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A GRAVITATIONAL EFFECTIVE ACTION ON A FINITE TRIANGULATION AS A DISCRETE MODEL OF CONTINUOUS CONCEPTS

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Abstract. We recall how the Gauss-Bonnet theorem can be interpreted as a finite dimensional index theorem. We describe the construction given in `hep-th/0512293` of a function that can be interpreted as a gravitational effective action on a triangulation. The variation of this function under local rescalings of the edge lengths sharing a vertex is the Euler density, and we use it to illustrate how continuous concepts can have natural discrete analogs.

1. Introduction

We want to study how we can translate concepts from continuum quantum field theories to discrete models with a finite number of degrees of freedom. The particular system that we explore is the theory of triangulations of a surface. This is a context that is well understood in both the discrete [1] and the continuous cases [2].

A nice example of such a translation is provided by the interpretation of the Euler character as an index. In the continuous case, we have the exterior derivative:

$$d:\ \omega_0 \to \omega_1 \to \omega_2\,, \tag{1}$$

where ω_p are p-forms, and the dual operator

$$*d*:\ \omega_2 \to \omega_1 \to \omega_0\,. \tag{2}$$

We now consider the operator $D \equiv d + *d*$ restricted to

$$D:\ \omega_0 \oplus \omega_2 \to \omega_1 \tag{3}$$

as well as its adjoint

$$D^\dagger:\ \omega_1 \to \omega_0 \oplus \omega_2\,. \tag{4}$$

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The index of D is defined to be difference of the dimensions of the kernels of D and $D^\dagger$, and is known to be proportional to the Euler character χ of the surface [3, 2]:

$$\dim\big(\mathrm{Ker}(D)\big) - \dim\big(\mathrm{Ker}(D^\dagger)\big) \propto \chi = 2(1-g)\,. \tag{5}$$

The discrete version of this well known story is somewhat less familiar. We consider a triangulation with V vertices v_i, E oriented edges e_{ij}, and F oriented faces f_{ijk}. The discrete analogs of p-forms ω_p are elements of a vector space E_p, where E_0 is associated to the vertices, E_1 is associated to the edges, and E_2 is associated to the faces. The sign associated to a edge or face depends on the orientation. There is an obvious notion of the exterior derivative d which obeys $d^2 = 0$; the dual operator in general will depend on a choice of metric as described in [4], but since the index is topological, we can ignore this dependence. Thus we choose our operators D and $D^\dagger$ as follows:

$$(D\omega)_{ij} = (\omega_i - \omega_j) \oplus (\omega_{ijk} - \omega_{ijk'})\,, \tag{6}$$

on an oriented edge e_{ij} between a vertex v_i and a vertex v_j or on an edge shared by oriented triangles f_{ijk} and $f_{ijk'}$. The adjoint operator is defined by, *e.g.*, (see [4] for further discussion)

$$(D^\dagger\omega)_i = \sum_{j\in<ij>} \omega_{ij}\,, \quad (D^\dagger\omega)_{ijk} = \omega_{ij} + \omega_{jk} + \omega_{ki}\,, \tag{7}$$

where the sum is over all the edges e_{ij} connected to a vertex v_i with a positive sign for edges leaving v_i and negative for edges coming into v_i, or over all the edges bounding the triangle f_{ijk} with a positive sign if their orientation is compatible with the orientation of the triangle; D and $D^\dagger$ are shown graphically in Figure 1. We may now compute the index of this discrete operator D; since it maps $E_0 \oplus E_2 \to E_1$, it is an $E \times (V + F)$ dimensional matrix, and the index is simply

$$V + F - E = \chi = 2(1-g)\,. \tag{8}$$

This is clearly topological, as it does not depend on the values of the entries of D, only on its dimension.

Not only does the Euler character χ make sense on a triangulation of a surface, but its density $\sqrt{g}R$, where $R\big(g(x)\big)$ is the scalar curvature[1] of the two dimensional metric g_{mn}, has a sensible analog as well: since $\chi = \frac{1}{2\pi}\sum_{i\in\{V\}}\epsilon_i$ where i runs over all vertices and ϵ_i is the defect at the i'th vertex, one can identify the defect $\epsilon_i = -\frac{1}{2}\sqrt{g}R_i$ with the curvature at the vertex [1]. Thus one has the correspondence:

$$-\frac{1}{4\pi}\int d^2x\ \sqrt{g}R = \chi \ \leftrightarrow\ \frac{1}{2\pi}\sum_{i\in\{V\}} \epsilon_i = V - E + F\,. \tag{9}$$

Anomalies are generally regarded as arising from the infinite numbers of degrees of freedom in continuous systems. For example, the action S of a scalar field Φ

[1] We use the convention that the scalar curvature is minus twice the Gaussian curvature, and hence is negative on the sphere [2].

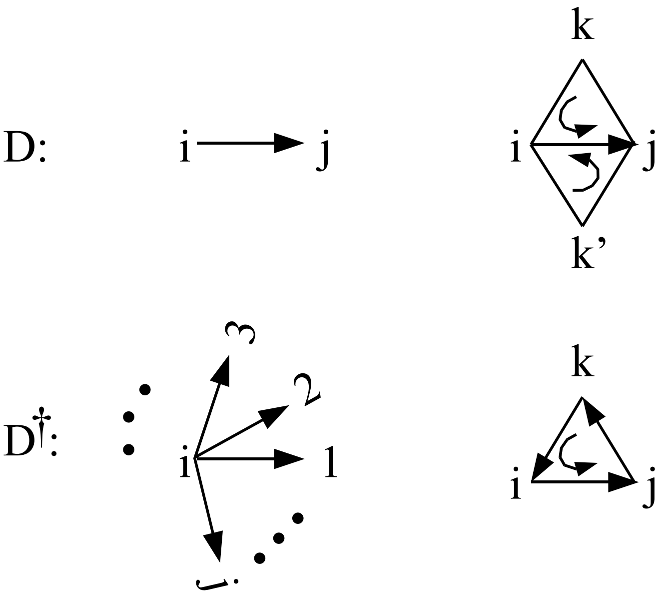

FIGURE 1. D and $D^\dagger$

coupled to a gravitational background on a two dimensional surface Σ,

$$S = \frac{1}{2}\int_\Sigma d^2x\sqrt{\det(g_{pq})}g^{mn}\partial_m\Phi\partial_n\Phi\,, \tag{10}$$

has a classical symmetry under rescalings of the metric on Σ: $g_{mn} \to \lambda(x)g_{mn}$. This implies

$$g_{mn}\frac{\partial S}{\partial g_{mn}} = 0\,. \tag{11}$$

Upon quantization, this symmetry is anomalous; that is, if one defines the quantum effective action Γ as

$$e^{-\Gamma[g]} = \int[d\Phi]\ e^{-S[\Phi,g]}\,, \tag{12}$$

then one finds [3, 2]

$$g_{mn}\frac{\partial\Gamma}{\partial g_{mn}} = -\frac{1}{24\pi}\sqrt{\det(g_{mn})}R\big(g(x)\big)\,, \tag{13}$$

If one integrates this over the surface, one finds

$$\int d^2x\ g_{mn}\frac{\partial\Gamma}{\partial g_{mn}} = -\frac{1}{24\pi}\int d^2x\ \sqrt{\det(g_{mn})}R\big(g(x)\big) = \frac{1}{6}\chi\,. \tag{14}$$

In [5], we extended this correspondence to the anomaly (13): we found an analog of the effective action Γ on a triangulation. That is, we found a function $\Gamma(l_{ij})$ of the edge lengths l_{ij} such that

$$\sum_{j\in<ij>} l_{ij}\frac{\partial\Gamma}{\partial l_{ij}} = \frac{1}{12\pi}\epsilon_i \tag{15}$$

for all vertices i (the sum is over all edges with one end at i).

We now present our construction; the remainder of the manuscript is taken verbatim from [5]. Our main result is

$$\Gamma = \frac{1}{12\pi}\Big[\sum_{\angle_{ijk}}\int_{\frac{\pi}{2}}^{\alpha_{ijk}}(y-\frac{\pi}{3})\cot(y)dy + \sum_{<ij>} 2k_{ij}\pi\,\ln\Big(\frac{l_{ij}}{l_0}\Big)\Big]\,, \tag{16}$$

where the first sum is over all internal angles α_{ijk}, second sum is over all edges $<ij>$ with lengths l_{ij} (the explicit factor of two arises because every edge is shared by two triangles), l_0 is a scale that we set to equal to one from now on, and the k_{ij} are constants associated to the edges that satisfy

$$\sum_{j\in<ij>} k_{ij} = 1-\frac{n_i}{6}\ , \qquad n_i = \sum_{j\in<ij>} 1 \tag{17}$$

at every vertex i; here n_i is the number of neighbors of the i'th vertex. Note that the conditions (17) do not in general determine the constants k_{ij} uniquely; one could add a subsidiary condition, *e.g.*, that $\sum k_{ij}^2$ is minimized, to remove this ambiguity. Note also that the total Euler character, which comes from a *uniform* scaling of all lengths and thus does not change the angles α_{ijk}, comes entirely from the last term, *i.e.*, from $\partial\Gamma/\partial l_0$.

The strategy that we use to find this solution is as follows: we first consider the simplest case, a triangulation of the sphere with three vertices, three edges, and two faces, and prove the integrability conditions needed for Γ to exist are satisfied. We then find Γ for this case and show that it immediately generalizes to all triangulations with a certain homogeneity property, and finally generalize Γ to an arbitrary triangulation.

It would be interesting to complete the correspondence, and find a way to compute the result (16) as the anomalous effective action corresponding to a discrete analog of, e.g., the scalar action (10); a promising approach might be the work of S. Wilson on triangulated manifolds [4].

2. Integrability

We begin with a triangulation of the sphere with three vertices, three edges, and two (identical) faces (a triangular "pillow"); we label the edges by their lengths a, b, c and the opposite internal angles of the triangles by α, β and γ, respectively. We also abbreviate

$$a\frac{\partial\Gamma}{\partial a} \equiv D_a(\Gamma)\,, \quad \text{etc.,} \tag{18}$$

and, for simplicity, drop an overall factor of $1/(12\pi)$ in Γ. The defect at the vertex α in this case is just $2(\pi-\alpha)$, *etc.* Using this notation, the equations that we want Γ to satisfy in the triangle are:

$$\begin{aligned} D_a(\Gamma) + D_b(\Gamma) &= 2(\pi-\gamma)\,, \\ D_a(\Gamma) + D_c(\Gamma) &= 2(\pi-\beta)\,, \\ D_b(\Gamma) + D_c(\Gamma) &= 2(\pi-\alpha)\,. \end{aligned} \tag{19}$$

If we add the first two equations, and subtract the third, we get $D_a(\Gamma) = (\pi-\gamma-\beta+\alpha)$. Since $\gamma+\beta+\alpha$ is π, we get

$$D_a(\Gamma) = 2\alpha\,, \quad D_b(\Gamma) = 2\beta\,, \quad D_c(\Gamma) = 2\gamma\,. \tag{20}$$

The function Γ can exist only if

$$D_b\big(D_a(\Gamma)\big) = D_a\big(D_b(\Gamma)\big)\,. \tag{21}$$

Using

$$\alpha = \arccos\left(\frac{-a^2+b^2+c^2}{2bc}\right)\,, \qquad \beta = \arccos\left(\frac{a^2-b^2+c^2}{2ac}\right)\,, \tag{22}$$

it is easy to see that (21) is satisfied. Thus the integrability conditions are satisfied for the triangle.

The tetrahedron and octahedron give results similar to those of the triangle. However, for a general triangulation, it is not easy to decouple the integrability conditions and reduce them to equations that may be checked straightforwardly. Instead, we construct Γ explicitly.

3. The effective action

Because the triangle is the simplest case, and can be related to any other system, we have examined it in detail. As noted above, in this case the defect at a vertex is directly related to the internal angle at that vertex; this suggests that in the general case, where the defect is related to the sum of the internal angles at a vertex, Γ should be just the sum of the Γ for each triangle. This is almost correct.

The basic strategy for the triangle was to rewrite the differential equations for Γ in terms of new variables: the angles α, β, and the edge length c between them. One can integrate some of the equations and finally arrive at Γ on a single triangle Δ:

$$\Gamma_\Delta = \sum_i \Big[(\alpha_i - \frac{\pi}{3})\ln\big(\sin(\alpha_i)\big) - \int_{\frac{\pi}{2}}^{\alpha_i} \ln\big(\sin(y)\big)dy + k_i\pi\ln(a_i)\Big]\,, \tag{23}$$

where $\{a_1, a_2, a_3; \alpha_1, \alpha_2, \alpha_3\} = \{a, b, c; \alpha, \beta, \gamma\}$, and k_i are constants associated to each edge. This can be simplified by integration by parts:

$$\Gamma_\Delta = \sum_i \Big[\int_{\frac{\pi}{2}}^{\alpha_i} (y - \frac{\pi}{3})\cot(y)dy + k_i\pi\ln(a_i)\Big]\,. \tag{24}$$

Note that all terms are expressed in terms of the internal angles α_i except for the last term, which explicitly involves a_i. To prove that this is correct (and to determine k_i), we differentiate Γ:

$$
\begin{aligned}
a_i \frac{\partial \Gamma}{\partial a_i} \equiv D_{a_i}\Gamma &= \sum_j \Big[(\alpha_j - \frac{\pi}{3})\cot(\alpha_j) D_{a_i}\alpha_j\Big] + k_i\pi \\
&= \sum_j \Big[-(\alpha_j - \frac{\pi}{3})\frac{\cos(\alpha_j) D_{a_i}\cos(\alpha_j)}{1-\cos^2(\alpha_j)}\Big] + k_i\pi\,. \qquad (25)
\end{aligned}
$$

Then the contribution of one triangle to the defect at vertex 1 is given by

$$
\begin{aligned}
(D_b+D_c)\Gamma = &-(\alpha - \frac{\pi}{3})\frac{\cos(\alpha)(D_b+D_c)\cos(\alpha)}{1-\cos^2(\alpha)} - (\beta - \frac{\pi}{3})\frac{\cos(\beta)(D_b+D_c)\cos(\beta)}{1-\cos^2(\beta)} \\
&-(\gamma - \frac{\pi}{3})\frac{\cos(\gamma)(D_b+D_c)cos(\gamma)}{1-\cos^2(\gamma)} + (k_b+k_c)\pi
\end{aligned}
$$

which can be explicitly evaluated using $\gamma = \pi - \alpha - \beta$ and (22), and gives:

$$
(D_b + D_c)\Gamma = \frac{\pi}{3} - \alpha + k_b\pi + k_c\pi\,. \qquad (26)
$$

This calculation works just as well for any triangle in a general triangulation to give Γ on any surface.

The constants k_i are assigned to every edge i. Clearly, in the case of the triangular pillow, there is a trivial solution to $k_i = \frac{1}{3}$ for all i (recall that $\pi - \alpha$ is half the defect at the vertex, but there are two triangles meeting at each vertex to sum over in this case). More generally, for any locally homogeneous triangulation in which all vertices have n nearest neighbors, we can choose

$$
k_i = \frac{1}{n} - \frac{1}{6}\,. \qquad (27)
$$

However, for a general triangulation, the edge may connect vertices with different numbers of edges connecting to them. In this case, it is not necessarily trivial to find the appropriate values for all the k_i.

4. A Problem in Graph Theory

On a general triangulated surface, the we have the condition

$$
\sum_{j\in<ij>} k_{ij} = 1 - \frac{n_i}{6} \qquad (28)
$$

at every vertex, where n_i is the number of neighbors of the i'th vertex. This means we want to find labels for the edges of a graph such that the sum at each vertex is the same. The equations (28) are a system V linear equations on the E variables k_{ij}, where V is the total number of vertices and E is the total number of edges; note that $E \geq V$ for all triangulations. We can rewrite this in terms of the $V \times E$ dimensional matrix that describes the connections between vertices. This matrix has exactly two ones, which correspond to two vertices, in each column, which corresponds to the edge connecting the vertices.

We are happy to thank L. Motl [6] for the following proof that a solution to these equations always exists. The system of equations would not have a solution only if there were a linear combination of the rows that vanishes, that is, if there existed a vector that is perpendicular to all the columns. Because every column contains exactly two ones, it suffices to consider a submatrix that defines a triangle:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} . \tag{29}$$

Since this matrix is nondegenerate, no nontrivial vector is annihilated by it. Since every edge sits on a triangle, no such vector can exist for the whole triangulation. Therefore, there must always be at least one way to label the edges.

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