ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 307 – 319

A PROPERTY OF WALLACH'S FLAG MANIFOLDS

TERESA ARIAS-MARCO

Dedicated to my advisors Prof. O. Kowalski and Prof. A. M. Naveira.

ABSTRACT. In this note we study the Ledger conditions on the families of flag manifold $(M^6 = SU(3)/SU(1) \times SU(1) \times SU(1), g_{(c_1,c_2,c_3)})$, $\big(M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2), g_{(c_1,c_2,c_3)}\big)$, constructed by N. R. Wallach in [14]. In both cases, we conclude that every member of the both families of Riemannian flag manifolds is a D'Atri space if and only if it is naturally reductive. Therefore, we finish the study of M^6 made by D'Atri and Nickerson in [7]. Moreover, we correct and improve the result given by the author and A. M. Naveira in [3] about M^{12}.

1. INTRODUCTION

A *Riemannian homogeneous space* $(G/H, g)$ with its origin $p = \{H\}$ is always a *reductive homogeneous space* in the following sense (cf. [9, vol.II, p.190]): we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There is a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ which is obviously $\mathrm{Ad}(H)$-invariant.

The following definition is well known from [9, Chapter X, sections 2, 3]:

Definition 1. A Riemannian homogeneous space $(G/H, g)$ is said to be *naturally reductive* if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of $\mathfrak{g}$ satisfying the condition

$$(1) \qquad \langle [X, Z]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle = 0 \qquad \text{for all} \quad X, Y, Z \in \mathfrak{m}\,,$$

Here the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$.

2000 *Mathematics Subject Classification.* 53C21, 53B21, 53C25, 53C30.
Key words and phrases. Riemannian manifold, naturally reductive Riemannian homogeneous space, D'Atri space, flag manifold.

It is also well-known that the condition (1) is equivalent to the following more geometrical property:

$$\text{(2)}\qquad \begin{array}{c}\text{For any vector } X \in \mathfrak{m}\backslash\{0\}, \text{ the curve } \gamma(t) = \tau(\exp tX)(p)\\ \text{is a geodesic with respect to the Riemannian connection.}\end{array}$$

Here exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$ respectively. Thus, for a naturally reductive homogeneous space every geodesic on $(G/H, g)$ is an orbit of a one-parameter subgroup of the group of isometries.

The property of being a *D'Atri space* (i.e., a space with volume-preserving symmetries) is equivalent to the infinite number of curvature identities called the *odd Ledger conditions* L_{2k+1}, $k \geq 1$ (see [6] and [13]). In particular, the first two non-trivial odd Ledger conditions are

$$\text{(3)}\qquad L_3 : (\nabla_X \rho)(X, X) = 0\,,$$

$$\text{(4)}\qquad L_5 : \sum_{a,b=1}^{n} \mathcal{R}_{XE_aXE_b}(\nabla_X \mathcal{R})_{XE_aXE_b} = 0\,,$$

where X is any tangent vector at any point $m \in M$ and $\{E_1, \ldots, E_n\}$ is any orthonormal basis of T_mM. Here $\mathcal{R}$ denotes the curvature tensor and ρ the Ricci tensor of (M, g), respectively, and $n = \dim M$. The condition L_3 is very important. Thus, a Riemannian manifold (M, g) satisfying the first odd Ledger condition is said to be of *type* $\mathcal{A}$ (see [12]).

D'Atri spaces have been a topic of interest in Riemannian geometry since they were introduced by J. E. D'Atri and H. K. Nickerson [6], [7] and studied extensively by J. E. D'Atri in [5]. In [6], [7] it was proved that *all naturally reductive spaces are D'Atri spaces*, and another more simple proof was provided in [5]. See [11] for a survey about the whole topic. In addition, the classification of all 3-dimensional D'Atri spaces is well-known. It was done by O. Kowalski in [10] concluding that all of them are locally naturally reductive. Besides, the first attempts to classify all 4-dimensional *homogeneous* D'Atri spaces were done by F. Podesta, A. Spiro and P. Bueken, L. Vanhecke, in the papers [12] and [4] (which are mutually complementary), respectively. The previous authors started with the corresponding classification of all spaces of type $\mathcal{A}$, but the classification given in [12] was incomplete as the author claimed in [1]. Later, the author and O. Kowalski in [2] obtained the complete classification of all homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form and, as a consequence, they proved correctly that *all homogeneous 4-dimensional D'Atri spaces are locally naturally reductive.*

On the other hand, N. R. Wallach in [14] constructed a family of Riemannian flag manifolds in the complex plane, $(M^6, g_{(c_1,c_2,c_3)})$, in the quaternionic plane, $(M^{12}, g_{(c_1,c_2,c_3)})$, and also in the octonionic plane $(M^{24}, g_{(c_1,c_2,c_3)})$ as examples of reductive homogeneous spaces. Here, c_1, c_2 and c_3 are positive real constants.

As concerns the first one, M^6, D'Atri and Nickerson in [7] proved that if two of the parameters c_1, c_2, c_3 are equal, the corresponding Riemannian space is of type $\mathcal{A}$. Moreover, for the case $c_1 = c_2 = 1$, $c_3 = 2$ they affirmed (without explicit argument) that the second odd Ledger condition L_5 is not satisfied.

Now, we shall finish their study of the L_5 condition over the manifold M^6. Of course, with all the relevant arguments. Further, we shall extend the study of the two-first odd Ledger conditions L_3, L_5 to the other Wallach's flag manifold M^{12}. Moreover, we shall correct the result given by the author and A. M. Naveira in [3] where this problem over the manifold M^{12} was studied for the first time. In both cases, we shall conclude that every member of both families of Riemannian flag manifolds is a D'Atri space if and only if it is naturally reductive.

Many symbolic computations are required to make this study. Thus, to organize them in the most systematic way, we use the software MATHEMATICA 5.2 throughout this work. We put stress on the full transparency of this procedure.

However, we shall not treat along this paper the 24-dimensional family of flag manifolds $\big(F_4/Spin(8), g_{(c_1,c_2,c_3)}\big)$.

2. Preliminaries

Let $(M = G/H, g)$ a reductive Riemannian homogeneous space. In agreement with the notation of section before let us recall, following [9, vol.2,p.201], that the Riemannian connection for g is given by

$$\nabla_X Y = \tfrac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y)\,, \tag{5}$$

where $U(X,Y)$ is the symmetric bilinear mapping of $\mathfrak{m}\times\mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2\langle U(X,Y),Z\rangle = \langle X,[Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}},Y\rangle\,, \tag{6}$$

for all X, Y, $Z\in\mathfrak{m}$.

Note that the space M becomes naturally reductive if and only if $U\equiv 0$.

Let $\mathcal{R}$ denote the curvature tensor of the Riemannian connection ∇. Following [7] we have

$$\begin{aligned}\mathcal{R}(X,Y)Z = &-[[X,Y]_{\mathfrak{h}},Z] - \frac{1}{2}[[X,Y]_{\mathfrak{m}},Z]_{\mathfrak{m}} - U([X,Y]_{\mathfrak{m}},Z)\\ &+\frac{1}{4}[X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + \frac{1}{2}[X,U(Y,Z)]_{\mathfrak{m}} + U(X,U(Y,Z))\\ &+\frac{1}{2}U(X,[Y,Z]_{\mathfrak{m}}) - \frac{1}{4}[Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[Y,U(X,Z)]_{\mathfrak{m}}\\ &-U(Y,U(X,Z)) - \frac{1}{2}U(Y,[X,Z]_{\mathfrak{m}})\,,\end{aligned} \tag{7}$$

for all X, Y, $Z\in\mathfrak{m}$.

In addition, in [7] the authors showed how the Ledger conditions can be reformulated on reductive homogeneous spaces without explicit use of covariant derivatives. Their theorem below covers only the first two non-trivial odd conditions (3) and (4), but it is useful for checking concrete examples as in the next section.

Theorem 1. *Let $M^n = G/H$ be a reductive Riemannian homogeneous space. Let $\{E_1,\dots,E_n\}$ be an orthonormal basis of $\mathfrak{m}$ and let ρ denote the Ricci curvature* *tensor of the Riemannian connection. Then, the first two odd Ledger's conditions can be reformulated in the following way:*

$$L_3 \equiv \rho(X, U(X,X)) = \sum_{a=1}^{n} \langle \mathcal{R}(E_a, X)U(X,X), E_a\rangle = 0\,, \tag{8}$$

$$L_5 \equiv \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, X)X, X)U(X,X), E_a\rangle = 0 \tag{9}$$

for all $X \in \mathfrak{m}$. *Or, equivalently*

$$L_3 \equiv \rho(X, U(Y,Z)) + \rho(Y, U(Z,X)) + \rho(Z, U(X,Y)) = 0\,, \tag{10}$$

$$\begin{aligned} L_5 \equiv & \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, X)Y, Z)U(V,W), E_a\rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, Y)Z, V)U(W,X), E_a\rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, Z)V, W)U(X,Y), E_a\rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, V)W, X)U(Y,Z), E_a\rangle \\ & + \sum_{a=1}^{n} \langle \mathcal{R}(\mathcal{R}(E_a, W)X, Y)U(Z,V), E_a\rangle = 0 \end{aligned} \tag{11}$$

for all $X, Y, Z, V, W \in \mathfrak{m}$.

In order to obtain examples using Theorem 1, we compute U from (6) and the curvature tensor $\mathcal{R}$ at the point p from (7).

3. Two families of flag manifolds

Let $SU(n)$ be the special unitary group and $Sp(n)$ be the symplectic group.

In the natural way, both $M^6 = SU(3)/SU(1) \times SU(1) \times SU(1)$ and $M^{12} = Sp(3)/SU(2) \times SU(2) \times SU(2)$ admit a reductive homogeneous decomposition [15].

Moreover, N. R. Wallach constructs an infinite number of metrics with strictly positive sectional curvature over the previous spaces [14].

Let $G = SU(3)$ or $Sp(3)$, and let $H = (SU(1) \times SU(1) \times SU(1))$ or $(Sp(1) \times Sp(1) \times Sp(1) \equiv SU(2) \times SU(2) \times SU(2))$. In agreement with the notation before, the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ or $\mathfrak{sp}(3)$ and $\mathfrak{h}$ is the subalgebra of diagonal matrices. To simplify notation, we use the same letter $\mathcal{K}$ for the complex plane $\mathbb{C}$ and for the quaternionic plane $\mathbb{H}$. Let us define $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ by

$$\mathfrak{m} = V_1 \oplus V_2 \oplus V_3\,,$$

where

$$V_1 = \left\{ \begin{bmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, z \in \mathcal{K} \right\}, \qquad V_2 = \left\{ \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \end{bmatrix}, z \in \mathcal{K} \right\}$$

and

$$V_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{bmatrix}, z \in \mathcal{K} \right\}.$$

Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{m}$ given by

$$\langle X, Y \rangle = \begin{cases} 0 & \text{if } X \in V_i,\, Y \in V_j,\, i \neq j\,, \\ -c_i \text{ Trace } XY & \text{if } X, Y \in V_i,\, i = 1,2,3\,. \end{cases} \tag{12}$$

where c_1, c_2 and c_3 are positive real parameters.

These spaces were introduced by N. R. Wallach in [14] where he also calculated from the formulas (6) and (12) that

$$U(X, Y) = \begin{cases} 0 & \text{if } X, Y \in V_i,\, i = 1,2,3\,, \\ -\frac{c_i - c_j}{2c_k}[X, Y] & \text{if } X \in V_i,\, Y \in V_j,\, i \neq j \neq k\,. \end{cases} \tag{13}$$

Obviously, *the decomposition is naturally reductive if and only if* $c_1 = c_2 = c_3$.

3.1. **Case $\mathcal{K} = \mathbb{C}$.** For this case, the corresponding flag manifold is $M^6 = SU(3)/SU(1) \times SU(1) \times SU(1)$. Further, we know that J. E. D'Atri and H. K. Nickerson in [7] proved that if at least two of the parameters c_1, c_2, c_3 are equal, the corresponding Riemannian space is of type $\mathcal{A}$. Moreover, for the case $c_1 = c_2 = 1$, $c_3 = 2$ they affirmed (without giving any argument) that the second odd Ledger condition L_5 is not satisfied. Now, we shall finish the study of the L_5 condition over the manifold M^6. For the convenience of the reader we repeat the relevant material from [7], thus making our exposition self-contained.

First, we define a basis $\{E_1, JE_1, E_2, JE_2, E_3, JE_3\}$ for $\mathfrak{m}$ taking $z = 1, i$ in V_1, $z = 1, -i$ in V_2 and $z = -1, -i$ in V_3, respectively. Note that implicitly we have defined the invariant almost complex structure $J : \mathfrak{m} \to \mathfrak{m}$ by

$$J \begin{pmatrix} 0 & a_{12} & a_{13} \\ -\bar{a_{12}} & 0 & a_{23} \\ -\bar{a_{13}} & -\bar{a_{23}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & ia_{12} & -ia_{13} \\ i\bar{a_{12}} & 0 & ia_{23} \\ -i\bar{a_{13}} & i\bar{a_{23}} & 0 \end{pmatrix}$$

i.e. for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{h}$, it satisfies

$$J^2 X = -X, \quad J[Y, X]_{\mathfrak{m}} = [Y, JX]_{\mathfrak{m}}\,.$$

Afterwards, we define a basis $\{K_1, K_2, K_3\}$ for $\mathfrak{h}$ taking

$$K_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then, we get that the multiplication table for $\mathfrak{m}$ is given by

$$[E_l, JE_l] = 2K_l\,, \quad l = 1,2,3\,.$$

$$[E_l, E_m] = -[JE_l, JE_m] = E_n\,, \quad [E_l, E_m] = -[JE_l, JE_m] = E_n\,,$$

where (l, m, n) is a cyclic permutation of $(1,2,3)$. Moreover, we get

$$[K_l, E_l] = 2JE_l\,, \quad [K_l, JE_l] = -2E_l\,, \quad l = 1,2,3\,,$$

$$[K_l, E_m] = -2JE_m\,, \quad [K_l, JE_m] = E_m\,, \quad l \neq m,\ l, m \in \{1,2,3\}\,.$$

The curvature tensor can be computed from (7) with respect to this basis. The non-trivial cases are the following formulas (14) and the formulas obtained from (14) by using the operator J:

$$\begin{aligned} \mathcal{R}(E_l, JE_l)E_l &= -4JE_l\,, \\ \mathcal{R}(E_l, JE_l)E_m &= 2\mathcal{R}(E_l, E_m)JE_l \\ &= -2\mathcal{R}(JE_l, E_m)E_l = \tfrac{4-(c_l-c_m-c_n)^2}{2c_m c_n} JE_m\,, \\ \mathcal{R}(E_l, E_m)E_l &= \mathcal{R}(JE_l, E_m)JE_l = \Big(\tfrac{(c_n-c_l)}{c_m} - \tfrac{(c_l-c_m-c_n)^2}{4c_m c_n}\Big)E_m\,, \end{aligned} \tag{14}$$

for l, m, n distinct and $l, m, n \in \{1,2,3\}$.

Further, we obtain easily from (14) that the only non-trivial terms of the Ricci tensor are

$$\rho(E_l, E_l) = \rho(JE_l, JE_l) = \tfrac{(6c_m c_n + c_l^2 - c_m^2 - c_n^2)}{c_m c_n} \tag{15}$$

for l, m, n distinct and $l, m, n \in \{1,2,3\}$.

Now, we shall use (13) and (15) to compute the Ledger condition L_3, (10). The equation (10) has a purely algebraic character because the family of metrics $g_{(c_1,c_2,c_3)}$ is left-invariant. Hence, we can substitute for X, Y, Z every triplet chosen from the basis of $\mathfrak{m}$ (with possible repetition). Thus, the condition (10) is equivalent to a system of algebraic equations. Finally, we have obtained, after a lengthy by routine calculation, that the only non-trivial equation appears when

$$(X, Y, Z) \in \big\{(E_l, E_m, E_n), (E_l, JE_m, JE_n) \mid l, m, n \in \{1,2,3\}, n \neq l \neq m \neq n\big\}\,.$$

To be precise, the L_3 condition is equivalent to

$$\frac{(c_1-c_2)(c_1-c_3)(c_2-c_3)}{c_1c_2c_3} = 0\,. \tag{16}$$

We conclude that *every member of the family of Riemannian flag manifolds* $(M^6, g_{(c_1,c_2,c_3)})$ *is of type* $\mathcal{A}$ *if and only if at least two of the parameters* c_1, c_2, c_3, *are equal.*

To finish, we shall prove that *the Ledger condition* L_5 *is satisfied if and only if* $c_1 = c_2 = c_3$.

Case $c_1 = c_l,\ l = 2,3$.

Let us put $X = E_2, Y = E_3, Z = V = W = E_1$ in (11). Thus, for $l = 2$ we obtain using (12), (13) and (14) that (11) can be written in the form

$$(x-1)(9x^2 + 24x + 80) = 0\,, \quad \text{for} \quad x = \tfrac{c_3}{c_1}\,. \tag{17}$$

Analogously for $l = 3$, we obtain that (11) can be written in the form

$$(x-1)(3x^2+8x+96)=0\,, \quad \text{for} \quad x=\tfrac{c_2}{c_1}\,. \tag{18}$$

In both equations (17), (18), the second order equation has negative discriminant. Then, if $c_1 = c_l$, $l = 2, 3$, the only possible real solution is $c_1 = c_2 = c_3$.

Case $c_2 = c_3$.

Let us put in (11) first $X = E_2, Y = JE_3, Z = W = E_1, V = JE_1$ and later $X = E_2, Y = JE_3, Z = JE_1, V = W = E_1$. Thus, we obtain a system of equations of the form

$$\begin{aligned}&(x-1)(x-4)(x^2+2x+4)=0,\\&(x-1)(x^2-4x-2)=0,\end{aligned} \tag{19}$$

respectively, where $x = \frac{c_1}{c_2}$. Here, the only solution of the system is $x = 1$. Then, if $c_2 = c_3$, the only possible solution is $c_1 = c_2 = c_3$.

As a conclusion, *every member of the family of Riemannian flag manifolds* $(M^6, g_{(c_1,c_2,c_3)})$ *is a D'Atri space if and only if it is naturally reductive.*

3.2. **Case** $\mathcal{K} = \mathbb{H}$**.** In this case, we shall make the study of the two-first odd Ledger conditions L_3, L_5 on the other Wallach's flag manifold, i.e. the twelve dimensional manifold $M^{12} = Sp(3)/SU(2)\times SU(2)\times SU(2)$. Moreover, we correct the result given in [3] where this problem was studied for the first time.

From now on, we will denote by j_l, $l = 1, 2, 3$ the three quaternionic imaginary units i, j, k, respectively.

First, we shall define a basis for $\mathfrak{m}$. Let us introduce three invariants almost-complex structures $J_l : \mathfrak{m} \to \mathfrak{m}$, $l = 1, 2, 3$, by

$$J_l \begin{pmatrix} 0 & a_{12} & a_{13} \\ -\bar{a_{12}} & 0 & a_{23} \\ -\bar{a_{13}} & -\bar{a_{23}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & j_l a_{12} & -j_l a_{13} \\ j_l \bar{a_{12}} & 0 & j_l a_{23} \\ -j_l \bar{a_{13}} & j_l \bar{a_{23}} & 0 \end{pmatrix}$$

for $l = 1, 2$ and

$$J_3 \begin{pmatrix} 0 & a_{12} & a_{13} \\ -\bar{a_{12}} & 0 & a_{23} \\ -\bar{a_{13}} & -\bar{a_{23}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & j_3 a_{12} & j_3 a_{13} \\ j_3 \bar{a_{12}} & 0 & j_3 a_{23} \\ j_3 \bar{a_{13}} & j_3 \bar{a_{23}} & 0 \end{pmatrix},$$

i.e. for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{h}$, they satisfy

$$J_l^2 X = -X\,, \quad J_l[Y,X]_{\mathfrak{m}} = [Y, J_l X]_{\mathfrak{m}} \quad \text{for} \quad l = 1,2,3\,,$$

$$J_l J_m X = -J_m J_l X = J_n X \quad \text{where} \quad (l,m,n) \quad \text{is a cyclic permutation of} \quad (1,2,3)\,.$$

On the other hand, it is easy to prove that the structures J_l, $l = 1, 2$ are nearly-Kähler (i.e. they satisfy $(\nabla_X J_l)X = 0$ for $X \in \mathfrak{m}$) and the structure J_3 is Hermitian (i.e. $(\nabla_X J_3)Y - (\nabla_{J_3X} J_3)J_3Y = 0$ for $X, Y \in \mathfrak{m}$), [8].

Finally, we define the adapted basis

$$\{E_1, J_1E_1, J_2E_1, J_3E_1, E_2, J_1E_2, J_2E_2, J_3E_2, E_3, J_1E_3, J_2E_3, J_3E_3\}$$

for $\mathfrak{m} = V_1 \oplus V_2 \oplus V_3$. In particular, we take for generating V_1 the elements

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_l E_1 = \begin{pmatrix} 0 & j_l & 0 \\ j_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad l = 1, 2, 3,$$

for generating V_2 the elements

$$E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_l E_2 = \begin{pmatrix} 0 & 0 & -j_l \\ 0 & 0 & 0 \\ -j_l & 0 & 0 \end{pmatrix}, \quad l = 1, 2,$$

$$J_3 E_2 = \begin{pmatrix} 0 & 0 & j_3 \\ 0 & 0 & 0 \\ j_3 & 0 & 0 \end{pmatrix},$$

and for generating V_3 the elements

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_l E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j_l \\ 0 & j_l & 0 \end{pmatrix}, \quad l = 1, 2, 3.$$

Thus, we get an adapted basis for $\mathfrak{m}$ such that

$$[E_l, E_m] = -[J_p E_l, J_p E_m] = -E_n, \quad [E_l, J_p E_m] = [J_p E_l, E_m] = J_p E_n,$$

where $p = 1, 2$ and (l, m, n) is a cyclic permutation of $(1, 2, 3)$,

$$[J_3 E_1, J_3 E_2] = -E_3, \quad [J_3 E_2, J_3 E_3] = -E_1, \quad [J_3 E_3, J_3 E_1] = E_2,$$
$$[E_1, J_3 E_2] = -[J_3 E_1, E_2] = -J_3 E_3, \quad [E_2, J_3 E_3] = -[J_3 E_2, E_3] = J_3 E_1,$$
$$[E_3, J_3 E_1] = [J_3 E_3, E_1] = -J_3 E_2,$$

$$[J_p E_3, J_q E_1] = -[J_q E_3, J_p E_1] = J_r E_2 \text{ for } (p, q, r) \in \{(1, 2, 3), (1, 3, 2), (3, 2, 1)\},$$
$$[J_1 E_l, J_2 E_m] = -[J_2 E_l, J_1 E_m] = -J_3 E_n \text{ for } (l, m, n) \in \{(1, 2, 3), (2, 3, 1)\},$$
$$[J_1 E_l, J_3 E_m] = [J_3 E_l, J_1 E_m] = J_2 E_n \text{ for } (l, m, n) \in \{(2, 1, 3), (2, 3, 1)\},$$
$$[J_2 E_l, J_3 E_m] = [J_3 E_l, J_2 E_m] = J_1 E_n \text{ for } (l, m, n) \in \{(1, 2, 3), (3, 2, 1)\}.$$

Now we introduce a basis $\{K_{lp} : l, p = 1, 2, 3\}$ for $\mathfrak{h}$. More explicitly, we take

$$K_{1l} = \begin{pmatrix} j_l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ K_{2l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j_l & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ K_{3l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j_l \end{pmatrix}, \quad l = 1, 2, 3.$$

Then, we get

$$
\begin{aligned}
[E_1, J_pE_1] &= 2(K_{1p} - K_{2p}) \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_1, J_qE_1] &= 2(K_{1r} - K_{2r}) \ \text{ for } \ (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}\,, \\
[E_2, J_pE_2] &= 2(-K_{1p} + K_{3p}) \ \text{ for } \ p = 1,2\,, \\
[E_2, J_3E_2] &= 2(K_{13} - K_{33})\,, \\
[J_pE_2, J_qE_2] &= 2(K_{1r} + K_{3r}) \ \text{ for } \ (p,q,r) \in \{(1,2,3),(1,3,2),(3,2,1)\}\,, \\
[E_3, J_pE_3] &= 2(K_{2p} - K_{3p}) \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_3, J_qE_3] &= 2(K_{2r} - K_{3r}) \ \text{ for } \ (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}\,, \\
[E_1, K_{1p}] &= -[E_1, K_{2p}] = -J_pE_1,\ [E_1, K_{3p}] = 0 \ \text{ for } \ p = 1,2,3\,, \\
[E_2, K_{1p}] &= -[E_2, K_{3p}] = J_pE_2\,,\ [E_2, K_{2p}] = 0 \ \text{ for } \ p = 1,2\,, \\
[E_2, K_{13}] &= -[E_2, K_{33}] = -J_3E_2\,,\ [E_2, K_{23}] = 0\,, \\
[E_3, K_{2p}] &= -[E_1, K_{3p}] = -J_pE_3\,,\ [E_3, K_{1p}] = 0 \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_1, K_{1p}] &= -[J_pE_1, K_{2p}] = E_1\,,\ [J_pE_1, K_{3p}] = 0 \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_2, K_{1p}] &= -[J_pE_2, K_{3p}] = -E_2\,,\ [J_pE_2, K_{2p}] = 0 \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_3, K_{2p}] &= -[J_pE_3, K_{3p}] = E_3\,,\ [J_pE_3, K_{1p}] = 0 \ \text{ for } \ p = 1,2,3\,, \\
[J_pE_1, K_{lq}] &= -[J_qE_1, K_{lp}] = J_rE_1\,,\ l = 1,2\,,\ [J_pE_1, K_{3q}] = [J_qE_1, K_{3p}] = 0 \\
&\quad \text{for } \ (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}\,, \\
[J_pE_2, K_{2q}] &= [J_qE_2, K_{2p}] = 0 \ \text{ for } \ (p,q) \in \{(1,2),(1,3),(2,3)\}\,, \\
[J_2E_2, K_{l1}] &= -[J_1E_2, K_{l2}] = J_3E_2\,,\ l = 1,3\,, \\
[J_2E_2, K_{l3}] &= [J_3E_2, K_{l2}] = J_1E_2\,,\ l = 1,3\,, \\
[J_1E_2, K_{l3}] &= [J_3E_2, K_{l1}] = -J_2E_2\,,\ l = 1,3\,, \\
[J_pE_3, K_{lq}] &= -[J_qE_3, K_{lp}] = J_rE_3\,,\ l = 2,3\,,\ [J_pE_3, K_{1q}] = [J_qE_3, K_{1p}] = 0 \\
&\quad \text{for } \ (p,q,r) \in \{(1,2,3),(2,3,1),(3,1,2)\}\,.
\end{aligned}
$$

The curvature tensor can be computed from (7) with respect to this basis. Let us denote by J_0 the identity and let us put $A = c_1^2 + (c_2 - c_3)^2 - 2c_1(c_2 + c_3)$. The non-trivial cases are the following formulas

$$
\begin{aligned}
\mathcal{R}(J_qE_l, J_pE_l)J_pE_l &= 4J_qE_l\,,\ p \neq q\,, \\
\mathcal{R}(J_qE_l, J_pE_m)J_pE_m &= \tfrac{-3c_n^2 + (c_l - c_m)^2 + 2c_n(c_l + c_m)}{4c_lc_n} J_qE_l\,,
\end{aligned}
$$

for distinct $l, m, n \in \{1,2,3\}$, $p, q \in \{0,1,2,3\}$,

$$\begin{aligned}\mathcal{R}(E_l,E_m)J_pE_m &= -\mathcal{R}(E_l,J_pE_m)E_m = -J_p(\mathcal{R}(J_pE_l,E_m)J_pE_m)\\ &= J_p(\mathcal{R}(J_pE_l,J_pE_m)E_m) = \tfrac{A}{4c_lc_n}J_pE_l\,,\ p=1,2\,,\\ \mathcal{R}(E_l,E_m)J_3E_m &= -\mathcal{R}(E_l,J_3E_m)E_m = -J_3(\mathcal{R}(J_3E_l,E_m)J_3E_m)\\ &= J_3(\mathcal{R}(J_3E_l,J_3E_m)E_m) = \tfrac{(-1)^{l+m}A}{4c_lc_n}J_3E_l\,,\end{aligned}$$

for distinct $l,m,n\in\{1,2,3\}$,

$$\begin{aligned}\mathcal{R}(E_l,J_pE_m)J_qE_m &= -\mathcal{R}(E_l,J_qE_m)J_pE_m = \tfrac{(-1)^{(p+r+n!+1)}A}{4c_lc_n}J_rE_l\,,\\ &\qquad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(E_l,J_pE_m)J_qE_m &= -\mathcal{R}(E_l,J_qE_m)J_pE_m = \tfrac{(-1)^{(l!+1)}A}{4c_lc_n}J_rE_l\,,\\ &\qquad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for $(p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}\mathcal{R}(J_pE_l,E_m)J_qE_m &= -\mathcal{R}(J_pE_l,J_qE_m)E_m = \tfrac{(-1)^{(q+r+n!)}A}{4c_lc_n}J_rE_l\,,\\ \mathcal{R}(J_rE_l,E_m)J_qE_m &= -\mathcal{R}(J_rE_l,J_qE_m)E_m = \tfrac{(-1)^{(q+r+n!+1)}A}{4c_lc_n}J_pE_l\,,\\ &\qquad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(J_pE_l,E_m)J_qE_m &= -\mathcal{R}(J_pE_l,J_qE_m)E_m = \tfrac{(-1)^{(l!+1)}A}{4c_lc_n}J_rE_l\,,\\ \mathcal{R}(J_rE_l,E_m)J_qE_m &= -\mathcal{R}(J_rE_l,J_qE_m)E_m = \tfrac{(-1)^{(l!)}A}{4c_lc_n}J_pE_l\,,\\ &\qquad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for $(p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_m)J_pE_m &= -\mathcal{R}(J_pE_l,J_pE_m)J_qE_m = \tfrac{(-1)^{q}A}{4c_lc_n}J_qE_l\,,\\ &\qquad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(J_pE_l,J_qE_m)J_pE_m &= -\mathcal{R}(J_pE_l,J_pE_m)J_qE_m = \tfrac{A}{4c_lc_n}J_qE_l\,,\\ &\qquad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for distinct $p,q\in\{1,2,3\}$,

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_m)J_rE_m &= -\mathcal{R}(J_pE_l,J_rE_m)J_qE_m = \tfrac{(-1)^{(r+n!)}A}{4c_lc_n}E_l\,,\\ &\qquad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(J_pE_l,J_qE_m)J_rE_m &= -\mathcal{R}(J_pE_l,J_rE_m)J_qE_m = \tfrac{(-1)^{l!}A}{4c_lc_n}E_l\,,\\ &\qquad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for $(p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}\mathcal{R}(E_l,J_pE_l)J_qE_m &= \tfrac{(-1)^{(r+n!)}A}{2c_mc_n}J_rE_m\,,\\ \mathcal{R}(E_l,J_pE_l)J_rE_m &= \tfrac{(-1)^{(r+n!+1)}A}{2c_mc_n}J_qE_m\,,\\ &\quad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(E_l,J_pE_l)J_qE_m &= \tfrac{(-1)^{l!}A}{2c_mc_n}J_rE_m\,,\\ \mathcal{R}(E_l,J_pE_l)J_rE_m &= \tfrac{(-1)^{(l!+1)}A}{2c_mc_n}J_qE_m\,,\\ &\quad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for $(p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}\mathcal{R}(E_l,J_pE_l)E_m &= -J_p(\mathcal{R}(E_l,J_pE_l)J_pE_m)=\tfrac{-A}{2c_mc_n}J_pE_m\,,\ p=1,2\,,\\ \mathcal{R}(E_l,J_3E_l)E_m &= -J_3(\mathcal{R}(E_l,J_3E_l)J_3E_m)=\tfrac{(-1)^{(l+m+1)}A}{2c_mc_n}J_3E_m\,,\end{aligned}$$

for distinct $l,m,n\in\{1,2,3\}$,

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_l)E_m &= -J_r(\mathcal{R}(J_pE_l,J_qE_l)J_rE_m)=\tfrac{(-1)^{(q+n!)}A}{2c_mc_n}J_rE_m\,,\\ &\quad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(J_pE_l,J_qE_l)E_m &= -J_r(\mathcal{R}(J_pE_l,J_qE_l)J_rE_m)=\tfrac{(-1)^{(m!)}A}{2c_mc_n}J_rE_m\,,\\ &\quad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for $(p,q,r)\in\{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}\mathcal{R}(J_pE_l,J_qE_l)J_pE_m &= \tfrac{(-1)^{r}A}{2c_mc_n}J_qE_m\,,\ r=\max(\{p,q\})\,,\\ &\quad (l,m,n)\in\{(1,2,3),(2,1,3),(2,3,1),(3,2,1)\}\,,\\ \mathcal{R}(J_pE_l,J_qE_l)J_pE_m &= \tfrac{A}{2c_mc_n}J_qE_m\,,\\ &\quad (l,m,n)\in\{(1,3,2),(3,1,2)\}\,,\end{aligned}$$

for distinct $p,q\in\{1,2,3\}$.

Further, we obtain easily from the previous formulas that the only non-trivial terms of the Ricci tensor are

$$\rho(E_l,E_l)=\rho(J_pE_l,J_pE_l)=\tfrac{2(8c_mc_n+c_l^2-c_m^2-c_n^2)}{c_mc_n} \tag{20}$$

for l,m,n distinct and $p,l,m,n\in\{1,2,3\}$.

Now, we shall use (13) and (20) to compute the Ledger condition L_3, (10). The equation (10) has a purely algebraic character because the family of metrics $g_{(c_1,c_2,c_3)}$ is left-invariant. Hence, we can substitute for X,Y,Z every triplet chosen from the basis of $\mathfrak{m}$ (with possible repetition). Thus, the condition (10) is equivalent to a system of algebraic equations. Finally, we have obtained after a lengthy by routine calculation, that the only non-trivial equation appears when

$$\begin{aligned}(X,Y,Z) \in &\{(E_l,E_m,E_n),(E_l,J_pE_m,J_pE_n),(J_lE_l,J_mE_m,J_nE_n),(J_lE_l,J_nE_m,\\ &J_mE_n),(J_nE_l,J_lE_m,J_mE_n) \mid p,l,m,n \in \{1,2,3\}, n \neq l \neq m \neq n\}\,.\end{aligned}$$

To be precise, the L_3 condition is equivalent to

$$\frac{(c_1-c_2)(c_1-c_3)(c_2-c_3)}{c_1c_2c_3} = 0\,. \tag{21}$$

We conclude that *every member of the family of Riemannian flag manifolds* $(M^{12}, g_{(c_1,c_2,c_3)})$ *is of type* $\mathcal{A}$ *if and only if at least two of the parameters* c_1, c_2, c_3, *are equal.*

To finish, we shall prove that *the* L_5 *Ledger condition is satisfied if and only if* $c_1 = c_2 = c_3$.

Case $c_1 = c_l,\ \ l = 2,3.$

Let us put $X = E_2, Y = E_3, Z = V = W = E_1$ in (11). Thus, for $l = 2$ we obtain using (12), (13) and (**??**) that (11) can be written in the form

$$(x-1)(9x^2+48x+112) = 0\,, \quad \text{for} \quad x = \tfrac{c_3}{c_1}\,. \tag{22}$$

Analogously for $l = 3$, we obtain that (11) can be written in the form

$$(x-1)(x^2+3x+36) = 0\,, \quad \text{for} \quad x = \tfrac{c_2}{c_1}\,. \tag{23}$$

In both equations (22), (23), the second order equation has negative discriminant. Then, if $c_1 = c_l$, $l = 2,3$, the only possible real solution is $c_1 = c_2 = c_3$.

Case $c_2 = c_3$.

Let us put in (11) first $X = E_2, Y = J_1E_3, Z = W = E_1, V = J_1E_1$ and later $X = E_2, Y = E_3, Z = V = W = E_1$. Thus, we obtain a system of equations of the form

$$\begin{aligned}&(x-1)(x-4)(3x^2-6x+4) = 0,\\ &(x-1)(7x^2-46x+48) = 0,\end{aligned} \tag{24}$$

respectively, where $x = \frac{c_1}{c_2}$. Here, the only solution of the system is $x = 1$. Then, if $c_2 = c_3$, the only possible solution is $c_1 = c_2 = c_3$.

As a conclusion, *every member of the family of Riemannian flag manifolds* $(M^{12}, g_{(c_1,c_2,c_3)})$ *is a D'Atri space if and only if it is naturally reductive.*

Acknowledgments. The author's work has been partially supported by D.G.I. (Spain) and FEDER Project MTM 2004-06015-C02-01, the network MTM2006-27480-E/, by a grant ACOMP07/088 from Agencia Valenciana de Ciencia y Tecnologí and by a Predoctoral Research Grant from Programa FPU del Ministerio de Educación y Ciencia of Spain.

The author wishes to thank to O. Kowalski and A. M. Naveira for their valuable hints.

References

[1] Arias-Marco, T., *The classification of 4-dimensional homogeneous D'Atri spaces revisited*, Differential Geometry and its Applications **25** (2007), 29–34.

[2] Arias-Marco, T., Kowalski, O., *The classification of 4-dimensional homogeneous D'Atri spaces*, to appear in Czechoslovak Math. J.

[3] Arias-Marco, T., Naveira, A. M., *A note on a family of reductive Riemannian homogeneous spaces whose geodesic symmetries fail to be divergence-preserving*, Proceedings of the XI Fall Workshop on Geometry and Physics. Publicaciones de la RSME **6** (2004), 35–45.

[4] Bueken, P., Vanhecke, L., *Three- and Four-dimensional Einstein-like manifolds and homogeneity*, Geom. Dedicata **75** (1999), 123–136.

[5] D'Atri, J. E., *Geodesic spheres and symmetries in naturally reductive homogeneous spaces*, Michigan Math. J. **22** (1975), 71–76.

[6] D'Atri, J. E., Nickerson, H. K., *Divergence preserving geodesic symmetries*, J. Differential Geom. **3** (1969), 467–476.

[7] D'Atri, J. E., Nickerson, H. K., *Geodesic symmetries in spaces with special curvature tensors*, J. Differenatial Geom. **9** (1974), 251–262.

[8] Gray, A., Hervella, L. M., *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123**(4) (1980), 35–58.

[9] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry, Vols. I and II*, Interscience, New York, 1963 and 1969.

[10] Kowalski, O., *Spaces with volume-preserving symmetries and related classes of Riemannian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo Speciale, (1983), 131–158.

[11] Kowalski, O., Prüfer, F., Vanhecke, L., *D'Atri Spaces*, Progr. Nonlinear Differential Equations Appl. **20** (1996), 241–284.

[12] Podestà, F., Spiro, A., *Four-dimensional Einstein-like manifolds and curvature homogeneity*, Geom. Dedicata **54** (1995), 225–243.

[13] Szabó, Z. I., *Spectral theory for operator families on Riemannian manifolds*, Proc. Sympos. Pure Math. **54**(3) (1993), 615–665.

[14] Wallach, N. R., *Compact homogeneous Riemannian manifols with strictly positive curvature*, Ann. of Math. **96** (1972), 276–293.

[15] Wolf, J., Gray, A., *Homogeneous spaces defined by Lie group automorphisms, I, II*, J. Differential Geom. **2** (1968), 77–114, 115–159.

Department of Geometry and Topology, University of Valencia
Vicente Andrés Estellés 1, 46100 Burjasot, Valencia, Spain
E-mail: Teresa.Arias@uv.es