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# LECTURES ON GENERALIZED COMPLEX GEOMETRY AND SUPERSYMMETRY

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Abstract. These are the lecture notes from the 26th Winter School "Geometry and Physics", Czech Republic, Srní, January 14 – 21, 2006. These lectures are an introduction into the realm of generalized geometry based on the tangent plus the cotangent bundle. In particular we discuss the relation of this geometry to physics, namely to two-dimensional field theories. We explain in detail the relation between generalized complex geometry and supersymmetry. We briefly review the generalized Kähler and generalized Calabi-Yau manifolds and explain their appearance in physics.

## Introduction

These are the notes for the lectures presented at the 26th Winter School "Geometry and Physics", Srní, Czech Republic, January 14–21, 2006. The principal aim in these lectures has been to present, in a manner intelligible to both physicists and mathematicians, the basic facts about the generalized complex geometry and its relevance to string theory. Obviously, given the constraints of time, the discussion of many subjects is somewhat abbreviated.

In [11] Nigel Hitchin introduced the notion of generalized complex structure and generalized Calabi-Yau manifold. The essential idea is to take a manifold $M$ and replace the tangent bundle $TM$ by $TM \oplus T^*M$, the tangent plus the cotangent bundle. The generalized complex structure is a unification of symplectic and complex geometries and is the complex analog of a Dirac structure, a concept introduced by Courant and Weinstein [6], [7]. These mathematical structures can be mapped into string theory. In a sense they can be derived and motivated from certain aspects of string theory. The main goal of these lectures is to show the appearance of generalized geometry in string theory. The subject is still in the progress and some issues remain unresolved. In an effort to make a self-consistent presentation we choose to concentrate on Hamiltonian aspects of the world-sheet theory and we leave aside other aspects which are equally important.

The lectures are organized as follows. In Lecture 1 we introduce the relevant mathematical concepts such as Lie algebroid, Dirac structure and generalized complex structure. In the next Lecture we explain the appearance of these structures in string theory, in particular from the world-sheet point of view. We choose the Hamiltonian formalism as most natural for the present purpose. In the last Lecture we review more advanced topics, such as generalized Kähler and generalized Calabi-Yau manifolds. We briefly comment on their appearance in string theory.

Let us make a comment on notation. Quite often we use the same letter for a bundle morphism and a corresponding map between the spaces of sections. Hopefully it will not irritate the mathematicians and will not lead not any confusion.

## Contents



## Lecture 1

This Lecture is devoted to a review of the relevant mathematical concepts, such as Lie algebroid, Courant bracket, Dirac structure and generalized complex geometry (also its real analog). The presentation is rather sketchy and we leave many technical details aside.

For further reading on the Lie algebroids we recommend [21] and [5]. On details of generalized complex geometry the reader may consult [10].

1.1. **Lie algebroid.** Any course on the differential geometry starts from the introduction of $TM$, the tangent bundle of smooth manifold $M$. The sections of $TM$ are the vector fields. One of the most important properties of $TM$ is that there exists a natural Lie bracket $\{\ ,\ \}$ between vector fields. The existence of a Lie bracket between vectors fields allows the introduction of many interesting geometrical structures. Let us consider the example of the complex structure:

**Example 1.1.** An almost complex structure $J$ on $M$ can be defined as a linear map (endomorphism) $J : TM \to TM$ such that $J^2 = -1$. This allows us to introduce the projectors

$$\pi_{\pm} = \frac{1}{2}(1 \pm iJ)\,, \qquad \pi_+ + \pi_- = 1\,,$$

which induce a decomposition of complexified tangent space

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

into a holomorphic and an antiholomorphic part, $\pi_- v = v\ v \in T^{(1,0)}M$ and $\pi_+ w = w\ w \in T^{(0,1)}M$. The almost complex structure $J$ is integrable if the subbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are involutive with respect to the Lie bracket, i.e. if

$$\pi_-\{\pi_+ v, \pi_+ w\} = 0\,, \qquad \pi_+\{\pi_- v, \pi_- w\} = 0$$

for any $v, w \in \Gamma(TM)$. The manifold $M$ with such an integrable $J$ is called a complex manifold.

From this example we see that the Lie bracket plays a crucial role in the definition of integrability of a complex structure $J$.

$TM$ is vector bundle with a Lie bracket. One can try to define a generalization of $TM$ as a vector bundle with a Lie bracket. Thus we come now to the definition of a Lie algebroid

**Definition 1.2.** A Lie algebroid is a vector bundle $\mathbb{L}$ over a manifold $M$ together with a bundle map (the anchor) $\rho : \mathbb{L} \to TM$ and a Lie bracket $\{\ ,\ \}$ on the space $\Gamma(\mathbb{L})$ of sections of $\mathbb{L}$ satisfying

$$\rho(\{v, k\}) = \{\rho(v), \rho(k)\}\,, \qquad v, k \in \Gamma(\mathbb{L})$$
$$\{v, fk\} = f\{v, k\} + (\rho(v)f)k\,, \qquad v, k \in \Gamma(\mathbb{L})\,,\ \ f \in C^\infty(M)$$

In this definition $\rho(v)$ is a vector field and $(\rho(v)f)$ is the action of the vector field on the function $f$, i.e. the Lie derivative of $f$ along $\rho(v)$. Thus the set of sections $\Gamma(\mathbb{L})$ is a Lie algebra and there exists a Lie algebra homomorphism from $\Gamma(\mathbb{L})$ to $\Gamma(TM)$.

To illustrate the definition 1.2 we consider the following examples

**Example 1.3.** The tangent bundle $TM$ is a Lie algebroid with $\rho = \mathrm{id}$.

**Example 1.4.** Any integrable subbundle $\mathbb{L}$ of $TM$ is Lie algebroid. The anchor map is inclusion

$$\mathbb{L} \hookrightarrow TM$$

and the Lie bracket on $\Gamma(\mathbb{L})$ is given by the restriction of the ordinary Lie bracket to $\mathbb{L}$.

The notion of a Lie algebroid can obviously be complexified. For a complex Lie algebroid $\mathbb{L}$ we can use the same definition 1.2 but with $\mathbb{L}$ being a complex vector bundle and the anchor map $\rho : \mathbb{L} \to TM \otimes \mathbb{C}$.

**Example 1.5.** In example 1.1 for the complex manifold $M$, $T^{(1,0)}M$ is an example of a complex Lie algebroid with the anchor given by inclusion

$$T^{(1,0)}M \hookrightarrow TM \otimes \mathbb{C}\,.$$

It is instructive to rewrite the definition of Lie algebroid in local coordinates. On a trivializing chart we can choose the local coordinates $X^\mu$ ($\mu = 1, \dots, \dim M$) and a basis $e^A$ ($A = 1, \dots, \operatorname{rank} \mathbb{L}$) on the fiber. In these local coordinates we introduce the anchor $\rho^{\mu A}$ and the structure constants according to

$$\rho(e^A)(X) = \rho^{\mu A}(X) \partial_\mu\,, \qquad \{e^A, e^B\} = f^{AB}_{\ \ C} e^C\,.$$

The compatibility conditions from the definition 1.2 imply the following equation

$$\rho^{\nu A} \partial_\nu \rho^{\mu B} - \rho^{\nu B} \partial_\nu \rho^{\mu A} = f^{AB}_{\ \ C} \rho^{\mu C}$$

$$\rho^{\mu [D} \partial_\mu f^{AB]}_{\ \ \ C} + f^{[AB}_{\ \ \ L} f^{D]L}_{\ \ \ C} = 0$$

where [ ] stands for the antisymmetrization.

To any real Lie algebroid we can associate a characteristic foliation which is defined as follows. The image of anchor map $\rho$

$$\Delta = \rho(\mathbb{L}) \subset TM$$

is spanned by the smooth vector fields and thus it defines a smooth distribution. Moreover this distribution is involutive with the respect to the Lie bracket on $TM$. If the rank of this distribution is constant then we can use the Frobenius theorem and there exists a corresponding foliation on $M$. However tha rank of $D$ does not have to be a constant and one should use the generalization of the Frobenius theorem due to Sussmann [24]. Thus for any real Lie algebroid $\Delta_D = \rho(\mathbb{L})$ is integrable distribution in sense of Sussmann and there exists a generalized foliation.

For a complex Lie algebroid the situation is a bit more involved. The image of the anchor map

$$\rho(\mathbb{L}) = E \subset TM \otimes \mathbb{C}$$

defines two real distribution

$$E + \bar{E} = \theta \otimes \mathbb{C} \qquad E \cap \bar{E} = \Delta \otimes \mathbb{C}\,.$$

If $E + \bar{E} = TM \otimes \mathbb{C}$ then $\Delta$ is a smooth real distribution in the sense of Sussmann which defines a generalized foliation.

1.2. **Geometry of $TM \oplus T^*M$.** At this point it would be natural to ask the following question. How can one generate interesting examples of real and complex Lie algebroids? In this subsection we consider the tangent plus cotangent bundle $TM \oplus T^*M$ or its complexification, $(TM \oplus T^*M) \otimes \mathbb{C}$ and later we will show how one can construct Lie algebroids as subbundles of $TM \oplus T^*M$.

The section of tangent plus cotangent bundle, $TM \oplus T^*M$, is a pair of objects, a vector field $v$ and a one-form $\xi$. We adopt the following notation for a section:

$v+\xi \in \Gamma(TM \oplus T^*M)$. There exists a natural symmetric pairing which is given by

$$\langle v+\xi, s+\lambda \rangle = \frac{1}{2}(i_v\lambda + i_s\xi)\,, \tag{1.1}$$

where $i_v\lambda$ is the contraction of a vector field $v$ with one-form $\lambda$. In the local coordinates $(dx^\mu, \partial_\mu)$ the pairing (1.1) can be rewritten in matrix form as

$$\langle A, B\rangle = \langle v+\xi, s+\lambda\rangle = \frac{1}{2}\begin{pmatrix} v & \xi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} s \\ \lambda \end{pmatrix} = A^t \mathcal{I} B\,, \tag{1.2}$$

where

$$\mathcal{I} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a metric in a local coordinates $(dx^\mu, \partial_\mu)$. $\mathcal{I}$ has signature $(d,d)$ and thus here is natural action of $O(d,d)$ which preserves the pairing.

The subbundle $\mathbb{L} \subset TM \oplus T^*M$ is called isotropic if $\langle A, B\rangle = 0$ for all $A, B \in \Gamma(\mathbb{L})$. $\mathbb{L}$ is called maximally isotropic if

$$\langle A, B\rangle = 0\,, \qquad \forall A \in \Gamma(\mathbb{L})$$

implies that $B \in \Gamma(\mathbb{L})$.

There is no canonical Lie bracket defined on the sections of $TM \oplus T^*M$. However one can introduce the following bracket

$$[v+\xi, s+\lambda]_c = \{v, s\} + \mathcal{L}_v\lambda - \mathcal{L}_s\omega - \frac{1}{2}d(i_v\lambda - i_s\xi)\,, \tag{1.3}$$

which is called the Courant bracket. In (1.3) $\mathcal{L}_v$ stands for the Lie derivative along $v$ and $d$ is de Rham differential on the forms. The Courant bracket is antisymmetric and it does not satisfy the Jacobi identity. Nevertheless it is interesting to examine how it fails to satisfy the Jacobi identity. Introducing the Jacobiator

$$\mathrm{Jac}\,(A,B,C) = \big[[A,B]_c, C\big]_c + \big[[B,C]_c, A\big]_c + \big[[C,A]_c B\big]_c \tag{1.4}$$

one can prove the following proposition

**Proposition 1.6.**

$$\mathrm{Jac}\,(A,B,C) = d\big(\mathrm{Nij}\,(A,B,C)\big)$$

where

$$\mathrm{Nij}\,(A,B,C) = \frac{1}{3}\big(\langle [A,B]_c, C\rangle + \langle [B,C]_c, A\rangle + \langle [C,A]_c, B\rangle\big)$$

and where $A, B, C \in \Gamma(TM \oplus T^*M)$.

**Proof.** Let us sketch the main steps of the proof. We define the Dorfman bracket

$$(v+\omega) * (s+\lambda) = \{v, s\} + \mathcal{L}_v\lambda - i_s d\omega\,,$$

such that its antisymmetrization

$$[A, B]_c = A * B - B * A$$

produces the Courant bracket. From the definitions of the Courant and Dorfman brackets we can also deduce the following relation

$$[A,B]_c = A * B - d\langle A, B\rangle \,.$$

It is crucial that the Dorfman bracket satisfies a kind of Leibniz rule

$$A * (B * C) = (A * B) * C + B * (A * C)\,,$$

which can be derived directly from the definition of the Dorfman bracket. The combination of two last expressions leads to the formula for the Jacobiator in the proposition. $\square$

Next we would like to investigate the symmetries of the Courant bracket. Recall that the symmetries of the Lie bracket on $TM$ are described in terms of bundle automorphism

$$\begin{tikzcd} TM \arrow[r, "F"] \arrow[d] & TM \arrow[d] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

such that

$$F\big(\{v,k\}\big) = \big\{F(v), F(k)\big\}\,.$$

For the Lie bracket on $TM$ the only symmetry is diffeomorphism, i.e. $F = f_*$.

Analogously we look for the symmetries of the Courant bracket as bundle automorphism

$$\begin{tikzcd} TM \oplus T^*M \arrow[r, "F"] \arrow[d] & TM \oplus T^*M \arrow[d] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

such that

$$\big[F(A), F(B)\big]_c = F\big([A,B]_c\big)\,, \qquad A, B \in \Gamma(TM \oplus T^*M)$$

and in addition we require that it preserves the natural pairing $\langle\ ,\ \rangle$. Obviously $\mathrm{Diff}(M)$ is the symmetry of the Courant bracket with $F = f_* \oplus f^*$. However there exists an additional symmetry. For any two-form $b \in \Omega^2(M)$ we can define the transformation

$$e^b(v+\lambda) \equiv v + \lambda + i_v b\,, \tag{1.5}$$

which preserves the pairing. Under this transformation the Courant bracket transforms as follows

$$\big[e^b(v+\xi), e^b(s+\lambda)\big]_c = e^b\big([v+\xi, s+\lambda]\big) + i_v i_s db\,. \tag{1.6}$$

If $db = 0$ then we have a an orthogonal symmetry of the Courant bracket. Thus we arrive to the following proposition [10]:

**Proposition 1.7.** The group of orthogonal Courant automorphisms of $TM \oplus T^*M$ is semi-direct product of Diff $(M)$ and $\Omega^2_{\mathrm{closed}}(M)$.

$TM \oplus T^*M$ equipped with the natural pairing $\langle \, , \, \rangle$ and the Courant bracket $[\, , \,]_c$ is an example of the Courant algebroid. In general the Courant algebroid is a vector bundle with the bracket $[\, , \,]_c$ and the pairing $\langle \, , \, \rangle$ which satisfy the same properties we have described in this subsection.

1.3. **Dirac structures.** In this subsection we will use the properties of $TM \oplus T^*M$ in order to construct the examples of real and complex Lie algebroids.

The proposition 1.6 implies the following immediate corollary

**Corollary 1.8.** For maximally isotropic subbundle $\mathbb{L}$ of $TM \oplus T^*M$ or $(TM \oplus T^*M) \otimes \mathbb{C}$ the following three statements are equivalent

* $\mathbb{L}$ is involutive
* $\mathrm{Nij}\,|_{\mathbb{L}} = 0$
* $\mathrm{Jac}\,|_{\mathbb{L}} = 0$

Here we call $\mathbb{L}$ involutive if for any $A, B \in \Gamma(\mathbb{L})$ the bracket $[A, B]_c \in \Gamma(\mathbb{L})$.

**Definition 1.9.** An involutive maximally isotropic subbundle $\mathbb{L}$ of $TM \oplus T^*M$ (or $(TM \oplus T^*M) \otimes \mathbb{C}$) is called a real (complex) Dirac structure.

It follows from corollary 1.8 that $\mathbb{L}$ is a Lie algebroid with the bracket given by the restriction of the Courant bracket to $\mathbb{L}$. Since $\mathrm{Jac}\,|_L = 0$ the bracket $[\, , \,]_c|_{\mathbb{L}}$ is a Lie bracket. The anchor map is given by a natural projection to $TM$.

Let us consider some examples of Dirac structures

**Example 1.10.** The tangent bundle $TM \subset TM \oplus T^*M$ is a Dirac structure since $TM$ is a maximally isotropic subbundle. Moreover the restriction of the Courant bracket to $TM$ is the standard Lie bracket on $TM$ and thus it is an involutive subbundle.

**Example 1.11.** Take a two-form $\omega \in \Omega^2(M)$ and consider the following subbundle of $TM \oplus T^*M$

$$\mathbb{L} = e^{\omega}(TM) = \{v + i_v \omega, v \in TM\}.$$

This subbundle is maximally isotropic since $\omega$ is a two-form. Moreover one can show that $\mathbb{L}$ is involutive if $d\omega = 0$. Thus if $\omega$ is a presymplectic structure[1] then $\mathbb{L}$ is an example of a real Dirac structure.

**Example 1.12.** Instead we can take an antisymmetric bivector $\beta \in \Gamma(\wedge^2 TM)$ and define the subbundle

$$\mathbb{L} = \{i_\beta \lambda + \lambda, \lambda \in T^*M\},$$

where $i_\beta \lambda$ is a contraction of bivector $\beta$ with one-form $\lambda$. $\mathbb{L}$ is involutive when $\beta$ is a Poisson structure[2]. Thus for a Poisson manifold $\mathbb{L}$ is a real Dirac structure.

[1] The two-form $\omega$ is called a symplectic structure if $d\omega = 0$ and $\exists\, \omega^{-1}$. If two-form is just closed then it is called a presymplectic structure.

[2] The antisymmetric bivector $\beta^{\mu\nu}$ is called Poisson if it satisfies $\beta^{\mu\nu}\partial_\nu \beta^{\rho\sigma} + \beta^{\rho\nu}\partial_\nu \beta^{\sigma\mu} + \beta^{\sigma\nu}\partial_\nu \beta^{\mu\rho} = 0$. The name of $\beta$ is justified by the fact that $\{f, g\} = (\partial_\mu f)\beta^{\mu\nu}(\partial_\nu g)$ defines a Poisson bracket for $f, g \in C^\infty(M)$.

**Example 1.13.** Let $M$ to be a complex manifold and consider the following subbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$

$$\mathbb{L} = T^{(0,1)}M \oplus T^{*(1,0)}M$$

with the sections being antiholomorphic vector fields plus holomorphic forms. $\mathbb{L}$ is maximally isotropic and involutive (this follows immediately when $[\ ,\ ]_c|_{\mathbb{L}}$ is written explicitly). Thus for a complex manifold, $\mathbb{L}$ is an example of a complex Dirac structure.

1.4. **Generalized complex structures.** In this subsection we present the central notion for us, a generalized complex structure. We will present the different but equivalent definitions and discuss some basic examples of a generalized complex structure.

We have defined all basic notions needed for the definition of a generalized complex structure

**Definition 1.14.** The generalized complex structure is a complex Dirac structure $\mathbb{L} \subset (TM \oplus T^*M) \otimes \mathbb{C}$ such that $\mathbb{L} \cap \bar{\mathbb{L}} = \{0\}$ .

In other words a generalized complex structure gives us a decomposition

$$(TM \oplus T^*M) \otimes \mathbb{C} = \mathbb{L} \oplus \bar{\mathbb{L}}$$

where $\mathbb{L}$ and $\bar{\mathbb{L}}$ are complex Dirac structures.

There exist an alternative definition however. Namely we can mimic the standard description of the usual complex structure which can be defined as an endomorphism $J : TM \to TM$ with additional properties, see Example 1.1.

Thus in analogy we define the endomorphism

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M\,,$$

such that

$$\mathcal{J}^2 = -1_{2d}\,. \tag{1.7}$$

There exist projectors

$$\Pi_\pm = \frac{1}{2}\,(1_{2d} \pm i\mathcal{J})$$

such that $\Pi_+$ is projector for $\bar{\mathbb{L}}$ and $\Pi_-$ is the projector for $\mathbb{L}$. However $\mathbb{L}$ ($\bar{\mathbb{L}}$) is a maximally isotropic subbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$. Thus we need to impose a compatibility condition between the natural pairing and $\mathcal{J}$ in order to insure that $\mathbb{L}$ and $\bar{\mathbb{L}}$ are maximally isotropic spaces. Isotropy of $\mathbb{L}$ implies that for any sections $A, B \in \Gamma((T \oplus T^*) \otimes \mathbb{C})$

$$\langle \Pi_- A, \Pi_- B\rangle = A^t \Pi_-^t \mathcal{I} \Pi_- B = \frac{1}{4} A^t (\mathcal{I} + i\mathcal{J}^t\mathcal{I} + i\mathcal{I}\mathcal{J} - \mathcal{J}^t\mathcal{I}\mathcal{J})B = 0$$

which produces the following condition

$$\mathcal{J}^t\mathcal{I} = -\mathcal{I}\mathcal{J}\,. \tag{1.8}$$

If there exists a $\mathcal{J}$ satisfying (1.7) and (1.8) then we refer to $\mathcal{J}$ as an almost generalized complex structure. Next we have to add the integrability conditions, namely that $\mathbb{L}$ and $\bar{\mathbb{L}}$ are involutive with respect to the Courant bracket, i.e.

$$\Pi_{\mp}[\Pi_{\pm}A, \Pi_{\pm}B]_c = 0 \tag{1.9}$$

for any sections $A, B \in \Gamma(TM \oplus T^*M)$. Thus $\mathbb{L}$ is $+i$-egeinbundle of $\mathcal{J}$ and $\bar{\mathbb{L}}$ is $-i$-egeinbundle of $\mathcal{J}$. To summarize a generalized complex structure can be defined as an endomorphism $\mathcal{J}$ with the properties (1.7), (1.8) and (1.9).

An endomorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ satisfying (1.8) can be written in the form

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^t \end{pmatrix} \tag{1.10}$$

with $J : TM \to TM$, $P : T^*M \to TM$, $L : TM \to T^*M$ and $J^t : T^*M \to TM$. Indeed $J$ can be identified with a $(1,1)$-tensor, $L$ with a two-form and $P$ with an antisymmetric bivector. Imposing further the conditions (1.7) and (1.9) we arrive to the set of algebraic and differential conditions on the tensors $J$, $L$ and $P$ which were first studied in [17].

To illustrate the definition of a generalized complex structure we consider a few examples.

**Example 1.15.** Consider $\mathcal{J}$ of the following form

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} .$$

Such $\mathcal{J}$ is a generalized complex structure if and only if $J$ is a complex structure. The corresponding Dirac structure is

$$\mathbb{L} = T^{(0,1)}M \oplus T^{*(1,0)}M$$

as in example 1.13.

**Example 1.16.** Consider a $\mathcal{J}$ of the form

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} .$$

Such $\mathcal{J}$ is a generalized complex structure if and only if $\omega$ is a symplectic structure. The corresponding Dirac structure is defined as follows

$$\mathbb{L} = \{v - i(i_v\omega),\, v \in TM \otimes \mathbb{C}\} .$$

**Example 1.17.** Consider a generic generalized complex structure $\mathcal{J}$ written in the form (1.10). Investigation of the conditions (1.7) and (1.9) leads to the fact that $P$ is a Poisson tensor. Furthermore one can show that locally there is a symplectic foliation with a transverse complex structure. Thus locally a generalized complex manifold is a product a symplectic and complex manifolds [10]. The dimension of the generalized complex manifold is even.

1.5. **Generalized product structure.** Both complex structure and generalized complex structures have real analogs. In this subsection we will discuss them briefly. Some of the observations presented in this subsection are original. However they follow rather straightforwardly from a slight modification of the complex case.

The complex structure described in the example 1.1 has a real analog which is called a product structure [25].

**Example 1.18.** An almost product structure $\Pi$ on $M$ can be defined as a map $\Pi : TM \to TM$ such that $\Pi^2 = 1$. This allows us to introduce the projectors

$$\pi_{\pm} = \frac{1}{2}(1 \pm \Pi)\,, \qquad \pi_+ + \pi_- = 1\,,$$

which induce the decomposition of real tangent space

$$TM = T^+M \oplus T^-M$$

into two parts, $\pi_- v = v$ $v \in T^+M$ and $\pi_+ w = w$ $w \in T^-M$. The dimension of $T^+M$ can be different from the dimension of $T^-M$ and thus the manifold $M$ does not have to be even dimensional. The almost product structure $\Pi$ is integrable if the subbundles $T^+M$ and $T^-M$ are involutive with respect to the Lie bracket, i.e.

$$\pi_-\{\pi_+ v, \pi_+ w\} = 0\,, \qquad \pi_+\{\pi_- v, \pi_- w\} = 0$$

for any $v, w \in \Gamma(TM)$. We refer to an integrable almost product structure as product structure. A manifold $M$ with such integrable $\Pi$ is called a locally product manifold.

There exists always the trivial example of such structure $\Pi = \mathrm{id}$.

Obviously the definition 1.14 of generalized complex structure also has a real analog.

**Definition 1.19.** A generalized product structure is a pair of real Dirac structures $\mathbb{L}_{\pm}$ such that $\mathbb{L}_+ \cap \mathbb{L}_- = \{0\}$. In other words

$$TM \oplus T^*M = \mathbb{L}_+ \oplus \mathbb{L}_-$$

Indeed the definitions 1.14 and 1.19 are examples of complex and real Lie bialgebroids [19]. However we will not discuss this structure here.

Analogously to the complex case we can define an almost generalized product structure by means of an endomorphims

$$\mathcal{R} : TM \oplus T^*M \to TM \oplus T^*M$$

such that

$$\mathcal{R}^2 = 1_{2d}\,, \tag{1.11}$$

and

$$\mathcal{R}^t \mathcal{I} = -\mathcal{I}\mathcal{R}\,. \tag{1.12}$$

The corresponding projectors

$$p_{\pm} = \frac{1}{2}(1_{2d} \pm \mathcal{R})$$

define two maximally isotropic subspaces $\mathbb{L}_+$ and $\mathbb{L}_-$. The integrability conditions are given by

$$p_{\mp}[p_{\pm}A, p_{\pm}B]_c = 0\,, \tag{1.13}$$

where $A$ and $B$ are any sections of $TM \oplus T^*M$. In analogy with (1.10) we can write an endormorphism which satisfies (1.12) as follows

$$\mathcal{R} = \begin{pmatrix} \Pi & \tilde{P} \\ \tilde{L} & -\Pi^t \end{pmatrix}\,, \tag{1.14}$$

where $\Pi$ is a $(1,1)$-tensor, $\tilde{P}$ is an antisymmetric bivector and $\tilde{L}$ is a two-form. The conditions (1.11) and (1.13) imply similar algebraic and the same differential conditions for the tensors $\Pi$, $\tilde{L}$ and $\tilde{P}$ as in [17].

Let us give a few examples of a generalized product structure.

**Example 1.20.** Consider $\mathcal{R}$ of the following form

$$\mathcal{R} = \begin{pmatrix} \Pi & 0 \\ 0 & -\Pi^t \end{pmatrix}\,.$$

Such an $\mathcal{R}$ is a generalized product structure if and only if $\Pi$ is a standard product structure. This example justifies the name, we have proposed: a generalized product structure. The Dirac structure $\mathbb{L}_+$ is

$$\mathbb{L}_+ = T^+M \oplus T^{*-}M\,,$$

where $\lambda \in T^{*-}M$ if $\pi_+\lambda = \lambda$, see Example 1.18.

**Example 1.21.** Consider an $\mathcal{R}$ of the form

$$\mathcal{R} = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}\,.$$

Such an $\mathcal{R}$ is a generalized product structure if and only if $\omega$ is a symplectic structure.

For the generic generalized product structure $\mathcal{R}$ (1.14) $\tilde{P}$ is a Poisson structure. Generalizing the complex case one can show that locally there is a symplectic foliation with a transverse product structure. Thus locally a generalized product manifold is a product of symplectic and locally product manifolds.

1.6. **Twisted case.** Indeed one can construct on $TM \oplus T^*M$ more than one bracket with the same properties as the Courant bracket. Namely the different brackets are parametrized by a closed three form $H \in \Omega^3(M)$, $dH = 0$ and are defined as follows

$$[v+\xi, s+\lambda]_H = [v+\xi, s+\lambda]_c + i_v i_s H\,. \tag{1.15}$$

We refer to this bracket as the twisted Courant bracket. This bracket has the same properties as the Courant bracket. If $H = db$ then the last term on the right hand side of (1.15) can be generated by non-closed b-transform, see (1.6).

Thus we can define a twisted Dirac structure, a twisted generalized complex structure and a twisted generalized product structure. In all definitions the Courant bracket $[\ ,\ ]_c$ should be replaced by the twisted Courant bracket $[\ ,\ ]_H$. For example, a twisted generalized complex structure $\mathcal{J}$ satisfies (1.7) and (1.8) and now the integrability is defined with respect to twisted Courant bracket as

$$\Pi_{\mp}\big[\Pi_{\pm}(v+\xi), \Pi_{\pm}(s+\lambda)\big]_H = 0\,. \tag{1.16}$$

There is a nice relation of the twisted version to gerbes [10, 13]. However due to lack of time we will have to leave it aside.

## Lecture 2

In this Lecture we turn our attention to physics. In particular we would like to show that the mathematical notions introduced in Lecture 1 appear naturally in the context of string theory. Here we focus on the classical aspect of the hamiltonian formalism for the world-sheet theory.

2.7. **String phase space $T^*LM$.** A wide class of sigma models share the following phase space description. For the world-sheet $\Sigma = S^1 \times \mathbb{R}$ the phase space can be identified with a cotangent bundle $T^*LM$ of the loop space $LM = \{X : S^1 \to M\}$. Using local coordinates $X^\mu(\sigma)$ and their conjugate momenta $p_\mu(\sigma)$ the standard symplectic form on $T^*LM$ is given by

$$\omega = \int\limits_{S^1} d\sigma\ \delta X^\mu \wedge \delta p_\mu, \tag{2.17}$$

where $\delta$ is de Rham differential on $T^*LM$ and $\sigma$ is a coordinate along $S^1$. The symplectic form (2.17) can be twisted by a closed three form $H \in \Omega^3(M)$, $dH = 0$ as follows

$$\omega = \int\limits_{S^1} d\sigma\ (\delta X^\mu \wedge \delta p_\mu + H_{\mu\nu\rho}\partial X^\mu \delta X^\nu \wedge \delta X^\rho)\,, \tag{2.18}$$

where $\partial \equiv \partial_\sigma$ is derivative with respect to $\sigma$. For both symplectic structures the following transformation is canonical

$$X^\mu \ \to\ X^\mu\,, \qquad p_\mu \ \to\ p_\mu + b_{\mu\nu}\partial X^\nu \tag{2.19}$$

associated with a closed two form, $b \in \Omega^2(M)$, $db = 0$. There are also canonical transformations which correspond to $\mathrm{Diff}(M)$ when $X$ transforms as a coordinate and $p$ as a section of the cotangent bundle $T^*M$. In fact the group of local canonical transformations[3] for $T^*LM$ is a semidirect product of $\mathrm{Diff}(M)$ and $\Omega^2_{\mathrm{closed}}(M)$. Therefore we come to the following proposition

**Proposition 2.22.** The group of local canonical transformations on $T^*LM$ is isomorphic to the group of orthogonal automorphisms of Courant bracket.

See the proposition 1.7 and the discussion of the symmetries on the Courant bracket in the previous Lecture. The proposition 2.22 is a first indication that the geometry of $T^*LM$ is related to the generalized geometry of $TM \oplus T^*M$.

2.8. **Courant bracket and $T^*LM$.** Indeed the Courant bracket by itself can be "derived" from $T^*LM$. Here we present a nice observation on the relation between the Courant bracket and the Poisson bracket on $C^\infty(T^*LM)$ which is due to [1].

Let us define for any section $(v+\xi) \in \Gamma(TM \oplus T^*M)$ (or its complexified version) a current (an element of $C^\infty(T^*LM)$) as follows

$$J_\epsilon(v+\xi) = \int\limits_{S^1} d\sigma\ \epsilon(v^\mu p_\mu + \xi_\mu \partial X^\mu)\,, \tag{2.20}$$

where $\epsilon \in C^\infty(S^1)$ is a test function. Using the symplectic structure (2.17) we can calculate the Poisson bracket between two currents

$$\big\{J_{\epsilon_1}(A), J_{\epsilon_2}(B)\big\} = -J_{\epsilon_1\epsilon_2}\big([A,B]_c\big) + \int_{S^1} d\sigma\ (\epsilon_1\partial\epsilon_2 - \epsilon_2\partial\epsilon_1)\langle A, B\rangle\,, \tag{2.21}$$

where $A, B \in \Gamma(TM \oplus T^*M)$. On the right hand side of (2.21) the Courant bracket and natural pairing on $TM \oplus T^*M$ appear. It is important to stress that the Poisson bracket $\{\ ,\ \}$ is associative while the Courant bracket $[\ ,\ ]_c$ is not.

If we consider $\mathbb{L}$ to be a real (complex) Dirac structure (see definition 1.9) then for $A, B \in \Gamma(\mathbb{L})$

$$\big\{J_{\epsilon_1}(A), J_{\epsilon_2}(B)\big\} = -J_{\epsilon_1\epsilon_2}\big([A,B]_c|_{\mathbb{L}}\big)\,, \tag{2.22}$$

where $[\ ,\ ]_c|_{\mathbb{L}}$ is the restriction of the Courant bracket to $\mathbb{L}$. Due to the isotropy of $\mathbb{L}$ the last term on the right hand side of (2.21) vanishes and $[\ ,\ ]_c|_{\mathbb{L}}$ is a Lie bracket on $\Gamma(\mathbb{L})$. Thus there is a natural relation between the Dirac structures and the current algebras.

For any real (complex) Dirac structure $\mathbb{L}$ we can define the set of constraints in $T^*LM$

$$v^\mu p_\mu + \xi_\mu \partial X^\mu = 0\,, \tag{2.23}$$

where $(v+\xi) \in \Gamma(\mathbb{L})$. The conditions (2.23) are first class constraints due to (2.22), i.e. they define a coisotropic submanifold of $T^*LM$. Moreover the number of independent constraints is equal to $\dim \mathbb{L} = \dim M$ and thus the constraints (2.23)

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[3] By local canonical transformation we mean those canonical transformations where the new pair $(\tilde{X}, \tilde{p})$ is given as a local expression in terms of the old one $(X, p)$. For example, in the discussion of T-duality one uses non-local canonical transformations, i.e. $\tilde{X}$ is a non-local expression in terms of $X$.

correspond to a topological field theory (TFT). Since $\mathbb{L}$ is maximally isotropic it then follows from (2.23) that

$$\begin{pmatrix} \partial X \\ p \end{pmatrix} \in X^*(\mathbb{L}), \tag{2.24}$$

i.e. $\partial X + p$ take values in the subbundle $\mathbb{L}$ (more precisely, in the pullback of $\mathbb{L}$). The set (2.24) is equivalent to (2.23). Thus with any real (complex) Dirac structure we can associate a classical TFT.

Also we could calculate the bracket (2.21) between the currents using the symplectic structure (2.18) with $H$. In this case the Courant bracket should be replaced by the twisted Courant bracket. Moreover we have to consider the twisted Dirac structure instead of a Dirac structure. Otherwise all statement will remain true.

2.9. **String super phase space $T^*\mathcal{L}M$.** Next we would like to extend our construction and add odd partners to the fields $(X, p)$. This will allow us to introduce more structure.

Let $S^{1,1}$ be a "supercircle" with coordinates $(\sigma, \theta)$, where $\sigma$ is a coordinate along $S^1$ and $\theta$ is odd parter of $\sigma$ such that $\theta^2 = 0$. Then the corresponding superloop space is the space of maps, $\mathcal{L}M = \{\Phi : S^{1,1} \to M\}$. The phase space is given by the cotangent bundle $\Pi T^*\mathcal{L}M$ of $\mathcal{L}M$, however with reversed parity on the fibers. In what follows we use the letter "$\Pi$" to describe the reversed parity on the fibers. Equivalently we can describe the space $\Pi T^*\mathcal{L}M$ as the space of maps

$$\Pi T S^1 \to \Pi T^* M,$$

where the supermanifold $\Pi T S^1$ ($\equiv S^{1,1}$) is the tangent bundle of $S^1$ with reversed parity of the fiber and the supermanifold $\Pi T^*M$ is the cotangent bundle of $M$ with reversed parity on the fiber.

In local coordinates we have a scalar superfield $\Phi^\mu(\sigma, \theta)$ and a conjugate momentum, spinorial superfield $S_\mu(\sigma, \theta)$ with the following expansion

$$\Phi^\mu(\sigma,\theta) = X^\mu(\sigma) + \theta\lambda^\mu(\sigma), \qquad S_\mu(\sigma,\theta) = \rho_\mu(\sigma) + i\theta p_\mu(\sigma), \tag{2.25}$$

where $\lambda$ and $\rho$ are fermions. $S$ is a section of the pullback $X^*(\Pi T^*M)$ of the cotangent bundle of $M$, considered as an odd bundle. The corresponding symplectic structure on $\Pi T^*\mathcal{L}M$ is

$$\omega = i \int\limits_{S^{1,1}} d\sigma d\theta \, (\delta S_\mu \wedge \delta\Phi^\mu - H_{\mu\nu\rho} D\Phi^\mu \delta\Phi^\nu \wedge \delta\Phi^\rho), \tag{2.26}$$

such that after integration over $\theta$ the bosonic part of (2.26) coincides with (2.18).

The above symplectic structure makes $C^\infty(\Pi T^*\mathcal{L}M)$ (the space of smooth functionals on $\Pi T^*\mathcal{L}M$) into superPoisson algebra. The space $C^\infty(\Pi T^*\mathcal{L}M)$ has a natural $\mathbb{Z}_2$ grading with $|F| = 0$ for even and $|F| = 1$ for odd functionals. For a functional $F(S, \phi)$ we define the left and right functional derivatives as follows

$$\delta F = \int d\sigma d\theta \left( \frac{F \overleftarrow{\delta}}{\delta S_\mu} \delta S^\mu + \frac{F \overleftarrow{\delta}}{\delta \phi^\mu} \delta\phi^\mu \right) = \int d\sigma d\theta \left( \delta S_\mu \frac{\overrightarrow{\delta} F}{\delta S_\mu} + \delta\phi^\mu \frac{\overrightarrow{\delta} F}{\delta\phi^\mu} \right). \tag{2.27}$$

Using this definition the Poisson bracket corresponding to (2.26) with $H = 0$ is given by

$$\{F, G\} = i \int d\sigma d\theta \left( \frac{F\overleftarrow{\delta}}{\delta S_\mu} \frac{\overrightarrow{\delta} G}{\delta \phi^\mu} - \frac{F\overleftarrow{\delta}}{\delta \phi^\mu} \frac{\overrightarrow{\delta} G}{\delta S_\mu} \right) . \tag{2.28}$$

and with $H \neq 0$

$$\{F, G\}_H = i \int d\sigma d\theta \left( \frac{F\overleftarrow{\delta}}{\delta S_\mu} \frac{\overrightarrow{\delta} G}{\delta \phi^\mu} - \frac{F\overleftarrow{\delta}}{\delta \phi^\mu} \frac{\overrightarrow{\delta} G}{\delta S_\mu} + 2 \frac{F\overleftarrow{\delta}}{\delta S_\nu} H_{\mu\nu\rho} D\phi^\mu \frac{\overrightarrow{\delta} G}{\delta S_\rho} \right) . \tag{2.29}$$

These brackets $\{\ ,\ \}$ and $\{\ ,\ \}_H$ satisfy the appropriate graded versions of antisymmetry, of the Leibnitz rule and of the Jacobi identity

$$\{F, G\} = -(-1)^{|F||G|}\{G, F\} , \tag{2.30}$$

$$\{F, GH\} = \{F, G\}H + (-1)^{|F||G|}G\{F, H\} , \tag{2.31}$$

$$(-1)^{|H||F|}\{F, \{G, H\}\} + (-1)^{|F||G|}\{G, \{H, F\}\} + (-1)^{|G||H|}\{H, \{F, G\}\} = 0 . \tag{2.32}$$

Next on $\Pi TS^1$ we have two natural operations, $D$ and $Q$. The derivative $D$ is defined as

$$D = \frac{\partial}{\partial \theta} + i\theta\partial \tag{2.33}$$

and the operator $Q$ as

$$Q = \frac{\partial}{\partial \theta} - i\theta\partial . \tag{2.34}$$

$D$ and $Q$ satisfy the following algebra

$$D^2 = i\partial , \qquad Q^2 = -i\partial , \qquad DQ + QD = 0 . \tag{2.35}$$

Here $\partial$ stands for the derivative along the loop, i.e. along $\sigma$.

Again as in the purely bosonic case (see the proposition 2.22) the group of local canonical transformations of $\Pi T^*\mathcal{L}M$ is a semidirect product of $\mathrm{Diff}(M)$ and $\Omega^2(M)$. The $b$-transform now is given by

$$\Phi^\mu \ \rightarrow\ \Phi^\mu , \qquad S_\mu \ \rightarrow\ S_\mu - b_{\mu\nu} D\Phi^\nu , \tag{2.36}$$

with $b \in \Omega^2_{\text{closed}}(M)$. Moreover the discussion from subsection 2.8 can be generalized to the supercase.

Consider first $C^\infty(\Pi T^*\mathcal{L}M)$ with $\{\ ,\ \}$. By construction of $\Pi T^*\mathcal{L}M$ there exists the following generator

$$\mathbf{Q}_1(\epsilon) = - \int_{S^{1,1}} d\sigma\, d\theta\ \epsilon S_\mu Q\Phi^\mu , \tag{2.37}$$

where $Q$ is the operator introduced in (2.34) and $\epsilon$ is an odd parameter (odd test function). Using (2.26) we can calculate the Poisson brackets for these generators

$$\big\{\mathbf{Q}_1(\epsilon), \mathbf{Q}_1(\tilde{\epsilon})\big\} = \mathbf{P}(2\epsilon\tilde{\epsilon})\,, \tag{2.38}$$

where $P$ is the generator of translations along $\sigma$

$$\mathbf{P}(a) = \int\limits_{S^{1,1}} d\sigma\, d\theta\ aS_\mu \partial\Phi^\mu \tag{2.39}$$

with $a$ being an even parameter. In physics such a generator $\mathbf{Q}_1(\epsilon)$ is called a supersymmetry generator and it has the meaning of a square root of the translations, see (2.38). Furthermore we call it a manifest supersymmetry since it exits as part of the superspace formalism. One can construct a similar generator of manifest supersymmetry on $C^\infty(\Pi T^*\mathcal{L}M)$ with $\{\ ,\ \}_H$.

2.10. **Extended supersymmetry and generalized complex structure.** Consider $C^\infty(\Pi T^*\mathcal{L}M)$ with $\{\ ,\ \}$. We look for a second supersymmetry generator. The second supersymmetry should be generated by some $\mathbf{Q}_2(\epsilon)$ such that it satisfies the following brackets

$$\big\{\mathbf{Q}_1(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\big\} = 0\,, \qquad \big\{\mathbf{Q}_2(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\big\} = \mathbf{P}(2\epsilon\tilde{\epsilon})\,. \tag{2.40}$$

If on $\big(C^\infty(\Pi T^*\mathcal{L}M),\ \{\ ,\ \}\big)$ there exist two generators which satisfy (2.38) and (2.40) then we say that there exists an $N = 2$ supersymmetry.

By dimensional arguments, there is a unique ansatz for the generator $\mathbf{Q}_2(\epsilon)$ on $\Pi T^*\mathcal{L}M$ which does not involve any dimensionful parameters

$$\mathbf{Q}_2(\epsilon) = -\frac{1}{2}\int\limits_{S^{1,1}} d\sigma\, d\theta\ \epsilon\big(2D\Phi^\rho S_\nu J^\nu_{\ \rho} + D\Phi^\nu D\Phi^\rho L_{\nu\rho} + S_\nu S_\rho P^{\nu\rho}\big)\,. \tag{2.41}$$

We can combine $D\Phi$ and $S$ into a single object

$$\Lambda = \begin{pmatrix} D\Phi \\ S \end{pmatrix}\,, \tag{2.42}$$

which can be thought of as a section of the pullback of $X^*(\Pi(TM \oplus T^*M))$. The tensors in (2.41) can be combined into a single object

$$\mathcal{J} = \begin{pmatrix} -J & P \\ L & J^t \end{pmatrix}\,, \tag{2.43}$$

which is understood now as $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$. With this new notation we can rewrite (2.41) as follows

$$\mathbf{Q}_2(\epsilon) = -\frac{1}{2}\int\limits_{S^{1,1}} d\sigma d\theta\ \epsilon\langle\Lambda, \mathcal{J}\Lambda\rangle\,, \tag{2.44}$$

where $\langle\ ,\ \rangle$ is understood as the induced pairing on $X^*(\Pi(TM \oplus T^*M))$. The following proposition from [26] tells us when there exists $N = 2$ supersymmetry.

**Proposition 2.23.** $\Pi T^*\mathcal{L}M$ admits $N = 2$ supersymmetry if and only if $M$ is a generalized complex manifold.

*Proof.* We have to impose the algebra (2.40) on $\mathbf{Q}_2(\epsilon)$. The calculation of the second bracket is lengthy but straightforward and the corresponding coordinate expressions are given in [17]. Therefore we give only the final result of the calculation. Thus the algebra (2.40) satisfied if and only if

$$\mathcal{J}^2 = -1_{2d} \qquad \Pi_{\mp}[\Pi_{\pm}(X+\eta), \Pi_{\pm}(Y+\eta)]_c = 0\,, \tag{2.45}$$

where $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm i\mathcal{J})$. Thus (2.45) together with the fact that $\mathcal{J}$ (see (2.43)) respects the natural pairing ($\mathcal{J}^t\mathcal{I} = -\mathcal{I}\mathcal{J}$) implies that $\mathcal{J}$ is a generalized complex structure. $\Pi_{\pm}$ project to two maximally isotropic involutive subbundles $\mathbb{L}$ and $\bar{\mathbb{L}}$ such that $(T\oplus T^*)\otimes\mathbb{C} = \mathbb{L}\oplus\bar{\mathbb{L}}$. Thus we have shown that $\Pi T^*\mathcal{L}M$ admits $N=2$ supersymmetry if and only if $M$ is a generalized complex manifold. Our derivation is algebraic in nature and does not depend on the details of the model. □

The canonical transformations of $\Pi T^*\mathcal{L}M$ cannot change any brackets. Thus the canonical transformation corresponding to a b-transform (2.36)

$$\begin{pmatrix} D\Phi \\ S \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}\begin{pmatrix} D\Phi \\ S \end{pmatrix} \tag{2.46}$$

induces the following transformation of the generalized complex structure

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\mathcal{J}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{2.47}$$

and thus gives rise to a new extended supersymmetry generator. Therefore $\mathcal{J}_b$ is again the generalized complex structure. This is a physical explanation of the behavior of generalized complex structure under $b$-transform.

Using $\delta_i(\epsilon)\bullet = \{\mathbf{Q}_i(\epsilon), \bullet\}$ we can write down the explicit form for the second supersymmetry transformations as follows

$$\delta_2(\epsilon)\Phi^\mu = i\epsilon D\Phi^\nu J^\mu_{\ \nu} - i\epsilon S_\nu P^{\mu\nu} \tag{2.48}$$

$$\begin{aligned}\delta_2(\epsilon)S_\mu = {} & i\epsilon D(S_\nu J^\nu_{\ \mu}) - \frac{i}{2}\epsilon S_\nu S_\rho P^{\nu\rho}_{\ \ ,\mu} + i\epsilon D(D\Phi^\nu L_{\mu\nu}) \\ & + i\epsilon S_\nu D\Phi^\rho J^\nu_{\ \rho,\mu} - \frac{i}{2}\epsilon D\Phi^\nu D\Phi^\rho L_{\nu\rho,\mu}.\end{aligned} \tag{2.49}$$

Indeed it coincides with the supersymmetry transformation analyzed in [17].

Also we could look for $N=2$ supersymmetry for $C^\infty(\Pi T^*\mathcal{L}M)$ with $\{\ ,\ \}_H$. Indeed the result is exactly the same but now we have to have a twisted generalized complex manifold.

Another comment: We may change the $N=2$ supersymmetry algebra (2.38) and (2.40) slightly. Namely we can replace the last bracket in (2.40) by

$$\big\{\mathbf{Q}_2(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\big\} = -\mathbf{P}(2\epsilon\tilde{\epsilon})\,. \tag{2.50}$$

This new algebra is sometimes called $N=2$ pseudo-supersymmetry. In this case we still use the ansatz (2.41) for $\mathbf{Q}_2$. However now we get

**Proposition 2.24.** $\Pi T^*\mathcal{L}M$ admits $N=2$ pseudo-supersymmetry if and only if $M$ is a generalized product manifold.

The proof of this statement is exactly the same as before. The only difference is that the condition $\mathcal{J}^2 = -1_{2d}$ get replaced by $\mathcal{J}^2 = 1_{2d}$.

2.11. **BRST interpretation.** Alternatively we can relate the generalized complex structure to an odd differential $\mathbf{s}$ on $C^\infty(\Pi T^*\mathcal{L}M)$ and thus we enter the realm of Hamiltonian BRST formalism. This formalism was developed to quantize theories with the first-class constraints.

Indeed the supersymmetry generators (2.37) and (2.41) can be thought of as odd transformations (by putting formally $\epsilon = 1$) which square to the translation generator. Thus we can define the odd generator

$$\mathbf{q} = \mathbf{Q}_1(1) + i\mathbf{Q}_2(1) = - \int\limits_{S^{1,1}} d\sigma d\theta\ (S_\mu Q\Phi^\mu + iD\Phi^\rho S_\nu J^\nu_{\ \rho} + \frac{i}{2} D\Phi^\nu D\Phi^\rho L_{\nu\rho} + \frac{i}{2} S_\nu S_\rho P^{\nu\rho})\,, \tag{2.51}$$

which is called the BRST generator. The odd generator $\mathbf{q}$ generates to the following transformation $\mathbf{s}$

$$\mathbf{s}\Phi^\mu = \{\mathbf{q}, \Phi^\mu\} = Q\Phi^\mu + iD\Phi^\nu J^\mu_{\ \nu} - iS_\nu P^{\mu\nu}\,, \tag{2.52}$$

$$\begin{aligned}\mathbf{s}S_\mu = \{\mathbf{q}, S_\mu\} = QS_\mu + iD(S_\nu J^\nu_{\ \mu}) - \frac{i}{2} S_\nu S_\rho P^{\nu\rho}_{\ \ \ ,\mu} \\ + iD(D\Phi^\nu L_{\mu\nu}) + iS_\nu D\Phi^\rho J^\nu_{\ \rho,\mu} - \frac{i}{2} D\Phi^\nu D\Phi^\rho L_{\nu\rho,\mu}\,,\end{aligned} \tag{2.53}$$

which is nilpotent due the properties of manifest and nonmanifest supersymmetry trasnformations. Thus $\mathbf{s}^2 = 0$ if and only if $\mathcal{J}$ defined in (2.43) is a generalized complex structure. In doing the calculations one should remember that now $\mathbf{s}$ is odd operation and whenever it passes through an odd object (e.g., $D$, $Q$ and $S$) there is extra minus. The existence of odd nilpotent operation (2.52)-(2.53) is typical for models with an $N = 2$ supersymmetry algebra and corresponds to a topological twist of the $N = 2$ algebra.

We can also repeat the argument for the $N = 2$ pseudo-supersymmetry algebra and now define the odd BRST generator as follows

$$\mathbf{q} = \mathbf{Q}_1(1) + \mathbf{Q}_2(1)\,. \tag{2.54}$$

This $\mathbf{q}$ generates an odd nilpotent symmetry if there exists a generalized product structure.

We can equally well work with the twisted bracket $\{\ ,\ \}_H$ and all results will be still valid provided that we insert the word "twisted" in appropriate places. We can summarize our discussion in the following proposition.

**Proposition 2.25.** The superPoisson algebra $C^\infty(\Pi T^*\mathcal{L}M)$ with $\{\ ,\ \}$ ($\{\ ,\ \}_H$) admits odd derivation $\mathbf{s}$ if and only if there exists on $M$ either (twisted) generalized complex or (twisted) generalized product structures.
In other words the existence of an odd derivation $\mathbf{s}$ on $C^\infty(\Pi T^*\mathcal{L}M)$ is related to real (complex) Lie bialgebroid structure on $TM \oplus T^*M$.

The space $\Pi T^*\mathcal{L}M$ with odd nilpotent generator $\mathbf{q}$ can be interpreted as an extended phase space for a set of the first-class constraints in $T^*LM$. The appropriate linear combinations of $\rho$ and $\lambda$ are interpreted as ghosts and antighosts. The differential $\mathbf{s}$ on $C^\infty(\Pi T^*\mathcal{L}M)$ induces the cohomology $H^\bullet_{\mathbf{s}}$ which is also a superPoisson algebra.

It is instructive to expand the transformations (2.52)-(2.53) in components. In particular if we look at the bosonic fixed points of the BRST action we arrive at the following constraint

$$(1_{2d} + i\mathcal{J}) \begin{pmatrix} \partial X \\ p \end{pmatrix} = 0\,,$$

which is exactly the same as the condition (2.24). Thus we got the BRST complex for the first-class constraints given by (2.23). These constraints correspond to TFTs as we have discussed, although the BRST complex above requires more structure than just simply a (twisted) Dirac structure.

2.12. **Generalized complex submanifolds.** So far we have discussed the hamiltonian formalism for two dimensional field theory without boundaries. All previous discussion can be generalized to the case hamiltonian system with boundaries.

We start from the notion of a generalized submanifold. Consider a manifold $M$ with a closed three form $H$ which specifies the Courant bracket.

**Definition 2.26.** The data $(D, B)$ is called a generalized submanifold if $D$ is a submanifold of $M$ and $B \in \Omega^2(D)$ is a two-from on $D$ such that $H|_D = dB$. For any generalized submanifold we define a generalized tangent bundle

$$\tau_D^B = \{v + \xi \in TD \oplus T^*M|_D,\ \ \xi|_D = i_v B\}\,.$$

**Example 2.27.** Consider a manifold $M$ with $H = 0$, then any submanifold $D$ of $M$ is a generalized submanifold with $B = 0$. The corresponding generalized tangent bundle is

$$\tau_D^0 = \{v + \xi \in TD \oplus N^*D\}$$

with $N^*D$ being a conormal bundle of $D$. Also we can consider $(D, B)$, a submanifold with a closed two-form on it, $B \in \Omega^2(D)$, $dB = 0$. Such a pair $(D, B)$ is a generalized submanifold with generalized tangent bundle

$$\tau_D^B = e^B \tau_D^0\,,$$

where the action of $e^B$ is defined in (1.5).

The pure bosonic model is defined as follows. Instead of the loop space $LM$ we now consider the path space

$$PM = \big\{X : [0,1] \to M,\ \ X(0) \in D_0,\ \ X(1) \in D_1\big\}$$

where the end points are confined to prescribed submanifolds of $M$. The phase space will be the cotangent bundle $T^*PM$ of path space. However to write down a symplectic structure on $T^*PM$ we have to require that $D_0$ and $D_1$ give rise to generalized submanifolds, $(D_0, B^0)$ and $(D_1, B^1)$, respectively. Thus the symplectic structure on $T^*PM$ is

$$\omega = \int_0^1 d\sigma \;\; (\delta X^\mu \wedge \delta p_\mu + H_{\mu\nu\rho} \partial X^\mu \delta X^\nu \wedge \delta X^\rho)$$
$$+ B^0_{\mu\nu}(X(0))\delta X^\mu(0) \wedge \delta X^\nu(0) - B^1_{\mu\nu}(X(1))\delta X^\mu(1) \wedge \delta X^\nu(1)\,,$$

where $\delta$ is de Rham differential on $T^*PM$. It is crucial that $(D_0, B^0)$ and $(D_1, B^1)$ are generalized submanifolds for $\omega$ to be closed.

Next we have to introduce the super-version of $T^*PM$. This can be done in different ways. For example we can define the cotangent bundle $\Pi T^*\mathcal{P}M$ of superpath space as the set of maps

$$\Pi TP \to \Pi T^*M$$

with the appropriate boundary conditions which can be written as

$$\Lambda(1) \in X^*(\Pi \tau^{B^1}_{D_1})\,, \qquad \Lambda(0) \in X^*(\Pi \tau^{B^0}_{D_0})$$

with $\Lambda$ defined in (2.42). These boundary conditions are motivated by the cancellation of unwanted boundary terms in the calculations [26].

Next we define a natural class of submanifold of a (twisted) generalized complex submanifold $M$.

**Definition 2.28.** A generalized submanifold $(D, B)$ is called a generalized complex submanifold if $\tau^B_D$ is stable under $\mathcal{J}$, i.e. if

$$\mathcal{J}\tau^B_D \subset \tau^B_D\,.$$

Finally we would like to realize the $N = 2$ supersymmetry algebra which has been discussed in previous subsections. The most of the analysis is completely identical to the previous discussion. The novelty is the additional boundary terms in the calculations. We present the final result and skip all technicalities.

**Proposition 2.29.** $\Pi T^*\mathcal{P}M$ admits $N = 2$ supersymmetry if and only if $M$ is a (twisted) generalized complex manifold and $(D_i, B^i)$ are generalized complex submanifolds of $M$.

It is quite easy to generalize this result to the real case when we talk about $N = 2$ pseudo-supersymmetry. The correct notion would be a generalized product submanifold, i.e. such generalized submanifold $(D, B)$ when $\tau^B_D$ is stable under $\mathcal{R}$ (see the definition 1.19 and the discussion afterwards). This is quite straightforward and we will not discuss it here.

## Lecture 3

In this Lecture we review more advanced topics such as (twisted) generalized Kähler geometry and (twisted) generalized Calabi-Yau manifolds. In our presentation we will be rather sketchy and give some of the statement without much elaboration. We concentrate only on the complex case, although obviously there exists a real version [2].

On physics side we would like to explain briefly that the generalized Kähler geometry naturally arises when we specify the model, i.e. we choose a concrete Hamiltonian in $C^\infty(\Pi T^*\mathcal{L}M)$, while the generalized Calabi-Yau conditions arise when one tries to quantize this model.

3.13. **Generalized Kähler manifolds.** $TM \oplus T^*M$ has a natural pairing $\langle\ ,\ \rangle$. However one can introduce the analog of the usual positive definite metric.

**Definition 3.30.** A generalized metric is a subbundle $C_+ \subset TM \oplus T^*M$ of rank d ($\dim M = d$) on which the induced metric is positive definite

In other words we have splitting

$$TM \oplus T^*M = C_+ \oplus C_-\,,$$

such that there exists a positive metric on $TM \oplus T^*M$ given by

$$\langle\ ,\ \rangle|_{C_+} - \langle\ ,\ \rangle|_{C_-}\,.$$

Alternatively the splitting into $C_\pm$ can be described by an endomorphims

$$\mathcal{G} : TM \oplus T^*M \to TM \oplus T^*M\,, \qquad \mathcal{G}^2 = 1\,, \qquad \mathcal{G}^t\mathcal{I} = \mathcal{I}\mathcal{G}\,,$$

such that $\frac{1}{2}(1_{2d} \pm \mathcal{G})$ projects out $C_\pm$. In order to write $\mathcal{G}$ explicitly we need the following proposition [10]:

**Proposition 3.31.** $C_\pm$ is the graph of $(b \pm g) : T \to T^*$ where $g$ is Riemannian metric and $b$ is two form.

As a result $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}\,. \tag{3.55}$$

Thus the standard metric $g$ together with the two-form $b$ give rise to a generalized metric as in the definition 3.30.

Now we can define the following interesting construction.

**Definition 3.32.** A (twisted) generalized Kähler structure is a pair $\mathcal{J}_1$, $\mathcal{J}_2$ of commuting (twisted) generalized complex structures such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric (generalized metric) on $TM \oplus T^*M$.

Indeed this is the generalization of the Kähler geometry as can been seen from the following example.

**Example 3.33.** A Kähler manifold is a complex hermitian manifold $(J, g)$ with a closed Kähler form $\omega = gJ$. A Kähler manifold is an example of a generalized Kähler manifold where $\mathcal{J}_1$ is given by example 1.15 and $\mathcal{J}_2$ by example 1.16. Since the corresponding symplectic structure $\omega$ is a Kähler form, two generalized complex structures commute and their product is

$$\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\,.$$

This example justifies the name, a generalized Kähler geometry.

For (twisted) generalized Kähler manifold there are the following decompositions of complexified tangent and cotangent bundle

$$(TM \oplus T^*M) \otimes \mathbb{C} = \mathbb{L}_1 \oplus \bar{\mathbb{L}}_1 = \mathbb{L}_2 \oplus \bar{\mathbb{L}}_2 ,$$

where the first decomposition corresponds to $\mathcal{J}_1$ and second to $\mathcal{J}_2$. Since $[\mathcal{J}_1, \mathcal{J}_2] = 0$ we can do both decompositions simultaneously

$$(TM \oplus T^*M) \otimes \mathbb{C} = \mathbb{L}_1^+ \oplus \mathbb{L}_1^- \oplus \bar{\mathbb{L}}_1^+ \oplus \bar{\mathbb{L}}_1^- ,$$

where the space $\mathbb{L}_1$ ($+i$-egeinbundle of $\mathcal{J}_1$) can be decomposed into $\mathbb{L}_1^\pm$, $\pm i$-egeinbundle of $\mathcal{J}_2$. In its turn the generalized metric subbundles are defined as

$$C_\pm \otimes \mathbb{C} = \mathbb{L}_1^\pm \otimes \bar{\mathbb{L}}_1^\pm .$$

One may wonder if there exists an alternative geometrical description for a (twisted) generalized Kähler manifolds. Indeed there is one.

**Definition 3.34.** The Gates-Hull-Roček geometry is the following geometrical data: two complex structures $J_\pm$, metric $g$ and closed three form $H$ which satisfy

$$J_\pm^t g J_\pm = g$$

$$\nabla^{(\pm)} J_\pm = 0$$

with the connections defined as $\Gamma^{(\pm)} = \Gamma \pm g^{-1}H$, where $\Gamma$ is a Levi-Civita connection for $g$.

This geometry was originally derived by looking at the general $N = (2,2)$ supersymmetric sigma model [8]. In [10] the equivalence of these two seemingly unrelated descriptions has been proven.

**Proposition 3.35.** The Gates-Hull-Roček geometry is equivalent to a twisted generalized Kähler geometry.

As we have discussed briefly a generalized complex manifold locally looks like a product of symplectic and complex manifolds. The local structure of (twisted) generalized Kähler manifolds is somewhat involved. Namely the local structure is given by the set of symplectic foliations arising from two real Poisson structures [20] and holomorphic Poisson structure [12]. Moreover one can show that in analogy with Kähler geometry there exists a generalized Kähler potential which encodes all local geometry in terms of a single function [18].

3.14. $N = (2,2)$ **sigma model.** In the previous Lecture we have discussed the relation between (twisted) generalized complex geometry and $N = 2$ supersymmetry algebra on $\Pi T^*\mathcal{L}M$. Our discussion has been model independent. A choice of concrete model corresponds to a choice of Hamiltonian function $\mathcal{H}(a) \in C^\infty(\Pi T^*\mathcal{L}M)$ which generates a time evolution of a system. Then the natural question to ask if the model is invariant under the $N = 2$ supersymmetry, namely

$$\big\{\mathbf{Q}_2(\epsilon), \mathcal{H}(a)\big\} = 0 \,, \tag{3.56}$$

where $\mathbf{Q}_2(\epsilon)$ is defined in (2.41) with the corresponding (twisted) generalized complex structure $\mathcal{J}$.

To be concrete we can choose the Hamiltonian which corresponds to $N = (2,2)$ sigma model used by Gates, Hull and Roček in [8]

$$\mathcal{H}(a) = \frac{1}{2}\int d\sigma\, d\theta\ a\left(i\partial\phi^\mu D\phi^\nu g_{\mu\nu} + S_\mu D S_\nu g^{\mu\nu} + S_\sigma D\phi^\nu S_\gamma g^{\lambda\gamma}\Gamma^\sigma{}_{\nu\lambda}\right.$$

$$\left. - \frac{1}{3}H^{\mu\nu\rho}S_\mu S_\nu S_\rho + D\phi^\mu D\phi^\nu S_\rho H_{\mu\nu}{}^\rho\ \right), \tag{3.57}$$

where $a$ is just an even test function. This Hamiltonian has been derived in [3]. This Hamiltonian is invariant under the $N = 2$ supersupersymmetry if

$$\mathcal{J}_1 = \mathcal{J}\,, \qquad \mathcal{J}_2 = \mathcal{J}G$$

is a (twisted) generalized Kähler structure, see the definition 3.32. For the Hamiltonian (3.57) $\mathcal{G}$ is defined by (3.55) by $g$ and $b = 0$, $H$ corresponds the closed three-form which is used in the definition of the twisted Courant bracket. Indeed on a (twisted) generalized Kähler manifold $\mathcal{H}$ is invariant under supersymmetries corresponding to both (twisted) generalized complex structures, $\mathcal{J}_1$ and $\mathcal{J}_2$.

Also the Hamiltonian (3.57) can be interpreted in the context of TFTs. Namely $\mathcal{H}$ is the gauge fixed Hamiltonian for the TFT we have discussed in subsection 2.11 with $\mathbf{s}$ being the BRST-transformations defined in (2.52)-(2.53). The Hamiltonian (3.57) is BRST-exact

$$\mathcal{H} = \mathbf{s}\left(\frac{i}{4}\int d\sigma d\theta\ \langle\Lambda, \mathcal{J}\mathcal{G}\Lambda\rangle\right) = \mathbf{s}\left(\frac{i}{4}\int d\sigma d\theta\ \langle\Lambda, \mathcal{J}_2\Lambda\rangle\right).$$

Moreover the translation operator $\mathbf{P}$ is given by

$$\mathbf{P} = \mathbf{s}\left(\frac{i}{4}\int d\sigma d\theta\ \langle\Lambda, \mathcal{J}\Lambda\rangle\right) = \mathbf{s}\left(\frac{i}{4}\int d\sigma d\theta\ \langle\Lambda, \mathcal{J}_1\Lambda\rangle\right).$$

The $N = (2,2)$ theory (3.57) is invariant under two extended supersymmetries associated to generalized complex structures, $\mathcal{J}_1$ and $\mathcal{J}_2$. Thus there are two possible BRST symmetries and correspondingly two TFTs associated either to $\mathcal{J}_1$ or to $\mathcal{J}_2$. In the literature these two TFTs are called either A or B topological twists of the $N = (2,2)$ supersymmetric theory.

Indeed one can choose a different Hamiltonian function on $\Pi T^*\mathcal{L}M$ and arrive to different geometries which involve the generalized complex structure, e.g. see [4].

3.15. **Generalized Calabi-Yau manifolds.** In this subsection we define the notion of generalized Calabi-Yau manifold. To do this we have to introduce a few new concepts.

We can define the action of a section $(v+\xi) \in \Gamma(TM \oplus T^*M)$ on a differential form $\phi \in \Omega(M) = \wedge^\bullet T^*M$

$$(v+\xi)\cdot\phi \equiv i_v\rho + \xi\wedge\phi\,.$$

Using this action we arrive at the following identity

$$\{A, B\}_+ \cdot \phi \equiv A\cdot(B\cdot\phi) + B\cdot(A\cdot\phi) = 2\langle A, B\rangle\phi\,,$$

which gives us the representation of Clifford algebra, $Cl(TM \oplus T^*M)$, on the differential forms. Thus we can view differential forms as spinors for $TM \oplus T^*M$ and moreover there are no topological obstructions for their existence. In further discussion we refer to a differential form as a spinor.

The decomposition for spinors corresponds to decomposing forms into even and odd degrees,

$$\Omega(M) = \wedge^{\bullet} T^*M = \wedge^{\text{even}} T^*M \oplus \wedge^{\text{odd}} T^*M \,.$$

We would like to stress that in all present discussion we do not consider a form a definite degree, we may consider a sum of the forms of different degrees. Also on $\Omega(M)$ there exists $\mathrm{Spin}\,(d,d)$-invariant bilinear form $(\ ,\ )$,

$$(\phi, r) = [\phi \wedge \sigma(r)]|_{\text{top}} \,, \tag{3.58}$$

where $\phi, r \in \Omega(M)$ and $\sigma$ is anti-automorphism which reverses the wedge product. In the formula (3.58) $[\dots]|_{\text{top}}$ stands for the projection to the top form.

**Definition 3.36.** For any form $\phi \in \Omega(M)$ we define a null space

$$\mathbb{L}_\phi = \big\{ A \in \Gamma(TM \oplus T^*M), A \cdot \phi = 0 \big\}$$

Indeed the null space $\mathbb{L}_\phi$ is isotropic since

$$2\langle A, B\rangle \phi = A \cdot (B \cdot \phi) + B \cdot (A \cdot \phi) = 0 \,.$$

**Definition 3.37.** A spinor $\phi \in \Omega(M)$ is called pure when $\mathbb{L}_\phi$ is a maximally isotropic subbundle of $TM \oplus T^*M$ (or its complexification).

**Proposition 3.38.** $\mathbb{L}_\phi$ and $\mathbb{L}_r$ satisfy $\mathbb{L}_\phi \cap \mathbb{L}_r = 0$ if and only if

$$(\phi, r) \neq 0 \,,$$

where $(\ ,\ )$ is bilinear form defined in (3.58).

Obviously all this can be complexified.

If we take a pure spinor $\phi$ on $(TM \oplus T^*M) \otimes \mathbb{C}$ such that $(\phi, \bar{\phi}) \neq 0$ then the complexified tangent plus cotangent bundle can be decomposed into the corresponding null spaces

$$(TM \oplus T^*M) \otimes \mathbb{C} = \mathbb{L}_\phi \oplus \mathbb{L}_{\bar{\phi}} = \mathbb{L}_\phi \oplus \bar{\mathbb{L}}_\phi \,.$$

Therefore we have an almost generalized complex structure.

The following definition is due to Hitchin [11]. However we follow the terminology proposed in [15].

**Definition 3.39.** A weak generalized Calabi-Yau manifold is a manifold with a pure spinor $\phi$ such that $(\phi, \bar{\phi}) \neq 0$ and $d\phi = 0$.

A weak generalized Calabi-Yau manifold is generalized complex manifold since $\mathbb{L}_\phi$ and $\mathbb{L}_{\bar{\phi}}$ are complex Dirac structures. The condition $d\phi = 0$ implies the involutivity of $\mathbb{L}_\phi$. There is also a twisted weak generalized Calabi-Yau manifold where in the definition 3.39 the condition $d\phi = 0$ is replaced by the condition $d\phi + H \wedge \phi = 0$. The twisted weak generalized Calabi-Yau manifold is a twisted generalized complex manifold.

**Example 3.40.** In Example 1.16 we have considered the symplectic manifold and have argued that there exists the generalized complex structure. Indeed a symplectic manifold is a weak generalized Calabi-Yau manifold with a pure spinor given by

$$\phi = e^{i\omega} = 1 + i\omega + \frac{i^2}{2}\omega \wedge \omega + \cdots + \frac{i^n}{n!}\omega \wedge \cdots \wedge \omega$$

with the last term on the right hand side corresponding to a top form.

**Example 3.41.** A complex manifold is a generalized complex manifold, see example 1.15. However it is not a weak generalized Calabi-Yau manifold automatically. We have to require the existence of a closed holomorphic volume form (the same as a closed holomorphic top form nowhere vanishing)

$$\phi = \Omega^{(n,0)}$$

which corresponds to a pure spinor.

We would like to stress that any weak generalized Calabi-Yau manifold is a generalized complex manifold, but not vice versa.

**Definition 3.42.** A generalized Calabi-Yau manifold is a manifold with two closed pure spinors, $\phi_1$ and $\phi_2$ such that

$$(\phi_1, \bar{\phi}_1) = c(\phi_2, \bar{\phi}_2) \neq 0$$

and they give rise to a generalized Kähler structure.

Also we can define a twisted Calabi-Yau manifold where in the above definitions the spinors satisfy $(d+H\wedge)\phi_i = 0$ and they give rise to a twisted generalized Kähler geometry.

**Definition 3.43.** A standard Calabi-Yau manifold is a Kähler manifold (see the example 3.33) with a closed holomorphic volume form $\Omega^{(n,0)}$. This gives us an example of generalized Calabi-Yau manifold with $\phi_1 = e^{i\omega}$ and $\phi_2 = \Omega^{(n,0)}$.

3.16. **Quantum $N = (2,2)$ sigma model.** In this subsection we would like to discuss very briefly the quantization of $N = (2,2)$ sigma model given by (3.57) and its corresponding TFTs cousins. In all generality this problem is a hard one and remains unresolved. Although it is always simpler to quantize TFTs. However by now we understand that for a $N = (2,2)$ sigma model to make sense at the quantum level we have to require the generalized Calabi-Yau conditions. We are going briefly sketch the argument which was presented essentially in [14].

We start our discussion from the TFT associated to a generalized complex structure. It is not simple to quantize a theory in all generality. However it is convenient to look first at the semiclassical approximation. It means that we can ignore $\sigma$ dependence and all loops collapse to a point on $M$. Thus we replace[4] $\Pi T^*\mathcal{L}M$ by $T^*(\Pi TM) \approx T(\Pi T^*M)$ and try to quantize this simpler theory. In particular we have to interpret the generator $\mathbf{q}$ (2.51) restricted to $T^*(\Pi TM)$. For

[4] $\Pi T^*\mathcal{L}M$ collapses to $T^*(\Pi TM)$ since $\sigma$ dependence disappear but $\theta$-dependence is still there. See [23] for the detailed discussion of $T^*(\Pi TM)$ and related matters.

this we have to expand the generator $\mathbf{q}$ (2.51) in components and drop all terms which contain the derivatives with respect to $\sigma$. Moreover it is useful to rotate odd basis $(\lambda^\mu, \rho_\mu) \in \Gamma\big(\Pi(TM \oplus T^*M)\big)$ to a new one $(\xi^A, \bar{\xi}_A) \in \Gamma\big(\Pi(\mathbb{L} \oplus \bar{\mathbb{L}})\big)$ which is adopted to $\pm i$-egeinbundles of $\mathcal{J}$. $\xi_A$ correspond to ghosts and $\bar{\xi}^A$ to antighosts. After these manipulations $\mathbf{q}$ can be written as follows

$$\mathbf{q} \sim p_\mu \rho^{\mu A}(X)\xi_A + f^{AB}_{\ \ C}(X)\xi_A\xi_B\bar{\xi}^C, \tag{3.59}$$

where we have ignored the irrelevant overall numerical factor. Now in new odd basis our phase space is $T^*(\Pi\mathbb{L}) \approx T^*(\Pi\bar{\mathbb{L}})$. We remember that $\mathbb{L}$ is a Lie algebroid and thus $\rho^{\mu A}(X)$ and $f^{AB}_{\ \ C}(X)$ are the anchor map and structure constants defined in subsection 1.1. This reduced $\mathbf{q}$ acts naturally on $\wedge^\bullet \bar{\mathbb{L}} = C^\infty(\Pi\mathbb{L})$ and gives rise to so-called Lie algebroid cohomology $H(d_L)$. In TFT we would associate the set of local observables to the elements of $H(d_L)$.

Also in any quantum field theory we have to build a Hilbert space of states. If we regard $(\lambda^\mu, \rho_\mu)$ as a set of creation and annihilation operators then the corresponding Fock space will be given by $\Omega(M)$. Alternatively we could choose $(\xi^A, \bar{\xi}_A)$ as a set of creation and annihilation operators. This choice would induce the natural grading

$$\Omega(M) = U_0 \oplus (\bar{\mathbb{L}} \cdot U_0) \oplus (\wedge^2\bar{\mathbb{L}} \cdot U_0) \oplus \cdots \oplus (\wedge^d\bar{\mathbb{L}} \cdot U_0)\,,$$

where $U_0$ is a vacuum state over which we build the Fock space. Mathematically we could choose $U_0$ to be a pure spinor line (i.e., we use the existence of pure spinor only locally). The operator $\mathbf{q}$ now acts on $\Omega(M)$ and it induces another cohomohology $H(\bar{\partial})$, which corresponds to a Hilbert space.

Next we cite the following theorem without a proof.

**Proposition 3.44.** For a (twisted) weak generalized Calabi-Yau manifold we have an isomorphism of two cohomologies

$$H(d_L) \sim H(\bar{\partial})$$

For the TFT the isomorphism of these two cohomologies is interpreted as operator-state correspondence, for each local observable we can associate a state in a Hilbert space and vice versa. Thus if we want to have the operator-state correspondence the corresponding TFT should be defined over a (twisted) weak Calabi-Yau manifold. Indeed there are more interesting structures in this TFT about which we do not have time to talk, see [15], [22].

Let us finish with a few comments about the $N = (2, 2)$ sigma model. The above analysis of states in TFT corresponds to analysis of the ground states in the $N = (2, 2)$ sigma model. At the level of ground states there should be also the operator-state correspondence and thus we have to require a (twisted) weak Calabi-Yau structure for both $\mathcal{J}_1$ and $\mathcal{J}_2$. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ correspond to a (twisted) generalized Kähler structure we arrive to the definition of (twisted) generalized Calabi-Yau manifold. Thus we can conclude with the following proposition.

**Proposition 3.45.** The quantum $N = (2, 2)$ sigma model requires $M$ to be a (twisted) generalized Calabi-Yau manifold, see the definition 3.42.

3.17. **Summary.** In these lecture notes we made an attempt to introduce the concepts of the generalized geometry and its relevance for the string theory. We concentrated our attention on the Hamiltonian approach to the world-sheet theory. Due to lack of time we did not discuss other important issues within the world-sheet theory, see the contribution [16] to the same volume for a review and the references.

Another topic which we did not touch at all concerns the space-time aspects of the generalized geometry, see [9] for the review and references. Eventually the world-sheet point of view is ultimately related to the space-time aspects of the problem.

Finally we have to stress that presently the subject is actively developing and there are still many unresolved problems.

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