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SPECTRAL PROPERTIES OF A CERTAIN CLASS OF CARLEMAN OPERATORS

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Abstract. The object of the present work is to construct all the generalized spectral functions of a certain class of Carleman operators in the Hilbert space $L^2(X,\mu)$ and establish the corresponding expansion theorems, when the deficiency indices are (1,1). This is done by constructing the generalized resolvents of A and then using the Stieltjes inversion formula.

1. Preliminaries

The set of generalized resolvents of a symmetric operator A with defect indices $(1,1)$ was first derived independently by Naimark [15] and Krein [10]. The case of defect indices (m,m), $m \in \mathbb{N}$ is due to Krein [11]. Saakjan [19] extended Krein's formula to the general case of defect indices (m,m), $m \in \mathbb{N} \cup \{\infty\}$. In another form, the generalized resolvent formula for symmetric operators (including the case of non-densely defined operators) has been obtained by Straus [20, 21].

Let H be a Hilbert space endowed with the inner product $(\cdot,\cdot)$, and let $A\colon D(A) \subset H \longrightarrow H$ be a densely defined closed linear operator whose range is denoted $R(A)$.

1.1. **Basic Spectral Properties.** We say that $\lambda \in \mathbb{C}$ is a regular point of the operator A if the resolvent $R_\lambda = (A-\lambda I)^{-1}$ exists and is a bounded operator defined everywhere in H (I denotes the identity operator in H). In this case we say that λ belongs to $\rho(A)$, the resolvent set of A. R_λ is an analytic operator function of λ on $\rho(A)$. The number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A if there exists an $f \in D(A)$ for which $f \neq 0$ and $Af = \lambda f$. In this case, the operator $A - \lambda I$ is not injective, i.e., $\ker(A-\lambda I) \neq \{0\}$. The complement of $\rho(A)$, in the complex plane, is denoted by $\sigma(A)$ and is called the spectrum of A.
A resolution of the identity [1] is a one-parameter family $\{E_t\}$, $-\infty < t < \infty$, of orthogonal projection operators acting on a Hilbert space H, such that

i) $E_s \leq E_t$ if $s \leq t$ (monotonicity);

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ii) E_t is strongly left continuous, i.e. $E_{t-0} = E_t$ for every $t \in \mathbb{R}$;
iii) $E_t \xrightarrow{s} 0$ as $t \to -\infty$ and $E_t \xrightarrow{s} I$ as $t \to \infty$; here 0 and I are the zero and the identity operator on the space H.

Condition ii) can be replaced by the condition of strong right continuity at every point $t \in \mathbb{R}$.

From this it follows that, for each fixed $f \in H$, the function $\rho_f \colon \mathbb{R} \to [0, 1)$ given by

$$\rho_f(t) = \big(E(t)f, f\big) = \|E(t)f\|^2 \tag{1.1}$$

is is bounded, non-decreasing, left continuous and

$$\lim_{t\to\infty} \rho_f(t) = \|f\|^2, \quad \lim_{t\to-\infty} \rho_f(t) = 0. \tag{1.2}$$

In [1] is proven that for each resolution of the identity E_t $(-\infty \le t \le +\infty)$ corresponds a uniquely defined self adjoint operator $\overset{\circ}{A}$, admitting the following integral representation

$$\overset{\circ}{A} = \int_{-\infty}^{+\infty} t\, dE_t\,, \tag{1.3}$$

where the integral is understood as the strong limit of the integral sums for each $f \in D(\overset{\circ}{A})$, and

$$D(\overset{\circ}{A}) = \left\{ f : \int_{-\infty}^{+\infty} t^2 d\,(E_t\ f, f) < \infty \right\} \tag{1.4}$$

is satisfied. The resolvent $\overset{\circ}{R}_\lambda$ and the spectral function E_t of a self adjoint operator $\overset{\circ}{A}$ are bound by the relation

$$\overset{\circ}{R}_\lambda = \int_{-\infty}^{+\infty} \frac{dE_t}{t-\lambda}\,, \quad \lambda \in \rho(\overset{\circ}{A})\,, \tag{1.5}$$

in the sense of strong limit.
The resolution of the identity given by the operator A completely determines the spectral properties of that operator, namely:
α) a real number t_0 is a regular point of A if and only if it is a point of constancy, that is, if there is an $\varepsilon > 0$ such that $E_{t_0+\varepsilon} - E_{t_0-\varepsilon} = 0$;
β) a real number t_0 is an eigenvalue of A if and only if λ is a jump point of E_t, that is, $E_{t_0+0} - E_{t_0} \neq 0$.
Hence the resolution of the identity determined by the operator is also called the spectral function of this operator.

1.2. **Deficiency indices.** The defect number is the dimension of the orthogonal complement to $R(A)$

$$d_A = \dim\big(H \ominus R(A)\big) = \dim \operatorname{Ker}(A^*)\,,$$

where A^* is the adjoint operator of A and $\operatorname{Ker}(A^*) = \{f \in D(A^*) \colon A^*f = 0\}$, $D(A^*)$ being the domain of A^*.

Let A be a symmetric operator, $\widetilde{A}$ its symmetric extension, then the following relation holds

$$A \subset \widetilde{A} \subset \widetilde{A}^* \subset A^* \, . \tag{1.6}$$

The interest of (1.6) resides in the following conclusion: all symmetrical extension of A comes of a restriction of the domain of A^*. So $D(\widetilde{A})$ is a subspace between $D(A)$ and $D(A^*)$. To construct the extensions $\widetilde{A}$ it is therefore well to examine the structure of the space $D(A^*)$. Let's put

$$\mathcal{N}_\lambda = \ker (A^* - \lambda I) \quad \text{and} \quad \mathcal{N}_{\bar\lambda} = \ker (A^* - \bar\lambda I) \, , \qquad (\Im m \lambda > 0) \ ,$$

with respective dimensions n_+, n_-. They are called the deficiency indices of the operator A and will be denoted by the ordered pair (n_+, n_-). It being, in the Hilbert space $D(A^*)$ we have the following hilbertienne decomposition [4]

$$D(A^*) = D(A) \oplus \mathcal{N}_\lambda \oplus \mathcal{N}_{\bar\lambda} \, . \tag{1.7}$$

A possesses self adjoint extensions [6] if and only if $n_+ = n_-$. We get in this case all self adjoints extensions of A from all isometric Cayley transforms $V = (A - \lambda I)(A - \bar\lambda I)^{-1}$ defined from $\mathcal{N}_{\bar\lambda}$ to $\mathcal{N}_\lambda$.

1.3. **Generalized resolvents formulas.** In the general case, every symmetric operator A can be prolonged in a selfadjoint operator A^+ defined in a wide space H^+ containing H. If E_t^+ (respectively R_λ^+) is the spectral function (respectively the resolvent) of A^+ and P^+ the operator of projection of H^+ on H then the functions operators $\mathbf{E}_t = P^+ E_t^+$ and $\mathbf{R}_\lambda = P^+ R_\lambda^+$ are said, respectively, generalized spectral function and generalized resolvent of the operator A. They are joined by the relation

$$\mathbf{R}_\lambda = \int_\alpha^\beta \frac{d\mathbf{E}_t}{t - \lambda} \, , \quad \lambda \in \rho\,(A) \ , \tag{1.8}$$

in addition, for all real numbers α, β ($\alpha < \beta$), we have the Stieltjes inversion formula

$$\big([\mathbf{E}_\alpha - \mathbf{E}_\beta] f, g\big) = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_\alpha^\beta \big([\mathbf{R}_{\sigma + i\tau} - \mathbf{R}_{\sigma - i\tau}] f, g\big) d\sigma \, , \quad f, g \in H \, . \tag{1.9}$$

Moreover, for all f of $D\,(A)$:

$$Af = \int_{-\infty}^{+\infty} t \, d\mathbf{E}_t f \, .$$

The generalized spectral function $\mathbf{E}_t$ satisfy the same conditions (ii) and (iii) of E_t but the first is replaced by
(i') $\mathbf{E}_{t_2} - \mathbf{E}_{t_1}$, where $t_2 > t_1$, is a bounded positive operator.

The restriction $P^+ A^+$ is said quasi selfadjoint extension of the operator A. It is from this notion that Straus [21] developed his theory of generalized resolvent of a symmetric operator. Let's designate by $\mathcal{F}_\lambda$ the class of all quasi selfadjoint linear operators defined on $\mathcal{N}_\lambda$ and that apply $\mathcal{N}_\lambda$ to $\mathcal{N}_{\bar{\lambda}}$. The set of generalized resolvents is defined by

$$\begin{cases} \mathbf{R}_\lambda = \left(A_{F(\lambda)} - \lambda I\right)^{-1} \\ \mathbf{R}_{\bar{\lambda}} = \mathbf{R}^*_\lambda \end{cases} \qquad \Im m\lambda \Im m\lambda_\circ > 0\,, \tag{1.10}$$

where $\lambda_\circ$ is a non real point, $F(\lambda)$ an analytic function operator in the half plane $(\Im m\lambda\Im m\lambda_\circ > 0)$ to value in $\mathcal{F}_{\lambda_\circ}$ and $A_{F(\lambda)}$ $(\Im m\lambda\Im m\lambda_\circ > 0)$ a quasi selfadjoint extension of the operator A defined by

$$D\big(A_{F(\lambda)}\big) = D(A) \oplus \left[F(\lambda) - I\right]\mathcal{N}_{\lambda_\circ}\,,$$

$$A_{F(\lambda)}\big(f + F(\lambda)\varphi - \varphi\big) = Af + \lambda_\circ F(\lambda)\varphi - \bar{\lambda_\circ}\varphi$$

with $f \in D(A)$ and $\varphi \in \mathcal{N}_{\lambda_\circ}$. The adjoint operator $A^*_{F(\lambda)}$ is defined by

$$D\big(A^*_{F(\lambda)}\big) = D(A) \oplus \left[F^*(\lambda) - I\right]\mathcal{N}_{\bar{\lambda_\circ}}\,,$$

$$A^*_{F(\lambda)}\big(f + F^*(\lambda)\psi - \psi\big) = Af + \bar{\lambda_\circ}F^*(\lambda)\psi - \bar{\lambda_\circ}\psi$$

with $f \in D(A)$ and $\psi \in \mathcal{N}_{\bar{\lambda_\circ}}$.

1.4. **Some convergences.** We call t a continuity point of E_t if $E_{t+0} - E_t = 0$.

We call [1] convergence in the mean the convergence in the space $L^2(X,\mu)$ and we denote by

$$f(x) = l.i.m.f_n(x)\ ,$$

if

$$\lim_{n\to\infty}\int_X |f(x) - f_n(x)|^2\ dx = 0\,, \quad \text{almost everywhere in } X\,.$$

($l.i.m.$ is an abbreviation for limes in medio, i.e. limit in the mean).

2. Carleman operators

One can find necessary information about Carleman operators, for example, in [5, 9, 22, 23, 24]. In this section we shall present only part of it. Let X be an arbitrary set, μ a σ-fini measure on X (μ is defined on a σ-algebra of subsets of X, we don't indicate this σ-algebra), $L_2(X,\mu)$ the Hilbert space of square integrable functions with respect to μ. Instead of writing 'μ-measurable', 'μ-almost everywhere' and '$(d\mu(x))$' we write 'measurable', 'a e' and 'dx'.

Definition 1 ([24])**.** A linear operator $A\colon D(A) \longrightarrow L_2(X,\mu)$, where the domain $D(A)$ is a dense linear manifold in $L_2(X,\mu)$, is said to be **integral** if there exists a measurable function K on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x,y)\,f(y)\,dy \quad \text{a e}\,. \tag{2.1}$$

A kernel K on $X \times X$ is said to be Carleman if $K(x,y) \in L_2(X,\mu)$ for almost every fixed x, that is to say

$$\int_X |K(x,y)|^2 \, dy < \infty \quad \text{a e}. \tag{2.2}$$

An integral operator A with a kernel K is called **Carleman operator** if K is a Carleman kernel. Every Carleman kernel K defines a Carleman function k from X to $L_2(X,\mu)$ by $k(x) = \overline{K(x,\cdot)}$ for all x in X for which $K(x,\cdot) \in L_2(X,\mu)$.

Self-adjoint Carleman operators have generalized eigenfunction expansions, which can be used in the study of linear elliptic operators, see [14]. A general reference for Carleman operators on L_2-spaces is [8]. The notion of a Carleman operator has been extended in many directions. By replacing L_2 by an arbitrary Banach function space one obtains the so-called generalized Carleman operators (see [18]) and by considering Bochner integrals and abstract Banach spaces one is lead to the so-called Carleman and Korotkov operators on a Banach space ([7]).

Now we consider the class of integral operators (2.1) that we go studied here generated by the following symmetric Carleman kernel

$$K(x,y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)}, \tag{2.3}$$

where the overbar denotes complex conjugation. $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ such that

$$\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty \quad \text{a e}, \tag{2.4}$$

and $\{a_p\}_{p=0}^{\infty}$ a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2 |\psi_p(x)|^2 < \infty \quad \text{a e}. \tag{2.5}$$

We called $\{\psi_p(x)\}_{p=0}^{\infty}$ a Carleman sequence. Let $L(\psi)$ be the closed set of linear combinations of elements of the orthogonal sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is lucid that the orthogonal complement $L^{\perp}(\psi) = L_2(X,\mu) \ominus L(\psi)$ is contained in $D(A)$ and annul the operator A.
The following lemma [3] tells us when the Carleman operator A possesse equal deficiency indices.

Lemma 1 ([3])**.** The operator A possesses equal deficiency indices $n_+(A) = n_-(A) = m$, $(m < \infty)$, if and only if there exist sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$, $(k = 1,2,\ldots,m)$, verifying

1) For all k

$$\theta_k(x)=\sum_{p=0}^{\infty}\gamma_p^{(k)}\psi_p(x)\in L^{\perp}(\psi)\quad (k=1,2,\dots,m) \tag{2.6}$$

2) For all λ ($\Im m\lambda\neq 0$)

$$\sum_{p=0}^{\infty}\left|\frac{\gamma_p^{(k)}}{a_p-\lambda}\right|^2<\infty\,,\quad (k=1,2,\dots,m) \tag{2.7}$$

3) The linear space of the sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$, ($k=1,2,\dots,m$), verifying 1) and 2) is m dimension.

3. Generalized resolvents

We first prove the following important lemma.

Lemma 2. Let B be a closed symmetric operator, ψ the eigenvector of B belonging to the eigenvalue b. Then $\psi\in D(B)$ if and only if for a certain λ ($\Im m\lambda\neq 0$) and for all φ_λ and $\varphi_{\bar\lambda}$

$$(\varphi_\lambda,\psi)=(\varphi_{\bar\lambda},\ \psi)=0\,,$$

where φ_λ and $\varphi_{\bar\lambda}$ belong respectively to the defect spaces $\mathcal{N}_{\bar\lambda}$ and $\mathcal{N}_\lambda$.

Proof. Let $\psi\in D(B)$ and $\varphi_\lambda\in\mathcal{N}_{\bar\lambda}$ ($\Im m\lambda\neq 0$), then

$$(b\psi,\varphi_\lambda)=(B\psi,\varphi_\lambda)=(\psi,B^*\varphi_\lambda)=\bar\lambda(\psi,\varphi_\lambda)\ .$$

Therefore,

$$\left(b-\bar\lambda\right)(\psi,\varphi_\lambda)=0$$

and as $b-\bar\lambda\neq 0$, it follows that $(\psi,\varphi_\lambda)=0$. Now let h be an arbitrary element of $D(B^*)$. By the hilbertienne decomposition we have

$$h=f+\alpha\varphi_\lambda+\beta\varphi_{\bar\lambda}\,,$$

with $f\in D(B)$, $\varphi_\lambda\in\mathcal{N}_{\bar\lambda}$, $\varphi_{\bar\lambda}\in\mathcal{N}_\lambda$, and α, β two complex numbers. Then,

$$(B^*h,\psi)=(Bf,\psi)=(f,b\psi)=(h,b\psi)\ ,$$

that is to say $\psi\in D(B)$. □

Now we suppose that the symmetric Carleman operator A (2.1) $--$ (2.3) possesse equal deficiency indices $n_+(A)=n_-(A)=1$. By Lemma 1 there exist a sequence $\{\gamma_p\}_{p=0}^{\infty}$ such that:

$$\sum_{p=0}^{\infty}|\gamma_p|^2=\infty$$

and verifying the three conditions of the quoted lemma. By (2.6) and (2.7) we conclude that the function

$$\varphi_\lambda(x)=\sum_{p=0}^{\infty}\frac{\gamma_p}{a_p-\lambda}\psi_p(x) \tag{3.1}$$

belongs to the defect space $\mathcal{N}_{\bar{\lambda}}$ of the operator A. In what follows, to facilitate the writing, we will designate by A the restriction of A on the subspace $L(\psi)$.

Now we consider the following integral equation

$$\int_X \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)} Y(y)\, dy - \lambda Y(x) = f(x)\ . \tag{3.2}$$

Let $f(x) = \sum\limits_{p=0}^{\infty} c_p \psi_p(x) \left(\sum\limits_{p=0}^{\infty} |c_p|^2 < \infty \right)$, then the solution of the equation (3.2) will be the function

$$Y(x,\lambda) = \sum_{p=0}^{\infty} \frac{c_p}{a_p - \lambda} \psi_p(x)\ . \tag{3.3}$$

Let's notice that the formula (3.3) gives the resolvent of the self-adjoint extension $\overset{\circ}{A}$ of the operator A which possesses a complete system of eigenfunctions $\{\psi_k(x)\}$ of the space $L(\psi)$. The resolvent $\overset{\circ}{R}_\lambda$ of the operator $\overset{\circ}{A}$ is an integral operator defined on the space $L(\psi)$:

$$\overset{\circ}{R}_\lambda f = \int_X \overset{\circ}{K}(x,y;\lambda) f(y)\, dy\,, \tag{3.4}$$

where

$$\overset{\circ}{K}(x,y;\lambda) = \sum_{p=0}^{\infty} \frac{1}{a_p - \lambda} \psi_p(x) \overline{\psi_p(y)}\,.$$

Any solution of the equation (3.2) in $D(A^*)$ admits the following representation

$$Y(x,\lambda) = \overset{\circ}{R}_\lambda f(x) + c\varphi_\lambda(x)\ , \tag{3.5}$$

where c is an any complex number.

Let's put $\lambda_\circ = i$, then $F(\lambda)$ (subsection 1.3) can be given by the formula

$$F(\lambda)\, \varphi_{-i} = \omega(\lambda)\, \varphi_i$$

with $\omega(\lambda)$ an analytic function in the upper half plan and $|\omega(\lambda)| \leq 1$.

The operator $A_{F(\lambda)}$ is defined on the set $D\left(A_{F(\lambda)}\right)$ as

$$\begin{cases} f = x + \omega(\lambda)\, \varphi_i - \varphi_{-i} \big(x \in D(A)\big)\,, \\ A_{F(\lambda)} f = Ax + i\omega(\lambda)\, \varphi_i + \varphi_{-i}\,, \end{cases} \tag{3.6}$$

then

$$\begin{aligned} D\big(A_{F(\lambda)}\big) &= \big\{ g \in L(\psi) : g = f + [\omega(\lambda)\, \varphi_i - \varphi_{-i}]\, c,\ f \in D(A) \big\}\,, \\ D\big(A^*_{F(\lambda)}\big) &= \big\{ h \in L(\psi) : g = f + \big[\overline{\omega(\lambda)} \varphi_{-i} - \varphi_i\big] c,\ f \in D(A) \big\}\,. \end{aligned} \tag{3.7}$$

We introduce the following function

$$\nu_\lambda = \overline{\omega(\lambda)} \varphi_{-i} - \varphi_i\,,$$

then $D\left(A_{F(\lambda)}\right)$ is defined as the set of $y \in D(A^*)$ for which

$$(A^* y, \nu_\lambda) = (y, A^* \nu_\lambda)\ .$$

While choosing in (3.5) for all $\lambda\,(\Im m\lambda > 0)$ $c = C\,(\lambda)$, as we have the equality

$$(A^*Y, \nu_\lambda) = (Y, A^*\nu_\lambda)\ , \tag{3.8}$$

we get a formula giving the set of generalized resolvents in terms of analytic functions $\omega\,(\lambda)$. By (3.8) we have

$$C\,(\lambda) = \frac{[1-\omega\,(\lambda)]\,(f, \varphi_{\bar\lambda})}{[\omega\,(\lambda)\,\chi\,(\lambda) - 1]\,(\lambda + i)\,(\varphi_\lambda, \varphi_i)} \quad (\Im m\lambda > 0)\ , \tag{3.9}$$

where

$$\chi\,(\lambda) = \frac{\lambda - i}{\lambda + i}\frac{(\varphi_\lambda, \varphi_{-i})}{(\varphi_\lambda, \varphi_i)} \tag{3.10}$$

denote the characteristic function [1] of operator A. If we substitute (3.9) in (3.5), we get the formula of generalized resolvents

$$\mathbf{R}_\lambda f = \overset{\circ}{R}_\lambda f + \frac{1-\omega\,(\lambda)}{\omega\,(\lambda)\,\chi\,(\lambda) - 1}\frac{(f, \varphi_{\bar\lambda})}{(\lambda + i)\,(\varphi_\lambda, \varphi_i)}\varphi_\lambda \quad (\Im m\lambda > 0)\ . \tag{3.11}$$

While taking account that $\mathbf{R}_{\bar\lambda} = \mathbf{R}^*_\lambda$, it is easy to have

$$\mathbf{R}_{\bar\lambda} f = \overset{\circ}{R}_{\bar\lambda} f + \frac{1-\overline{\omega\,(\lambda)}}{\overline{\omega\,(\lambda)\chi\,(\lambda)} - 1}\frac{(f, \varphi_\lambda)}{\left(\overline{\lambda} - i\right)\left(\varphi_{\overline{\lambda}}, \varphi_{-i}\right)}\varphi_{\overline{\lambda}} \quad (\Im m\lambda > 0)\ . \tag{3.12}$$

So we have demonstrated

Theorem 1. *Formulas* (3.11) *and* (3.12) *establish a bijective correspondence between the set of generalized resolvents of the operator A and the set of the analytic functions $\omega\,(\lambda)$ as $|\omega\,(\lambda)| \le 1$ $(\Im m\lambda > 0)$. These formulas define the resolvent of a selfadjoint extension of A in the space $L\,(\psi)$ if and only if, $\omega\,(\lambda) = \varkappa$*(constant), $|\varkappa| = 1$.

4. Generalized spectral functions

Let's consider the function $\chi\,(\lambda)$ given by the formula (3.10):

$$\chi\,(\lambda) = \frac{\lambda - i}{\lambda + i}\,\frac{\sum\limits_{p=0}^{\infty}\frac{\gamma_p^2}{(a_p-\lambda)(a_p-i)}}{\sum\limits_{p=0}^{\infty}\frac{\gamma_p^2}{(a_p-\lambda)(a_p+i)}}\,,$$

it's an analytic function in the half plane $\Pi^+ = \{\lambda \in \mathbb{C} : \Im m\lambda \ge 0\}$ and take its values on the unit disk $D = \{\zeta \in \mathbb{C} : |\zeta| \le 1\}$, so that the real axis $\mathbb{R}$ turns into the unit circle $C = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Thus, for all $p = 0, 1, 2, \ldots,$ $\chi\,(a_p) = 1$.
Let's put

$$\zeta = \frac{\lambda - i}{\lambda + i}\,.$$

We can write [1] $\chi\,(\lambda)$ under the form

$$\chi\,(\lambda) = \chi\Big(i\frac{1+\zeta}{1-\zeta}\Big) = \omega(\zeta) = \frac{\zeta((\overset{\circ}{U} - \zeta I)^{-1}\varphi_i, \varphi_i)}{((\overset{\circ}{U} - \zeta I)^{-1}\overset{\circ}{U}\varphi_i, \varphi_i)} = \frac{\Phi\,(\zeta) - \|\varphi_i\|^2}{\Phi\,(\zeta) + \|\varphi_i\|^2}\,,$$

where

$$\overset{\circ}{U} = (\overset{\circ}{A} - iI)(\overset{\circ}{A} + iI^{-1})$$

is the unitary Cayley transform of the self-adjoint operator $\overset{\circ}{A}$ and

$$\Phi(\zeta) = \int_0^{2\pi} \frac{e^{is} + \zeta}{e^{is} - \zeta}\, d(\overset{\circ}{E}_s \varphi_i, \varphi_i)\,,$$

$\overset{\circ}{E}_s$ being the resolution of the identity of the unitary operator $\overset{\circ}{U}$. For $|\zeta| = 1$, we have

$$\Re e\, [\Phi\,(\zeta)] = 0\,. \tag{4.1}$$

From the equality

$$(\varphi_\lambda, \varphi_i) = \frac{i}{\lambda + i}\left[\Phi\,(\zeta) + \|\varphi_i\|^2\,\right] \tag{4.2}$$

we conclude that

$$(\varphi_\lambda, \varphi_i) \neq 0\ \forall\ \lambda\,, \quad \Im m\lambda \geq 0\,.$$

Formulas (4.1) and (4.2) imply that

$$\Im m\, [(\sigma + i)\,(\varphi_\sigma, \varphi_i)] = \|\varphi_i\|^2 \quad (\Im m\sigma = 0)\ . \tag{4.3}$$

Now, we introduce the following useful lemmas:

Lemma 3. For all $f,\ g \in H$, the functions $(\overset{\circ}{R}_\lambda f, g), (\varphi_\lambda, \varphi_i)\,, \left(f, \varphi_{\overline{\lambda}}\right)$ and (φ_λ, g) are regular on all the complex plane except to points a_p $(p = 0, 1, 2, \dots)$, where they admit simple poles. Besides, the following equalities are true:

$$\operatorname*{res}_{\lambda = a_p} (\overset{\circ}{R}_\lambda f, g) = \operatorname*{res}_{\lambda = a_p} \frac{\left(f, \varphi_{\overline{\lambda}}\right)(\varphi_\lambda, g)}{(\lambda - i)\,(\varphi_\lambda, \varphi_{-i})} = (f, \psi_p)\,(\psi_p, g)\ ,$$

$$\operatorname*{res}_{\lambda = a_p} \frac{\left(f, \varphi_{\overline{\lambda}}\right)(\varphi_\lambda, g)}{(\lambda - i)\,(\varphi_\lambda, \varphi_{-i})} = \operatorname*{res}_{\lambda = a_p} \frac{(f, \varphi_\lambda)\left(\varphi_{\overline{\lambda}}, g\right)}{\left(\overline{\lambda} - i\right)\left(\varphi_{\overline{\lambda}}, \varphi_i\right)} = (f, \psi_p)\,(\psi_p, g)\ .$$

Proof. The fact that the mentioned functions are regular on the complex plane except to poles a_p $(p = 0, 1, 2, \dots)$ result from formulas (3.1) and

$$(\overset{\circ}{R}_\lambda f, g) = \sum_{p=0}^{\infty} \frac{(f, \psi_p)\,(\psi_p, g)}{a_p - \lambda}.$$

Furthermore we have:

$$\operatorname*{res}_{\lambda = a_p} (\overset{\circ}{R}_\lambda f, g) = (f, \psi_p)\,(\psi_p, g)\ ,$$

it is easy to see that the function

$$\frac{\left(f, \varphi_{\overline{\lambda}}\right)(\varphi_\lambda, g)}{(\lambda - i)\,(\varphi_\lambda, \varphi_{-i})} = \frac{\left[\sum_{p=0}^{\infty} \frac{\gamma_p (f, \psi_p)}{a_p - \overline{\lambda}}\right]\left[\sum_{p=0}^{\infty} \frac{\gamma_p (\psi_p, g)}{a_p - \lambda}\right]}{(\lambda - i) \sum\limits_{p=0}^{\infty} \frac{\gamma_p^2}{(a_p - \lambda)(a_p - i)}}\,,$$

admits the same residue to the point $\lambda = a_p$.
The second equality can be verified in the same way. ☐

Lemma 4 ([21])**.** Let $\varphi(\lambda)$ an analytic function in the half-plane Π^+ with a positive imaginary part and $\psi(\lambda)$ an analytic function in a certain domain containing the interval $[\alpha, \beta]$. Then we have the formula

$$\frac{1}{2\pi i}\lim_{\tau\to+0}\int_\alpha^\beta \left[\varphi(\lambda)\psi(\lambda)-\overline{\varphi(\lambda)\psi(\lambda)}\right]d\sigma=\int_\alpha^\beta\psi(\sigma)\,d\rho(\sigma)\quad(\lambda=\sigma+i\tau)\ ,$$

with

$$\rho(\sigma)=\frac{1}{\pi}\lim_{\tau\to+0}\int_0^\sigma \Im m\varphi(t+i\tau)\ dt\,.$$

Let $\omega(\lambda)$ be an arbitrary analytic function who applies the half-plane Π^+ on the unit disk D. It is known that the spectral function $\mathbf{E}_t$ is uniform and that we can get it by the formula of Stieltjes (1.9):
for all $f(s)$ and $g(s)$ of L and for all reals α and β $(\alpha<\beta)$ we have the equality

$$(\mathbf{E}_{\alpha,\beta}f,g)=\frac{1}{2\pi i}\lim_{\tau\to+0}\int_\alpha^\beta\big([\mathbf{R}_{\sigma+i\tau}-\mathbf{R}_{\sigma-i\tau}]f,g\big)\,d\sigma$$

with

$$\mathbf{E}_{\alpha,\beta}=(\mathbf{E}_\beta+\mathbf{E}_{\beta+0})/2-(\mathbf{E}_\alpha+\mathbf{E}_{\alpha+0})/2.$$

Let's consider the difference

$$\mathbf{R}_\lambda f-\mathbf{R}_{\bar\lambda}f=\left[\overset{\circ}{R}_\lambda f-\overset{\circ}{R}_{\bar\lambda}f\right]+\left[C(\lambda)\varphi_\lambda-C\left(\overline{\lambda}\right)\varphi_{\overline{\lambda}}\right]\ . \tag{4.4}$$

While holding in account (3.3) and (3.4), we get

$$\lim_{\tau\to+0}\int_\alpha^\beta\left[\overset{\circ}{R}_\lambda f-\overset{\circ}{R}_{\bar\lambda}f\right]d\sigma=\sum_{\alpha_k\in(\alpha,\beta)}^{\infty}c_k\psi_k(s)\quad(\lambda=\sigma+i\tau)\ ,$$

where

$$f(s)=\sum_{k=0}^{\infty}c_k\psi_k(s)\ . \tag{4.5}$$

Let's consider the second member of (4.4):

$$\begin{aligned}C(\lambda)\varphi_\lambda-C\left(\bar\lambda\right)\varphi_{\overline{\lambda}}&=\frac{-i}{\omega(\lambda)\chi(\lambda)-1}\left[\frac{1}{(\lambda-i)(\varphi_\lambda,\varphi_{-i})}-\frac{1}{(\lambda+i)(\varphi_\lambda,\varphi_i)}\right]\\&\times\frac{1}{i}\left(f,\varphi_{\overline{\lambda}}\right)\varphi_\lambda-\frac{i}{\overline{\omega(\lambda)\chi(\lambda)}-1}\left[\frac{1}{\left(\overline{\lambda}-i\right)\left(\varphi_{\overline{\lambda}},\varphi_{-i}\right)}-\frac{1}{\left(\overline{\lambda}+i\right)\left(\varphi_{\overline{\lambda}},\varphi_i\right)}\right]\\&\times\frac{1}{i}(f,\varphi_\lambda)\varphi_{\overline{\lambda}}-\left[\frac{\left(f,\varphi_{\overline{\lambda}}\right)\varphi_\lambda}{(\lambda-i)(\varphi_\lambda,\varphi_{-i})}-\frac{(f,\varphi_\lambda)\varphi_{\overline{\lambda}}}{\left(\overline{\lambda}+i\right)\left(\varphi_{\overline{\lambda}},\varphi_i\right)}\right].\end{aligned}$$

Let's put

$$\frac{(f,\varphi_{\bar\lambda})\varphi_\lambda}{(\lambda-i)(\varphi_\lambda,\varphi_{-i})}=f_1(\lambda)\ ;\quad\frac{(f,\varphi_\lambda)\varphi_{\overline{\lambda}}}{\left(\overline{\lambda}+i\right)(\varphi_{\bar\lambda},\varphi_i)}=f_2(\lambda)\qquad(\Im m\lambda>0)\ .$$

Then $(\lambda = \sigma + i\tau)$

$$\frac{1}{2\pi i}\lim_{\tau\to+0}\int_{\alpha}^{\beta}\left[f_1(\lambda) - f_2(\lambda)\right]d\sigma = \frac{1}{2\pi i}\int_{\alpha}^{\beta}\frac{2i\Im m\left[(\sigma+i)\left(\varphi_\sigma,\varphi_i\right)\right]\left(f,\varphi_\sigma\right)}{\left(\sigma^2+1\right)\left|\left(\varphi_\sigma,\varphi_i\right)\right|^2}d\sigma$$

$$+\sum_{\alpha_k\in(\alpha,\beta)}^{\infty} c_k\psi_k(s)\ ,$$

c_k being coefficients in the development (4.5).

Now, we notice that for all analytic function $\omega(\lambda)$ in the half-plane Π^+ as $|\omega(\lambda)| \le 1$, we obtain

$$\Im m\frac{i}{\omega(\lambda)\chi(\lambda)-1} > 0 \quad (\Im m\lambda > 0)\ .$$

After this, while using the Lemma 2 and the equality (4.3), we get

$$\mathbf{E}_{\alpha,\beta}f = \frac{1}{2\pi i}\lim_{\tau\to+0}\int_{\alpha}^{\beta}\left[\mathbf{R}_\lambda - \mathbf{R}_{\bar\lambda}\right]f\,d\sigma = \int_{\alpha}^{\beta}\frac{\left(f,\varphi_\sigma\right)\varphi_\sigma}{\left(\sigma^2+1\right)|(\varphi_\sigma,\mathring{\varphi}_i)|^2}d\rho(\sigma)\ , \tag{4.6}$$

with

$$\rho(\sigma) = \frac{1}{\pi}\lim_{\tau\to+0}\int_0^{\sigma}\left[\Im m\frac{-2i}{\omega(\lambda)\chi(\lambda)-1} - 1\right]dt, \quad (\lambda = t + i\tau) \tag{4.7}$$

and

$$\mathring{\varphi}_i(s) = \frac{\varphi_i(s)}{\|\varphi_i\|}\ .$$

The function $\rho(\sigma)$ is decreasing because

$$\Re e\frac{1}{\omega(\lambda)\chi(\lambda)-1} \ge \frac{1}{1+|\omega(\lambda)\chi(\lambda)|} \ge \frac{1}{2}.$$

Thus, we have demonstrated the theorem

Theorem 2. *Let $\omega(\lambda)$ be an analytic function in the half-plane Π^+ and E_t $(-\infty < t < +\infty)$ the spectral function of the operator A. Then for all $f(s)$ of $L(\psi)$ and for all reals α and β $(\alpha < \beta)$ we have the relation* (4.6) *and the following equalities*

$$(E_{\alpha,\beta}f,f) = \int_{\alpha}^{\beta}\frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)|(\varphi_\sigma,\mathring{\varphi}_i)|^2}d\rho(\sigma)\,,$$

$$f(s) = \underset{\substack{\alpha\to-\infty\\ \beta\to+\infty}}{l.i.m.}\int_{\alpha}^{\beta}\frac{(f,\varphi_\sigma)\varphi_\sigma(s)}{\left(\sigma^2+1\right)\left|\left(\varphi_\sigma,\mathring{\varphi}_i\right)\right|^2}d\rho(\sigma)\ ,$$

$$(f,f) = \int_{-\infty}^{+\infty}\frac{\left|(f,\varphi_\sigma)\right|^2}{\left(\sigma^2+1\right)\left|\left(\varphi_\sigma,\mathring{\varphi}_i\right)\right|^2}d\rho(\sigma)\ ,$$

where $\rho(\sigma)$ is defined by the formula (4.7) *for $\lambda = \sigma + i\tau$, $\Im m\lambda > 0$.*

Corollary 1. *In order that* t $(-\infty < t < +\infty)$ *be a continuity point of the spectral function* E_t *of the operator* A *it is necessary and sufficient that it is a continuity point of the function* $\rho(\sigma)$.

Let's consider the formula (4.7). The function $\chi(\lambda)$ applies all interval $(a_{p_k}, a_{p_{k+1}})$ (we suppose that a_{p_k} and $a_{p_{k+1}}$ are neighboring points) in the unit disk. The homographic transform $\frac{1+z}{1-z}$ applies the circle $|z| = r \leq 1$ in the not euclidean circle of center i such that the image of $r = 0$ will be the point i and the image of $r = 1$ will be the real axis $\mathbb{R}$. Therefore, for $\omega(\lambda) = 1$, $\rho(\sigma)$ is a jumps function with points jumps a_{p_k} and for $\omega(\lambda) = \varkappa$ ($\varkappa =$ constant with $|\varkappa| < 1$), $\rho(\sigma)$ is absolutely continuous.

With the help of the self-adjoints extensions $(\omega(\lambda) = \varkappa = \exp(i\varphi))$ $\rho(\sigma)$ will be a jumps function with points jumps σ_p for whom $\chi(\sigma_p) = \exp(-i\varphi)$.

Of the pace of the function $\rho(\sigma)$ we are convinced of the following findings.

Corollary 2. *The quasi-self-adjoint extension associated to the analytical function* $\omega(\lambda)$ $(|\omega(\lambda)| \leq 1$ *in* Π^+ *and* $|\omega(\sigma)| = 1$ *for* $\Im m\sigma = 0)$ *admits a merely point spetrum.*

Corollary 3. *The interval* (c, d) $(-\infty \leq c < d \leq +\infty)$ *doesn't contain the spectrum points of the self-adjoint extension of the operator* A *generated by the functions* $\omega(\lambda)$ *if and only if* $\omega(\lambda)$ *verify the following conditions:*

a) $\omega(\lambda)$ *is analytic in* Π^+ *and* $|\omega(\lambda)| \leq 1$ $(\Im m\lambda > 0)$;
b) $\omega(\lambda)$ *admits an extension by continuity from* Π^+ *on* (c, d);
c) $|\omega(\sigma)| = 1$, *if* $\sigma \in (c, d)$;
d) $\omega(\sigma) \neq \overline{\chi(\sigma)}$ *for* $\sigma \in (c, d)$.

If we suppose in (2.3) that $a_p > 0$, then A will be a positive operator. Thus the Corollary 3 give the criteria to get the positive spectral functions. In particular self-adjoint extension possessed a positive spectral function if it is generated by functions $\omega(\lambda) = \varkappa = \exp(i\varphi)$, $0 \leq \varphi \leq \varphi_0$, $\chi(0) = \exp(-i\varphi_0)$.

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