ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 105–114

COMPLETE SPACELIKE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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Abstract. In this paper, we characterize the n-dimensional ($n \geq 3$) complete spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple.We show that M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in S_1^{n+1} and is contained in an $(n-1)$-dimensional sphere $S^{n-1}\Big(c(s)\Big) = E^n(s) \cap S_1^{n+1}$ and is of constant curvature $\Big(\frac{d\{\log|\lambda^2-(1-R)|^{1/n}\}}{ds}\Big)^2 - \lambda^2 + 1$,where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$ and $u = \Big|\lambda^2 - (1-R)\Big|^{-\frac{1}{n}}$ satisfies the ordinary differental equation of order 2, $\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0$.

1. Introduction

Let R_1^{n+2} be the $(n+2)$-dimensional Lorentz-Minkowski space and S_1^{n+1} be the de Sitter space given by $S_1^{n+1} = \{p \in R_1^{n+2} \mid \langle p,p\rangle p = 1\}$. A hypersurface M^n of S_1^{n+1} is said to be spacelike if the induced metric on M^n from that of ambient space is positive definite. In [4] we investigated the spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures whose multiplicities are greater than 1. We showed that

Theorem 1.1 ([4]). *Let M^n be an n-dimensional complete spacelike hypersurface in S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then M^n is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$, $1 < k < n-1$.*

As we know that Otsuki [3] characterized the minimal hypersurfaces in a Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with two distinct principal curvatures

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2000 *Mathematics Subject Classification:* Primary: 53C42; Secondary: 53C20.

Key words and phrases: de Sitter space, spacelike hypersurface, scalar curvature, principal curvature, umbilical.

Project supported by NSF of Shaanxi Province (2003A02) and NSF of Shaanxi Educational Committee (2003JK215).

Received July 14, 2007. Editor J. Slovák.

one of which is simple and Cheng [2] investigated the n-dimensional oriented complete hypersurfaces ($n \geq 3$) in Euclidean space R^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple. It is natural and important to investigate the spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple. In this paper, we obtain the following

Theorem 1.2. *Let M^n be an n-dimensional ($n \geq 3$) complete spacelike hypersurface in a de Sitter space S_1^{n+1} with constant scalar curvature $n(n-1)R$ and with two distinct principal curvatures one of which is simple, then M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in S_1^{n+1} and is contained in an $(n-1)$-dimensional sphere $S^{n-1}\big(c(s)\big) = E^n(s) \cap S_1^{n+1}$ and is of constant curvature $\big(\frac{d\{\log|\lambda^2-(1-R)|^{1/n}\}}{ds}\big)^2 - \lambda^2 + 1$, where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$ and $u = \big|\lambda^2 - (1-R)\big|^{-\frac{1}{n}}$ satisfies the ordinary differental equation of order 2*

$$\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0\,.$$

2. Preliminaries

Let M^n be an n-dimensional spacelike hypersurfaces in S_1^{n+1}, we choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+1}$ in S_1^{n+1} such that at each point of $M^n, e_1, \dots, e_n$ span the tangent space of M^n and form an othonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1\,; \quad 1 \leq i, j, k, \dots \leq n\,.$$

Let $\omega_1, \dots, \omega_{n+1}$ be the dual frame field so that the semi-Riemannian metric of S_1^{n+1} is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

The structure equations of S_1^{n+1} are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}\,, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D\,, \tag{2.3}$$

$$K_{ABCD} = \epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,. \tag{2.4}$$

Restrict these forms to M^n, we have

$$\omega_{n+1} = 0\,. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij}\wedge\omega_j\,, \quad \omega_{ij}+\omega_{ji}=0\,, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l\,, \tag{2.8}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}) - (h_{ik}h_{jl}-h_{il}h_{jk})\,, \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i\otimes\omega_j \tag{2.10}$$

is the second fundamental form of M^n.

From the above equation, we have

$$n(n-1)R = n(n-1) - n^2H^2 + |h|^2\,, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature of M^n, H is the mean curvature, and $|h|^2 = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form of M^n.

The Codazzi equation and the Ricci identity are

$$h_{ijk} = h_{ikj}\,, \tag{2.12}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}\,, \tag{2.13}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

We choose $e_1,\ldots,e_n$ such that $h_{ij}=\lambda_i\delta_{ij}$. From (2.6) we have

$$\omega_{n+1i} = \lambda_i\omega_i\,, \quad i=1,2,\ldots,n\,. \tag{2.14}$$

Hence, we have from the structure equations of M^n

$$\begin{aligned} d\omega_{n+1i} &= d\lambda_i\wedge\omega_i + \lambda_i d\omega_i \\ &= d\lambda_i\wedge\omega_i + \lambda_i\sum_j \omega_{ij}\wedge\omega_j. \end{aligned} \tag{2.15}$$

On the other hand, we have on the curvature forms of S_1^{n+1},

$$\begin{aligned} \Omega_{n+1i} &= -\frac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C\wedge\omega_D \\ &= \frac{1}{2}\sum_{C,D}(\delta_{n+1C}\delta_{iD}-\delta_{n+1D}\delta_{iC})\omega_C\wedge\omega_D \\ &= \omega_{n+1}\wedge\omega_i = 0\,. \end{aligned} \tag{2.16}$$

Therefore, from the structure equations of S_1^{n+1}, we have

$$\begin{aligned} d\omega_{n+1i} &= \sum_j \omega_{n+1j} \wedge \omega_{ji} - \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} \\ &= \sum_j \lambda_j \omega_{ij} \wedge \omega_j . \end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we obtain

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0 \, . \tag{2.18}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij} \, . \tag{2.19}$$

Then $\psi_{ij} = \psi_{ji}$. (2.18) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0 \, . \tag{2.20}$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk}\omega_k \, , \tag{2.21}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj} \, . \tag{2.22}$$

3. Proof of theorem

We firstly have the following Proposition 3.1 due to [1], which original due to Otsuki [3] for Riemannian space forms.

Proposition 3.1 ([1])**.** *Let M^n be a spacelike hypersurface in S_1^{n+1} such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature λ is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of Theorem 1.2. Let M^n be an n-dimensional complete spacelike hypersurface with constant scalar curvature and with two distinct principal curvatures one of which is simple, that is, without lose of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda \, , \quad \lambda_n = \mu \, ,$$

where λ_i for $i = 1, 2, \ldots, n$ are the principal curvatures of M^n. Since the scalar curvature $n(n-1)R$ is constant, from (2.11), we obtain

$$n(n-1)(1-R) = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\mu \, . \tag{3.1}$$

If $\lambda = 0$ at some points, then $R = 1$ at these points from (3.1), since R is constant, we know $R = 1$ on M^n. Since these principal curvatures λ and μ are continuous on M^n, from (3.1) and $R = 1$ we obtain $\lambda = 0$ on M^n. Hence, from the Gauss equation, the sectional curvature of M^n $R_{ijij} = 1-\lambda\mu = 1 > 0$, by Myers' theorem we know that M^n is compact. From the result of Zheng [6, 5], we know that M^n is a totally umbilical spacelike hypersurface. This is impossible because we assumed that M^n is of two distinct principal curvatures. Hence, we can assume $\lambda \neq 0$ on M^n. From (3.1), we have

$$\mu = \frac{n(1-R)}{2\lambda} - \frac{(n-2)\lambda}{2}\,. \tag{3.2}$$

Therefore, we get

$$\lambda - \mu = n\frac{\lambda^2 - (1-R)}{2\lambda} \neq 0\,,$$

we know $\lambda^2 - (1-R) \neq 0$.

Let $u = \left|\lambda^2 - (1-R)\right|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\,\omega_k\,, \quad d\mu = \sum_{k=1}^{n} \mu_{,k}\,\omega_k\,. \tag{3.3}$$

From Proposition 3.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.4}$$

From (3.2), we have

$$d\mu = \Big[-\frac{n(1-R)}{2\lambda^2} - \frac{n-2}{2}\Big]\,d\lambda\,. \tag{3.5}$$

Hence, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.6}$$

In this case, we may consider locally λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.21) and (3.4), we have for $1 \leq j \leq n-1$,

$$\begin{aligned} d\lambda = d\lambda_j &= \sum_{k=1}^{n} Q_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda_{,n}\,\omega_n\,. \end{aligned} \tag{3.7}$$

Therefore, we have

$$Q_{jjk} = 0\,, \quad 1 \leq k \leq n-1\,, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\,. \tag{3.8}$$

By (2.21) and (3.6), we have

$$\begin{aligned} d\mu = d\lambda_n &= \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^{n} \mu_{,i}\,\omega_i = \mu_{,n}\,\omega_n\,. \end{aligned} \tag{3.9}$$

Hence, we obtain

$$Q_{nnk} = 0\,, \quad 1 \le k \le n-1\,, \quad \text{and} \quad Q_{nnn} = \mu_{,n}\ . \tag{3.10}$$

From (3.5), we get

$$Q_{nnn} = \mu_{,n} = \Big[-\frac{n(1-R)}{2\lambda^2} - \frac{n-2}{2}\Big]\lambda_{,n}\ . \tag{3.11}$$

From the definition of ψ_{ij}, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.21), if $i \neq j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$Q_{ijk} = 0\,, \quad \text{for any} \quad k\,. \tag{3.12}$$

By (2.21), (3.8), (3.10), (3.11) and (3.12), we get

$$\begin{aligned}\psi_{jn} &= \sum_{k=1}^{n} Q_{jnk}\omega_k \\ &= Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\,\omega_j\ .\end{aligned} \tag{3.13}$$

Since λ and μ are two distinct principal curvatures of M^n, we have

$$\lambda - \mu = n\frac{\lambda^2 - (1-R)}{2\lambda} \neq 0\,.$$

From (2.19), (3.2)and (3.13)we have

$$\begin{aligned}\omega_{jn} &= \frac{\psi_{jn}}{\lambda-\mu} = \frac{\lambda_{,n}}{\lambda-\mu}\omega_j \\ &= \frac{\lambda_{,n}}{\lambda - [\frac{n(1-R)}{2\lambda} - \frac{n-2}{2}\lambda]}\omega_j \\ &= \frac{2\lambda\lambda_{,n}}{n[\lambda^2-(1-R)]}\,\omega_j\ .\end{aligned} \tag{3.14}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1}\omega_k \wedge \omega_{kn} + \omega_{nn}\wedge\omega_n = 0\,.$$

Therefore, we may put $\omega_n = ds$. By (3.7) and (3.9), we get

$$d\lambda = \lambda_{,n}\,ds\,, \quad \lambda_{,n} = \frac{d\lambda}{ds}\,,$$

and

$$d\mu = \mu_{,n}\,ds\,, \quad \mu_{,n} = \frac{d\mu}{ds}\ .$$

Then we have

$$\omega_{jn} = \frac{2\lambda\lambda_{,n}}{n[\lambda^2-(1-R)]}\,\omega_j = \frac{2\lambda\frac{d\lambda}{ds}}{n[\lambda^2-(1-R)]}\,\omega_j = \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\,\omega_j\,. \tag{3.15}$$

(3.15) shows that the integral submanifold $M_1^{n-1}(x)$ corresponding to λ and s is umbilical in M^n and S_1^{n+1}.

From (3.15) and the structure equations of S_1^{n+1}, we have

$$\begin{aligned} d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} + \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} \\ &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn+1}\wedge\omega_{n+1n} - \omega_j\wedge\omega_n \\ &= \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (\lambda\mu+1)\omega_j\wedge ds\,. \end{aligned}$$

From (3.15) we have

$$\begin{aligned} d\omega_{jn} &= \frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}d\omega_j \\ &= \frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k \\ &= \Big\{-\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} + \Big[\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big]^2\Big\}\omega_j\wedge ds \\ &\quad + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k\,. \end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} - \Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\}^2 - (\lambda\mu+1) = 0\,. \tag{3.16}$$

From (3.2) we get

$$\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} - \Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\}^2 + \frac{n-2}{2}\big[\lambda^2-(1-R)\big] + R - 2 = 0\,. \tag{3.17}$$

Since we define $u = \big|\lambda^2-(1-R)\big|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0\,. \tag{3.18}$$

Since S_1^{n+1} is an $(n+1)$-dimensional de Sitter space of constant 1 in R_1^{n+2}. We consider the frame $e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}$ in R_1^{n+2}. Since the second fundamental form of S_1^{n+1} as the hypersurface R_1^{n+2} is given by $h_{AB} = -\sum_B \epsilon_B \delta_{AB}$, we have

$$\omega_{n+1n+2} = 0\,, \quad \text{and} \quad \omega_{in+2} = -\omega_i\,.$$

Then, from (2.14), (3.15) and (3.16), we have

$$\begin{aligned} de_i &= \sum_{j=1}^{n-1} \omega_{ij} e_j + \omega_{in} e_n + \omega_{in+1} e_{n+1} + \omega_{in+2} e_{n+2} \\ &= \sum_{j=1}^{n-1} \omega_{ij} e_j + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, \omega_i e_n - \lambda \omega_i e_{n+1} - e_{n+2}\omega_i \\ &= \sum_{j=1}^{n-1} \omega_{ij} e_j + \Big[\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big]\, \omega_i\,. \end{aligned}$$

On the other hand, by means of (3.16) we get

$$\begin{aligned} d\Big\{ &\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big\} = d\Big\{\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\Big\}\, e_n \\ &+ \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, de_n - d\lambda e_{n+1} - \lambda de_{n+1} - de_{n+2} \\ = &\Big\{\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} e_n - \frac{d\lambda}{ds} e_{n+1}\Big\}\, ds \\ &+ \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} \Big(\sum_{j=1}^{n-1} \omega_{nj} e_j + \omega_{nn+1} e_{n+1} + \omega_{nn+2} e_{n+2}\Big) \\ &- \lambda\Big(\sum_{j=1}^{n-1} \omega_{n+1j} e_j + \omega_{n+1n} e_n + \omega_{n+1n+2} e_{n+2}\Big) - \sum_{j=1}^{n-1} \omega_i e_j - \omega_n e_n \\ = &\Big\{\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} e_n - \frac{d\lambda}{ds} e_{n+1}\Big\}\, ds \\ &+ \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} \Big(\sum_{j=1}^{n-1} \omega_{nj} e_j - \mu \omega_n e_{n+1} - \omega_n e_{n+2}\Big) \\ &- \lambda\Big(\lambda \sum_{j=1}^{n-1} \omega_j e_j + \mu \omega_n e_n\Big) - \sum_{j=1}^{n-1} \omega_i e_j - \omega_n e_n \\ \equiv &\Big[\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} - \lambda\mu - 1\Big]\, e_n \omega_n \\ &- \Big\{\frac{d\lambda}{ds} + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, \mu\Big\} e_{n+1} \omega_n \end{aligned}$$

$$- \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} e_{n+2}\omega_n \qquad (\mathrm{mod}\{e_1,\dots,e_{n-1}\})$$
$$= \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big\}\, ds\,.$$

We put

$$W = e_1\wedge\cdots\wedge e_{n-1}\wedge\Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\, e_n - \lambda e_{n+1} - e_{n+2}\Big\}\,.$$

Therefore we have

$$dW = \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\, W ds\,, \tag{3.19}$$

(3.19) shows that n-vector W in R_1^{n+2} is constant along $M_1^{n-1}(x)$. Hence, there exists an n-dimensional linear subspace $E^n(s)$ in R_1^{n+2} containing $M_1^{n-1}(x)$. By (3.19), the n-vector field W depends only on s and by integrating it, we get

$$W = \Big\{\frac{\lambda^2(s)-(1-R)}{\lambda^2(s_0)-(1-R)}\Big\}^{\frac{1}{n}} = W(s_0)\,.$$

Hence, we have that $E^n(s)$ is parallel to $E^n(s_0)$ in R_1^{n+2}.

Since $\Omega_{ij} = -\omega_i\wedge\omega_j$, from (2.2) the curvature of $M_1^{n-1}(x)$ is given by

$$d\omega_{ij} - \sum_{k=1}^{n-1}\omega_{ik}\wedge\omega_{kj} = \omega_{in}\wedge\omega_{nj} - \omega_{in+1}\wedge\omega_{n+1j} - \omega_i\wedge\omega_j$$
$$= -\Big\{\Big(\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big)^2 - \lambda^2 + 1\Big\}\omega_i\wedge\omega_j\,.$$

Therefore we know that the curvature of $M_1^{n-1}(x)$ is $\big(\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\big)^2 - \lambda^2 + 1$ and $M_1^{n-1}(x)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s)\cap S_1^{n+1}$ of the intersection of S_1^{n+1} and an n-dimensional linear subspace $E^n(s)$ in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$. This completes the proof of Theorem 1.2.

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