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PARABOLIC GEOMETRIES AS CONFORMAL INFINITIES OF EINSTEIN METRICS

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The study of Einstein metrics is a central area of research in geometry and analysis, for both mathematical and physical reasons. The simplest examples of Einstein metrics are symmetric spaces. In this article, we study symmetric spaces of noncompact type and their Einstein deformations in the spirit of the physical AdS/CFT correspondence, i.e. focusing on the relationship between Einstein metrics and (generalized) conformal structures on the boundary at infinity.

For real hyperbolic space $H^{n+1}_{\mathbb{R}}$ and the associated class of asymptotically real hyperbolic geometries, this has been intensively studied. Let M^{n+1} be a compact smooth manifold with boundary $X^n = \partial M^{n+1}$. By definition, a metric g on the interior of a compact manifold with boundary is called conformally compact if it has the form $g = \rho^{-2}\overline{g}$ where ρ is a nonnegative function which vanishes simply precisely on the boundary and $\overline{g}$ is nondegenerate (and of some specified regularity) up to the boundary. Note that any such g is complete. The conformal class of the restriction of $\overline{g}$ to the boundary is well defined and called the conformal infinity of g. If $|\nabla\rho| = 1$ at $\rho = 0$ (all quantities taken with respect to $\overline{g}$), then g is also called asymptotically hyperbolic (AH), since in that case its sectional curvatures all tend to -1 at infinity. Real hyperbolic space is the simplest example; its conformal infinity is the standard conformal class on the sphere S^n. Fefferman and Graham [15] introduced a program to use these AH 'filling' metrics to obtain conformal invariants. More specifically, if one can associate to a conformal class on X a "canonical" AH filling, then the Riemannian invariants for that interior metric give conformal invariants of the boundary structure. They showed that at a formal level, the requirement that the filling metric be Einstein is (almost) well-posed. The first general examples of AH Einstein metrics were constructed by Graham and Lee [17] as perturbations of $H^{n+1}_{\mathbb{R}}$. This program has been quite successful and has received considerable impetus through the AdS/CFT correspondence [27] in physics.

This theory was extended by the first author [2] to Einstein metrics obtained by deforming the other rank one noncompact symmetric spaces (complex and quaternionic hyperbolic spaces); the conformal infinities for these are CR structures and quaternionic contact structures on the boundary at infinity, respectively.

The paper is in final form and no version of it will be submitted elsewhere.

The present article is an announcement of an ongoing project toward a very general correspondence between 'asymptotically symmetric' Einstein metrics and their conformal infinities. We explain here how this study can be extended to all symmetric spaces G/K of noncompact type: the role of the boundary at infinity of the symmetric space is played by the Furstenberg boundary, and the conformal infinity structure is a "parabolic geometry", i.e. a geometry modeled on G/P where P is a parabolic subgroup of G.

The correspondence between Einstein metrics and the conformal infinities has deep consequences in the rank one case. We refer to the recent book [5] for more about the mathematics of the AdS/CFT correspondence and the real AH setting, and the survey [6] for a discussion of the complex hyperbolic case. These examples strongly suggest that the extended correspondence in higher rank is the beginning of a similarly rich story, which should provide new insights on parabolic geometries as well as Einstein metrics.

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1. Symmetric spaces of noncompact type

Let $M = G/K$ be a symmetric space of noncompact type, where K is a maximal compact subgroup in the connected semisimple Lie group G. The Cartan involution on the Lie algebra $\mathfrak{g}$ induces the Cartan decomposition[1]

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}\,. \tag{1}$$

The space $\mathfrak{m}$ contains a maximal abelian subalgebra $\mathfrak{a}$, the dimension of which is denoted r and is called the rank of M. Then $\exp(\mathfrak{a})$ is a maximal flat in M. We choose a positive Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$, and denote by $\mathfrak{a}^{\mathrm{reg}}$ the set of regular elements of $\mathfrak{a}$, and K_0 the stabilizer in K of an element of $\mathfrak{a}^{\mathrm{reg}}$. A dense open set M^{reg} of M is identified with $\mathfrak{a}_+^{\mathrm{reg}} \times K/K_0$, via the map

$$\Phi : \mathfrak{a}_+^{\mathrm{reg}} \times K/K_0 \to M^{\mathrm{reg}}, \quad (a,k) \to k\exp(a)\,. \tag{2}$$

We are interested here in the Furstenberg boundary of M,

$$B = K/K_0 = G/P\,, \tag{3}$$

where P is a minimal parabolic subgroup in G. Note that $\dim M = n + r$ where $n = \dim B$, so this is not a topological boundary except when the rank is one. It is a classical fact of harmonic analysis on symmetric spaces (see [20, chapter V, §3]) that bounded harmonic functions on M are in 1-1 correspondence with their "boundary values" on B. The Poisson transform carries a function on B to the unique bounded harmonic function on M which is asymptotic to that function on B; we wish to generalize this to the (nonlinear) Einstein problem on M.

We recall here some basic facts about the root theory for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Since $\mathfrak{a}$ is abelian, the endomorphisms $(\operatorname{ad} a)_{a\in\mathfrak{a}}$ of $\mathfrak{g}$ are simultaneously

[1] The authors (especially the first one, from the French school) apologize for not using the standard Cartan notation $\mathfrak{k} \oplus \mathfrak{p}$, but we reserve the symbol $\mathfrak{p}$ here for a parabolic subalgebra.

diagonalizable; their eigenvalues are elements of $\mathfrak{a}^*$ called the roots. Let Δ be the set of all roots, and for $\alpha \in \Delta$ denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the eigenspace corresponding to α. We distinguish the zero root and its eigenspace $\mathfrak{g}_0 \supset \mathfrak{a}$. Since we have chosen a Weyl chamber in $\mathfrak{a}$, there is a decomposition $\Delta = \Delta^+ \cup \Delta^-$ into positive and negative roots.

The metric on the symmetric space can be written down explicitly, at least on M^{reg}, in terms of the root data. To describe this, note that if $a \in \mathfrak{a}$, then $(\mathrm{ad}\, a)^2$ preserves $\mathfrak{m}$, so there is a decomposition

$$\mathfrak{m} = \mathfrak{a} \oplus \Big(\bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha \Big), \tag{4}$$

where $\mathfrak{m}_\alpha \subset \mathfrak{m}$ is the eigenspace with eigenvalue α^2 for $(\mathrm{ad}\, a)^2$. The metric at the identity coset is given as

$$g = \kappa_{\mathfrak{a}} + \sum_{\alpha \in \Delta^+} \sinh^2(\alpha) \kappa_{\mathfrak{m}_\alpha}\,, \tag{5}$$

where $\kappa_{\mathfrak{a}}$ and $\kappa_{\mathfrak{m}_\alpha}$ are the restrictions of the Killing form κ to $\mathfrak{a}$ and $\mathfrak{m}_\alpha$, respectively, see [19, chap. II, lemma 5.25]. The differential of the map Φ defined in (2) identifies the tangent space at any regular point of M with $\mathfrak{m}$. This trivializes TM (at least over $M^{reg} \simeq \mathfrak{a}_+^{\mathrm{reg}} \times B$), and determines the metric everywhere.

2. Parabolic structures on the boundary

As indicated in (3), the Furstenberg boundary B is identified with K/K_0 and also G/P, where P is the parabolic subgroup with Lie algebra

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \big(\oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \big)\,. \tag{6}$$

This latter identification is independent of basepoint, while the identification with K/K_0 is not. There are several other purely geometric descriptions of B, as a G-homogeneous space: for example, B is also identified as the space of equivalence classes of Weyl chambers [28].

From this point of view, B is the model space for the so-called parabolic geometry of type $(\mathfrak{g}, P)$. B is the base of the P-frame bundle $G \to G/P$; the total space G carries a Cartan connection ω, which is a $\mathfrak{g}$-valued one form. (More specifically, this Maurer-Cartan form gives an identification of TG with right-invariant vector fields on G, $\omega : TG \to G \times \mathfrak{g}$.) In general, a Cartan connection on X is a P-equivariant $\mathfrak{g}$-valued one form on a P-frame bundle $\mathcal{G} \to X$, the restriction of which to the vertical (fibre-tangent) subbundle gives an isomorphism with $\mathfrak{p}$ at each point, and which satisfies other assumptions which we do not list here. In particular, one gets an identification

$$TX \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \simeq \mathcal{G} \times_P \mathfrak{n}, \tag{7}$$

where $\mathfrak{n}$ is the nilpotent Lie algebra defined by

$$\mathfrak{n} = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha\,. \tag{8}$$

We also use the nilpotent ideal of $\mathfrak{p}$,

$$\mathfrak{n}^* = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \,, \tag{9}$$

which is the dual of $\mathfrak{n}$ via the Killing form.

The decreasing sequence

$$\mathfrak{n}^* = \mathfrak{n}_0^* \supset \mathfrak{n}_1^* \supset \cdots \,, \quad \mathfrak{n}_{i+1}^* = [\mathfrak{n}^*, \mathfrak{n}_i^*] \,, \tag{10}$$

induces an associated increasing filtration of $\mathfrak{n}$ by

$$\mathfrak{n}_i = (\mathfrak{n}_i^*)^\perp \,, \quad i \geq 1 \tag{11}$$

and associated graded spaces

$$\mathrm{Gr}_i \, \mathfrak{n} = \mathfrak{n}_i / \mathfrak{n}_{i-1} \,. \tag{12}$$

This corresponds in the tangent space of $B = G/P$ to an increasing filtration of TB by distributions D_i, such that the vector field bracket satisfies $[D_i, D_j] \subset D_{i+j}$. (This of course means that if X_i and X_j are sections of D_i and D_j, then $[X_i, X_j]$ is a section of D_{i+j}.) There is an induced bracket $[\mathrm{Gr}_i \, D, \mathrm{Gr}_j \, D] \to \mathrm{Gr}_{i+j} \, D$ which corresponds precisely to the P-equivariant Lie algebra bracket $[\mathrm{Gr}_i \, \mathfrak{n}, \mathrm{Gr}_j \, \mathfrak{n}] \to \mathrm{Gr}_{i+j} \, \mathfrak{n}$. Moreover, the subgroup G_0 of P with Lie algebra $\mathfrak{g}_0$, which is called the Levi subgroup, acts on each $\mathrm{Gr}_i \, \mathfrak{n}$; this induces a G_0-structure on $\oplus \, \mathrm{Gr}_i \, D$.

The filtration $\{D_i\}$ has a refinement which will be important for us. The decomposition (8) is invariant with respect to K_0, but not P, and hence corresponds to a decomposition of the tangent bundle of $B = K/K_0$ into subbundles which depends on the choice of a basepoint in the symmetric space. On the other hand, the subspaces $\oplus_{\beta \leq \alpha} \mathfrak{g}_\beta \subset \mathfrak{n}$ are P-invariant, and thus give rise to G-invariant (and in particular basepoint independent) distributions D_α on B. The relation with the previous filtration $\{D_i\}$ is that

$$D_1 = \oplus_1^r D_{\alpha_i} \,, \tag{13}$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots, and more generally D_i is the sum of the D_α for all roots α of length at most i.

Usually, a parabolic structure is given from the more primitive data of a set of distributions D_i and G_0-structures, satisfying constraints on the brackets of vector fields. After a complicated process called "prolongation", one sometimes arrives at a Cartan connection satisfying nice conditions (regular, normal...). In this article, we do not need any precise fact about Cartan connections, but we keep this in mind as a general setting for these geometries. In fact, we introduce several simple concepts from parabolic geometry, but shall not try to interpret them in terms of Cartan connections.

3. Hyperbolic spaces

In this section, we review the geometry of rank one symmetric spaces of noncompact type. These are the real, complex and quaternionic hyperbolic spaces and the Cayley hyperbolic plane. As homogeneous spaces G/K, these are

$$(14)\qquad \begin{aligned} H^m_{\mathbb{R}} &= SO_{1,m}/SO_m\,, & H^m_{\mathbb{C}} &= SU_{1,m}/U_m\,,\\ H^m_{\mathbb{H}} &= Sp_{1,m}/Sp_1Sp_m\,, & H^2_{\mathbb{O}} &= F_4^{-20}/Spin_9\,. \end{aligned}$$

Note that

$$(15)\qquad \dim_{\mathbb{R}} H^m_{\mathbb{K}} = m\dim_{\mathbb{R}}\mathbb{K} := n+1\,.$$

As above, the boundary at infinity, S^n, can be represented either as G/P for some minimal parabolic $P \subset G$, or as K/K_0, where

$$(16)\qquad K_0 = SO_{m-1},\ U_{m-1},\ Sp_1Sp_{m-1} \quad\text{or}\quad Spin_7\,,$$

respectively.

The nilpotent Lie algebra $\mathfrak{n}$ in (8) is the $\mathbb{K}$-Heisenberg algebra

$$(17)\qquad \mathbb{K}^{m-1}\oplus \operatorname{Im}(\mathbb{K})\,,$$

with bracket $[x,y] = \operatorname{Im}\big(\sum_1^{m-1}\bar{x}_i y_i\big)$, for $x=(x_i)$, $y=(y_i)\in\mathbb{K}^{m-1}$. The Levi component of $\mathfrak{p}$ is $\mathfrak{g}_0 = \mathfrak{k}\oplus\mathfrak{a} = \mathfrak{k}\oplus\mathbb{R}A$, where we may take A to be the unique element of $\mathfrak{a}$ such that $\operatorname{ad}A$ acts with eigenvalues 0 on $\mathfrak{g}_0$, 1 and 2 on $\mathfrak{n}^*$, and -1 and -2 on $\mathfrak{n}$. (In fact, write $\mathfrak{g} = \oplus_{i=-2}^{+2}\mathfrak{g}_i$ where $\operatorname{ad}(A)|_{\mathfrak{g}_i} = i$; then $\mathfrak{n} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_{-2}$ with $\mathfrak{n}_1 = \mathfrak{g}_{-1} = \mathbb{K}^{m-1}$ and $\mathfrak{g}_{-2} = \operatorname{Im}(\mathbb{K})$.) The Levi group G_0 is the semidirect product of K_0 with $\exp(\mathbb{R}A)$. In the real case, G_0 is the conformal group CO_{m-1}, with its standard action on $\mathfrak{n} = \mathbb{R}^{m-1}$; in the complex and quaternionic cases, G_0 is the conformal extension CU_{m-1} and $CSp_{m-1}Sp_1$ of the corresponding group U_{m-1} and $Sp_{m-1}Sp_1$, with the standard action on $\mathfrak{n}_1 = \mathbb{K}^{m-1}$. For more details on the complex hyperbolic case (the quaternionic case being similar), we refer to [18].

All of this carries over to the geometry of $S^n = G/P$. This sphere carries a distribution D_1 induced by $\mathfrak{n}_1 = \mathbb{K}^{m-1}$ which is equipped with a G_0-structure, or in other words, a real, complex or quaternionic conformal structure. As in §2, there is no G-invariant supplement to D_1, but there is a K-invariant one, namely D_2, induced by $\mathfrak{g}_{-2} = \operatorname{Im}(\mathbb{K})$. Furthermore, Lie bracket induces a map

$$[\ ,\ ] : D_1\times D_1 \longrightarrow TS^n/D_1 \simeq D_2$$

which is isomorphic to the corresponding Lie algebra bracket in $\mathfrak{n}$.

The conformal structure on D_1 can also be seen from the formula (5) for the metric: normalizing the invariant form on $\mathfrak{g}$ so that $|A| = 1$ yields that

$$(18)\qquad g = d\alpha^2 + \sinh^2(\alpha)\kappa_1 + \sinh^2(2\alpha)\kappa_2$$

where κ_1 and κ_2 are the metrics on the distributions D_1 and D_2 and α is a linear coordinate on the one-dimensional flat. (With this normalization, the sectional curvatures take values between -4 and -1.) Define S^n_α to be the sphere of radius α; then the limit

$$(19)\qquad \lim_{\alpha\to\infty} 4e^{-2\alpha} g\big|_{S^n_\alpha}$$

is finite only on the subbundle $D_1 \subset TS^n$, and equals κ_1 there. There is no natural origin in the symmetric space, so $e^{-2\alpha}$ and κ_1 are defined only up to a multiplicative factor; this explains why D_1 inherits a conformal structure but not an actual metric.

To see this from a slightly different point of view, denote by $\eta \in \Omega^1(S^n)\otimes \mathrm{Im}(\mathbb{K})$ the connection 1-form of the Hopf bundle

$$\begin{array}{ccc} S^{\dim \mathbb{K}-1} & \longrightarrow & S^n \\ & & \downarrow \\ & & P_{\mathbb{K}}^{m-1} \end{array} \tag{20}$$

Then $D_1 = \ker \eta$ while D_2 is the vertical tangent bundle for this fibration, and (18) can be rewritten as

$$g = d\alpha^2 + \sinh^2(\alpha)\gamma + \sinh^2(2\alpha)\eta^2, \tag{21}$$

where γ is the pullback to D_1 of the invariant metric on $P_{\mathbb{K}}^{m-1}$. Moreover, γ and η satisfy the following compatibility conditions:

- in the complex case, the complex structure I, the metric γ on D_1 and the restriction of $d\eta$ to D_1 are related by $d\eta(\cdot,\cdot) = \gamma(I\cdot,\cdot)$ on D_1;
- in the quaternionic case, the quaternionic structure on D_1 is given by three complex structures (I_1, I_2, I_3), with $I_3 = I_1 I_2$, and if $\eta = (\eta_1, \eta_2, \eta_3)$ is the $\mathrm{Im}(\mathbb{H})$-valued contact form, then $d\eta_i(\cdot,\cdot) = \gamma(I_i\cdot,\cdot)$ on D_1.

(There is a similar formula in the octonionic case). Therefore, in each case, there is a compatibility between the G_0-structure on the pair (D_1, D_2) and the bracket structure $[\ ,\ ] : D_1 \times D_1 \to D_2$. This compatibility is the essential point in defining more general "asymptotically hyperbolic" metrics.

4. Asymptotically rank one symmetric Einstein metrics

In this section we recall the Einstein deformations of rank one symmetric spaces, as studied in [17, 2], cf. also [8].

Fix the $(n+1)$-dimensional hyperbolic space $H_{\mathbb{K}}^m = G/K$, with boundary $S^n = G/P = K/K_0$; the parabolic subgroup P has a Levi subgroup $G_0 = K_0 \rtimes \mathbb{R}$. We define a **$G$-conformal structure** on a manifold X^n to be a distribution $D \subset TX$ of codimension $\dim \mathbb{K} - 1$, equipped with a conformal class $[\gamma]$ on D, such that:

(i) at each point, the nilpotent Lie algebra $D \oplus TX/D$ induced by the vector field bracket $[\ ,\ ] : D \times D \to TX/D$ is isomorphic to the $\mathbb{K}$-Heisenberg algebra;

(ii) there is a G_0-structure on $D \oplus TX/D$, compatible with the aforementioned bracket and inducing the conformal class $[\gamma]$ on D.

Let us see what the definition means in each case.

When $\mathbb{K} = \mathbb{R}$, one has $D = TX$, so one simply has a $G_0 = CO_n$-structure on TX, i.e. a conformal class.

When $\mathbb{K} = \mathbb{C}$, D is a distribution of codimension 1. By condition (i), D is a contact distribution. If η is a 1-form such that $\ker \eta = D$, then $d\eta|_D$ is a symplectic form; condition (ii) prescribes a pair (γ, η), defined only up to conformal change $(\gamma, \eta) \to (f\gamma, f\eta)$, for any smooth $f > 0$, satisfying $\gamma(\cdot,\cdot) = d\eta(\cdot, I\cdot)$ for some complex structure I (which depends on the choice of conformal representative). (Note that the metric η^2 on TX/D becomes $f^2\eta^2$ under this conformal change.)

This is precisely the usual definition of an abstract strictly pseudoconvex CR structure. By Moser's lemma, any small deformation of the contact distribution D is equivalent to D by a diffeomorphism. Hence a deformation of the G-conformal structure in this case can always be reduced to a ($d\eta|_D$-compatible) deformation of the metric γ on D.

When $\mathbb{K} = \mathbb{H}$, D is a distribution of codimension 3, called a **quaternionic contact structure**, as introduced in [2], cf. also the survey [3]. Condition (i) is much stronger in this setting. Fixing a metric γ on D in the conformal class, we can choose an orthonormal basis $\eta = (\eta_1, \eta_2, \eta_3)$ of $(TX/D)^* = D^\perp \subset \Omega^1 X$ such that $\gamma(\cdot,\cdot) = d\eta_i(\cdot, I_i\cdot)$ for a triple of complex structures (I_1, I_2, I_3) on D satisfying the usual quaternion commutation relations. Again, the conformal equivalence is that $(\gamma, \eta) \sim (f\gamma, f\eta)$.

A very significant difference from the complex case is that in this setting the metric γ is completely determined by the distribution D; this is because the 3-dimensional subspace $\Omega_+ \subset \Omega^2 D$ generated by the $d\eta_i|_D$ uniquely fixes the $G_0 = CSp_{m-1}Sp_1$-structure on D. Hence condition (ii) becomes superfluous. On the other hand, the existence of such distributions is far from clear since they are solutions of a complicated differential system. It is true, but by no means obvious, that they exist and come in infinite dimensional families (see the references above).

Dimension 7 is a special case: then D is 4-dimensional, and $Sp_1Sp_1 = SO_4$, so $G_0 = CO_4$ and the G_0-structure on D is just a conformal class of metrics. Condition (i) reduces to a positivity condition, namely, that the wedge product on $\Omega_+ \subset \Omega^2 D$ must be positive. Any such subspace of $\Omega^2 D$ defines a conformal class on D (see [12, 11] for more about 7-dimensional quaternionic contact structures).

Finally, although the definition makes sense in the octonionic case, the differential system corresponding to condition (i) for the distribution D is so strong that there are no deformations of the standard structure on the sphere S^{15} (this follows from [34]).

Note that in each case, the G_0-structure is determined by the bracket and the conformal metric, i.e. by the data $(D, [\gamma])$.

Let M^{n+1} be a manifold with boundary X^n, and identify a neighborhood of X in M with $(\alpha_0, \infty) \times X$, with linear coordinate α on the first factor (so the boundary is $\alpha = \infty$). Let G/K be a rank one noncompact symmetric space. We say that a metric g on M is **asymptotically rank one symmetric** (or just asymptotically symmetric) of type G/K if there exists a G-conformal structure on X so that, as $\alpha \to \infty$,

$$g = d\alpha^2 + e^{2\alpha}\gamma + e^{4\alpha}\eta^2 + O(e^{-\alpha})\,, \tag{22}$$

where (γ, η^2) is a choice of metric on $D \oplus TX/D$ compatible with the G_0-conformal structure. To be more precise, if g_0 denotes the metric given as the sum of the first three terms on the right here, then the error term decays at the rate $e^{-\alpha}$ when measured with respect to g_0. We say that the G_0-conformal class of (γ, η) is the

conformal infinity of g. In this rank one setting, we also call these asymptotically real, complex, quaternionic, or octonionic hyperbolic.

There are several main problems concerning asymptotically rank one symmetric Einstein metrics of type G/K with prescribed conformal infinity on X:

a) The formal problem. Determine the complete (formal) series expansion for Einstein metrics of the form (22) corresponding to a given (γ, η);
b) The deformation problem. Given an asymptotically symmetric Einstein metric g with conformal infinity (γ, η), determine all nearby metrics g' of this same type, presumably in terms of deformations of the G-conformal structure on X;
c) The global existence problem. Given any G-conformal structure on X, determine if there is an asymptotically symmetric Einstein metric (either specifying the filling M or not) with this as its conformal infinity;
d) Regularity. Given any asymptotically symmetric Einstein metric g with (smooth) conformal infinity (γ, η), prove that in an appropriate 'gauge' (i.e. choice of coordinates near the boundary) it has an expansion as in a).

There are, of course, a host of related questions concerning finer aspects of the geometry and analysis of these special metrics.

The formal problem was studied first in the complex case: Fefferman [14] described a high order asymptotic expansion for the solution of the Monge-Ampère equation leads to a Kähler-Einstein metric on a strictly pseudoconvex domain of $\mathbb{C}^m$, (see the Cheng-Yau metric below). In the real case, LeBrun [22] constructed local selfdual real asymptotically hyperbolic Einstein metric in a small neighbourhood of any 3-manifold with prescribed conformal class, assuming all data is real analytic. A complete description of the formal asymptotics in the real case in all dimensions (with $\mathcal{C}^\infty$ data) was obtained by Fefferman and Graham [15].

As for existence, there are a number of explicit examples of asymptotically real hyperbolic Einstein metrics beyond the obvious cocompact quotients of hyperbolic space itself. The most natural ones are the SU_2-invariant selfdual Einstein metrics on B^4. Explicit formulæ for these (in terms of ϑ functions) were found by Hitchin [21], generalizing earlier work by Pedersen [33], who determined the metrics with an extra U_1-symmetry, which have the Berger metrics on S^3 as their conformal infinities. Quite a few other 4-dimensional examples appear in the physics literature, as the so-called Taub-Bolt metrics, the AdS toral black hole metrics, see [1] for some of these.

In the complex case, if the CR structure arises as the boundary of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^m$, then by the work of Cheng and Yau [10], Ω admits a complete Kähler-Einstein metric; this Cheng-Yau metric is asymptotically complex hyperbolic with conformal infinity the given CR structure on $\partial\Omega$. The first constructions of asymptotically quaternionic hyperbolic Einstein metrics (actually, quaternionic-Kähler metrics) can be found in the works of Galicki [16] and LeBrun [23].

The deformation theory was first studied in the real case in [17], and in the other rank one cases in [2], leading to the

Theorem A. *Any small deformation of the standard G_0-conformal structure on the sphere S^n is the conformal infinity of a complete asymptotically rank one symmetric Einstein metric on the ball B^{n+1}. Moreover, this metric is locally unique modulo diffeomorphisms which act as the identity on the boundary. The same result holds for perturbations of any nondegenerate asymptotically hyperbolic Einstein metric.*

Here, **non degenerate** means that the infinitesimal Einstein operator has trivial L^2-kernel, modulo the infinitesimal action of diffeomorphisms fixing the boundary. This condition is fulfilled when the metric has negative sectional curvature, but can be verified under less restrictive hypotheses too [24]. We refer to [2] for more details.

Anderson (see, for example, [1]) has also established a more global deformation theory for asymptotically real hyperbolic metrics with scalar positive conformal infinity in four dimensions.

To understand the ramifications of the deformation theory in the complex case, note that when $n \geq 5$, all integrable CR deformations of S^n are embeddable in $\mathbb{C}^m$, so for these, the corresponding asymptotically complex hyperbolic Einstein metrics are already known by the Cheng-Yau theorem; if the CR structure is not integrable, however, this theorem constructs new Einstein metrics which are not Kähler. The case of dimension 3 is special since the CR integrability condition is empty, but not all CR structures on S^3 are embeddable. Embeddable CR deformations of S^3 were studied in the 90's in various papers of Lempert, Epstein and Bland [25, 13, 7]. In this case, the filling Einstein metric is actually Kähler-Einstein. Left-invariant CR structures on S^3 provide examples of nonembeddable CR structures; explicit formulæ for the corresponding Einstein metrics were determined by Hitchin [21].

In the quaternionic setting, there is again a difference between the 7- and higher dimension cases. For $n > 7$, the Einstein metric is actually quaternionic Kähler [4], i.e. its holonomy group is $Sp_m Sp_1$, as is the case for the quaternionic hyperbolic metric itself. For $n = 7$, this is no longer true, and in fact there is a local obstruction on a quaternionic contact structure on any X^7 to be the local conformal infinity of a quaternionic Kähler metric [12, 11].

Finally, the true (nonformal) regularity theory has been obtained in the real setting [9] and for the Cheng-Yau metrics [26].

5. G-conformal structures

Fix a symmetric space G/K of noncompact type of rank $r > 1$, with Furstenberg boundary $B^n = G/P$. We let G_0 be the Levi component of P, and fix a flat $\mathfrak{a} \subset \mathfrak{m}$, as well as a Weyl chamber, and define the associated root system as in section 1.

Our next goal is to give a precise formulation of the boundary structures which serve as the conformal infinities for asymptotically symmetric metrics in this higher rank setting. Fixing the model symmetric space G/K, we shall define a G-conformal structure on an arbitrary n-manifold X (so the dimension of X is the same dimension as that of the Furstenberg boundary of G/K). Analogous to the rank one case, this consists of a set of distributions D_α on X, one for each positive root α, and a conformal class of metrics on each D_α, such that this entire set of data is compatible with the Lie-theoretic relations.

We say that a system of distributions $\{D_\alpha\}$ (indexed by the positive root system Δ_+ associated to G/K) is transverse ordered if for each $\alpha, \beta \in \Delta_+$,

$$D_\alpha \cap D_\beta = \sum_{\substack{\gamma \leq \alpha \\ \gamma \leq \beta}} D_\gamma \,.$$

In particular, $\alpha \leq \beta$ implies $D_\alpha \subset D_\beta$. To any such system we associate the graded vector bundle $\mathrm{Gr}_\alpha D = D_\alpha/(\oplus_{\beta<\alpha} D_\beta)$. When $\alpha+\beta$ is not a root, we also define $D_{\alpha+\beta} = D_\alpha + D_\beta$.

Definition. A **G-conformal structure** on a manifold X^n consists of:

- a transverse ordered system $(D_\alpha)_{\alpha\in\Delta_+}$ of distributions on X, and
- a G_0-structure on the graded bundle $\mathrm{Gr}\, D = \oplus_{\alpha\in\Delta_+} \mathrm{Gr}_\alpha D$,

satisfying the following properties:

(i) at each point, and for each $\alpha, \beta \in \Delta_+$, the vector field bracket satisfies $[\ ,\ ] : D_\alpha \times D_\beta \to D_{\alpha+\beta}$, and the induced algebraic bracket $[\mathrm{Gr}_\alpha D, \mathrm{Gr}_\beta D] \to \mathrm{Gr}_{\alpha+\beta} D$ coincides with the Lie algebra bracket $[\ ,\ ] : \mathfrak{g}_\alpha \times \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$;

(ii) the G_0-structure is compatible with the bracket structure.

Condition (i) is a **local integrability condition** for the G-conformal structure. Condition (ii) means that at each point we get an (algebraic) identification of $\mathrm{Gr}\, D$ with the graded space constructed from the model $\mathfrak{g}/\mathfrak{p}$, so that the bracket and the G_0-structure coincide.

The family of distributions (D_α) may be recovered from the G_0-structures and the rougher filtration (11), but we have stated the definition in such a way as to make explicit the interactions of the distributions under the bracket condition (i).

The Furstenberg boundary of G/K has a canonical G-conformal structure, as described in §2. What we are calling a G-conformal structure is a parabolic structure of type $(\mathfrak{g}, P)$, with the additional local integrability condition (i) on the bracket. It would be of interest to understand the meaning of this extra condition from the point of view of parabolic geometry (it corresponds to something between regular and torsion free).

We now illustrate this definition through several examples. In the rank one case, the roots are α and 2α (only α occurs in the real hyperbolic case), so the definition reduces exactly to the one in §4.

Now pass to the rank two case. There are two standard examples to understand: the product of two rank one spaces and SL_3/SO_3. In the case of a product of two rank one spaces, $G/K = G_1/K_1 \times G_2/K_2$, there are four positive roots, α_1, $2\alpha_1$, α_2, $2\alpha_2$ (or fewer if at least one of the factors is real hyperbolic). Therefore a G-conformal structure consists of the data of four distributions $D_{\alpha_1} \subset D_{2\alpha_1}$, $D_{\alpha_2} \subset D_{2\alpha_2}$, with $D_{2\alpha_1}$ and $D_{2\alpha_2}$ transverse and intersecting trivially. By conditions (i) and (ii), both $D_{2\alpha_1}$ and $D_{2\alpha_2}$ are integrable; the leaves of $D_{2\alpha_i}$ carry a

G_i-conformal structure with corresponding distribution D_{α_i}, and the brackets of sections of D_{α_1} and D_{α_2} lie in $D_{\alpha_1} + D_{\alpha_2}$. This last restriction means that D_{α_1} is invariant along the leaves of $D_{2\alpha_2}$ and vice versa.

To be even more specific, consider the symmetric space $H_{\mathbb{R}}^{n_1+1} \times H_{\mathbb{R}}^{n_2+1}$, corresponding to the group $G = SO_{1,n_1+1} \times SO_{1,n_2+1}$, and with Furstenberg boundary $X^n = S^{n_1} \times S^{n_2}$. A G-conformal structure on X consists of two transverse integrable distributions D_i of dimension n_i, $i = 1, 2$, each endowed with a conformal class $[\gamma_i]$ over X.

The other rank 2 case we discuss is SL_3/SO_3. This has three positive roots α_1, α_2 and $\alpha_1 + \alpha_2$; the Lie algebra $\mathfrak{n}$ is realized as the lower triangular 3×3 real matrices, and G_0 can be identified with the group of diagonal matrices in SL_3. An SL_3-conformal structure on X^3 consists of two one-dimensional distributions D_{α_1} and D_{α_2} such that $D_{\alpha_1} + D_{\alpha_2}$ is a contact distribution. Condition (ii) gives no extra information since any two conformal structures on a line are equivalent. To state this slightly differently, an SL_3-conformal structure consists of a contact distribution on X^3 along with two one-dimensional subbundles of the contact plane field which are everywhere transverse. For SL_3/SO_3 itself, the Furstenberg boundary is $X^3 = K/K_0 = SO_3/(SO_3 \cap G_0)$ (which is a quotient of $P_{\mathbb{R}}^3$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$); this can also be thought of as the manifold of complete flags $D \subset P \subset \mathbb{R}^3$. There are two natural maps $B \to P_{\mathbb{R}}^2$, given by $(D, P) \to D$ and $(D, P) \to P$, which can be identified with finite quotients of two different Hopf fibrations on S^3, and the two line bundles on B are the tangent spaces to the fibres of the two maps.

6. Asymptotically symmetric metrics of higher rank

We wish to formulate a generalization of (5) for an asymptotically symmetric metric g with conformal infinity $\mathfrak{c} = \{D_\alpha, [\gamma_\alpha]\}_{\alpha \in \Delta_+}$, which is a G-conformal structure on a manifold X. In the rank one setting, such a metric is defined on the collar neighbourhood $[\alpha_0, \infty) \times X$, and there are no other a priori restrictions on the filling manifold M, so long as it has this as its end. In higher rank, however, as a first step we shall define this metric on $\mathfrak{a}_+^{\mathrm{reg}} \times X$, where $\mathfrak{a}$ is the Cartan subspace of the symmetric space G/K. This is only a very small neighbourhood of infinity in the "filling manifold" M, as we shall illustrate below even when $M = G/K$. In general, we would need to assume that the filling manifold M has a compactification $\overline{M}$ as a manifold with corners, which has the property that the arrangement of boundary faces (of all codimension) has the same combinatorics of intersection as the system of boundary faces of the symmetric space G/K itself. The precise definition is formulated inductively, but is sufficiently complicated to state that in the following discussion we shall mostly assume that $M = G/K$, with a metric which is a deformation of the exact symmetric one.

In this section we begin by defining this initial approximation to a locally symmetric metric modelled on that of G/K and with conformal infinity $\mathfrak{c}$. As stated above, this will be defined on $\mathfrak{a}_+^{\mathrm{reg}} \times X$. We also note a few of its geometric properties. We then need to discuss the compactification of $M = G/K$ and show how to extend this metric to an entire neighbourhood of infinity. We build up to this in stages, first discussing the two simple rank two cases and then the general case.

So, fix the G-conformal structure $\mathfrak{c}$ on X. We first choose metrics γ_α representing the prescribed conformal classes on each $\mathrm{Gr}_\alpha D$. More abstractly, this is a reduction of the G_0-structure on $\mathrm{Gr}\, D$ to a K_0-structure. The G_0-structure intertwines the various conformal classes on each $\mathrm{Gr}_\alpha D$. In particular, the relationship between two different reductions (or choices of conformal representatives) is determined by a set of positive functions $\{f_1, \dots, f_r\}$ which are the conformal factors relating the two choices of metric on each distribution D_{α_i} associated to the simple roots $\alpha_1, \dots, \alpha_r$ only. The conformal factor relating the two metrics on any other component $\mathrm{Gr}_\alpha D$, $\alpha = \sum_1^r n_i \alpha_i$, is $\prod_1^r f_i^{n_i}$.

In order to consider the γ_α as metrics on subbundles of TX, we must choose a family of supplementary subbundles for every inclusion $D_\alpha \subset D_\beta$, $\alpha < \beta$, which are all mutually compatible, so as to identify TX with $\mathrm{Gr}\, D$. Having made these choices, we define the asymptotically symmetric metric as a multi-warped product

$$g_{\mathfrak{c}} = \kappa_{\mathfrak{a}} + \sum_{\alpha \in \Delta_+} e^{2\alpha} \gamma_\alpha \,, \tag{23}$$

where the first term on the right is the Euclidean metric on $\mathfrak{a}$ (restricted to $\mathfrak{a}_+^{\mathrm{reg}}$).

It is necessary to understand the effect on $g_{\mathfrak{c}}$ of the various choices we have made. First, alterating the various supplementary subspaces gives a new metric $g_{\mathfrak{c}}'$. It is not hard to check that

$$|g_{\mathfrak{c}}' - g_{\mathfrak{c}}|_{g_{\mathfrak{c}}} = O\left(e^{-\alpha_1} + \ldots + e^{-\alpha_r}\right) \,.$$

In other words, the difference between these two metrics tends to zero (when measured with respect to either one of them) when approaching infinity in $\mathfrak{a}_+^{\mathrm{reg}}$ in such a way that all the positive roots tend to infinity. Next, under the conformal change $\gamma_\alpha' = \prod_1^r f_i^{n_i} \gamma_\alpha$ (as described above) where $\alpha = \sum_1^r n_i \alpha_i$, then defining $\alpha' = \alpha - \frac{1}{2} \log f_i$ yields

$$\begin{aligned} g_{\mathfrak{c}} &= \sum \kappa_{ij} \big(d\alpha_i' + \frac{df_i}{2f_i}\big)\big(d\alpha_j' + \frac{df_j}{2f_j}\big) + \sum_{\alpha \in \Delta_+} e^{2\alpha'} \gamma_\alpha' \\ &= \sum \kappa_{ij} d\alpha_i' d\alpha_j' + \sum_{\alpha \in \Delta_+} e^{2\alpha'} \gamma_\alpha' + O(e^{-\alpha_1} + e^{-\alpha_2} + \cdots + e^{-\alpha_r}) \,, \end{aligned} \tag{24}$$

so the error term again tends to 0 in the regular directions of $\mathfrak{a}$. A further calculation shows that the curvature tensor of $g_{\mathfrak{c}}$ differs from that of the symmetric metric by a term which is $O(e^{-\alpha_1} + e^{-\alpha_2} + \cdots + e^{-\alpha_r})$. We have only stated this loosely since, in this general setting, these tensors live on different manifolds. However, this has a precise consequence which is easy to state and most relevant for our purposes:

Theorem B. *Let $-\lambda$ be the Einstein constant for the symmetric metric on G/K. Then for any G-conformal structure $\mathfrak{c}$,*

$$\mathrm{Ric}(g_{\mathfrak{c}}) + \lambda g_{\mathfrak{c}} = O\left(e^{-\alpha_1} + e^{-\alpha_2} + \cdots + e^{-\alpha_r}\right) \,.$$

As already indicated, the error terms which appear in each of these statements show that $g_{\mathfrak{c}}$ is only a suitable asymptotically symmetric metric when all of the roots α_i are large. We explain how to go further, starting with the rank 2 examples.

Product spaces: Suppose that M is the product of two rank one symmetric spaces $M_1 \times M_2$. Slightly more generally, and because it involves no extra work, we allow the factors M_i to be any asymptotically rank one symmetric space of dimension $n_i + 1$, with metric g_i and boundary $X_i = \partial M_i$. For simplicity of exposition here, we shall assume that these are asymptotically real hyperbolic (i.e. Poincaré-Einstein), but all of the discussion below extends immediately to the other asymptotically rank one settings. Let g_i be the metric on M_i, normalized so that the sectional curvatures approach -1 at infinity, and denote by $[\gamma_i]$ its conformal infinity. Then

$$g = n_1 g_1 + n_2 g_2 \tag{25}$$

is an Einstein metric on M with $\mathrm{Ric}(g) = -g$. Its conformal infinity on the corner $X = X_1 \times X_2$ is the $G_1 \times G_2$-conformal structure (with $G_i = SO(n_i+1, 1)$) consists of the pair of distributions $D_1 = TX_1 \times \{0\}$, $D_2 = \{0\} \times TX_2$, along with the conformal classes $[\gamma_i]$ on D_i.

Write the initial asymptotically hyperbolic metric on each factor as

$$g_j = \frac{dx_j^2 + \gamma_j}{x_j^2}\,,$$

where $x_j > 0$ is a boundary defining function for M_j and is related to the root α_j by $x_j = e^{-\alpha_j}$. The product metric is then

$$g = n_1 \frac{dx_1^2 + \gamma_1}{x_1^2} + n_2 \frac{dx_2^2 + \gamma_2}{x_2^2}, \tag{26}$$

or equivalently

$$n_1 d\alpha_1^2 + n_2 d\alpha_2^2 + n_1 e^{2\alpha_1}\gamma_1 + n_2 e^{2\alpha_2}\gamma_2\,;$$

since we can rescale, replacing α_i by $\sqrt{n_i}\alpha_i$, we see that this is indeed of the correct form (23) when $\alpha_1, \alpha_2 \to \infty$, i.e. as $x_1, x_2 \to 0$.

The manifold M has two codimension one boundary faces,

$$\partial_1 M = M_1 \times X_2, \qquad \partial_2 M = X_1 \times M_2\,.$$

Each of these faces has a product structure, which we think of as a fibration with fibres the manifold with boundary M_i and base the boundary of the other factor. The metric g induces the nondegenerate metric $n_i g_i$ on the fibres of the face $\partial_i M$. Said differently, there is an integrable distribution $\tilde{D}_i$ on each of these boundary hypersurfaces $\partial_i M$ which "fills" D_i, and which is precisely the subspace of $T\partial_i M$ on which g is finite.

While it is not unreasonable to call any metric of the form (26) asymptotically product hyperbolic (or asymptotically symmetric of type $G_1/K_1 \times G_2/K_2$), we shall reserve this name for any metric of this form which has the additional property that it is finite on some distribution $\tilde{D}_i \subset T\partial_i M$, and that this distribution is integrable and $\tilde{D}_i|_X \cap TX = D_i$.

Now consider the problem of whether a small deformation of the initial product $G_1 \times G_2$-conformal structure $(D_1, D_2, [\gamma_1], [\gamma_2])$ on X can be filled by asymptotically product hyperbolic metrics in the preceding sense. This deformation comprises a small perturbation of the two distributions and a small perturbation of the conformal classes on each. In order for this to remain a $G_1 \times G_2$-conformal structure, it is clearly necessary that the perturbed distributions remain integrable. This is already a strong hypothesis. For example, on $B = S^{n_1} \times S^{n_2}$, assuming that both $n_1, n_2 \geq 2$, this implies that (D_1, D_2) remains diffeomorphic to the standard pair $(TS^{n_1} \times \{0\}, \{0\} \times TS^{n_2})$. Hence for such a deformation we may as well assume that (D_1, D_2) remains fixed and only the conformal classes vary. In this case, we can clearly extend these distributions to the integrable distributions $\tilde{D}_i$ over $\partial_i M$ so that these hypersurface boundaries are fibred by their leaves, as required. This same local rigidity holds for distributions on $X = X_1 \times X_2$ provided this space is simply connected. If X is not simply connected, we must impose the extra condition on the perturbations of D_1 and D_2, that not only do they remain integrable, but their leaves remain compact, so that X is still a product. Thus we shall need to impose on this deformed G-conformal structure on X that it is also **globally integrable** in the sense that the two distributions D_1 and D_2 are the tangent spaces for the factors in $X = X_1 \times X_2$. (This global integrability is to be distinguished from the local integrability condition in the definition of G-conformal structures.) In other words, the globally integrable deformations of the G-conformal structure on X reduce simply to deformations of the pair of conformal classes $([\gamma_1], [\gamma_2])$ on the fixed pair of distributions (D_1, D_2), and any one of these is the conformal infinity of an asymptotically product hyperbolic metric.

An arbitrary asymptotically product hyperbolic metric of this type will usually not be asymptotically Einstein near the entire boundary. We return to this point in the next section.

The case of SL_3/SO_3:

Although we shall be more brief here, we again describe what we mean by an asymptotically symmetric metric associated to a G-conformal structure, particularly when it is a small deformation of the standard structure on the Furstenberg boundary of $M = SL_3/SO_3$.

First note that M has a compactification as a manifold with corners of codimension 2, which we denote $\overline{M}$. At least in the regular part, this is obtained by adding boundary faces where each simple root equals infinity; setting $x_j = e^{-\alpha_j}$, $j = 1, 2$, then we add the faces $\{x_1 = 0\}$ and $\{x_2 = 0\}$. It must be verified that this definition extends across the Weyl chamber walls, and we refer to [31] for all further details on this. The corner of $\overline{M}$ is the Furstenberg boundary, $B = SO_3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The standard G-conformal structure on B consists of a pair of one-dimensional distributions D_1 and D_2 such that their sum is contact. There are two hypersurface boundaries of this manifold, $\partial_i M$, $i = 1, 2$. These faces are again the total spaces of fibrations, with fibre a closed disk (which realizes the compactification of the two-dimensional hyperbolic space) and base $P^2_{\mathbb{R}}$. The symmetric metric on

M is finite along these boundaries precisely on the tangent spaces of these fibres, and the induced metric is the standard one on the hyperbolic plane.

Following the same reasoning as in the product case, we require that the one-dimensional distributions in a more general SL_3-conformal structure on B remain integrable (which is immediate since they are one-dimensional), but also that they extend to integrable rank 2 distributions on $\partial_i M$ with leaves diffeomorphic to the disk. With this extra global integrability condition, we can construct an asymptotically symmetric metric with this prescribed conformal infinity, and which is finite precisely along the tangent spaces of the fibres on each $\partial_i M$.

The general case: The two special cases above motivate the general case. We first compactify a general rank r symmetric space $M = G/K$ to a manifold with corners of codimension r. This is done by adding boundary faces $\{x_j = 0\}$, $j = 1, \ldots, r$, where $x_j = e^{\alpha_j}$. The Furstenberg boundary B is the highest codimension corner, where all the $x_j = 0$. See [32] for details on this. This compactification $\overline{M}$ has a rich geometric structure. The most important point here is that each boundary hypersurface $\partial_j M$ is the total space of a fibration, where the fibres are the compactifications of symmetric spaces of lower rank and the base is a compact locally symmetric space. By induction on the rank, one can ascertain that all boundary faces of arbitrary codimension have a similar structure.

The full details on the extensions of the family of distributions $\{D_\alpha\}$ to the entire boundary skeleton are complicated, but the main idea is clear: we require that these extensions are integrable, with compact leaves diffeomorphic to those in the model case. A metric is asymptotically symmetric if these fibres are the leaves of the distributions determined by its successive levels of singularity.

7. Asymptotically symmetric Einstein metrics

Let $M = G/K$ be an arbitrary symmetric space of noncompact type. The Furstenberg boundary $B = G/P$ has a canonical G-conformal structure. Our eventual goal is to carry out the

Program. *Given any G-conformal structure on B which is a small deformation of the canonical one, and which satisfies the appropriate global integrability condition, prove that there exists a complete, asymptotically symmetric Einstein metric on M with this prescribed G-conformal structure as its conformal infinity.*

This will establish the existence of a nonlinear Poisson transform for Einstein metrics.

Even more ambitiously, one would like to prove existence and study properties of these metrics on other manifolds, or for G-conformal structures which are far from the standard one. However, this has proved to be extremely difficult even in the real hyperbolic case, so we confine ourselves to this perturbative problem.

In this section we give some indication of the steps needed to accomplish this program, focusing attention mostly on the product rank one case, where this has already been carried out, cf. [8].

The overall strategy is a familiar one: for each admissible G-conformal structure on B we associate an asymptotically symmetric Einstein metric on M. Using the implicit function theorem, these metrics are then perturbed to exact Einstein metrics. Both steps require understanding the solvability of certain systems of nonlinear elliptic PDE on asymptotically symmetric spaces. In this perturbative setting, this reduces in turn to understanding the solvability of the corresponding linearized equations. For this we draw on and extend some recent techniques in geometric microlocal analysis developed by the second author and Andras Vasy [30, 29, 31, 32].

We now provide more details in the product case. As in the last section, we restrict to the real hyperbolic case, but simultaneously allow the factors to be more general Poincaré-Einstein metrics. Recall from §4 that a Poincaré-Einstein metric is nondegenerate if the infinitesimal (gauged) Einstein operator has no L^2-kernel. We shall assume that both factors (M_i, g_i) are nondegenerate in this sense. Let $[\widetilde{\gamma}_i]$ be any conformal class on X_i near to the conformal infinity of g_i.

We begin with the initial asymptotically symmetric Einstein metric

$$\widetilde{g} = n_1 \frac{dx_1^2 + \widetilde{\gamma}_1}{x_1^2} + n_2 \frac{dx_2^2 + \widetilde{\gamma}_2}{x_2^2}\,; \tag{27}$$

by Theorem B this satisfies

$$\operatorname{Ric}(\widetilde{g}) + \lambda \widetilde{g} = O(x_1 + x_2)\,, \tag{28}$$

and by construction has conformal infinity $([\tilde{\gamma}_1], [\tilde{\gamma}_2])$. Our first task is to find a better approximation, i.e. a metric $\widehat{g}$ which is asymptotically Einstein not only near the corner but also near the hypersurface boundaries $\{x_i = 0\} = \partial_i M$, $i = 1, 2$. The idea is that we define $\widehat{g}$ near $\partial_2 M$ by

$$\widehat{g}_2 = n_1 \frac{dx_1^2 + \widehat{\gamma}_1}{x_1^2} + n_2 h_2\,, \tag{29}$$

where $\widehat{\gamma}_1 \in \Gamma(\partial_1 M, S^2 T^* D_1)$ is a metric in the conformal class $[\widehat{\gamma}_1]$ on $D_1 \subset T\partial_1 M = TX_1 \oplus TM_2$, and $h_2 \in \Gamma(\partial_1 M, S^2 T^* M_2)$ is a metric along the M_2 fibres of $\partial_2 M$. Clearly, $\widehat{\gamma}_1$ and h_2 must be asymptotic near X to $\widetilde{\gamma}_1$ and $(dx_2^2 + \widetilde{\gamma}_2)/x_2^2$, respectively. A calculation shows that $\widehat{g}_2$ satisfies

$$\operatorname{Ric}(\widehat{g}_2) + \lambda \widehat{g}_2 = O(x_1) \tag{30}$$

if and only if $([\widehat{\gamma}_1], h_2)$ satisfies a coupled nonlinear system on $\partial_2 M$, which is elliptic on each of the fibres $\{p_1\} \times M_2$ (once an appropriate gauge has been chosen). A minor generalization of Theorem A guarantees the existence of solutions to this system with prescribed conformal infinity on each slice $\{p_1\} \times M_2$, for all sufficiently small deformations of the initial conformal infinity data. This uses crucially the nondegeneracy of g_2, as well as the global integrability of the G-conformal structure (which ensures that the leaves of the distribution D_2 appear as boundaries of the slices $\{p_1\} \times M_2$).

On the other hypersurface boundary $\partial_1 M$ we solve the analogous system of equations, yielding a metric

$$\widehat{g}_1 = n_1 h_1 + n_2 \frac{dx_2^2 + \widehat{\gamma}_2}{x_2^2} \tag{31}$$

which is asymptotically Einstein near $x_2 \to 0$, and which has the correct asymptotic behaviour.

It follows rather simply from the construction that these metrics are compatible as $x_1, x_2 \to 0$, i.e. near the corner X, which shows that we may define a metric $\widehat{g}$ by patching these together, and $\widehat{g}$ is the asymptotically Einstein metric we need. It satisfies

$$\mathrm{Ric}(\widehat{g}) + \lambda \widehat{g} = O\big(\inf(x_1, x_2)\big) \tag{32}$$

along the entire boundary of M.

For the other step, we employ the implicit function theorem for the gauged Einstein equation. We refer to [2] for a general description of the Einstein operator, the Bianchi gauge, and the corresponding nonlinear gauged Einstein operator. Its linearization is a geometric Laplacian of the form

$$L = \nabla^* \nabla - 2\overset{\circ}{R},$$

where the second term is the curvature tensor acting as a self-adjoint operator on symmetric 2-tensors.

By its geometric naturality, the linearization L^g computed at product metric g splits into the sum of linearizations $L^{g_1} + L^{g_2}$, with the two terms acting on the different factors of M. The basic problem is to show that L^g is Fredholm acting between two weighted Hölder spaces, and in fact is an isomorphism if both factors are nondegenerate. The corresponding statement for the action of L^g on weighted Sobolev spaces is one of the main theorems of [30]. In order to obtain the same conclusion on Hölder spaces, it is necessary to examine the pointwise behaviour of the Schwartz kernel of the inverse of L^g. Fortunately, this behaviour is also examined closely in [30]. We shall not describe this rather lengthy analysis in much detail. The first key idea is to represent the inverse of L^g as a certain contour integral in the spectral plane involving the resolvents $(L^{g_i} - \lambda)^{-1}$ of the two factors. We then use the well-known structure of the resolvents of geometric Laplacians on asymptotically hyperbolic spaces to analyze this contour integral. We refer to [30] for details.

Altogether, this proves the

Theorem C. *Let (M_1, g_1) and (M_2, g_2) be nondegenerate Poincaré-Einstein manifolds. Then any sufficiently small globally integrable deformation of the G-conformal structure on $\partial M_1 \times \partial M_2$ is the conformal infinity of an Einstein asymptotically symmetric deformation of the product metric on $M_1 \times M_2$.*

There are trivial examples of such deformations, namely where we vary the asymptotically hyperbolic Einstein metrics on M_1 and M_2 independently. This corresponds to a variation of the conformal class $[\gamma_1]$ on $TX_1 \times \{0\}$ which is invariant in the X_2 direction, and a variation of the conformal class $[\gamma_2]$ on $\{0\} \times TX_2$

which is invariant in the X_1 direction. However, there are many other admissible deformations since $[\gamma_1]$ and $[\gamma_2]$ may depend on both factors of X. Hence this theorem gives the existence of a large family of nonproduct Einstein metrics on M.

In the other product rank one cases, the same ideas apply almost identically. There are some additional technical complications when one of the factors is quaternionic; these result from the fact that the various asymptotically symmetric metrics corresponding to deformations of the initial G-conformal structure are not necessarily mutually quasi-isometric. This necessitates the construction of the inverse of the linearized operator L^g not only when g is exactly of product type but also when it is a small perturbation of a product metric. This uses even more strongly the full force of these geometric microlocal methods.

Finally, if we wish to carry out the same steps on a general symmetric space G/K, we face several new complications. The first is combinatorial, though not particularly easy. To construct the asymptotically symmetric metric with a prescribed conformal infinity which is asymptotically Einstein metric near the entire boundary of G/K, we must solve the appropriate coupled nonlinear elliptic systems on the fibres of each of the boundary faces of the compactification of G/K. Since these fibres are all symmetric spaces of lower rank, this should be possible using the main theorem and induction on the rank. The second complication is of a more analytic nature. Unlike the product case, the asymptotically symmetric metrics are no longer even approximately of product type near the (closed) boundary faces of the compactification of G/K. In other words, it is no longer possible to directly use the contour integral representation to understand the inverse of the linearized Einstein operator. This complication already appears in [32]; as is shown there, L^g is modeled by product operators in certain neighbourhoods of each face (these are quite delicate to define), and one may patch together the inverses of these product operators to obtain a good global parametrix for L^g. We refer to [32] for all further details.

References

[1] Anderson, M., *Geometric aspects of the AdS/CFT correspondence*, AdS/CFT correspondence: Einstein metrics and their conformal boundaries (O. Biquard, éd.), IRMA Lectures Math. Theoret. Phys. **8**, EMS (2005), 1–31.

[2] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000), vi+109.

[3] ———, *Quaternionic contact structures*, Proceedings of the Second Meeting on Quaternionic Structures in Mathematics and Physics, (Roma 1999) (S. Marchiafava, P. Piccinni et M. Pontecorvo, éds.), World Sci. (2001), available electronically in the EMS Electronic Library, 23–30.

[4] ———, *Métriques autoduales sur la boule*, Invent. Math. **148**, No. 3 (2002), 545–607.

[5] ——— (éd.), *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, IRMA Lectures Math. Theoret. Phys. **8**, Zürich, 2005.

[6] ———, *Cauchy-Riemann 3-Manifolds and Einstein Fillings, Perspectives in Riemannian Geometry*, CRM Proc. Lecture Notes **40**, AMS (2006), 27–46.

[7] Bland, J. S., – *Contact geometry and CR structures on S^3*, Acta Math. **172**, No. 1 (1994), 1–49.

[8] Biquard , O. et R. Mazzeo, *A nonlinear Poisson transform for Einstein metrics on product spaces*, in preparation.

[9] Chruściel, P. T., Delay, E., Lee, J. M. et Skinner, D. N., *Boundary regularity of conformally compact Einstein metrics*, J. Differential Geom. **69**, No. 1 (2005), 111–136.

[10] Cheng, S. Y. et Yau, S. T., *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33**, No. 4 (1980), 507–544.

[11] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier **56**, No. 4 (2006), 851–885.

[12] ———, *Géométrie quaternionienne en basses dimensions*, Prépublication de l'Institut de Recherche Mathématique Avancée, 2004/23, Université Louis Pasteur. Institut de Recherche Mathématique Avancée, Strasbourg, 2004, Thèse, Université Louis Pasteur, Strasbourg, 2004.

[13] Epstein, C. L., *CR-structures on three-dimensional circle bundles*, Invent. Math. **109**, No. 2 (1992), 351–403.

[14] Fefferman, C. L., *Monge-Ampère equations, the Bergman kernel, and geometry of pseudo-convex domains*, Ann. of Math. **103**, No. 2 (1976), 395–416.

[15] Fefferman, C. et Graham, C. R., *Conformal invariants*, Astérisque (1985), no. hors série, 95–116, The mathematical heritage of Élie Cartan (Lyon, 1984).

[16] Galicki, K., *Multi-centre metrics with negative cosmological constant*, Classical Quantum Gravity **8**, No. 8 (1991), 1529–1543.

[17] Graham, C. R. et Lee, J. M., *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87**, No. 2, (1991), 186–225.

[18] Goldman, W. M., *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1999, Oxford Science Publications.

[19] Helgason, S., *Groups and geometric analysis*, Pure Appl. Math. vol. 113, Academic Press Inc., Orlando, FL, 1984, Integral geometry, invariant differential operators, and spherical functions.

[20] ———, *Geometric analysis on symmetric spaces*, Math. Surveys Monogr. **39** (1994).

[21] Hitchin, N. J., *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42**, No. 1 (1995), 30–112.

[22] LeBrun, C., *$\mathcal{H}$-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380**, No. 1778 (1982), 171–185.

[23] ———, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63**, No. 3, (1991), 723–743.

[24] Lee, J. M., *Fredholm operators and Einstein metrics on conformally compact manifolds*, Mem. Amer. Math. Soc., **183** (2006).

[25] Lempert, L., *On three-dimensional Cauchy-Riemann manifolds*, J. Amer. Math. Soc. **5**, No. 4 (1992), 923–969.

[26] Lee, J. M. et Melrose, R., *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math. **148** (1982), 159–192.

[27] Maldacena, J., *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, No. 2 (1998), 231–252.

[28] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Ann. of Math. Stud. No. 78, 1973.

[29] Mazzeo, R. et Vasy, A., *Scattering theory on $SL(3)/SO(3)$: connections with quantum 3-body scattering*, Proc. London Math. Soc., to appear.

[30] ———, *Resolvents and Martin boundaries of product spaces*, Geom. Funct. Anal. **12**, No. 5 (2002), 1018–1079.

[31] ———, *Analytic continuation of the resolvent of the Laplacian on* SL(3)/SO(3), Amer. J. Math. **126**, No. 4 (2004), 821–844.

[32] ———, *Analytic continuation of the resolvent of the Laplacian on symmetric spaces of noncompact type*, J. Funct. Anal. **228**, No. 2 (2005), 311–368.

[33] Pedersen, H., *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274**, No. 1 (1986), 35–59.

[34] Yamaguchi, K., *Differential systems associated with simple graded Lie algebras*, Progress in differential geometry, Adv. Stud. Pure Math. **22** (1993), 413–494.

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