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THE LOCAL EQUIVALENCE PROBLEM IN CR GEOMETRY

MARTIN KOLÁŘ

Abstract. This article is dedicated to the centenary of the local CR equivalence problem, formulated by Henri Poincaré in 1907. The first part gives an account of Poincaré's heuristic counting arguments, suggesting existence of infinitely many local CR invariants. Then we sketch the beautiful completion of Poincaré's approach to the problem in the work of Chern and Moser on Levi nondegenerate hypersurfaces. The last part is an overview of recent progress in solving the problem on Levi degenerate manifolds.

1. Introduction

There are two fundamental facts which link analysis and geometry in one complex variable. The local, almost obvious one, states that every real analytic arc can be straightened by an invertible holomorphic map. The global one is the Riemann mapping theorem - any simply connected subdomain of the complex plane is biholomorphically equivalent to the open unit disc.

In March of 1907, Rendiconti del Circolo Matematico di Palermo published an article "Les fonctions analytique de deux variables et la représentation conforme". In the first section, titled "Énnoncé du problèm" Poincaré asks the same questions in two complex variables. He formulates the following problem, which was to become one of the cornerstones of CR geometry:

"Soit alors dans l'espace zz' une portion de surface à 3 dimensions s et sur cette surface un point m. Soit dans l'espace des ZZ' une portion de surface à 3 dimensions S et sur cette surface un point M. Est-il possible de déterminer les fonctions Z at Z' de telle façon qu'elle soint régulière dans in voisinage du point m, que le point ZZ' soit en M quand le point zz' est en m et qu'il décrive S quand le point zz' décrit s? C'est le *problèm local.*"

Poincaré then defines the global and the mixed version of the problem. We will not attempt to survey all the development in complex analysis inspired by

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the three equivalence problems. Instead, this paper confines itself to the local problem, and results directly related to its solution in dimension two.

Since the notation and terminology, being a century old, make the original article less accessible today, we give in Section 2 a review of its two heuristic counting arguments, reformulated in modern language. After introducing the first CR invariant, the Levi form, we sketch a solution of the local problem for Levi nondegenerate hypersurfaces, obtained by S. S. Chern and J. K. Moser. Their construction of normal forms completes Poincaré's approach for this class of manifolds.

In Section 5 we consider Levi degenerate hypersurfaces and the second CR invariant - the type of the point - introduced by J. J. Kohn in [K]. Then we review the important result on convergence of formal equivalences for hypersurfaces of finite type, due to M. S. Baouendi, P. Ebenfelt and L. P. Rothschild. Normal forms for finite type hypersurfaces are described in Section 6. In Section 7 we consider applications to a classification of local symmetry groups and the jet determination problem. Open problems, mainly for points of infinite type, are also discussed.

There are several excellent surveys on closely related topics in the literature. In particular, we mention the articles of R. O. Wells Jr., Isaev-Krantz and Baouendi-Ebenfelt-Rothschild.

2. Two counting arguments

In more familiar notation, the local Poincaré problem asks if for two given pieces of hypersurface M_1 and M_2 in $\mathbb{C}^2$, and points $p_1 \in M_1$ and $p_2 \in M_2$ there exists a biholorphic map in a neighbourhood of p_1 which maps p_1 to p_2 and M_1 to M_2.

Poincaré gives two heuristic counting arguments which suggest that the problem does not always have a solution. In the first one, he essentially derives the tangential Cauchy-Riemann equation.

To review his argument, consider holomorphic coordinates (z, w), where $z = x + iy$, $w = u + iv$, and a biholomorphic transformation

$$z^* = f(z, w)\,, \qquad w^* = g(z, w)\,, \tag{1}$$

where $f = f_1 + if_2$ and $g = g_1 + ig_2$. The components of f and g satisfy the ordinary Cauchy-Riemann equations:

$$\frac{df_1}{dx} = \frac{df_2}{dy}\,, \qquad \frac{df_1}{dy} = -\frac{df_2}{dx}$$

and three other such systems, one for f replaced by g and two other for derivatives with respect to u and v. That gives eight equations.

Now let M_1 be given by

$$v = \Phi(x, y, u) \tag{2}$$

and M_2 by

$$v^* = \Phi^*(x^*, y^*, u^*)\,. \tag{3}$$

Assuming that the point q in a neighbourhood of p stays on the hypersurface M_1, we express all functions on M_1 in terms of the three variables x, y, u. Differentiation with respect to these three variables will be denoted by ∂, while differentiation with respect to all four variables, considered independent, will be denoted by ordinary d. We have

$$\frac{\partial f_1}{\partial x} = \frac{df_1}{dx} + \frac{df_1}{dv}\frac{d\Phi}{dx}$$

and eleven analogous equations obtained by replacing x by y and u, and f_1 by f_2, g_1 and g_2.

Now consider all the twenty equations obtained above and eliminate the sixteen ordinary d-derivatives $\frac{df_1}{dx}, \frac{df_1}{dy}, \dots$. This leaves us with a system of four linear partial differential equations for the twelve ∂-derivatives $\frac{\partial f_1}{\partial x}, \dots$, which we call system S.

On the other hand, if the image of q is to stay on M_2, we have also

$$\frac{\partial g_2}{\partial x} = \frac{d\Phi^*}{dx^*}\frac{\partial f_1}{\partial x} + \frac{d\Phi^*}{dy^*}\frac{\partial f_2}{\partial x} + \frac{d\Phi^*}{du^*}\frac{\partial g_1}{\partial x}\,,$$

and replacing x by y and u gives two other equations. Substituting these expressions into S, we arrive at a system of four differential equations for three unknown functions, f_1, f_2, g_1, and their partial derivatives with respect to x, y and u. Hence, in general, it will be impossible to find a solution.

The second counting argument considers a refined version of the local equivalence problem. For given $p_1 \in M_1$, $p_2 \in M_2$ and $n \in \mathbb{N}$, we ask if there is a local biholomorphic map taking p_1 to p_2, such that the image of M_1, denoted M_1', has n-th order of contact with M_2 at p_2. Without any loss of generality, assume $p_1 = p_2 = 0$.

Again, let M_1 be given by a defining equation of the form (2) and M_2 by (3). Consider the Taylor expansion of Φ^* up to order n. It involves

$$N = \frac{(n+1)(n+2)(n+3)}{6} - 1$$

arbitrary real coefficients. Next, consider a transformation of the form (1). Specifying f and g up to order n involves $2\big(\binom{n+2}{2} - 1\big)$ complex coefficients, which is

$$N' = 2(n+1)(n+2) - 4 = 2n^2 + 6n$$

real coefficients. Now we write the equation for the image of M_1 in a parametric form with parameters x, y, u,

$$x^* + iy^* = f\big(x, y, u, \Phi(x,y,u)\big)\,, \qquad u^* + iv^* = g\big(x, y, u, \Phi(x,y,u)\big)\,. \tag{4}$$

In order to check if M_1' and M_2 have contact of order n, we substitute those values into the defining equation $v^* = \Phi(x^*, y^*, u^*)$ for M_2. The order of contact is obtained if all the resulting coefficients up to order n are zero. Considering the coefficients of M_1 and M_2 as given, and those of f and g as unknown, we have N equations for N' unknowns. It remains to calculate that $N > N'$ if and only if

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 > 2n^2 + 6n$$

which gives

$$n^2 > 6n + 25\,.$$

The first integer which satisfies this relation is $n = 9$. Hence for $n \geq 9$ we cannot, in general, get contact of order n.

Poincaré also gives a heuristic definition of local invariants. They are defined relative to a chosen reference hypersurface, by considering infinitesimal perturbations of the hypersurface and of the identity transformation. In a loose sense, the definition contains the ideas of using model hypersurfaces and a certain linearization of the biholomorphism, which are central for the construction of Chern and Moser, and for later results on Levi degenerate hypersurfaces.

It is an interesting fact that although Poincaré gives this definition, he does not get to the point of actually calculating the lowest order invariant - the Levi form. It was done two years later by E. E. Levi.

3. The first invariant

In fact, the first CR invariant appears already at order two. In order to define it, we now consider a smooth hypersurface $M \subseteq \mathbb{C}^n$, $n \geq 2$, and a point $p \in M$. Let $r \in C^\infty$ be a local defining function, i.e., for a neighbourhood U of p

$$M \cap U = \big\{ z \in U \mid r(z) = 0 \big\}\,,$$

and $\nabla r \neq 0$ in $M \cap U$.

The Levi form on M at p is the Hermitian form defined by

$$L_p(\zeta) = \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z \partial \bar z}(p) \zeta_i \bar\zeta_j\,,$$

for $\zeta = (\zeta_1, \dots \zeta_n)$ in the complex tangent space to M at p,

$$T_p^{\mathbb{C}} M = \Big\{ \zeta \in \mathbb{C}^n : \sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(p) \zeta_i = 0 \Big\}\,.$$

It is easily checked the that the signature of the Levi form is a biholomorphic invariant of (an oriented) hypersurface.

In $\mathbb{C}^2$, the complex tangent space is one dimensional, so the Levi form is a scalar. Hence the first invariant of a hypersurface can be thought of as taking values in the three point set $\{-1, 0, 1\}$. In the sequel, we consider this two dimensional case.

Not surprisingly, the equivalence problem was first considered in the nondegenerate case, when the Levi form is nonzero. The first substantial progress was made by B. Segre in 1931. In [S] he defined a set of invariants, which he thought to form a complete set. A year later, E. Cartan showed that in fact the set was not complete, and provided himself a complete solution to the problem, as an application of his general method of moving frames. His intrinsic approach is different from the extrinsic one, using a reformulation of the problem in terms of differential forms. For a detailed exposition of Cartan's solution we refer the reader to the book of H. Jacobowitz ([J]).

The direct approach was again taken up in the first part of the celebrated paper of Chern and Moser [CM]. This part, originating in the work of the second author, solves the local Poincaré problem for Levi nondegenerate hypersurfaces, in arbitrary dimension, by a construction of normal coordinates.

4. Chern-Moser normal form

By his observations, Poincaré originated the extrinsic approach to the problem, which directly analyzes the action of the group of local biholomorphisms on the defining equation of the hypersurface.

In this section we consider the case when the Levi form is nondegenerate, i.e. the first invariant is nonzero, and sketch the solution of the Poincaré problem obtained in [CM].

We will use again local holomorphic coordinates (z, w), centered at p, such that the hyperplane $\{v = 0\}$ is tangent to M at p. The complex tangent at p is given by $\{w = 0\}$.

M is locally described as a graph of a function $v = \Phi(x, y, u)$. Assuming that M is real analytic, Φ is the sum of its Taylor expansion starting with 2-nd order terms, which we will express in terms of z, $\bar{z}$, u.

The first step in normalizing Φ treats the leading second order terms. We have

$$v = \operatorname{Re} \alpha z^2 + A|z|^2 + o\big(|z|^2, u\big)\,.$$

By a change of variable $w^* = w + \beta z^2$ we may eliminate the harmonic term, taking $\beta = i\alpha$. By definition, A is the value of the Levi form at p, corresponding to the defining function $r = \Phi - v$, so $A \neq 0$. By a suitable scaling in the z-variable and a change of sign in w, if necessary, we make $A = 1$. Then we can write (with stars omitted)

$$v = |z|^2 + F(z, \bar{z}, u)\,, \tag{5}$$

where F is real analytic, with Taylor expansion

$$F(z, \bar{z}, u) = \sum_{i+j+m \geq 2} a_{ijm} z^i \bar{z}^j u^m\,, \tag{6}$$

where $a_{ijm} = \overline{a_{jim}}$ and $a_{110} = a_{200} = 0$. In the next step we will consider only transformations which preserve this form and normalize the higher order part $F(z, \bar{z}, u)$.

The model hypersurface is defined using the leading term, as

$$S = \big\{(z, w) \in \mathbb{C}^2 \mid v = |z|^2\big\}\,.$$

S is an unbounded version of the unit sphere in $\mathbb{C}^2$.

It has a five dimensional group of local automorphisms, consisting of transformations of the form

$$z^* = \frac{\delta e^{i\theta}(z + aw)}{(1 - 2i\bar{a}z - (\mu + i|a|^2)w)}\,, \qquad w^* = \frac{\delta^2 w}{(1 - 2i\bar{a}z - (\mu + i|a|^2)w)}\,, \tag{7}$$

where $a \in \mathbb{C}$, $\delta \in \mathbb{R}^*$ and $\mu, \theta \in \mathbb{R}$. We will denote this group by $\mathcal{H}$.

One of the ideas in [CM] is to consider power series expansions along real analytic curves transversal to the complex tangent space at p, rather than the ordinary expansion. It reflects the inhomogeneity of the real tangent space and the special role played by the transverse coordinate u. Hence we will consider partial Taylor expansion of F in z, $\bar{z}$. Denoting

$$F_{ij}(u) = \sum_{m=0}^{\infty} a_{ijm} u^m\,,$$

we have

$$F(z, \bar{z}, u) = \sum_{i,j=0}^{\infty} F_{ij}(u) z^i \bar{z}^j\,.$$

We will subject the defining equation to a general biholomorphic transformation

$$z^* = z + f(z, w)\,, \qquad w^* = w + g(z, w)\,, \tag{8}$$

where f and g are represented by power series

$$f(z, w) = \sum_{i,j=0}^{\infty} f_{ij} z^i w^j\,, \qquad g(z, w) = \sum_{i,j=0}^{\infty} g_{ij} z^i w^j\,.$$

The only requirement on (8) is that it preserves form (5). Along with the partial Taylor expansion of F, we will consider the corresponding expansions of f and g. Denote

$$f_k(w) = \sum_{j=0}^{\infty} f_{kj} w^j\,, \qquad g_k(w) = \sum_{j=0}^{\infty} g_{kj} w^j\,,$$

so that

$$f(z, w) = \sum_{k=0}^{\infty} f_k(w) z^k\,, \qquad g(z, w) = \sum_{k=0}^{\infty} g_k(w) z^k\,.$$

Now we can formulate the normalizing conditions on F.

Theorem 4.1 ([CM]). *There exists a biholomorphic change of coordinates such that the defining equation in the new coordinates satisfies*

$$\begin{aligned} &F_{j,0} = 0\,, && j = 0, 1, \dots\,,\\ &F_{1,j} = 0\,, && j = 1, 2, 3, \dots\,,\\ &F_{2,2} = 0\,, &&\\ &F_{3,3} = 0\,, &&\\ &F_{3,2} = 0\,. && \end{aligned} \tag{9}$$

This transformation is determined uniquely, up to a natural action of the symmetry group $\mathcal{H}$.

In order to give a few (heuristic) remarks about the proof, consider the change of variables formula, obtained by substituting (8) into $|z^*|^2+F^* = v^*$, and restricting variables to M,

$$\begin{aligned}&\big|z+f\big(z,u+i(|z|^2+F)\big)\big|^2+F^*\big(z+f(z,u+i(|z|^2+F)),\overline{z+f(\dots)}\,,\\&\mathrm{Re}\ g(z,u+i(|z|^2+F)))\big)=|z|^2+F+\mathrm{Im}\ g\big(z,u+i(|z|^2+F)\big)\,,\end{aligned}$$

where the argument of F is $(z,\bar z,u)$. It is viewed as an equality of two power series in $z,\bar z,u$. By multiplying out we can in principle obtain relations between various coefficients of F^* and F,f,g. Separating the leading linear term leads to the Chern-Moser operator,

$$L(f,g)=\mathrm{Re}\left(2\bar z f(z,u+i|z|^2)+ig(z,u+i|z|^2)\right)\,.$$

The first two conditions in (9) are relatively easy to satisfy. Vanishing of the harmonic terms $F_{j,0}$ determines all parts of g, except for Re g_0. Note that while $F_{j0}=0$ for $j\geq 1$ is a complex condition, $F_{00}=0$ is a real condition, which determines only one part of g_0, namely Im g_0.

The coefficients F_{1j} of $z\bar z^j$ for $j\geq 2$ are essentially absorbed into the leading term $z\bar z$ by the substitution $z^*=z+f_j(w)z^j$. This determines f_j for all $j\geq 2$.

In fact, in a more geometric setting, one can prove that for any real analytic curve transversal to $T_p^{\mathbb{C}}M$ there is a biholomorphic transformation which attains the first two conditions and in the same time maps the given curve into the u-axis. This curve can be chosen in such a way that $F_{32}=0$, which determines f_0. There is exactly one such curve in any direction transverse to the complex tangent space at p. This non-uniqueness corresponds to the parameter a in (7).

The remaining three conditions, $F_{11}=F_{22}=F_{33}=0$ then determine f_1 and Re g_0. Geometrically, $F_{33}=0$ corresponds to a choice of a preferred parametrization of the curve. There is a projective one parameter family of such parametrizations, corresponding to the parameter μ in (7). Similarly, $f_1=0$ corresponds to a choice of a preferred section of $T_q^{\mathbb{C}}M$ along the u-axis, which is mapped into the unit section by the normalization mapping. There is a unique such section for every initial condition given by each vector in $T_p^{\mathbb{C}}M$. This non-uniqueness corresponds to the parameters δ and θ in (7).

5. Kohn's finite type

When the Levi form vanishes, i.e.

$$L_p=0\,,$$

one would like to find the next nontrivial invariant. The second invariant, type of the point, was defined in the pioneering work of J. J. Kohn ([K]).

It can be defined as the maximal order of contact between complex curves and M at p (originally, it was defined in terms of commutators of CR vector fields, see [K], [D]).

Note that M is Levi nondegenerate at p if and only if p is a point of finite type two. From now on we assume that p is a point of finite type k, where $k > 2$.

Since the structure of Levi degenerate points near a point of finite type is already quite diverse, it doesn't seem reasonable to expect any kind of uniform geometric theory for such hypersurfaces. On the other hand, the possibility of constructing a formal normal form theory is still very attractive, since for many applications convergence is not necessary.

The first attempt to construct normal forms for Levi degenerate hypersurfaces is due to P. Wong, who considered a class of hypersurfaces of finite type four, given by

$$v = |z|^4 + a|z|^2 \mathrm{Re}\, z^2 + |z|^2 u^2 + \dots ,$$

where dots denote terms of order higher than four. Here $0 \le a < \frac{4}{3}$, in particular p is an isolated weakly pseudoconvex point. His construction uses in an essential way both the leading fourth order term in $z, \bar z$ and the additional term $|z|^2 u^2$ which controls the Levi form along the u-axis. Further results on the equivalence problem and normal form constructions were obtained in [S], [E], [Ju], [BB], [BE].

As a first step in normalizing the defining equation $v = \Phi(z, \bar z, u)$, we consider again the low order harmonic terms. One can show easily that $p \in M$ is a point of finite type k if and only if there exist (uniquely determined) complex numbers $\alpha_2, \dots, \alpha_k$ such that after the change of variable

$$w^* = w + \sum_{i=2}^{k} \alpha_i z^i$$

the defining equation has form

$$v^* = P(z, \bar z) + o\big(|z|^k, u^*\big)\,, \tag{10}$$

where P is a nonzero real valued homogeneous polynomial of degree k

$$P(z, \bar z) = \sum_{j=1}^{k-1} a_j z^j \bar z^{k-j}\,. \tag{11}$$

Here $a_j \in \mathbb{C}$ and $a_j = \overline{a_{k-j}}$, since P is real valued. Dropping stars we rewrite (10) as

$$v = P(z, \bar z) + F(z, \bar z, u) \tag{12}$$

and define the homogeneous model to M at p,

$$M_H = \big\{(z, w) \in \mathbb{C}^2 \mid v = P(z, \bar z)\big\}\,.$$

P is uniquely defined up to a linear change of variables

$$w^* = \delta w\,, \qquad z^* = \beta z\,,$$

where $\delta \in \mathbb{R}^*$ and $\beta \in \mathbb{C}^*$. The subgroup of all such transformations which preserve M_H will be again denoted by $\mathcal{H}$. It is straightforward to determine this group explicitly. In most cases $\mathcal{H}$ is the full local automorphism group of M_H (see Section 7).

6. Local equivalence of finite type hypesurfaces

We will find a solution to the local equivalence problem for finite type hypersurfaces by a generalization of Chern-Moser's construction. Convergence of the normalizing map will not be proved. In fact it seems plausible that it need not converge. For the application we will need the essential result on convergence of formal equivalences between finite type hypersurfaces, due to Baouendi, Ebenfelt and Rothschild.

Theorem 6.1 ([BER]). *Let M_1, M_2 be two real analytic hypersurfaces in $\mathbb{C}^2$ and $p_1 \in M_1$, $p_2 \in M_2$ be points of finite type. Let ϕ be a formal equivalence between (M_1, p_1) and (M_2, p_2). Then ϕ is convergent.*

Proving this result involves intricate analysis of Segre varieties. We refer the reader to [BER] for a detailed description of this technique.

In order to prove that the coefficients of the power series in normal form provide a complete set of invariants, we have to show that two hypersurfaces which are assigned the same power series are indeed biholomorphically equivalent. The composition of the first normalization mapping with the inverse of the second one gives a formal equivalence of the two hypersurfaces. The result of [BER] implies convergence of this formal equivalence.

In view of the convergence result, it is enough to find a normalization on the level of formal power series. The defining function Φ and the transformation (8) will be interpreted in this sense, the action of (8) on Φ being given by the transformation rule (15) below.

Starting with the finite type hypersurface (12), we give a sketch of the normal form construction. The most important information carried by the model is its essential type, denoted by l. It can be defined as the lowest index in (11) for which $a_l \neq 0$, hence $1 \leq l \leq \frac{k}{2}$.

The equivalence problem now splits into three cases, depending on the form of the model. The most symmetric, circular case, corresponding to $2l = k$ and $P = a_l|z|^k$. The tubular case, when P is equivalent to $(\mathrm{Re}\, z)^k$, which corresponds to a tube domain. All other hypersurfaces can be treated together, as the generic case.

In order to formulate the normal form conditions in the generic case, we need a natural scalar product on the vector space of homogeneous polynomials of degree $k-1$ without a harmonic term. If $Q = \sum_{j=1}^{k-2} \alpha_j z^j \bar{z}^{k-1-j}$ and $S = \sum_{j=1}^{k-2} \beta_j z^j \bar{z}^{k-1-j}$, we denote

$$(Q, S) = \sum_{j=1}^{k-2} \alpha_j \bar{\beta}_j \, .$$

This notation is applied also to polynomials which contain a harmonic term, which is ignored. We extend this notation also to polynomials whose coefficients are functions of u. In particular, for $S = P_z = \sum_{j=1}^{k-2} ja_j z^{j-1}\bar{z}^{k-j}$ we denote

$$(F_{k-1}, P_z) = \sum_{j=1}^{k-2} F_{j,k-1-j}(j+1)\bar{a}_{j+1}\,.$$

In the generic case we get the following normal form conditions.

Theorem 6.2 ([Ko1]). *There exists a formal change of coordinates such that the new defining equation satisfies*

$$\begin{aligned} F_{j0} &= 0\,, \qquad j = 1, 2, \dots, \\ F_{k-l+j,l} &= 0\,, \qquad j = 1, 2, \dots, \\ F_{k-l,l} &= 0\,, \\ F_{2k-2l,2l} &= 0\,, \\ (F_{k-1}, P_z) &= 0\,. \end{aligned}$$

It is determined uniquely up to the action of the symmetry group $\mathcal{H}$.

Since now $\mathcal{H}$ contains only linear transformations, its action on normal forms is straightforward.

We give again a few (heuristic) remarks about the proof. The first two conditions are satisfied in a similar way as in the Levi nondegenerate case. The first condition determines g_k for $k \geq 1$. The difference here is that we don't impose the real condition $F_{00} = 0$ (all the information about M which is used and all the normal form conditions are complex). The second condition is again satisfied by absorbing the terms $F_{k-l-j,l}$ into the leading term $z^{k-l}\bar{z}^l$, which determines f_k for $k \geq 2$.

The third and fourth condition, $F_{k-l,l} = F_{2k-2l,2l} = 0$, determine f_1 and g_0. The scalar product condition determines f_0. In general, there seems to be no geometric interpretation of these conditions.

The construction in the circular case is similar to the nondegenerate case. The model is now

$$S_k = \{(z,w) \in \mathbb{C}^2 \mid v = |z|^k\}\,. \tag{13}$$

The local automorphism group of S_k is three dimensional, consisting of transformations of the form

$$f(z,w) = \frac{\delta e^{i\theta} z}{(1+\mu w)^{\frac{1}{l}}}\,, \qquad g((z,w) = \frac{\delta^k w}{1+\mu w}\,, \tag{14}$$

with $\delta > 0$, and $\theta, \mu \in \mathbb{R}$.

One obtains the following normal form conditions:

$$
\begin{aligned}
F_{j0} &= 0\,, & j &= 0,1,\dots,\\
F_{l,l+j} &= 0\,, & j &= 0,1,2,\dots,\\
F_{2l,2l} &= 0\,,\\
F_{3l,3l} &= 0\,,\\
F_{2l,2l-1} &= 0\,.
\end{aligned}
$$

and the same conclusion as in Theorem 6.2. For normal forms in the tubular case see [Ko1].

A fundamental tool for proving Theorem 6.2 is again the change of variables formula

$$\Phi^*(z+f,\bar z+\bar f,u+\mathrm{Re}\ g)=\Phi(z,\bar z,u)+\mathrm{Im}\ g\big(z,u+i\Phi(z,\bar z,u)\big)\,, \tag{15}$$

where f and Re g are also evaluated at $(z,u+i\Phi(z,\bar z,u))$.

It becomes manageable if we assign weights to the variables, namely weight one to $z,\bar z$ and weight k to u. In this formula we separate the leading linear term. Denoting weights by subscripts, for terms of weight $\mu>k$ we get

$$
\begin{aligned}
&\Phi^*_\mu(z,\bar z,u)+2\mathrm{Re}\ P_z(z,\bar z)f_{\mu-k+1}(z,u+iP(z,\bar z))\\
&\qquad=\Phi_\mu(z,\bar z,u)+\mathrm{Im}\ g_\mu(z,u+iP(z,\bar z))+\dots
\end{aligned} \tag{16}
$$

where dots denote terms depending on $f_{\nu-k+1},g_\nu,F_\nu,F^*_\nu$ for $\nu<\mu$, and $P_z=\frac{\partial P}{\partial z}$. From this we obtain the generalized Chern-Moser operator

$$L(f,g)=\mathrm{Re}\{ig(z,u+iP(z,\bar z))+2P_zf(z,u+iP(z,\bar z))\}\,.$$

Careful analysis of this operator is an essential part in the proof of Theorem 6.2.

7. Applications and open problems

The normal form construction gives immediately substantial information about local automorphism groups and finite jet determination. We will denote by $\mathrm{Aut}(M,p)$ the group of local automorphisms of M at p (i.e. local biholomorphic transformations preserving M and p). By Theorem 6.2, the dimension of $\mathrm{Aut}(M,p)$ is less then or equal to the dimension of $\mathrm{Aut}(M_H,p)$. In particular, in the generic case it is less or equal to one, and less or equal to three in the circular case.

This result was sharpened in [Ko2], by further analysis of the circular case and a refinement of the normal forms, which leads to a full classification of local symmetry groups for finite type hypersurfaces.

Proposition 7.1. *For a given hypersurface exactly one of the following possibilities occurs.*

1. $\mathrm{Aut}(M,p)$ *has real dimension three. This happens if and only if* M *is equivalent to* S_k.
2. $\mathrm{Aut}(M,p)$ *has real dimension one and is noncompact, isomorphic to* $\mathbb{R}^+\oplus\mathbb{Z}_m$. *This happens if and only if* M *is a model hypersurface with* $l<\frac{k}{2}$.

(3) $\mathrm{Aut}(M,p)$ *has real dimension one and is compact, isomorphic to* S^1. *This happens if and only if the defining equation of* M *in normal coordinates has form*

$$v = G(|z|^2, u).$$

(4) $\mathrm{Aut}(M,p)$ *is finite, isomorphic to* $\mathbb{Z}_m$. *This happens in all remaining cases.*

The last case includes the trivial symmetry group, when $m = 1$. It is also possible to determine the integer m in the second and fourth cases in terms of the defining function in normal coordinates.

It can be seen from (14) that local automorphisms of S_k are determined by their 2-jets. For all other hypersurfaces, 1-jets are sufficient ([Ko2]).

Proposition 7.2. *Let* M *be a hypersurface which is not equivalent to* S_k. *Then local automorphisms are determined by their 1-jets.*

This result proves a conjecture formulated recently by Dmitri Zaitsev in the finite type case. His conjecture states that 1-jets suffice for all hypersurfaces in dimension two, except for those which are biholomorphic to the sphere at a generic point (which is the case of S_k).

The local equivalence problem still remains open for points of infinite type. If p is of infinite type, the order of contact of M with complex curves at p is unbounded. It follows from real analyticity that M actually has to contain a complex curve. In the terminology of CR geometry, M is not CR minimal, since it contains a proper submanifold of the same CR dimension as M, namely the complex curve. In terms of local coordinates, p is of infinite type if in suitable coordinates the hypersurface is given by

$$v = u^s P(z, \bar{z}) + o(|z|^k, u^s)\,.$$

Here P is again a polynomial of degree k of the form (10). The numbers s and k are invariants of M.

It is not known if formal equivalences of such hypersurfaces are necessarily convergent. One additional difficulty is the fact that the 2-jet determination property for local automorphisms does not hold. For every integer k there is an infinite type hypersurface whose local automorphisms are not determined by their k-jets (see [Kow], [Z]). In accord with Zaitsev's conjecture, all known examples are obtained by blowing up the sphere.

On the other hand, there are some promising positive recent results for certain classes of infinite type hypersurfaces (see e.g. [ELZ]).

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Department of Mathematical Analysis, Masaryk University
Janáčkovo nám. 2a, 602 00 Brno
E-mail: mkolar@math.muni.cz