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$\varphi(\boldsymbol{Ric})$-VECTOR FIELDS IN RIEMANNIAN SPACES

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Abstract. In this paper we study vector fields in Riemannian spaces, which satisfy $\nabla\varphi = \mu$, $\boldsymbol{Ric}$, $\mu = \text{const}$. We investigate the properties of these fields and the conditions of their coexistence with concircular vector fields. It is shown that in Riemannian spaces, noncollinear concircular and $\varphi(\boldsymbol{Ric})$-vector fields cannot exist simultaneously. It was found that Riemannian spaces with $\varphi(\boldsymbol{Ric})$-vector fields of constant length have constant scalar curvature. The conditions for the existence of $\varphi(\boldsymbol{Ric})$-vector fields in symmetric spaces are given.

1. Introduction

In this paper we study a class of curvature-determined vector fields in Riemannian spaces, which are in some sense modifications of concircular vector fields. Concircular vector fields are characterized by the property that their covariant derivative is proportional to the unity tensor, $\nabla\xi = \rho\cdot\mathrm{Id}$, with ρ being a function on the manifold [3, 5, 6, 7, 8, 9, 10]. Examples of Riemannian spaces with concircular vector fields, in the language of geometry called equidistant spaces, are the well-known spatially homogeneous and isotropic cosmological models of space-time (pseudo-Riemannian manifolds with Friedmann-Lemaitre-Robertson-Walker metric).

Riemannian spaces with concircular vector fields were studied in the work of H. W. Brinkmann, A. Fialkov, K. Yano, H. de Vries on conformal mappings and of N. S. Sinjukov, A. S. Solodovnikov, D. I. Rosenfeld, J. Mikeš, V. A. Kiosak, I. Hinterleitner, G. Hall and others on geodesic mappings.

Einstein spaces, another class of cosmological models, are characterized by the proportionality of the Ricci tensor to the metric tensor, so that in these spaces concircular vector fields could equally well be defined by $\nabla\xi = \rho\cdot\boldsymbol{Ric}$. This inspires us to a general investigation of vector fields satisfying the latter relation and the conditions for their existence in general (i.e. non-Einstein) Riemannian spaces, with the specialisation $\rho = \mu = \text{const}$.

2. $\varphi(\boldsymbol{Ric})$-vector fields

Definition 2.1. A $\varphi(\boldsymbol{Ric})$-*vector field* is a vector field φ on an n dimensional Riemannian manifold (M^n, g) with metric g and Levi-Civita connection ∇, which

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satisfies the condition

$$\nabla \boldsymbol{\varphi} = \mu \ \boldsymbol{Ric}\,, \tag{2.1}$$

where μ is some constant and $\boldsymbol{Ric}$ is the Ricci tensor.

Obviously, when (M^n, g) is an Einstein space, the vector field $\boldsymbol{\varphi}$ is concircular. Moreover, when $\mu = 0$, the vector field $\boldsymbol{\varphi}$ is covariantly constant.

In the following we suppose that $\mu \neq 0$ and (M^n, g) is neither an Einstein space nor a vacuum solution of the Einstein equations.

In a locally coordinate neighbourhood $U(x)$ equation (2.1) is written as

$$\varphi^h_{,i} = \mu\, R^h_i\,, \tag{2.2}$$

where φ^i and R^h_i are components of $\boldsymbol{\varphi}$ and $\boldsymbol{Ric}$, respectively " , " denotes covariant derivative in (M^n, g). After lowering indices (2.2) has the form

$$\varphi_{i,j} = \mu\, R_{ij}\,, \tag{2.3}$$

where $\varphi_i = \varphi^\alpha\, g_{i\alpha}$ and $R_{ij} = g_{i\alpha}\, R^\alpha_j$.

Here and in the following g_{ij} are the components of the metric tensor and g^{ij} are the components of its inverse.

As it is well known, the components of the Ricci tensor are symmetric in Riemannian spaces. For this reason $\varphi_{i,j} = \varphi_{j,i}$.

We make the remark that from the symmetry of the covariant derivative follows the local existence of a function $\varphi(x)$, the gradient of which is φ_i, $\varphi_i = \partial\varphi(x)/\partial x^i$.

As well known, the Ricci identities for a vector field φ have the following form $\varphi^h_{,jk} - \varphi^h_{,kj} = -\varphi^\alpha R^h_{\alpha jk}$.

Using them we obtain from the antisymmetric covariant derivative of (2.3)

$$\varphi_\alpha R^\alpha_{ijk} = \mu \cdot (R_{ij,k} - R_{ik,j})$$

and applying the Bianchi identity to the right hand side yields the integrability condition of (2.3) in the form:

$$\varphi_\alpha R^\alpha_{ijk} = \mu\, R^\alpha_{ijk,\alpha}\,, \tag{2.4}$$

where R^h_{ijk} is the Riemannian tensor of (M^n, g).

This equation is contracted by g^{ij}, and from the contracted Bianchi identity $R^\alpha_{k,\alpha} = \frac{1}{2}\, R_{,k}$, we can see

$$\varphi_\alpha R^\alpha_k = \frac{\mu}{2}\, R_{,k}\,, \tag{2.5}$$

where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature.

Now we consider $\boldsymbol{\varphi}(\boldsymbol{Ric})$-vector fields of constant length $|\boldsymbol{\varphi}| = \sqrt{|\varphi^\alpha \varphi^\beta g_{\alpha\beta}|}$ or

$$\varphi^\alpha \varphi^\beta g_{\alpha\beta} = \text{const.} \tag{2.6}$$

Differentiating (2.6) and making use of (2.2) we obtain $\mu \cdot \varphi_\alpha R^\alpha_i = 0$. Due to the assumption $\mu \neq 0$ we have $\varphi_\alpha R^\alpha_i = 0$. With the condition (2.5) this leads to

$$R_{,k} = 0\,,$$

this means that the scalar curvature of (M^n, g) is necessarily constant. In this way we have proved the following theorem.

Theorem 2.1. *Riemannian or pseudo-Riemannian spaces* (M^n, g) *with a* $\varphi(\mathbf{Ric})$*-vector field of constant length have constant scalar curvature.*

3. $\varphi(\mathbf{Ric})$ AND CONCIRCULAR VECTOR FIELDS

As we have already mentioned, a nonzero vector field $\boldsymbol{\xi}$ is called concircular if $\nabla\boldsymbol{\xi} = \rho \cdot \mathrm{Id}$, where ρ is a function on (M^n, g). In coordinates this condition has the form $\xi^h_{,i} = \rho\,\delta^h_i$. Introducing $\xi_i = g_{i\alpha}\xi^\alpha$, we can rewrite this as

$$\xi_{i,j} = \rho\, g_{ij}\,. \tag{3.1}$$

When $\rho = \text{const.}$, the vector field $\boldsymbol{\xi}$ is called *convergent*, in the case $\rho = 0$ it is *covariantly constant.*

We will study the question whether a $\varphi(\mathbf{Ric})$-vector field and a concircular field $\boldsymbol{\xi}$ can exist simultaneously.

We prove the following theorem

Theorem 3.1. *If a* $\varphi(\mathbf{Ric})$*-vector field exists together with a non-collinear concircular vector field on a Riemannian manifold, then the latter one is necessarily covariantly constant.*

Proof. Assume a $\varphi(\mathbf{Ric})$-vector field $\boldsymbol{\varphi}$ and a concircular vector field $\boldsymbol{\xi}$, characterized by the respective equations (2.3) and (3.1). After some easy calculations, the integrability conditions of (3.1) can be written in the form

$$\xi_\alpha R^\alpha_{ijk} = c(\xi)(g_{ij}\xi_k - g_{ik}\xi_j)\,, \tag{3.2}$$

where $c(\xi)$ is some function, depending on the function ξ, which in turn locally generates the covector ξ_i, i.e. $\xi_i = \partial\xi(x)/\partial x^i$.

By differentiating the last formula and by using (3.1) and certain properties of the Riemann tensor we obtain

$$\rho\, R_{lijk} + \xi_\alpha R^\alpha_{ijk,l} = c'(\xi)\,(g_{ij}\xi_k\xi_l - g_{ik}\xi_j\xi_l) + c(\xi)\,\rho\,(g_{ij}g_{kl} - g_{ik}g_{jl})\,,$$

where $R_{lijk} = g_{\alpha l}R^\alpha_{ijk}$, and $c'(\xi)$ is the derivation of the function $c(\xi)$.

After contracting the indices k and l, taking into account (2.4) and (3.2), we have

$$\mu\,\rho\, R_{ij} = (\mu\, c\,\xi_\alpha\xi^\alpha + (n-1)\,\rho\,\mu\, c + c\,\xi_\alpha\varphi^\alpha)\, g_{ij} + c\;\xi_j\varphi_i - \mu\, c'\;\xi_i\xi_j\,. \tag{3.3}$$

Alternating (3.3) we see that

$$c\,(\xi_j\varphi_i - \xi_i\varphi_j) = 0\,.$$

Because the vanishing of the expression in parentheses would mean collinearity of the vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\xi}$, which is excluded, c must be equal to zero.

But then (3.3) acquires the form $\mu\rho R_{ij} = 0$. Because $\mu R_{ij} \neq 0$, ρ must vanish. We now conclude from (3.1) that $\nabla\boldsymbol{\xi} = 0$, i.e. $\boldsymbol{\xi}$ is a covariantly constant vector field, which establishes the theorem. □

4. $\boldsymbol{\varphi}(\boldsymbol{Ric})$-vector fields in symmetric spaces

We consider $\boldsymbol{\varphi}(\boldsymbol{Ric})$-vector fields in symmetric spaces (M^n, g), characterized by the covariant constance of the Riemann tensor $\nabla R = 0$, written in coordinates as

$$R^h_{ijk,l} = 0\,. \tag{4.1}$$

We start from equations (2.2) which characterize $\boldsymbol{\varphi}(\boldsymbol{Ric})$-vector fields. In symmetric spaces the integrability conditions (2.4) of these equations simplify due to (4.1)

$$\varphi_\alpha R^\alpha_{ijk} = 0\,. \tag{4.2}$$

Application of (2.2), (4.1) and the fact that $\mu \neq 0$, to the differential prolongations of (4.2) yields

$$R_{\alpha l} R^\alpha_{ijk} = 0\,. \tag{4.3}$$

Formula (4.3) has an intrinsic character in the space (M^n, g). Its index-free form is $\boldsymbol{Ric}(X, R(Y,Z)V) = 0$ for all tangent fields X, Y, Z, V. In symmetric spaces its differential prolongation is satisfied identically.

As we can easily see, it follows from the theory of partial differential equations that in a symmetric space, satisfying condition (4.3), the equations

$$\varphi^h_{,i} = \mu\, R^h_i\,, \quad \mu = \text{const.}\,,$$

have locally a unique solution for an arbitrary constant μ and arbitrary initial conditions

$$\varphi^h(x_0) = \overset{0}{\varphi}{}^h\,,$$

which in the point x_0 fulfill the condition

$$\overset{0}{\varphi}{}_\alpha R^\alpha_{ijk}(x_0) = 0\,.$$

From (4.3) it follows that the last conditions have a nontrivial solution $\varphi^h(x_0)$. In his way we can formulate the following theorem.

Theorem 4.1. *In a non-Einstein symmetric pseudo-Riemannian space (M^n, g) with $\boldsymbol{Ric}\big(X, R(Y,Z)V\big) = 0$ for all tangent fields X, Y, Z, V there exists locally a $\boldsymbol{\varphi}(\boldsymbol{Ric})$-vector field.*

We make the remark that in symmetric spaces no concircular vector fields other than covariantly constant ones exist. We can easily convince ourselves of the existence of symmetric spaces satisfying conditions (4.3). An example of a non-Einstein symmetric pseudo-Riemannian space (M^n, g) satisfying the condition (4.3) is a space with a metric of the following form

$$\mathrm{d}s^2 = \exp(2x_1)\big\{2\mathrm{d}x^1\mathrm{d}x^2 + e_3(\mathrm{d}x^3)^2 + \dots + e_n(\mathrm{d}x^n)^2\big\}\,,$$

where $e_i = \pm 1$, $i = 3, \dots, n$.

Lemma 4.1. *Classical Riemannian spaces (M^n, g) with positive definite metric and the above properties (i.e. non-Einstein symmetric true Riemannian spaces which satisfy conditions* (4.3)*) do not exist.*

Proof. After contraction of the formulae (4.3) follows $R_{\alpha\beta}R_{\gamma\mu}g^{\alpha\gamma}g^{\beta\mu} = 0$, this means when the metric is positive definite, then $R_{ij} = 0$, in contradiction to the assumption $R_{ij} \neq 0$. □

5. A simple example of a $\varphi(\boldsymbol{Ric})$-vector field

Our example is a non-isotropic generalization of an equidistant space, motivated by the Kasner vacuum metric in general relativity [4]. For simplicity we have restricted ourselves to a $2+1$ dimensional Riemannian space with diagonal metric in the coordinates x^1, x^2, x^3,

$$\mathrm{d}s^2 = -(\mathrm{d}x^1)^2 + f(x^1)(\mathrm{d}x^2)^2 + g(x^1)(\mathrm{d}x^3)^2\,, \tag{5.1}$$

where f and g are C^2 functions of the first coordinate.

We assume the existence of a $\varphi(\boldsymbol{Ric})$-vector field in the x^1 direction in the form

$$\varphi^i = \big(\varphi^1(x^1), 0, 0\big)\,,$$

depending only on the coordinate x^1, too. After some calculations based on (2.3) with $\mu = 1$, we obtain a one-parameter family of solutions for the metric components

$$f(x^1) = (x^1)^{\,2\cos\theta}\,, \qquad g(x^1) = (x^1)^{\,2\sin\theta}\,, \tag{5.2}$$

with the parameter θ conveniently restricted by $\theta \in \langle 0, 2\pi\rangle$.

The non-vanishing components of the Ricci tensor are

$$R_{11} = (1-\cos\theta-\sin\theta)\,(x^1)^{-2}\,, \qquad R_{22} = \cos\theta\,(1-\cos\theta-\sin\theta)\,(x^1)^{-2(1-\cos\theta)}\,, \tag{5.3}$$

$$R_{33} = (1-\cos\theta)\,(\sin\theta-1-\cos\theta)\,(x^1)^{-2(1-\sin\theta)} \tag{5.4}$$

and the scalar curvature is

$$R = -2(1-\cos\theta)\,(1-\sin\theta)\,(x^1)^{-2}\,. \tag{5.5}$$

In this space the $\varphi(\boldsymbol{Ric})$-vector field is given by the component

$$\varphi^1 = (1-\cos\theta-\sin\theta)(x^1)^{-1}\,. \tag{5.6}$$

The Riemannian space with the above metric (5.1) with components (5.2) provides a nontrivial example of a space with a $\varphi(\boldsymbol{Ric})$-vector field, which is neither equidistant, nor an Einstein space, nor a space of constant curvature. The functional form (5.2) of the components is of Kasner type, but, of course, the metric is not a vacuum solution.

The vector field $\boldsymbol{\varphi} \neq 0$ if and only if $\theta \neq 0,\ \pi/2$. In the special cases for $\theta = \pi$, $3\pi/2$, $\pi/4$, $5\pi/4$ the space is equidistant. From the expression (5.5) we see that in the generic case the metric (5.1), (5.2) displays a curvature singularity at $x_1 = 0$, like the Kasner metric.

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