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COINCIDENCE FREE PAIRS OF MAPS

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Abstract. This paper centers around two basic problems of topological coincidence theory. First, try to measure (with the help of Nielsen and minimum numbers) how far a given pair of maps is from being loose, i.e. from being homotopic to a pair of coincidence free maps. Secondly, describe the set of loose pairs of homotopy classes. We give a brief (and necessarily very incomplete) survey of some old and new advances concerning the first problem. Then we attack the second problem mainly in the setting of homotopy groups. This leads also to a very natural filtration of all homotopy sets. Explicit calculations are carried out for maps into spheres and projective spaces.

1. Introduction

Let M be a closed smooth m-dimensional manifold.

In the first half of the 1920's S. Lefschetz established his celebrated results on fixed point theory which, in particular, yield the following.

Theorem 1.1. *Let $f : M \to M$ be a map. If $L(f) \neq 0$, then every map f' homotopic to f has at least one fixed point $x \in M$ (i.e. $f'(x) = x$).*

Here the Lefschetz number $L(f)$ can be defined to be the intersection number of the graph of f with the diagonal Δ in $M \times M$.

The theorem of Lefschetz was a groundbreaking achievement. Still, it left several questions open.

Question I. What is the minimum number of fixed points of maps which are homotopic to f?

This is the principal problem of topological fixed point theory (cf. [1], p. 9).

Question II. What can we say about the set of homotopy classes $[f]$ which contain a fixed point free selfmap f of M (apart from the necessary condition $L(f) = 0$)?

In 1927 the Danish mathematician J. Nielsen achieved decisive progress concerning the first of these questions by decomposing the fixed point set of f as

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follows. Two fixed points are called (*Nielsen*) *equivalent* if they can be joined by a path σ in M which is homotopic to $f \circ \sigma$ by a homotopy which keeps the endpoints fixed. Each of the resulting Nielsen equivalence classes of fixed points contributes (trivially or nontrivially) to the Lefschetz number $L(f)$. Define $N(f)$ to be the number of *essential* Nielsen classes (i.e. of those classes whose contribution to $L(f)$ is not zero). This *Nielsen number* $N(f)$ is a lower bound for the **m**inimum number of **f**ixed points

$$MF(f) := \min_{f' \sim f} \big(\#\{x \in M \mid f'(x) = x\} \big) \tag{1.2}$$

and agrees with it except when $m = 2$ and the Euler number $\chi(M)$ of M is strictly negative; however, in this exceptional case $MF(f) - N(f)$ can be arbitrarily large for suitable f (compare [18], [20], [7], [8], [9], and [10]; an excellent survey is given by R. Brown [1]).

Fixed point questions allow a very natural and interesting generalization. Given a pair of maps $f_1, f_2 : M \to N$ between smooth connected manifolds (where M is closed), let

$$C(f_1, f_2) := \big\{ x \in M \mid f_1(x) = f_2(x) \big\} \subset M \tag{1.3}$$

denote its coincidence set. Here the relevant **m**inimum number of **c**oincidence points is

$$MC(f_1, f_2) := \min \big\{ \#C(f_1', f_2') \mid f_1' \sim f_1, f_2' \sim f_2 \big\}. \tag{1.4}$$

According to a result of R. B. S. Brooks [3] the same minimum number is achieved if we vary only one of the maps f_1 or f_2 by a homotopy; in particular, $MF(f) = MC(f, \text{identity map})$ (compare 1.2).

Unlike fixed point theory, coincidence theory allows the domain M and the target N to be different manifolds of arbitrary positive dimensions m and n, resp. If $m > n$, then $MC(f_1, f_2)$ will be often infinite. Thus it makes sense to consider also the **m**inimum number of **c**oincidence **c**omponents

$$MCC(f_1, f_2) := \min \big\{ \#\pi_0\big(C(f_1', f_2')\big) \mid f_1' \sim f_1; f_2' \sim f_2 \big\} \tag{1.5}$$

which is always finite.

Remark 1.6. (i) As one can see rather easily the values of the minimum numbers MC and MCC remain unchanged if we replace the base point free maps and homotopies in 1.4 and 1.5 by base point preserving ones (requiring e.g. that $f_1(*) = *_1 \neq *_2 = f_2(*)$ where $* \in M$ and $*_1, *_2 \in N$ are given base points; cf. e.g. the appendix of [16]).

(ii) Clearly $MC(f_1, f_2) = MCC(f_1, f_2) = 0$ whenever $m < n$ (use an approximation of $(f_1, f_2) : M \to N \times N$ which is transverse to the diagonal).

In a series of recent papers (see, in particular, [16] and [17]) we studied the minimum numbers MC and MCC in arbitrary codimensions $m - n$. For this purpose we introduced a "strong"Nielsen number $N^{\#}(f_1, f_2)$ which generalizes Nielsen's original definition. Our approach is based on a careful analysis of the case when the pair (f_1, f_2) is generic. Here the coincidence locus $C(f_1, f_2)$ is a smooth closed $(m-n)$–dimensional submanifold of M, equipped with

(i) a canonical description of its (nonstabilized) normal bundle; and

(ii) the map $\widetilde{g}$ from $C(f_1, f_2)$ to the path space

$$E(f_1, f_2) := \{(x, \theta) \in M \times P(N) \mid \theta(0) = f_1(x), \theta(1) = f_2(x)\} \tag{1.7}$$

defined by

$$\widetilde{g}(x) = (x, \text{ constant path at } f_1(x) = f_2(x)),$$

$x \in C(f_1, f_2)$, (where $P(N)$ denotes the space of all continuous paths $\theta : I \to N$).

The space $E(f_1, f_2)$ has a very rich topology. Already its set $\pi_0(E(f_1, f_2))$ of pathcomponents can be huge (it is bijectively related to a wellstudied "Reidemeister set"; cf. [13], 2.1). The corresponding decomposition of $C(f_1, f_2)$ into the inverse images (under $\widetilde{g}$) of these pathcomponents generalizes the Nielsen decomposition of fixed point sets. But in higher codimensions $m - n > 0$ the map $\widetilde{g}$ into $E(f_1, f_2)$ can capture much further and deeper geometric information (which – surprising often – is related to various strong versions of Hopf invariants; see e.g. [13], 1.14, or [16], 7.2 and (64)).

Details of the definition of $N^{\#}(f_1, f_2)$ can be found in [16] where we proved also the following result.

Theorem 1.8. *Let $f_1, f_2 : M^m \to N^n$ be (continuous) maps between smooth connected manifolds of the indicated dimensions, M being closed. Then:*

(i) *The Nielsen number $N^{\#}(f_1, f_2) = N^{\#}(f_2, f_1)$ is finite and depends only on the homotopy classes of f_1 and f_2.*

(ii) $0 \leq N^{\#}(f_1, f_2) \leq MCC(f_1, f_2) \leq MC(f_1, f_2) \leq \infty$*; if $n \neq 2$ then also $MCC(f_1, f_2) \leq \#\pi_0(E(f_1, f_2))$; if $(m, n) \neq (2, 2)$ then $MC(f_1, f_2) \leq \#\pi_0(E(f_1, f_2))$ or $MC(f_1, f_2) = \infty$.*

(iii) *If $M = N$ and $f_2 =$ identity map, then $N^{\#}(f_1, f_2)$ coincides with the Nielsen number of f_1 as defined in classical fixed point theory.*

This allows us to compute our minimum numbers explicitly in various concrete cases.

Example 1.9. Spherical maps into spheres (compare e.g. [16], 1.12). Consider maps $f_1, f_2 : S^m \to S^n$ where $m, n \geq 1$, and let A denote the antipodal involution. Then

$$N^{\#}(f_1, f_2) = MCC(f_1, f_2) = \begin{cases} 0 & \text{if } \ f_1 \sim A \circ f_2\,; \\ \#\pi_0(E(f_1, f_2)) & \text{else}\,. \end{cases}$$

If $f_1 \not\sim A \circ f_2$ then $\#\pi_0(E(f_1, f_2))$ equals 1 (and $|d^0(f_1) - d^0(f_2)|$, resp.) according as $n \neq 1$ (or $m = n = 1$, resp.; here $d^0(f_i)$ denotes the usual degree).

Clearly $MC(f_1, f_2) \leq 1$ whenever $[f_1] - [A \circ f_2]$ lies in $E(\pi_{m-1}(S^{n-1}))$, the image of the Freudenthal suspension. On the other hand, it is wellknown that $MC(f_1, f_2)$ is infinite if $[f_1] - [A \circ f_2] \notin E(\pi_{m-1}(S^{n-1}))$ and $(m, n) \neq (1, 1)$.

Example 1.10. Spherical maps into projective spaces (cf. [17], 1.17). Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ denote the field of real, complex or quaternionic numbers, and let $d = 1, 2$ or 4 be its real dimension. Let $\mathbb{K}P(n')$ and $V_{n'+1,2}(\mathbb{K})$, resp., denote the corresponding space of lines and orthonormal 2-frames, resp., in $\mathbb{K}^{n'+1}$. The real dimension of $N = \mathbb{K}P(n')$ is $n := d \cdot n'$. Consider the diagram

$$\begin{array}{ccccccc} \cdots \to \pi_m\big(V_{n'+1,2}(\mathbb{K})\big) & \xrightarrow{p_{\mathbb{K}*}} & \pi_m(S^{n+d-1}) & \xrightarrow{\partial_{\mathbb{K}}} & \pi_{m-1}(S^{n-1}) \to \cdots \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle E} \\ & & \pi_m(\mathbb{K}P(n')) & & \pi_m(S^n) \end{array} \tag{1.11}$$

determined by the canonical fibrations p and $p_{\mathbb{K}}$; E denotes the Freudenthal suspension homomorphism.

We want to determine the minimum numbers of all pairs of maps $f_1, f_2 : S^m \to \mathbb{K}P(n')$, $m, n' \geq 1$. In view of Example 1.9 and Remark 1.6 (ii) we need not consider the cases where $n' = 1$ (or $m = 1$).

Lemma 1.12. *Assume $m, n' \geq 2$. Then p_* (cf. 1.11) is injective and*

$$\pi_m\big(\mathbb{K}P(n')\big) = p_*\big(\pi_m(S^{n+d-1})\big) \oplus \pi^c_m\big(\mathbb{K}P(n')\big)$$

where $\pi^c_m\big(\mathbb{K}P(n')\big) := \mathrm{incl}_*\big(\pi_m(\mathbb{K}P(n') - \{*\})\big)$ and incl *denotes the inclusion of $\mathbb{K}P(n')$, punctured at some point $*$. Hence, given $[f_i] \in \pi_m\big(\mathbb{K}P(n')\big)$, there is a unique homotopy class $[\widetilde{f}_i] \in \pi_m(S^{n+d-1})$ such that $p_*([\widetilde{f}_i]) - [f_i] \in \pi^c_m\big(\mathbb{K}P(n')\big)$, $i = 1, 2$. (Since $\pi^c_m\big(\mathbb{K}P(n')\big) \cong \pi_{m-1}(S^{d-1})$, we may assume that $\widetilde{f}_i$ is a genuine lifting of f_i when $\mathbb{K} = \mathbb{R}$ or when $m > 2$ and $\mathbb{K} = \mathbb{C}$).*

We see this by comparing the exact homotopy sequences of the fibrations

$$p : S^{n+d-1} \longrightarrow \mathbb{K}P(n') \tag{1.13}$$

and $p| : S^{n-1} \to \mathbb{K}P(n'-1)\big(\underset{\subset}{\sim} \mathbb{K}P(n') - \{*\}\big)$.

Theorem 1.14. *Assume $m, n' \geq 2$. Each pair of homotopy classes $[f_1], [f_2] \in \pi_m\big(\mathbb{K}P(n')\big)$ satisfies precisely one of the seven conditions which are listed in table 1.15, together with the corresponding Nielsen and minimum numbers. (Here we use the language of Lemma 1.12 and define also $[f'_i] := [p \circ \widetilde{f}_i] \in \pi_m\big(\mathbb{K}P(n')\big)$, $i = 1, 2$; moreover A denotes the antipodal map on S^{n+d-1}).*

Condition	$N^{\#}(f_1, f_2)$	$MCC(f_1, f_2)$	$MC(f_1, f_2)$
1) $f'_1 \sim f'_2$, $[\widetilde{f}_2] \in \ker \partial_{\mathbb{K}}$	0	0	0
2) $f'_1 \sim f'_2$, $[\widetilde{f}_2] \in \ker E \circ \partial_{\mathbb{K}} - \ker \partial_{\mathbb{K}}$	0	1	1
3) $\mathbb{K} = \mathbb{R}$, $f'_1 \sim f'_2$, $\widetilde{f}_2 \not\sim A \circ \widetilde{f}_2$	1	1	1
4) $\mathbb{K} = \mathbb{R}$, $f'_1 \not\sim f'_2$, $[\widetilde{f}_1] - [\widetilde{f}_2] \in E(\pi_{m-1}(S^{n-1}))$	2	2	2
5) $\mathbb{K} = \mathbb{R}$, $[\widetilde{f}_1] - [\widetilde{f}_2] \notin E(\pi_{m-1}(S^{n-1}))$	2	2	∞
6) $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, $[\widetilde{f}_1] = [\widetilde{f}_2] \notin \ker E \circ \partial_{\mathbb{K}}$	1	1	1
7) $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, $[\widetilde{f}_1] \neq [\widetilde{f}_2]$	1	1	∞

Table 1.15. Nielsen and minimum coincidence numbers of all pairs of maps $f_1, f_2 : S^m \to \mathbb{K}P(n')$, $m, n' \geq 2$: replace each (possibly base point free) homotopy class $[f_i]$ by a base point preserving representative and read off the values of $N^\#$ and $M(C)C$. (Here $f_1' \sim f_2'$ means that f_1', f_2' are homotopic in the base point free sense. For proofs see [17]).

This concludes our brief (and necessarily rather incomplete) survey of some of the developments triggered by our initial Question I. □

In this paper we start investigating a natural generalization of Question II.

Definition 1.16. Let M and N be smooth connected manifolds, M being closed. A pair of maps $f_1, f_2 : M \to N$ is called *loose* if it is homotopic to a coincidence free pair; in other words, if $MC(f_1, f_2) = 0$ or, equivalently $MCC(f_1, f_2) = 0$.

It makes no difference whether we use base point free or base point preserving homotopies in this definition (provided $f_1(*) \neq f_2(*)$ when $*$ is a given base point of M; cf. 1.6).

Question II′. What can we say about the set of homotopy classes of loose pairs?

We will concentrate on the case $M = S^m,\ m \geq 1$. Let $* \in S^m$ and $*_1 \neq *_2 \in N$ be given base points.

Consider the subgroups

$$\pi_m^c(N, *_i) \subset \pi_m^{(2)}(N, *_i) \subset \pi_m(N, *_i)\,, \qquad i = 1, 2\,,$$

where

(1.17)

$$\pi_m^c(N, *_i) := \{[f] \in \pi_m(N, *_i) \mid (f, *_{i\pm1}) \text{ is loose}\} = \mathrm{incl}_*(\pi_m(N - \{*_{i\pm1}\}, *_i))$$

and

$$\pi_m^{(2)}(N, *_i) := \{[f] \in \pi_m(N, *_i) \mid \exists[\overline{f}] \in \pi_m(N, *_{i\pm1})\text{s. t. } (f, \overline{f}) \text{ is loose}\}.$$

Here $*_i$ denotes also the constant map with the indicated value, and incl stands for the obvious inclusion. (Compare also Remark 3.7 below).

Theorem 1.18. *For $m \geq 1$ there is a welldefined group isomorphism*

$$c : \pi_m^{(2)}(N, *_1)/\pi_m^c(N, *_1) \longrightarrow \pi^{(2)}(N, *_2)/\pi_m^c(N, *_2)$$

which takes the coset $[[f]]$ of $[f] \in \pi_m^{(2)}(N, *_1)$ to the coset of any element $[\overline{f}] \in \pi_m^{(2)}(N, *_2)$ such that $(f, \overline{f})$ is loose.

A pair $([f_1], [f_2]) \in \pi_m(N, *_1) \times \pi_m(N, *_2)$ is loose if and only if $[f_i] \in \pi_m^{(2)}(N, *_i)$, $i = 1, 2$, and $c([[f_1]]) = [[f_2]]$.

In particular, if $\pi_m^c(N, *_i) = \pi_m(N, *_i)$ then all pairs of maps $f_1, f_2 : S^m \to N$ (base point preserving or not) are loose; this is the case e.g. when N is not compact or when $m < n$.

Special case 1.19. *If N allows a fixed point free selfmap $A : N \to N$ such that $A(*_1) = *_2$ then $\pi_m^{(2)}(N, *_i) = \pi_m(N, *_i)$ for all $m \geq 1$, and c is induced by A (i.e. $c([[f]]) = [[A \circ f]]$).*

Thus (the nontriviality of) $\pi_m(N,*_1)/\pi_m^{(2)}(N,*_1)$ is an obstruction to the existence of such a fixed point free selfmap. On the other hand, such selfmaps occur e.g. on the total space of every nontrivial covering map.

Example 1.20. ($\mathbf{N = S^n}$) : A pair of maps $f_1, f_2 : S^m \to S^n$ (base point preserving or not) is loose if and only if $f_1 \sim A \circ f_2$ where A denotes the antipodal map. Indeed, $\pi_m^c(S^n) = \{0\}$ and

$$c \;=\; A_* \;:\; \pi_m(S^n,*_1) \;\xrightarrow{\;\cong\;}\; \pi_m\big(S^n, A(*_1)\big)$$

(compare also [4], 2.10).

Corollary 1.21. *If N allows a nowhere vanishing vector field (e.g. if N is odd); then*

$$\pi_m^{(2)}(N,*_i) = \pi_m(N,*_i) \quad \textit{for all} \quad m \geq 1\,,\; i = 1,2\,,$$

and c is induced by a map A (as in 1.19*) which is homotopic to the identity.*

Indeed, the flow of the vector field yields the required fixed point free map A.

Theorem 1.18 and Corollary 1.21 suggest that Question II$'$ may be most interesting when N is a closed even-dimensional manifold.

Example 1.22. (Projective spaces). Consider the case $N = \mathbb{K}P(n')$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, $m, n' \geq 2$, as in 1.10 (and use the language of Lemma 1.12). Then a pair of maps $f_1, f_2 : S^m \to \mathbb{K}P(n')$ is loose precisely if the corresponding pair $(p \circ \widetilde{f}_1, p \circ \widetilde{f}_2)$ is loose or, equivalently, if the maps $p \circ \widetilde{f}_1, p \circ \widetilde{f}_2$ are homotopic and $\widetilde{f}_i : S^m \to S^{n+d-1}$ can be lifted to the Stiefel manifold $V_{n'+1,2}(\mathbb{K})$, $i = 1$ or 2 (compare 1.11). Thus here the isomorphism c (cf. 1.18) is induced by a selfmap A of $\mathbb{K}P(n')$ which is homotopic to the identity map but which can be fixed point free only when $\mathbb{K} = \mathbb{R}$ and n is odd, i.e. when the Lefschetz number $L(A) = \chi\big(\mathbb{K}P(n')\big)$ vanishes.

Problem 1.23. Is the group isomorphism c in Theorem 1.18 always induced by a selfmap of N?

It seems to be very desirable to determine $\pi_*^c(N)$, $\pi_*^{(2)}(N)$, c and hence the sets of loose pairs of homotopy classes (cf. Theorem 1.18) for many more concretely given closed sample manifolds, e.g. for Stiefel manifolds and Grassmannians. Here is a partial result in this direction.

Example 1.24. For every *even* integer $r \geq 4$, all pairs of maps $f_1, f_2 : S^m \to G_{r,2}(\mathbb{R})$ into the Grassmann manifold of 2-planes in $\mathbb{R}^r$ are loose.

Details of the proof of Theorem 1.18 and of its consequences will be given in Section 2.

Throughout our discussion a central role is played by the set of homotopy classes of those maps which occur in loose pairs (i.e. which are *not coincidence producing* in the terminology of Brown and Schirmer, cf. [2]). For arbitrary topological spaces X and Y this set turns out to be the first interesting term of a very natural descending filtration of the full homotopy set $[X, Y]$. In section 3 we study this filtration and determine it e.g. for the homotopy groups of spheres and projective spaces.

2. Loose pairs

Throughout this paper manifolds are required to be Hausdorff spaces having a countable basis and no boundary.

Proof of Theorem 1.18. If the pairs $(f, \overline{f})$, $(f, \overline{\overline{f}})$ and $(\hat{f}, *_2)$ are loose, then so are $(*_1 \sim f \cdot f^{-1}, \overline{f} \cdot \overline{\overline{f}}^{-1})$ and $(f \cdot \hat{f}, \overline{f} \cdot *_2 \sim \overline{f})$. Thus the coset $[[\overline{f}]] = [[\overline{\overline{f}}]]$ is determined by $[f] \in \pi_m^{(2)}(N, *_1)$; it does not depend on the choice of the class $[\overline{f}]$ (which makes $(f, \overline{f})$ loose) and not even on the choice of $[f]$ within its coset.

If the pairs $(f, \overline{f})$ and $(f', \overline{f}')$ are loose, then so is also $(f \cdot f', \overline{f} \cdot \overline{f}')$. Hence the bijection c which interchanges the roles of f_1 and f_2 in a loose pair (f_1, f_2) is compatible with the group structure.

If N is not compact or if $m < n$, then every map $f : S^m \to N$ can be deformed into the complement of a given point in N via a suitable isotopy or via transverse approximation. □

Next we turn to the situation of Example 1.22. Given arbitrary (not necessarily base point preserving) maps $f_i : S^m \to \mathbb{K}P(n')$, put $*_i := f_i(*)$ and choose $\widetilde{f}_i$ as in Lemma 1.12; then the summand $[p \circ \widetilde{f}_i] - [f_i]$ plays no role in looseness questions, $i = 1, 2$. If the pair $(p \circ \widetilde{f}_1, p \circ \widetilde{f}_2)$ is coincidence free, then the unit vectors $\widetilde{f}_1(x), \widetilde{f}_2(x)$ in $\mathbb{K}^{n'+1}$ are linearly independent for all $x \in S^m$; suitable rotations yield both a homotopy $\widetilde{f}_1 \sim \widetilde{f}_2$ and liftings to the Stiefel manifold of *orthonormal* 2-frames in $\mathbb{K}^{n'+1}$.

This argument shows also that the isomorphism

$$\pi_m(\mathbb{K}P(n')) \ / \ \pi_m^c(\mathbb{K}P(n')) \cong \pi_m(S^{n+d-1}) \tag{2.1}$$

induced by p (c.f. 1.12 and 1.13) makes $\pi_m^{(2)}(\mathbb{K}P(n'))/\pi_m^c\big(\mathbb{K}P(n')\big)$ correspond to $\operatorname{im}(p_{\mathbb{K}*}) = \ker \partial_{\mathbb{K}}$ in diagram 1.11. This yields an alternative proof of the calculations in Table 1.15 as far as Condition 1) is concerned. Furthermore there are easy examples (e.g. when $\mathbb{K} = \mathbb{R}$ and $m = n' \equiv 0(2)$, cf. 3.13) where $\partial_{\mathbb{K}}$ and hence $\pi_m\big(\mathbb{K}P(n')\big)/\pi_m^{(2)}\big(\mathbb{K}P(n')\big)$ is nontrivial so that every selfmap of $\mathbb{K}P(n')$ must have a fixed point (compare 1.19). Of course such questions can be settled more systematically by the Lefschetz fixed point theorem.

Finally we prove the statement in Example 1.24. In view of 1.6 (ii) we may assume that $m \geq 4$. According to Theorem 1.18 our claim is established once we see that

$$\operatorname{incl}_* : \pi_m(G_{r,2}(\mathbb{R}) - \{\text{point}\}) \longrightarrow \pi_m\big(G_{r,2}(\mathbb{R})\big)$$

is surjective. But this follows from

Lemma 2.2. *For all $m \geq 3$ and for all even integers $r = 2r' \geq 4$ we have the isomorphism*

$$e_* + u_* : \pi_m\big(\mathbb{R}P(r-2)\big) \oplus \pi_m\big(\mathbb{C}P(r'-1)\big) \ \xrightarrow{\ \cong\ } \ \pi_m\big(G_{r,2}(\mathbb{R})\big)$$

where $e(\lambda) = \lambda \oplus \mathbb{R}(0, \ldots, 0, 1)$, $\lambda \in \mathbb{R}P(r-2)$, *and* u *assigns the underlying real plane to any complex line.* (*Note that both embeddings* e *and* u *have codimensions* $r - 2 > 0$).

Proof. Scalar multiplication with the complex number i on $\mathbb{C}^{r'} \cong \mathbb{R}^r$ determines a section s of the fibration $S^{r-2} \subset V_{r,2}(\mathbb{R}) \to S^{r-1}$.

Thus the exact homotopy sequence splits and yields the isomorphism

$$\widetilde{e}_* + s_* : \pi_m(S^{r-2}) \oplus \pi_m(S^{r-1}) \longrightarrow \pi_m\bigl(V_{r,2}(\mathbb{R})\bigr) .$$

This implies our claim since the fiber $O(2)$ of the projection $V_{r,2}(\mathbb{R}) \to G_{r,2}(\mathbb{R})$ is aspherical. □

Problem 2.3. What is known about the groups π_*^c and $\pi_*^{(2)}$ of arbitrary Grassmannians $G_{r,k}(\mathbb{K})$, $r - 1 > k > 1$?

Note that in the special case $r = 2k$ there is the fixed point free involution $\perp$ on $G_{2k,k}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (take orthogonal complements). Thus $\pi_*^{(2)}\bigl(G_{2k,k}(\mathbb{K})\bigr)$ is the full homotopy group.

3. A filtration of homotopy sets

In this section we extend the definition of the group $\pi_*^{(2)}(N)$ (formed by those maps which occur in loose pairs) and obtain a very natural infinite descending filtration of arbitrary homotopy sets $[M, N]$.

Given any topological space N, consider the commuting diagram

(3.1)
$$\begin{tikzcd} N = \widetilde{C}_1(N) \arrow[dr, "\mathrm{id}=p_1"'] & \widetilde{C}_2(N) \arrow[l] \arrow[d, "p_2"] & \cdots \arrow[l] & \widetilde{C}_q(N) \arrow[l] \arrow[dll, "p_q"] & \cdots \arrow[l] \\ & N & & & \end{tikzcd}$$

of configuration spaces

$$\widetilde{C}_q(N) = \bigl\{(y_1, \ldots, y_q) \in N^q \mid y_i \neq y_j \text{ for } 1 \leq i \neq j \leq q\bigr\}$$

$q \geq 1$, and of projections which drop the last component(s) of an (ordered) configuration $(y_1, \ldots, y_q)$. For any topological space M this leads to the filtration

(3.2)
$$[M, N] = [M, N]^{(1)} \supset [M, N]^{(2)} \supset \cdots \supset [M, N]^{(q)} := p_{q*}([M, \widetilde{C}_q(N)]) \supset \ldots$$

Thus $[M, N]^{(q)}$ consists of the homotopy classes of those maps which fit into a q-tuple $f_1, \ldots, f_q : M \to N$ of maps without any (pairwise) coincidences.

Next, given a base point $* \in M$ and an infinite sequence $(*_1, *_2, \ldots)$ of pairwise distinct points in N, equip $\widetilde{C}_q(N)$ with the base point $(*_1, *_2, \ldots, *_q)$ and consider also the base point preserving version of the filtration 3.2.

Example 3.3 ($N = S^n$, $n \geq 1$)**.** For every point $y \in S^n$ use the stereographic projection σ_y from $S^n - \{y\}$ to the orthogonal complement of the line $\mathbb{R}y$ in $\mathbb{R}^{n+1}$ (i.e. to the tangent space $T_y(S^n)$) and obtain the following fiber preserving homeomorphisms and homotopy equivalences over S^n:

$$\widetilde{C}_2(S^n) = S^n \times S^n - \Delta \cong TS^n$$

and

$$\widetilde{C}_3(S^n) \sim TS^n - \text{ zero section } \stackrel{\supset}{\sim} V_{n+1,2}\,.$$

Here e.g. the vectors 0 and $v \neq 0$ in $T_y(S^n)$ correspond to the configurations $(y, -y) \in \widetilde{C}_2(S^n)$ and $(y, -y, \sigma_y^{-1}(v)) \in \widetilde{C}_3(S^n)$.

When $q \geq 3$ the projection $\widetilde{C}_q(S^n) \to \widetilde{C}_3(S^n)$ has a section which corresponds to the map $v \to (v, 2v, \ldots, (q-2)v)$.

Thus both in the base point free and in the base point preserving setting we have for every topological space M and $q \geq 3$

$$[M, S^n] = [M, S^n]^{(2)} \supset [M, S^n]^{(3)} = p_{\mathbb{R}*}([M, V_{n+1,2}(\mathbb{R})]) = [M, S^n]^{(q)}\,.$$

As in 1.11 $p_{\mathbb{R}} : V_{n+1,2}(\mathbb{R}) \to S^n$ denotes the standard projection from the Stiefel manifold $ST(S^n)$ of unit tangent vectors. If n is odd it has a section and $[M, S^n]^{(q)} = [M, S^n]$ for *all* $q \geq 1$. However, if n is even and e.g. $M = S^n$, then $[M, S^n]^{(2)} \neq [M, S^n]^{(3)}$.

Proposition 3.4. *Both in the base point free and base point preserving setting we have*

$$[M, N]^{(q)} = [M, N] \quad \textit{for all} \quad q \geq 1$$

if at least one of the following condition holds:
(i) *M is compact, but N is not – in the base point preserving setting we assume also that N is a connected topological manifold; or*
(ii) *N is a smooth manifold which allows a nowhere vanishing vector field; or*
(iii) *M and N are smooth manifolds such that* $\dim M < \dim N$.

Proof. (i) For every map $f : M \to N$ the complement $N - f(M)$ contains infinitely many points. Thus for $q \geq 2$ there exist (e.g. constant) maps $f_2, \ldots, f_q$ which, together with $f_1 = f$, define the required map into $\widetilde{C}_q(N)$. If N is a connected topological manifold we may assume that $f_i(*) = *_i$, $i = 1, \ldots, q$, e.g. after suitable isotopies.
(ii) We use the resulting flow φ and a suitable function $\varepsilon : N \longrightarrow (0, \infty)$ to define the pairwise coincidence-free selfmaps $\mathrm{id} = A_1, A_2, \ldots, A_q$ of N by

$$A_i(x) = \varphi(x, (i-1) \cdot \varepsilon(x)/q)\,, \quad x \in N\,.$$

Then $(f, A_2 \circ f, \ldots, A_q \circ f)$ is a lifting of f to $\widetilde{C}_q(N)$; suitable modifications (e.g. by finger moves) allow us to make it base point preserving.
(iii) After a transverse approximation f maps M into $N - \{*_2, \ldots, *_q\}$. □

We will be mainly interested in the case $M = S^m$, $m \geq 1$ (studied in the base point preserving setting). We obtain the nested sequence of subgroups

$$(3.5) \qquad \pi_m(N,*_1) = \pi_m^{(1)}(N,*_1) \supset \pi_m^{(2)}(N,*_1) \supset \cdots \supset \pi_m^{(q)}(N,*_1) \supset \dots$$

defined by

$$(3.6) \qquad \pi_m^{(q)}(N,*_1) := p_{q*}\big(\pi_m(\widetilde{C}_q(N),(*_1,\dots,*_q))\big)\,, \quad q \geq 1\,.$$

For $q = 2$ this agrees with the definition in 1.17 since p_2 is the first projection on $\widetilde{C}_2(N) = N \times N - \Delta$.

Remark 3.7. Assume that N is a topological manifold of dimension $n \geq 1$.

Then all the arrows in diagram 3.1 are projections of locally trivial fibrations (compare [5]). It follows from the homotopy lifting property that, given a loose pair (f_1, f_2), only one of the maps f_i, say f_2, has to be deformed to f_2' so that (f_1, f_2') is coincidence free (compare [3]). In particular for all $m \geq 1$

$$(3.8) \qquad \pi_m^c(N,*_1) = \mathrm{incl}_*(\pi_m(N - \{*_2\},*_1)) \subset \bigcap_{q\geq 1} \pi_m^{(q)}(N,*_1) =: \pi_m^{(\infty)}(N,*_1)$$

(compare 1.17 and 3.5; here incl denotes the inclusion of $N - \{*_2\} \cong N$–small ball around $*_2$).

Moreover we have the exact sequence

(3.9)

$$\cdots \to \pi_m\big(N \times N - \Delta,(*_1,*_2)\big) \xrightarrow{\ p_{2*}\ } \pi_m(N,*_1) \xrightarrow{\ \delta_m\ } \pi_{m-1}(N - \{*_1\},*_2)$$

where p_2 denotes the projection $(y_1,y_2) \to y_1$. Thus

$$(3.10) \qquad \pi_m^{(2)}(N,*_1) = \ker \delta_m\,.$$

In view of proposition 3.4 we will be particularly interested in the case where N is a closed connected manifold of even dimension $n \leq m$.

Example 3.11. $N = \mathbb{K}P(n')$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (compare 1.10). In view of 3.3 and 3.4 (iii) we need to consider only the case $m, n' \geq 2$. Thus we can (and will) use the terminology of 1.11 and 1.12.

Proposition 3.12. *Assume $n' \geq 2$. Then for all $q \geq 2$ (and $m \geq 1$)*

$$\pi_m^{(q)}(\mathbb{K}P(n')) = \pi_m^{(2)}(\mathbb{K}P(n')) = p_*(\ker \partial_{\mathbb{K}}) \oplus \pi_m^c(\mathbb{K}P(n'))\,;$$

the analogous result holds for base point free homotopy classes of arbitrary maps $f : S^m \to \mathbb{K}P(n')$.

Moreover let M be any paracompact space. If $\mathbb{K} = \mathbb{R}$ and $H^1(M;\mathbb{Z}_2) = 0$, or if $\mathbb{K} = \mathbb{C}$ and $H^2(M;\mathbb{Z}) = 0$, then for all $q \geq 2$

$$[M, \mathbb{K}P(n')]^{(q)} = [M, \mathbb{K}P(n')]^{(2)}\,;$$

this set coincides with the full homotopy set $[M, \mathbb{K}P(n')]$ if in addition n' is odd.

Proof. In each case we need to consider only maps f which allow a lifting $\widetilde{f}: M \to S^{n+d-1}$, i.e. $f = p\circ\widetilde{f}$ (compare 1.13). If $M = S^m$ this follows from 3.8; otherwise use characteristic classes to see that the pullback of the canonical line bundle over $\mathbb{K}P(n')$ is trivial.

Given liftings $\widetilde{f}, \widetilde{\widetilde{f}}$ such that the pair $(p\circ\widetilde{f}, p\circ\widetilde{\widetilde{f}})$ is coincidence free, $\widetilde{f}(x), \widetilde{\widetilde{f}}(x)$ are linearly independent unit vectors in $\mathbb{K}^{n'+1}$ for all $x \in M$. Thus $p\circ\widetilde{f}$ is the starting term of an (arbitrarily long) sequence of pairwise coincidence free maps $f_i : M \to \mathbb{K}P(n')$ defined by

$$f_i(x) = \mathbb{K}\big(\widetilde{f}(x) + (i-1)\widetilde{\widetilde{f}}(x)\big)\,, \quad x \in M,\ i \geq 1\,.$$

We conclude that $[M, \mathbb{K}P(n')]^{(2)} \subset [M, \mathbb{K}P(n')]^{(q)}$.

If n' is odd and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $p_{\mathbb{K}}$ (cf. 1.11) allows a section (via the complex or quaternionic scalar multiplication on $\mathbb{K}^{n'+1}$) and every map $f = p\circ\widetilde{f}$ occurs in a coincidence free pair as above. □

In contrast, when n' is even then $[M, \mathbb{K}P(n')]^{(2)}$ often turns out to be strictly smaller than the full homotopy set $[M, \mathbb{K}P(n')]$ (or, in the terminology of Brown and Schirmer, there are *coincidence producing* maps from M to $\mathbb{K}P(n')$, cf. [2]). Let us illustrate this for $\mathbb{K} = \mathbb{R}$.

Lemma 3.13. *For all $m, n > 1$ the diagram*

$$\begin{tikzcd} \pi_m(S^n) \arrow[r, "\partial_{\mathbb{R}}"] \arrow[dr, "(1+(-1)^n)E^\infty"'] & \pi_{m-1}(S^{n-1}) \arrow[d, "E^\infty"] \\ & \pi^S_{m-n} \end{tikzcd}$$

commutes up to a fixed sign. (*Here E^∞ denotes stable suspension.*)

In particular, in the stable dimension range $m < 2n-2$ (*where both arrows labelled E^∞ are isomorphisms*) *we have*

$$\ker \partial_{\mathbb{R}} = \big\{ z \in \pi_m(S^n) \mid (1 + (-1)^n)\cdot z = 0 \big\}\,.$$

Proof. Given $[f] \in \pi_m(S^n)$, the Freudenthal suspension of $\partial_{\mathbb{R}}([f])$ equals the selfintersection invariant $\pm\,\underline{\omega}(f,f)$ (cf. [17], 5.6 and 5.7). In turn

$$E^\infty\big(\underline{\omega}(f,f)\big) = \omega(f,f) = \pm\chi(S^n)\cdot E^\infty\big([f]\big)$$

in $\Omega_{m-n}(S^m;\varphi) \cong \pi^S_{m-n}$ (cf. [12], 1.4, 1.6, and 2.2; here we use the canonical stable trivializations of the tangent bundle TS^n and of the virtual coefficient bundle $\varphi = f^*(TS^n) - TS^m$). □

Example 3.14.

$$\pi_9^c\big(\mathbb{R}P(6)\big) = 0 \subset \pi_9^{(2)}\big(\mathbb{R}P(6)\big) \cong \mathbb{Z}_2 \subset \pi_9\big(\mathbb{R}P(6)\big) \cong \mathbb{Z}_{24}$$

and

$$\pi_{17}^c\big(\mathbb{R}P(10)\big) = 0 \subset \pi_{17}^{(2)}\big(\mathbb{R}P(10)\big) \cong \mathbb{Z}_2 \subset \pi_{17}\big(\mathbb{R}P(10)\big) \cong \mathbb{Z}_{240}$$

This follows from our results 1.12, 3.12, 3.13, and Toda's tables [19].

Remark 3.15. Assume that N is a topological manifold. For $i = 1, 2$ consider the fiber inclusion and the projection

$$(N - \{*_i\},\ *_{i\pm 1}) \xrightarrow[j_i]{\subset} (\widetilde{C}_2(N), (*_1, *_2)) \xrightarrow[p_{2,i}]{} (N, *_i)$$

of the locally trivial fibration defined by $p_{2,i}(y_1, y_2) = y_i$; its exact homotopy sequence (cf. 3.9) yields the isomorphism

$$p_{2,i*} : \pi_m(\widetilde{C}_2(N), (*_1, *_2))/(\operatorname{im} j_{1*} + \operatorname{im} j_{2*}) \xrightarrow{\cong} \pi_m^{(2)}(N, *_i)/\pi_m^c(N, *_i).$$

Then the composite $p_{2,2*} \circ p_{2,1*}^{-1}$ equals the group isomorphism c (cf. theorem 1.18) which is so central to our study of loose pairs of maps.

References

[1] Brown, R., *Wecken properties for manifolds*, Contemp. Math. **152** (1993), 9–21.

[2] Brown, R. and Schirmer, H., *Nielsen coincidence theory and coincidence–producing maps for manifolds with boundary*, Topology Appl. **46** (1992), 65–79.

[3] Brooks, R. B. S., *On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy*, Pacif. J. Math. **39** (1971), 45–52.

[4] Dold, A. and Gonçalves, D., *Self–coincidence of fibre maps*, Osaka J. Math. **42** (2005), 291–307.

[5] Fadell, E. and Neuwirth, L., *Configuration spaces*, Math. Scand. **10** (1962), 111–118.

[6] Geoghegan, R., *Nielsen Fixed Point Theory*, Handbook of geometric topology, (R.J. Daverman and R. B. Sher, Ed.), 500–521, Elsevier Science 2002.

[7] Jiang, B., *Fixed points and braids*, Invent. Math. **75** (1984), 69–74.

[8] Jiang, B., *Fixed points and braids. II*, Math. Ann. **272** (1985), 249–256.

[9] Jiang, B., *Commutativity and Wecken properties for fixed points of surfaces and 3–manifolds*, Topology Appl. **53** (1993), 221–228.

[10] Kelly, M., *The relative Nielsen number and boundary-preserving surface maps*, Pacific J. Math. **161**, No. 1, (1993), 139–153.

[11] Koschorke, U., *Vector fields and other vector bundle monomorphisms – a singularity approach*, Lect. Notes in Math. **847** (1981), Springer–Verlag.

[12] Koschorke, U., *Selfcoincidences in higher codimensions*, J. reine angew. Math. **576** (2004), 1–10.

[13] Koschorke, U., *Nielsen coincidence theory in arbitrary codimensions*, J. reine angew. Math. (to appear).

[14] Koschorke, U., *Linking and coincidence invariants*, Fundamenta Mathematicae **184** (2004), 187–203.

[15] Koschorke, U., *Geometric and homotopy theoretic methods in Nielsen coincidence theory*, Fixed Point Theory and App. (2006), (to appear).

[16] Koschorke, U., *Nonstabilized Nielsen coincidence invariants and Hopf–Ganea homomorphisms*, Geometry and Topology **10** (2006), 619–665.

[17] Koschorke, U., *Minimizing coincidence numbers of maps into projective spaces*, submitted to Geometry and Topology monographs (volume dedicated to Heiner Zieschang).
(The papers [12] – [17] can be found at http://www.math.uni-siegen.de/topology/publications.html)

[18] Nielsen, J., *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen*, Acta Math. **50** (1927), 189–358.

[19] Toda, H., *Composition methods in homotopy groups of spheres*, Princeton Univ. Press, 1962.

[20] Wecken, F., *Fixpunktklassen*, I, II, III, Math. Ann. **117** (1941), 659–671; **118** (1942), 216–234 and 544–577.

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