, p. 213], it follows, that a natural $(1,3)$-tensor field can be constructed only from $R[\widetilde{\Lambda}]$, T and $\widetilde{\nabla} T$. So the maximal order of $\Phi_2(\Lambda)$ is one and

$$\Phi_2(\Lambda) = \varrho(R[\widetilde{\Lambda}]) + \tau(T) + \sigma(\widetilde{\nabla} T)\,,$$

where ϱ, τ and σ are given by tensorial operations from the indicated tensor fields.

Let X, Y, Z be vector fields on M. Then from the first Bianchi identity and the symmetry of $\Phi_2(\Lambda)$ in the last two vector variables we obtain that ϱ is given by the tensor product of $R[\widetilde{\Lambda}]$ with the identity of TM and contractions. Such operators form a 4-parameter family

$$\varrho(X,Y,Z) = b_1\big(R(X,Z)(Y) + R(X,Y)(Z)\big) + b_2 X\big(c_2^1 R(Y,Z) + c_2^1 R(Z,Y)\big) \\ + b_3\big(Y c_1^1 R(X,Z) + Z c_1^1 R(X,Y)\big) + b_4\big(Y c_2^1 R(X,Z) + Z c_2^1 R(X,Y)\big)\,,$$

where $b_i \in \mathbb{R}$. In what follows c_j^i denotes the contraction on the indicated indices.

Similarly τ is given by the tensor product of T (two times), the identity of TM and contractions. Such operators form a 7-parameter family

$$\begin{aligned}\tau(X,Y,Z) = {} & d_1\big(T(T(X,Z),Y) + T(T(X,Y),Z)\big) \\ & + d_2\big(T(X,Y)c_1^1 T(Z) + T(X,Y)c_1^1 T(Z)\big) \\ & + d_3 X c_{31}^{12}(T\otimes T)(Y,Z) + d_4 X c_{13}^{12}(T\otimes T)(Y,Z) \\ & + d_5\big(Y c_{12}^{12}(T\otimes T)(X,Z) + Z c_{12}^{12}(T\otimes T)(X,Y)\big) \\ & + d_6\big(Y c_{31}^{12}(T\otimes T)(X,Z) + Z c_{31}^{12}(T\otimes T)(X,Y)\big) \\ & + d_7\big(Y c_{13}^{12}(T\otimes T)(X,Z) + Z c_{13}^{12}(T\otimes T)(X,Y)\big)\,, \quad d_i \in \mathbb{R}\,.\end{aligned}$$

Finally, σ is given by $\widetilde{\nabla} T$, the tensor product with the identity of TM and contractions. We have a 5-parameter family

$$\begin{aligned}\sigma(X,Y,Z) = {} & e_1\big((\widetilde{\nabla}_Z T)(X,Y) + (\widetilde{\nabla}_Y T)(X,Z)\big) \\ & + e_2 X\big(c_1^1(\widetilde{\nabla}_Z T)(Y) + c_1^1(\widetilde{\nabla}_Y T)(Z)\big) \\ & + e_3\big(Y c_1^1(\widetilde{\nabla}_Z T)(X) + Z c_1^1(\widetilde{\nabla}_Y T)(X)\big) \\ & + e_4\big(Y c_1^1(\widetilde{\nabla}_X T)(Z) + Z c_1^1(\widetilde{\nabla}_X T)(Y)\big) \\ & + e_5\big(Y c_3^1(\widetilde{\nabla} T)(X,Z) + Z c_3^1(\widetilde{\nabla} T)(X,Y)\big)\,, \quad e_i \in \mathbb{R}\,.\end{aligned}$$ □

Theorem 2.2. *All natural principal connections $\Lambda_2(\Lambda)$ on P^2M given by a classical connection Λ on M form a 19-parameter family.*

Proof. Every natural connection $\Lambda_2(\Lambda)$ on P^2M has form $\mathcal{P}^2\Lambda_1(\Lambda)+\Phi_2(\Lambda)$, which is over $\Lambda_1(\Lambda) = \Lambda + \Phi_1(\Lambda)$. So this theorem is the corollary of Proposition 2.1 and Lemma 2.1. □

Remark 2.1. Let as remark that without using the reduction theorem natural principal connections on the semi-holonomic frame bundle $\bar{P}^2M$ naturally given by Λ was studied in [11, p. 185].

Corollary 2.2. *All natural principal connections on P^2M given by a classical connection Λ on M and by a principal connection K on PE form a 20-parameter family.*

Proof. It is sufficient to consider that Φ_1 and Φ_2 are natural tensor fields given by Λ and K. By the reduction theorem for general linear connections, [6], and the homogeneous function theorem we have that $\Phi_1(\Lambda, K) = \Phi_1(\Lambda)$ given by

Proposition 2.1. On the other hand for $\Phi_2(\Lambda, K)$ we obtain that $\Phi_2(\Lambda, K)$ include also the tensor product of the contracted curvature tensor $R[K]$ of K and the identity of TM. By using the the symmetry condition we have

$$\Phi_2(\Lambda, K)(X, Y, Z) = \Phi_2(\Lambda)(X, Y, Z) + k(Yc_1^1 R[K](X, Z) + Zc_1^1 R[K](X, Y)),$$

$k \in \mathbb{R}$. □

Corollary 2.3. *All natural connections on P^2M given by a classical symmetric connection Λ on M and by a principal connection K on PE form a 5-parameter family.*

2.2. **Classification of natural connections on W^2PE.** We consider all connections and tensor fields naturally given by Λ and K and we will not denote this fact explicitly.

Proposition 2.2. *Any natural principal connection Γ_2 on W^2PE is in a bijective correspondence with the triplet*

$$\Gamma_2 \approx (\Lambda_2, \Gamma_1, \Psi_2),$$

where Λ_2 is a natural principal connection on P^2M, Γ_1 is a natural principal connection on W^1PE and Ψ_2 is natural tensor field $M \to E \otimes E^* \otimes T^*M \otimes S^2T^*M$. Moreover, Λ_2 and Γ_1 are over the same Λ_1 on P^1M.

Proof. Let us assume a natural connection Γ_2' projectable on Γ_1 and Λ_2 (over the same Λ_1). The difference of Γ_2' with any other natural connection Γ_2 over the same natural connections Γ_1 and Λ_2 is by Corollary 1.1 natural tensor field Ψ_2. □

From Proposition 2.2 it follows that for the classification of Γ_2 we need to classify all natural Λ_2 (see Corollary 2.2), all natural Γ_1 and all natural tensor fields Ψ_2.

First, we have to classify all natural principal connections Γ_1 on W^1PE. Let as recall that we have the correspondence $\Gamma_1 \approx (\Lambda_1, \Gamma_0, \Psi_1)$, [9]. Λ_1 is described in Proposition 2.1.

Lemma 2.2. *All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections Γ_0 on PE are of zero order and form a 1-parameter family*

$$\Gamma_0 = K + \Psi_0 = K + m\, Id_E \otimes \hat{T}, \quad m \in \mathbb{R}.$$

Proof. If we consider any natural principal connection Γ_0 on PE, then the difference $\Gamma_0 - K$ is a section $\Psi_0 \colon M \to E \otimes E^* \otimes T^*M$. So to classify all natural Γ_0 it is sufficient to classify natural Ψ_0. Then from the reduction theorem and the homogeneous function theorem we obtain that Ψ_0 is of zero order and is obtained as a multiple of the tensor product of the identity of E and $(0,1)$-tensor field given by the contraction of the torsion of Λ. □

Proposition 2.3. *All natural operators transforming a classical connection Λ on M and a principal connection K on PE into principal connections Γ_1 on W^1PE are of the maximal order 1 and form a 14-parameter family.*

Proof. We have the 3-parameter family of connections Λ_1 given by Proposition 2.1, the 1-parameter family of connections Γ_0 given by Lemma 2.2. First, we have to prove that there exists a natural connection Γ_1' projectable on Λ_1 and Γ_0. But it is exactly the flow lift $\mathcal{W}^1\Gamma_0$ with respect to Λ_1. Then the difference $\Gamma_1' - \Gamma_1$ is a natural tensor field $\Psi_1\colon M \to E \otimes E^* \otimes T^*M \otimes T^*M$. From the reduction theorems, [6], and the homogeneous function theorem, [9], follows, that Ψ_1 is given by tensorial operations on from $R[K]$, $R[\widetilde{\Lambda}]$, T, $\widetilde{\nabla}T$ and the identity of E. So the maximal order of Ψ_1 is 1 and from the Bianchi identity and the symmetry condition we obtain 10-parameter family. □

Remark 2.2. Let us remark that in [8] Proposition 2.3 was proved without using the reduction theorem, so the base of 10-parameter family Ψ_1 is different.

Lemma 2.3. *All natural tensor fields $\omega\colon M \to T^*M \otimes S^2T^*M$ of type $(0,3)$ given by Λ and K are of the maximal order two and form a 31-parameter family.*

Proof. From the reduction theorems for classical non-symmetric connections and general linear connections, [6], and the homogeneous function theorem we obtain that ω is given by the tensorial operations from $R[\widetilde{\Lambda}]$, $\widetilde{\nabla}R[\widetilde{\Lambda}]$, T, $\widetilde{\nabla}T$, $\widetilde{\nabla}^2T$, $R[K]$ and $\widetilde{\nabla}R[K]$ ($\widetilde{\nabla}$ is assumed with respect to K and $\widetilde{\Lambda}$, see [5]). So natural ω is of the maximal order two.

The first part of ω has order two and is given by the contracted $\widetilde{\nabla}R[K]$, $\widetilde{\nabla}R[\widetilde{\Lambda}]$ and $\widetilde{\nabla}^2T$. From the Bianchi and the Ricci identities and the symmetry condition we obtain a 6-parameter family.

The second part of ω has order one and is given by $R[K]$, $R[\widetilde{\Lambda}]$ and $\widetilde{\nabla}T$ multiplied tensorially with T and two contractions. In this way we obtain a 18-parameter family.

The last part of ω has order zero and is given by the tensor product of T (three times) only and three contractions. In this way we obtain a 7-parameter family. □

Lemma 2.4. *All natural tensor fields $\Psi_2\colon M \to E \otimes E^* \otimes T^*M \otimes S^2T^*M$ given by Λ and K are of the maximal order two and form a 34-parameter family.*

Proof. As in Lemma 2.3 we have that all natural Ψ_2 are given by tensorial operations from $R[\widetilde{\Lambda}]$, $\widetilde{\nabla}R[\widetilde{\Lambda}]$, T, $\widetilde{\nabla}T$, $\widetilde{\nabla}^2T$, $R[K]$ and $\widetilde{\nabla}R[K]$.

Then Ψ_2 has two parts. The first one is given by $\widetilde{\nabla}R[K]$ or by a composition of $R[K]$ and T. By using the symmetry property and the Bianchi identity we obtain 3-parameter family. Exactly we have

$$\begin{aligned}
&n_1\big((\widetilde{\nabla}_Z R[K])(X,Y) + (\widetilde{\nabla}_Y R[K])(X,Z)\big)+\\
&n_2\big(R[K](X,Y)c_1^1T(Z) + R[K](X,Z)c_1^1T(Y)\big)+\\
&n_3\big(R[K](T(X,Z),Y) + R[K](T(X,Y),Z)\big),
\end{aligned}$$

where X, Y, Z are vector fields on M.

The second part is of the form $Id_E \otimes \omega$, where ω is the natural tensor of type $(0,3)$ described in Lemma 2.3. All together we have a 34-parameter family. □

Theorem 2.3. *Natural principal connections on W^2PE given by a classical connection Λ on M and by a principal connection K on PE form a* 65*-parameter family.*

Proof. By Proposition 2.2 and by [9] we have

$$\Gamma_2 \approx (\Lambda_2, \Gamma_1, \Psi_2) \approx (\Lambda_1, \Phi_2, \Gamma_0, \Psi_1, \Psi_2) \approx (\Lambda, \Phi_1, \Phi_2, K, \Psi_0, \Psi_1, \Psi_2).$$

By Proposition 2.1 Φ_1 form a 3-parameter family, by proof of Corollary 2.2 Φ_2 form a 17-parameter family, by Lemma 2.2 Ψ_0 form a 1-parameter family, by [8] Ψ_1 form a 10-parameter family, and, finally, by Lemma 2.4 Ψ_2 form a 34-parameter family. All together we obtain a 65-parameter family of natural connections. $\square$

Corollary 2.4. *All natural connections on W^2PE given by a classical symmetric connection Λ on M and by a principal connection K on PE form a* 13*-parameter family.*

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