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SPECTRUM GENERATING ON TWISTOR BUNDLE

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Abstract. Spectrum generating technique introduced by Ólafsson, Ørsted, and one of the authors in the paper [5] provides an efficient way to construct certain intertwinors when K-types are of multiplicity at most one. Intertwinors on the twistor bundle over $S^1 \times S^{n-1}$ have some K-types of multiplicity 2. With some additional calculation along with the spectrum generating technique, we give explicit formulas for these intertwinors of all orders.

1. Introduction

It was shown in [5] that one can construct intertwining operators of principal series representations induced from maximal parabolic subgroups without too much effort when K-types occur with multiplicity at most one. On the differential form bundle over $S^1 \times S^{n-1}$, a double cover of the compactified Minkowski space, some K-types occur with multiplicity two. One of the authors showed that the spectrum generating technique can also handle this multiplicity 2 case provided that some extra computation is performed.

It is thus natural to do the same thing on general tensor-spinor bundle. Intertwinors on spinors like the Dirac operator have eigenspaces with multiplicity one over $S^1 \times S^{n-1}$and explicit spectral function was given in [7]. On twistors, however, the eigenspaces of the intertwinors including Rarita Schwinger operator have multiplicity two on some K-types. In this paper, we present the spectral function for these operators.

We briefly review conformal covariance and intertwining relation (for more details, see [2], [5]).

Let M be an n-dimensional spin manifold. We enlarge the structure group $\mathrm{Spin}(n)$ to $\mathrm{Spin}(n) \times \mathbb{R}_+$ in conformal geometry. $(V(\lambda), \lambda^r)$ are finite dimensional $\mathrm{Spin}(n) \times \mathbb{R}_+$ representations, where $(V(\lambda), \lambda)$ are finite dimensional representations of $\mathrm{Spin}(n)$ and $\lambda^r(h, \alpha) = \alpha^r \lambda(h)$ for $h \in \mathrm{Spin}(n)$ and $\alpha \in \mathbb{R}_+$. The corresponding associated vector bundles are $\mathbb{V}(\lambda) = P_{\mathrm{Spin}(n)} \times_\lambda V(\lambda)$ and

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$\mathbb{V}^r(\lambda) = P_{\mathrm{Spin}(n)\times\mathbb{R}_+} \times_{\lambda^r} V(\lambda)$ with structure groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}(n)\times\mathbb{R}_+$. r is called the conformal weight of $\mathbb{V}^r$. Tangent bundle TM carries conformal weight -1 and cotangent bundle T^*M carries conformal weight $+1$. In general, if V is a subbundle of $(TM)^{\otimes p}\otimes(T^*M)^{\otimes q}\otimes(\Sigma M)^{\otimes r}\otimes(\Sigma^*M)^{\otimes s}$, then V carries conformal weight $q-p$, where ΣM is the contravariant spinor bundle.

A conformal covariant of bidegree (a,b) is a $\mathrm{Spin}(n)\times\mathbb{R}_+$-equivariant differential operator $D:\mathbb{V}^r(\lambda)\to\mathbb{V}^s(\sigma)$ which is a polynomial in the metric g, its inverse g^{-1}, the volume element E, and the fundamental tensor-spinor γ with a conformal covariance law

$$\omega\in C^\infty(M),\quad \overline{g}=e^{2\omega}g,\quad \overline{E}=e^{n\omega}E,\quad \overline{\gamma}=e^{-\omega}\gamma\Rightarrow \overline{D}=e^{-b\omega}D\mu(e^{a\omega}),$$

where $\mu(e^{a\omega})$ is multiplication of $e^{a\omega}$.

Given a conformal covariant of bidegree (a,b), $D:\mathbb{V}^r(\lambda)\to\mathbb{V}^s(\sigma)$, we can assign new conformal weights to get $D:\mathbb{V}^{r'}(\lambda)\to\mathbb{V}^{s'}(\sigma)$ whose bidegree is then $(a-r'+r, b-s'+s)$. Calling this D again is an abuse of notation. If $r'=r+a$ and $s'=s+b$, then $D:\mathbb{V}^{r+a}(\lambda)\to\mathbb{V}^{s+b}(\sigma)$ becomes conformally invariant and we call $(a+r,b+s)$ the reduced conformal bidegree of D. To see how conformal covariants behave under a conformal transformation and a conformal vector field, we recall followings.

A diffeomorphism $h:M\to M$ is called a conformal transformation if $h\cdot g=e^{2\omega_h}g$, where "$\cdot$" is the natural action of h on tensor fields; in particular, $h\cdot=(h^{-1})^*$ on purely covariant tensors like g. A conformal vector field is a vector field X with $\mathcal{L}_Xg=2\omega_Xg$ for some $\omega_X\in C^\infty(M)$. A conformal covariant $D:\mathbb{V}^0(\lambda)\to\mathbb{V}^0(\sigma)$ of reduced bidegree (a,b) satisfies

$$D(e^{a\omega_h}h\cdot\varphi)=e^{b\omega_h}h\cdot(D(\varphi))\quad\text{and}\quad D(\mathcal{L}_X+a\omega_X)\varphi=(\mathcal{L}_X+b\omega_X)D\varphi\,.$$

for all $\varphi\in\Gamma(\mathbb{V}^0(\lambda))$. Thus if $D:\mathbb{V}^r(\lambda)\to\mathbb{V}^s(\sigma)$ of reduced bidegree (a,b), then

$$D(\mathcal{L}_X+(a-r)\omega_X)\varphi=(\mathcal{L}_X+(b-s)\omega_X)D\varphi \tag{1.1}$$

for $\varphi\in\Gamma(\mathbb{V}^r(\lambda))$ and $D\varphi\in\Gamma(\mathbb{V}^s(\sigma))$.

Note that conformal vector fields form a Lie algebra $\mathfrak{c}(M,g)$ and give rise to the principal series representation

$$U_a^\lambda:\mathfrak{c}(M,g)\to\mathrm{End}\Gamma(\mathbb{V}^0(\lambda))\quad\text{by}\quad X\mapsto\mathcal{L}_X+a\omega_X\,.$$

So a conformal covariant $D:\mathbb{V}^r(\lambda)\to\mathbb{V}^s(\sigma)$ of reduced bidegree (a,b) intertwines the principal series representation

$$DU^\lambda_{a-r}\varphi=U^\sigma_{b-s}D\varphi$$

for $\varphi\in\Gamma(\mathbb{V}^r(\lambda))$ and $D\varphi\in\Gamma(\mathbb{V}^s(\sigma))$.

2. Spinors and twistors

Let $M=S^1\times S^{n-1}$, n even, be a manifold endowed with the Lorentz metric $-dt^2+g_{S^{n-1}}$.

To get a fundamental tensor-spinor α for M from the corresponding object γ on S^{n-1}, let

$$\alpha^j = \begin{pmatrix} \gamma^j & 0 \\ 0 & -\gamma^j \end{pmatrix}, \qquad j = 1, \ldots, n-1,$$

and

$$\alpha^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since M is even-dimensional, there is a *chirality* operator χ_M, equal to some complex unit times $\alpha^0\tilde{\chi}_S$, where

$$\tilde{\chi}_S = \begin{pmatrix} \chi_S & 0 \\ 0 & -\chi_S \end{pmatrix},$$

χ_S being the chirality operator on S. The chirality operator is always normalized to have square 1; thus $(\chi_S)^2$ and $(\tilde{\chi}_S)^2$ are identity operators, and since $\alpha^0\alpha^0 = 1$, we have $(\alpha^0\tilde{\chi}_S)^2 = -1$. As a result, we may take

$$\chi_M = \pm\sqrt{-1}\alpha^0\tilde{\chi}_S.$$

A *spinor* on M can be viewed as a pair of time-dependent spinors on S^{n-1}, i.e., $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$, where φ and ψ are t-dependent spinors on S^{n-1}. But by chirality consideration ([6]), we get $\Xi = \pm 1$ spinors:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \Xi\psi/\sqrt{-1} \\ \psi \end{pmatrix}.$$

Recall that *twistors* are spinor-one-forms Φ_λ with $\alpha^\lambda\Phi_\lambda = 0$. Given a chirality Ξ, a twistor Ψ is determined by a t-dependent spinor-one-form ψ_j on S^{n-1} via

$$\Psi = dt \wedge \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix},$$

where

$$\begin{aligned} \varphi_j &= -\Xi\sqrt{-1}\psi_j, \\ \psi_0 &= \Xi\sqrt{-1}\gamma^k\psi_k, \\ \varphi_0 &= \gamma^k\psi_k. \end{aligned}$$

Let θ_j be a spinor-one-form on S^{n-1}. Then, it can be written as

$$\theta_j = \gamma_j\left(-\frac{1}{n-1}\gamma^i\theta_i\right) + \left(\theta_j + \frac{1}{n-1}\gamma_j\gamma^i\theta_i\right) =: \gamma_j\theta + \pi_j, \tag{2.2}$$

where θ is a spinor and π_j is a twistor on S^{n-1} since $\gamma^j(\theta_j + \frac{1}{n-1}\gamma_j\gamma^i\theta_i) = 0$. It turned out ([6]) that we can Hodge decompose the twistor bundle over the sphere so that a twistor π_j can be written as

$$\pi_j = \mathcal{T}_j\tau + (-\nabla^i\eta_{ij}),$$

where $\mathcal{T}_j\tau := \nabla_j\tau + \gamma_j D\tau$ (here D is the Dirac operator on the sphere) is the j-th component of the twistor operator applied to a spinor τ and η_{ij} is a spinor-two form with $\gamma^i\eta_{ij} = 0$.

Therefore, a twistor on M can be decomposed as follows:

$$\begin{pmatrix} -(n-1)\theta & -\Xi\sqrt{-1}\gamma_i\theta \\ -(n-1)\Xi\sqrt{-1}\theta & \gamma_i\theta \end{pmatrix} + \begin{pmatrix} 0 & -\Xi\sqrt{-1}\mathcal{T}_i\tau \\ 0 & \mathcal{T}_i\tau \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -\Xi\sqrt{-1}\nabla^j\eta_{ji} \\ 0 & \nabla^j\eta_{ji} \end{pmatrix} =: \langle\theta\rangle + \{\tau\} + [\eta]\,, \tag{2.3}$$

for some spinors θ, τ and some spinor-two form η.

3. Intertwining relation on twistors

Let us briefly review some standard materials on the conformal structure on the manifold $S^1 \times S^{n-1}$. Let $G = \mathrm{Spin}_0(2,n)$ and P the maximal parabolic subgroup for which G/P is the 4-fold cover of the compactified Minkowski space $(S^1 \times S^{n-1})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action comes from the product of antipodal maps on S^1 and on S^{n-1}. $G^{'}/P^{'}$, where $G^{'} = \mathrm{SO}_0(2,n)$ and P' its maximal parabolic subgroup, is the double cover $S^1 \times S^{n-1}$ of $(S^1 \times S^{n-1})/\mathbb{Z}_2$. Then G/P is the double cover of $S^1 \times S^{n-1}$ obtained from the standard covering of S^1 factor. The Lie algebra $\mathfrak{g}$ can be realized in homogeneous coordinates $(\xi_{-1}, \dots, \xi_n)$ ([1, 9]):

$$L_{\alpha\beta} = \varepsilon_\alpha\xi_\alpha\partial_\beta - \varepsilon_\beta\xi_\beta\partial_\alpha \quad \alpha, \beta = -1, \dots, n\,,$$

where $\partial_\alpha = \partial/\partial\xi_\alpha$, and $-\varepsilon_{-1} = -\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_n = 1$. The $L_{-1,0}$ generates SO(2) group of isometries and the $L_{\alpha\beta}$ for $\alpha, \beta = 1, \dots, n$ generate SO(n) group of isometries. If $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ is a Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{k}$ corresponds to the $\mathfrak{so}(2) \times \mathfrak{so}(n)$ and $\mathfrak{s}$ corresponds to the *proper* conformal vector fields:

$$\mathcal{L}_{L_{\alpha\beta}} g = 2\omega_{\alpha\beta} g, \quad \text{with } \omega_{\alpha\beta} \neq 0\,,$$

where $\mathcal{L}$ denotes Lie derivative. So they are just the ones with mixed indices: $L_{\alpha\beta}$ for $-1 \le \alpha \le 0 < \beta \le n$. Let t be the angular parameter on S^1 so that $\xi_{-1} = \cos t$ and $\xi_0 = \sin t$. And set $\xi_n = \cos\rho$ and complete a set of spherical angular coordinates $(\rho, \theta_1, \dots, \theta_{n-2})$ on S^{n-1} so that ∂_ρ is $g_{S^{n-1}}$-orthogonal to the ∂_{θ_i}. Then we get a typical conformal vector field T and its conformal factor ϖ:

$$\begin{aligned} L_{-1,n} &= \cos\rho\sin t\partial_t + \cos t\sin\rho\partial_\rho := T \\ \omega_{-1,n} &= \cos t\cos\rho := \varpi\,. \end{aligned}$$

Let $A = A_{2r}$ be an intertwinor of order $2r$. That is, an operator satisfying the intertwining relation ((1.1), [2, 3, 5])

$$A\left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} - r\right)\varpi\right) = \left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} + r\right)\varpi\right)A\,, \tag{3.4}$$

where $\tilde{\mathcal{L}}_T$ is the *reduced Lie derivative.* On a tensor-spinor with $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor content, this is

$$\tilde{\mathcal{L}}_T = \mathcal{L}_T + (p-q)\varpi\,.$$

So here (with only 1-form content), it is $\mathcal{L}_T - \varpi$. Note that we are using the convention where spinors do not have an internal weight; otherwise the spinor

content would influence the reduction.
Since intertwinors change chirality, we want to consider an exchange operator

$$\begin{aligned} E :&= \alpha^0(\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t)) \\ &= \alpha^0(1 - 2\varepsilon(dt)\iota(\partial_t))\,, \end{aligned}$$

where ι is the interior multiplication and ε is the exterior multiplication. It is immediate that $E^2 = \mathrm{Id}$. Because of the α^0 factor, E reverses chirality. To see that E takes twistors to twistors, note that, for a twistor Φ,

$$\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t) : \Phi_\lambda \mapsto \Phi_\lambda - 2\delta_\lambda{}^0\Phi_0\,.$$

Thus

$$\begin{aligned} \alpha^\lambda(E\Phi)_\lambda &= \alpha^\lambda\alpha^0(\Phi_\lambda - 2\delta_\lambda{}^0\Phi_0) \\ &= -2g^{\lambda 0}(\Phi_\lambda - 2\delta_\lambda{}^0\Phi_0) + 2\alpha^0\alpha^\lambda\delta_\lambda{}^0\Phi_0 \\ &= \underbrace{-2\Phi^0}_{2\Phi_0} + 4\underbrace{g^{00}}_{-1}\Phi_0 + 2\underbrace{\alpha^0\alpha^0}_{1}\Phi_0 \\ &= 0, \end{aligned}$$

as desired.

We want to convert the relation (3.4) for EA. So we will eventually need $\mathcal{L}_T E$. We have:

$$\begin{aligned} \mathcal{L}_T E &= \mathcal{L}_T\,\{\alpha(dt)(1 - 2\varepsilon(dt)\iota(\partial_t))\} \\ &= \{-\varpi\alpha(dt) + \alpha(d(Tt))\}(1 - 2\varepsilon^0\iota_0) \\ &\quad - 2\alpha^0\{\varepsilon(dt)\iota([T,\partial_t]) + \varepsilon(d(Tt)\iota(\partial_t)\}. \end{aligned}$$

But

$$\begin{aligned} Tt &= \cos\rho\sin t, \\ d(Tt) &= -\sin\rho\sin t\,d\rho + \cos\rho\cos t\,dt, \\ [T,\partial_t] &= -\cos\rho\cos t\,\partial_t + \sin t\sin\rho\,\partial_\rho\,. \end{aligned}$$

This reduces the above to

$$\begin{aligned} \mathcal{L}_T E &= \sin t\alpha(d\omega)(1 - 2\varepsilon^0\iota_0) - 2\sin t\alpha^0(\varepsilon^0\iota(Y) + \varepsilon(d\omega)\iota_0) \\ &= \sin t\sin\rho\{-\alpha^1(1 - 2\varepsilon^0\iota_0) - 2\alpha^0(\varepsilon^0\iota_1 - \varepsilon^1\iota_0)\}. \end{aligned} \tag{3.5}$$

By Kosmann ([8], eq(16)), the Lie and covariant derivatives on spinors are related by

$$\mathcal{L}_X - \nabla_X = -\tfrac{1}{4}\nabla_{[a}X_{b]}\gamma^a\gamma^b = -\tfrac{1}{8}(dX_\flat)_{ab}\gamma^a\gamma^b\,.$$

Note that

$$\begin{aligned} T_\flat &= -\cos\rho\sin t\,dt + \cos t\sin\rho\,d\rho, \\ dT_\flat &= 2\sin\rho\sin t\,d\rho\wedge dt\,. \end{aligned}$$

and

$$d\varpi = -T_{\flat,\mathrm{R}}\,,$$

where $\flat$,R is the musical isomorphism in the "Riemannian" metric. According to the above,

$$\mathcal{L}_T - \nabla_T = -\frac{1}{2}\sin\rho\sin t\alpha^1\alpha^0 \tag{3.6}$$

on spinors.

On a 1-form η,

$$\langle(\mathcal{L}_T - \nabla_T)\eta, X\rangle = -\langle\eta, (\mathcal{L}_T - \nabla_T)X\rangle\,,$$

since $\mathcal{L}_T - \nabla_T$ kills scalar functions. But by the symmetry of the pseudo-Riemannian connection,

$$[T, X] - \nabla_T X = -\nabla_X T\,.$$

We conclude that

$$(\mathcal{L}_T - \nabla_T)\eta = \langle\eta, \nabla T\rangle\,,$$

where in the last expression, $\langle\cdot,\cdot\rangle$ is the pairing of a 1-form with the contravariant part of a $\begin{pmatrix}1\\1\end{pmatrix}$-tensor:

$$((\mathcal{L}_T - \nabla_T)\eta)_\lambda = \eta_\mu\nabla_\lambda T^\mu\,.$$

Combining this with what we derived above for spinors (3.6), for a spinor-1-form Φ_λ, we have

$$((\mathcal{L}_T - \nabla_T)\Phi)_\lambda = \Phi_\mu\nabla_\lambda T^\mu - \frac{1}{2}\sin\rho\sin t\alpha^1\alpha^0\Phi_\lambda\,.$$

But ∇T *a priori* has projections in 3 irreducible bundles, TFS^2, Λ^0, and Λ^2 (after using the musical isomorphisms). By conformality, the TFS^2 part is gone. We expect a Λ^0 part, essentially ϖ. We also found the Λ^2 part above,

$$dT_\flat = 2\sin\rho\sin t\,d\rho\wedge dt\,.$$

More precisely, tracking the normalizations,

$$(\nabla T_\flat)_{\lambda\mu} = (\nabla T_\flat)_{(\lambda\mu)} + (\nabla T_\flat)_{[\lambda\mu]} = (\varpi g + \frac{1}{2}dT_\flat)_{\lambda\mu}\,.$$

Now note that

$$\begin{aligned}
\Phi_\mu\nabla_\lambda T^\mu &= \varpi g_\lambda{}^\mu\Phi_\mu + \frac{1}{2}((dT_\flat)_{\nu\mu}\varepsilon^\nu\iota^\mu\Phi)_\lambda\\
&= \varpi\Phi_\lambda + \frac{1}{2}(((dT_\flat)_{01}\varepsilon^0\iota^1 + (dT_\flat)_{10}\varepsilon^1\iota^0)\Phi)_\lambda\\
&= \varpi\Phi_\lambda + \frac{1}{2}((-2\sin\rho\sin t\varepsilon^0\iota^1 + 2\sin\rho\sin t\varepsilon^1\iota^0)\Phi)_\lambda\\
&= \varpi\Phi_\lambda - \sin\rho\sin t((\varepsilon^0\iota^1 - \varepsilon^1\iota^0)\Phi)_\lambda\\
&= \varpi\Phi_\lambda - \sin\rho\sin t((\varepsilon^0\iota_1 + \varepsilon^1\iota_0)\Phi)_\lambda\,.
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathcal{L}_T - \nabla_T &= \varpi - \sin\rho\sin t\left(\tfrac{1}{2}\alpha^1\alpha^0 + \varepsilon^0\iota_1 + \varepsilon^1\iota_0\right)\\
&=: \varpi - \sin\rho\sin tP\\
&=: \varpi - \mathcal{P}\,,
\end{aligned}$$

and

$$\tilde{\mathcal{L}}_T - \nabla_T = -\mathcal{P}\,.$$

An explicit calculation using (3.5) gives

$$(\mathcal{L}_T E)E = -2\mathcal{P}\,.$$

Since $E^2 = \mathrm{Id}$, we conclude that

$$\mathcal{L}_T E = -2\mathcal{P}E\,.$$

With the above, the intertwining relation for EA becomes

$$\begin{aligned}\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}+r\right)\varpi\right) EA &= E\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}+r\right)\varpi\right) A + (\mathcal{L}_T E)A \\ &= EA\left(\tilde{\mathcal{L}}_T + \left(\tfrac{n}{2}-r\right)\varpi\right) - 2\mathcal{P}EA,\end{aligned}$$

so that, with $B = EA$,

$$B\left(\nabla_T + \left(\frac{n}{2}-r\right)\varpi - \mathcal{P}\right) = \left(\nabla_T + \left(\frac{n}{2}+r\right)\varpi + \mathcal{P}\right)B\,.$$

To see what P does, let us define two convenient operations.

$$\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \overset{\mathbf{slot}}{\longmapsto} \psi_j\,,$$

where $u = \gamma^k\psi_k$.
Note that

$$\begin{aligned}&\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \overset{\iota_0}{\longmapsto} \begin{pmatrix} u \\ -\Xi u/\sqrt{-1} \end{pmatrix} \\ &\overset{\varepsilon^1}{\longmapsto} \begin{pmatrix} 0 & \varepsilon^1 u \\ 0 & -\Xi\varepsilon^1 u/\sqrt{-1} \end{pmatrix} \overset{\mathbf{slot}}{\longmapsto} -\Xi\varepsilon^1 u/\sqrt{-1}\,.\end{aligned}$$

As for the $\varepsilon^0\iota_1$ term, anything in the range of ε^0 has a **slot** of 0.

Finally,

$$\begin{aligned}&\psi_j \overset{\mathbf{expa}}{\longmapsto} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \overset{\alpha^0}{\longmapsto} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \\ &= \begin{pmatrix} -\Xi u/\sqrt{-1} & \psi_j \\ u & \Xi\psi_j/\sqrt{-1} \end{pmatrix} \overset{\alpha^1}{\longmapsto} \begin{pmatrix} -\Xi\gamma^1 u/\sqrt{-1} & \gamma^1\psi_j \\ -\gamma^1 u & -\Xi\gamma^1\psi_j/\sqrt{-1} \end{pmatrix}.\end{aligned}$$

So

$$\begin{aligned}\mathbf{slot}\,P\,\mathbf{expa} : \psi_j &\mapsto -\tfrac{1}{2}\Xi\gamma^1\psi_j/\sqrt{-1} - \Xi(\varepsilon^1 u)_j/\sqrt{-1} \\ &= -\tfrac{\Xi}{\sqrt{-1}}\left(\tfrac{1}{2}\gamma^1\psi_j + (\varepsilon^1 u)_j\right) = -\tfrac{\Xi}{\sqrt{-1}}\left(\tfrac{1}{2}\gamma^1\psi_j + \delta_j{}^1 u\right).\end{aligned}$$

Up to a factor of a complex unit, $\mathbf{slot}\,P\,\mathbf{expa}$ is

$$\frac{1}{2}\gamma^1\psi_j + \delta_j{}^1\gamma^k\psi_k\,.$$

We can also get this expression by successively taking the commutator of ϖ with ∂_t and the operator $\mathcal{D}$ defined by

$$\mathbf{slot}\,\mathcal{D}\,\mathbf{expa} : \psi_j \mapsto \frac{1}{2}\gamma^k\nabla_k\psi_j + \gamma^k\nabla_j\psi_k\,.$$

That is,

$$\mathcal{P} = \Xi\sqrt{-1}[\partial_t, [\mathcal{D}, \varpi]] \,.$$

Recall that $\mathcal{P} = \sin\rho \sin tP$.

After some straightforward computation, we get the block matrix for $\mathcal{D}$ relative to the decomposition $\{\langle\theta\rangle, \{\tau\}, [\eta]\}$ (2.3) as follows.

$$\begin{pmatrix} \dfrac{n+1}{2(n-1)}J_\theta & \dfrac{n-2}{4} - \dfrac{n-2}{(n-1)^2}J_\tau^2 & 0 \\ -n & \dfrac{n-3}{2(n-1)}J_\tau & 0 \\ 0 & 0 & \dfrac{1}{2}L \end{pmatrix},$$

where J_θ and J_τ are the Dirac eigenvalues of θ and τ on S^{n-1}, respectively and L is the Rarita-Schwinger eigenvalue of $[\eta]$ on S^{n-1}.

The spectrum generating relation takes the following form:

$$[N, \varpi] = 2\left(\nabla_T + \frac{n}{2}\varpi\right),$$

where $\nabla^{*,\mathrm{R}}\nabla := N$ is the Riemannian Bochner Laplacian. Therefore the relation (3.4) becomes

(3.7)

$$B\left(\frac{1}{2}[N, \varpi] - r\varpi - \Xi\sqrt{-1}[\partial_t, [\mathcal{D}, \varpi]]\right) = \left(\frac{1}{2}[N, \varpi] + r\varpi + \Xi\sqrt{-1}[\partial_t, [\mathcal{D}, \varpi]]\right) B \,.$$

As explained in detail in ([3]), the recursive numerical spectral data come from the compressed relation of the above.

4. Projections into isotypic summands

Let us denote the $K = \mathrm{Spin}(2) \times \mathrm{Spin}(n)$-type with highest weight as follows:

$$\mathcal{V}_\Xi(f; j, \tfrac{1}{2}+q, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{\varepsilon}{2}) := (f) \otimes \underbrace{(j, \tfrac{1}{2}+q, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{\varepsilon}{2})}_{n/2 \text{ entries}},$$

where $j \in \frac{1}{2} + q + \mathbb{N}$, $\varepsilon = \pm 1$, $q = 0$ or 1, and (f) is a Spin(2)-type generated by the function $e^{\sqrt{-1}ft}$ on S^1 factor.

Proper conformal vector fields and corresponding conformal factors map such a K-type to a sum of different K-types under the classical selection rule ([3]).

Consider a Ξ spinor $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$. Since $\varphi = \Xi\psi/\sqrt{-1}$, we have

$$\alpha^0 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ \Xi\psi/\sqrt{-1} \end{pmatrix}.$$

Here $\bullet$ denotes a top entry that is computable from the bottom entry, but whose value is not needed at the moment.

In addition,

$$\sin t \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ \sin t\psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -[\partial_t, \cos t]\psi \end{pmatrix},$$

$$\mathrm{Proj}^{f}_{f'} \sin t \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ \frac{f'-f}{\sqrt{-1}} \cos t|^{f}_{f'} \psi \end{pmatrix},$$
$$\sin\rho\alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -\sin\rho\gamma^1\psi \end{pmatrix} = \begin{pmatrix} \bullet \\ [D, \cos\rho]\psi \end{pmatrix},$$
$$\mathrm{Proj}^{a}_{b} \sin\rho\alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -\mathrm{Proj}^{a}_{b} \sin\rho\gamma^1\psi \end{pmatrix} = \begin{pmatrix} \bullet \\ (J_b - J_a)\cos\rho|^{a}_{b}\, \psi \end{pmatrix},$$

where $D = \gamma^i \nabla_i$ is the Dirac operator on S^{n-1}, a and b (resp., f and f') are abbreviated labels for the Spin(n)-types (resp., Spin(2)-types) in question and J_a (resp., J_b) is the Dirac eigenvalue on a (resp., b).

For the compressed relations of $\varpi = \cos t \cos\rho$ between Clifford range part, twistor range part, and divergence part (2.3), we note that $\cos\rho$ is the conformal factor corresponding to the conformal vector field $\sin\rho\partial_\rho$ on S^{n-1}. Clifford range piece is essentially spinor on S^{n-1} while twistor range piece and divergence piece are twistors on S^{n-1}. So, for example, $\varpi\langle\theta\rangle$ is a sum of Clifford pieces only. Thus we have:

$$\varpi \begin{pmatrix} \langle\theta\rangle \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \langle|\varpi|\theta\rangle \\ 0 \\ 0 \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} \langle\tilde{\theta}\rangle \\ 0 \\ 0 \end{pmatrix},$$
$$\varpi \begin{pmatrix} 0 \\ \{\tau\} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ |\varpi|\{\tau\} \\ |\varpi|\{\tau\} \end{pmatrix} = \begin{pmatrix} 0 \\ C\{|\varpi|\tau\} \\ |\varpi|\{\tau\} \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} 0 \\ C\{\tilde{\tau}\} \\ [\eta] \end{pmatrix}, \tag{4.8}$$
$$\varpi \begin{pmatrix} 0 \\ 0 \\ [\eta] \end{pmatrix} = \begin{pmatrix} 0 \\ |\varpi|[\eta] \\ |\varpi|[\eta] \end{pmatrix} \overset{\mathrm{Proj}}{\longmapsto}: \begin{pmatrix} 0 \\ \{\bar{\tau}\} \\ [\tilde{\eta}] \end{pmatrix},$$

where C is a quantity we will compute in the following lemma.

Lemma 4.1. *Let* $\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2})$ *and* $\beta = \mathcal{V}_\Xi(f'; j', \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon'}{2})$, $\varepsilon = \pm 1$. *Then we have*

$$|_\beta\varpi|_\alpha\{\tau\} = C_{ba}\{|_\beta\varpi|_\alpha\tau\},$$

where

$$C_{ba} = \frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\left(\frac{1}{2}J_b^2 + \frac{1}{2}J_a^2 - \frac{J_bJ_a}{n-1} - \frac{n(n-1)}{4}\right),$$

J_a *(resp.,* J_b*) is the Dirac eigenvalue on Spin(n)-type at* α *(resp., Spin(n)-type at* β*),* $\lambda_b(\mathcal{T}^*\mathcal{T})$ *is the eigenvalue of* $\mathcal{T}^*\mathcal{T}$ *on Spin(n)-type at* β*, and* $\mathcal{T}$ *is the twistor operator (with adjoint* $\mathcal{T}^*$*) over* S^{n-1}.

Proof. It suffices to show that

$$|_b\omega|_a\mathcal{T}\tau = C_{ba}\cdot\mathcal{T}(|_b\omega|_a\tau),$$

where $\omega = \cos\rho$. Let D be the Dirac operator on S^{n-1}. Then

$$\begin{aligned}[D^2, \omega]\tau &= [\nabla^*\nabla, \omega]\tau \ \text{ by Bochner identity} \\ &= (\nabla^*\nabla\omega)\tau - 2\nabla^k\omega\nabla_k\tau = (n-1)\omega\tau + 2\sin\rho\nabla_1\tau,\end{aligned}$$

Also

$$\begin{aligned}
\mathcal{T}^*(\omega\mathcal{T}\tau) &= -\nabla^j(\omega\nabla_j\tau + \tfrac{1}{n-1}\omega\gamma_j D\tau)\\
&= \sin\rho\nabla_1\tau + \omega\nabla^*\nabla\tau + \tfrac{1}{n-1}\sin\rho\gamma_1 D\tau - \tfrac{1}{n-1}\omega D^2\tau\\
&= \tfrac{1}{2}\left([D^2,\omega] - (n-1)\omega\right)\tau + \omega\left(D^2 - \tfrac{(n-1)(n-2)}{4}\right)\tau + \tfrac{1}{n-1}[\omega, D]D\tau\\
&\qquad -\tfrac{1}{n-1}\omega D^2\tau \quad \text{by the above and Bochner identity}\\
&= \tfrac{1}{2}D^2(\omega\tau) + \tfrac{1}{2}\omega D^2\tau - \tfrac{1}{n-1}D(\omega D\tau) - \tfrac{n(n-1)}{4}\omega\tau.
\end{aligned}$$

Therefore

$$\begin{aligned}
|_b\omega|_a\mathcal{T}\tau &= \mathcal{T}\left(\frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\mathcal{T}^*(|_b\omega|_a\mathcal{T}\tau)\right)\\
&= \mathcal{T}\left(\frac{1}{\lambda_b(\mathcal{T}^*\mathcal{T})}\left(\frac{1}{2}J_b^2 + \frac{1}{2}J_a^2 - \frac{1}{n-1}J_bJ_a - \frac{n(n-1)}{4}\right)|_b\omega|_a\tau\right).
\end{aligned}$$

□

Remark 1. Eigenvalues of D and $\mathcal{T}^*\mathcal{T}$ on S^{n-1} are known due to Branson ([4]).

With the above (4.8) at hand, we get

(4.9)

$$|_\beta[\mathcal{D},\varpi]|_\alpha\langle\theta\rangle = \begin{pmatrix} (\mathcal{D}_{11}^\beta - \mathcal{D}_{11}^\alpha)\langle\tilde\theta\rangle \\ (\mathcal{D}_{21}^\beta - C_{ba}\mathcal{D}_{21}^\alpha)\{\tilde\theta\} \\ -\mathcal{D}_{21}^\alpha[\eta] \end{pmatrix}, \text{ where } \begin{cases} \langle\tilde\theta\rangle = |_\beta\varpi|_\alpha\langle\theta\rangle \\ [\eta] = |_\beta\varpi|_\alpha\{\theta\} \end{cases},$$

$$|_\beta[\mathcal{D},\varpi]|_\alpha\{\tau\} = \begin{pmatrix} (C_{ba}\mathcal{D}_{12}^\beta - \mathcal{D}_{12}^\alpha)\langle\tilde\tau\rangle \\ C_{ba}(\mathcal{D}_{22}^\beta - \mathcal{D}_{22}^\alpha)\{\tilde\tau\} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{22}^\alpha)[\eta] \end{pmatrix}, \text{ where } \begin{cases} \{\tilde\tau\} = |_\beta\varpi|_\alpha\{\tau\} \\ [\eta] = |_\beta\varpi|_\alpha\{\tau\} \end{cases}, \text{ and}$$

$$|_\beta[\mathcal{D},\varpi]|_\alpha[\eta] = \begin{pmatrix} \mathcal{D}_{12}^\beta\langle\bar\tau\rangle \\ (\mathcal{D}_{22}^\beta - \mathcal{D}_{33}^\alpha)\{\bar\tau\} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{33}^\alpha)[\tilde\eta] \end{pmatrix}, \text{ where } \begin{cases} \{\bar\tau\} = |_\beta\varpi|_\alpha[\eta] \\ [\tilde\eta] = |_\beta\varpi|_\alpha[\eta] \end{cases}.$$

Here we use subscripts to refer to the specific entries of the $\mathcal{D}$ and superscripts to indicate where these entries are computed.

Let us now consider the compressed relation of (3.7) between K-types related by the selection rule.

Case 1: Multiplicity 2 ↔ 1

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}) \leftrightarrow \beta = \mathcal{V}_\Xi(f'; j, \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}).$$

Note that the operator B in block form looks

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}.$$

With

$$|_\alpha N|_\beta = f^2 - f'^2 - (n-2)$$

and (4.9), we get $\alpha \to \beta$ transition quantities

$$\beta \to \alpha: \qquad \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ E^- \end{pmatrix} = B^\beta_{33} \begin{pmatrix} -A_1 \\ E^+ \end{pmatrix} \text{ and}$$

$$\alpha \to \beta: \quad \begin{pmatrix} A_2 & -E^- \end{pmatrix} \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix} = B^\beta_{33} \begin{pmatrix} -A_2 & -E^+ \end{pmatrix},$$

where

$$\begin{aligned} A_1 &:= \Xi(f-f')\mathcal{D}^\alpha_{12}, \\ A_2 &:= -\Xi(f-f')\mathcal{D}^\alpha_{21}, \\ E^- &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} - r + \Xi(f-f')(\mathcal{D}^\alpha_{22} - \mathcal{D}^\beta_{33}), \\ E^+ &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} + r - \Xi(f-f')(\mathcal{D}^\alpha_{22} - \mathcal{D}^\beta_{33}). \end{aligned}$$

In particular, we can write all 2×2 entries of B^α in terms of B^α_{21} and B^β_{33}:

$$\begin{aligned} B^\alpha_{11} &= (E^- B^\alpha_{21} - A_2 B^\beta_{33})/A_2, \\ B^\alpha_{12} &= -A_1 B^\alpha_{21}/A_2, \text{ and} \\ B^\alpha_{22} &= (-A_1 B^\alpha_{21} + E^+ B^\beta_{33})/E^-. \end{aligned} \tag{4.10}$$

Thus if we can express B^α_{21} in terms of B^β_{33}, we can completely determine all entries in the 2×2 block.

Case 2: Multiplicity 2 $\leftrightarrow$ 2

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2}) \to \beta = \mathcal{V}_\Xi(f'; j' \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon'}{2}).$$

Here we have

$$|_\beta N|_\alpha = f'^2 - f^2 + J_b^2 - J_a^2.$$

So using (4.9), we get the transition quantities

$$\begin{pmatrix} B^\beta_{11} & B^\beta_{12} \\ B^\beta_{21} & B^\beta_{22} \end{pmatrix} \begin{pmatrix} F_1^- & G_2 \\ G_1 & C_{ba}F_2^- \end{pmatrix} = \begin{pmatrix} F_1^+ & -G_2 \\ -G_1 & C_{ba}F_2^+ \end{pmatrix} \begin{pmatrix} B^\alpha_{11} & B^\alpha_{12} \\ B^\alpha_{21} & B^\alpha_{22} \end{pmatrix}, \tag{4.11}$$

where

$$\begin{aligned} F_1^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f'-f)(\mathcal{D}^\beta_{11} - \mathcal{D}^\alpha_{11}), \\ F_1^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f'-f)(\mathcal{D}^\beta_{11} - \mathcal{D}^\alpha_{11}), \\ F_2^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f'-f)(\mathcal{D}^\beta_{22} - \mathcal{D}^\alpha_{22}), \\ F_2^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f'-f)(\mathcal{D}^\beta_{22} - \mathcal{D}^\alpha_{22}), \\ G_1 &:= \Xi(f'-f)(\mathcal{D}^\beta_{21} - C_{ba}\mathcal{D}^\alpha_{21}), \text{ and} \\ G_2 &:= \Xi(f'-f)(C_{ba}\mathcal{D}^\beta_{12} - \mathcal{D}^\alpha_{12}). \end{aligned}$$

Therefore we get determinant quotients of B on multiplicity 2 part.
Note the following diagram of reachable multiplicity 2 isotypic summands from $\mathcal{V}_\Xi(f;j,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$ under the selection rule:

$$\begin{array}{ccccc}
\mathcal{V}_\Xi(f-1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
 & \nwarrow & & \nearrow & \\
\mathcal{V}_\Xi(f-1;j,\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) & \leftarrow & \bullet & \rightarrow & \mathcal{V}_\Xi(f+1;j,\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) \\
 & \swarrow & & \searrow & \\
\mathcal{V}_\Xi(f-1;j-1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j-1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}).
\end{array}$$

The determinant quotients corresponding to the above diagram are:
(4.12)

$$\begin{pmatrix}
\frac{(-f+J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(-f+J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)} & \frac{(f+J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(f+J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} \\[2ex]
\frac{(-f+\frac{1}{2}-\Xi+r-\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi+r-\varepsilon\Xi J)}{(-f+\frac{1}{2}-\Xi-r+\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi-r+\varepsilon\Xi J)} & \frac{(f+\frac{1}{2}-\Xi+r+\varepsilon\Xi J)(f+\frac{1}{2}+\Xi+r+\varepsilon\Xi J)}{(f+\frac{1}{2}-\Xi-r-\varepsilon\Xi J)(f+\frac{1}{2}+\Xi-r-\Xi J)} \\[2ex]
\frac{(-f-J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(-f-J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} & \frac{(f-J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(f-J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)}
\end{pmatrix},$$

where $J=\varepsilon J_a$.

And these data can be put into the following Gamma function expression:

$$\frac{1}{4}\bullet\frac{\Gamma\left(\frac{1}{2}(f+J+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J-r-\frac{\varepsilon}{2}\Xi)\right)}$$

$$\bullet\frac{\Gamma\left(\frac{1}{2}(f+J+2+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+2+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J+2-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+2-r-\frac{\varepsilon}{2}\Xi)\right)}.$$

Case 3: Multiplicity 1 $\leftrightarrow$ 1

$$\alpha=\mathcal{V}_\Xi(f;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\leftarrow\beta=\mathcal{V}_\Xi(f';j'\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon'}{2}).$$

Again we have

$$|_\alpha N|_\beta=f^2-f'^2+J_a^2-J_b^2.$$

And the transition quantities are

$$B_{33}^\alpha P^-=P^+B_{33}^\beta, \tag{4.13}$$

where

$$\begin{aligned}
P^-&:=\tfrac{1}{2}(f^2-f'^2)+\tfrac{1}{2}(J_a^2-J_b^2)-r+\Xi(f-f')(\mathcal{D}_{33}^\alpha-\mathcal{D}_{33}^\beta)\text{ and}\\
P^+&:=\tfrac{1}{2}(f^2-f'^2)+\tfrac{1}{2}(J_a^2-J_b^2)+r-\Xi(f-f')(\mathcal{D}_{33}^\alpha-\mathcal{D}_{33}^\beta).
\end{aligned}$$

The diagram of reachable multiplicity 1 isotypic summands from

$$\mathcal{V}_\Xi(f;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})$$

under the selection rule looks:

$$\begin{array}{ccccc}
\mathcal{V}_\Xi(f-1;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
 & \nwarrow & & \nearrow & \\
\mathcal{V}_\Xi(f-1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) & \leftarrow & \bullet & \rightarrow & \mathcal{V}_\Xi(f+1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},-\frac{\varepsilon}{2}) \\
 & \swarrow & & \searrow & \\
\mathcal{V}_\Xi(f-1;j-1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & & & & \mathcal{V}_\Xi(f+1;j-1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\,.
\end{array}$$

And the eigenvalue quotients are:

$$\begin{pmatrix}
\dfrac{-f+J+1+r+\frac{\varepsilon}{2}\Xi}{-f+J+1-r-\frac{\varepsilon}{2}\Xi} & \dfrac{f+J+1+r-\frac{\varepsilon}{2}\Xi}{f+J+1-r+\frac{\varepsilon}{2}\Xi} \\[2ex]
\dfrac{-f+\frac{1}{2}+r-\varepsilon\Xi J}{-f+\frac{1}{2}-r+\varepsilon\Xi J} & \dfrac{f+\frac{1}{2}+r+\varepsilon\Xi J}{f+\frac{1}{2}-r-\varepsilon\Xi J} \\[2ex]
\dfrac{-f-J+1+r-\frac{\varepsilon}{2}\Xi}{-f-J+1-r+\frac{\varepsilon}{2}\Xi} & \dfrac{f-J+1+r+\frac{\varepsilon}{2}\Xi}{f-J+1-r-\frac{\varepsilon}{2}\Xi}
\end{pmatrix},$$

where $J=\varepsilon J_a$.

Thus, following the normalization on the multiplicity 2 part, we get the spectral function on the multiplicity 1 part:

(4.14)

$$Z(r;f,J,\Xi\varepsilon)=\frac{\varepsilon}{2}\Xi\frac{\Gamma\left(\frac{1}{2}(f+J+1+r-\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+1+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J+1-r+\frac{\varepsilon}{2}\Xi)\right)\Gamma\left(\frac{1}{2}(-f+J+1-r-\frac{\varepsilon}{2}\Xi)\right)}\,.$$

In particular,

$$Z(\frac{1}{2},f,J,\Xi\varepsilon)=-\frac{1}{4}(f-\Xi\varepsilon J)=\frac{1}{4}\sqrt{-1}\ \mathrm{eig}(E\mathcal{R};f,J,\Xi\varepsilon)\,,$$

where $E\mathcal{R}$ is the exchanged Rarita-Schwinger operator.

5. Interface between multiplicity 1 and 2 parts

Consider the following diagram:

$$\begin{array}{ccc}
\alpha_1=\mathcal{V}_\Xi(f;j,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & \rightarrow & \alpha_2=\mathcal{V}_\Xi(f+1;j+1,\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) \\
\updownarrow & & \updownarrow \\
\beta_1=\mathcal{V}_\Xi(f+1;j,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2}) & \leftarrow & \beta_2=\mathcal{V}_\Xi(f;j+1,\frac{3}{2},\frac{1}{2},\cdots,\frac{1}{2},\frac{\varepsilon}{2})\,.
\end{array}$$

Then (4.11) reads

$$B^{\alpha_2}M_1=M_2B^{\alpha_1}\,.$$

So

$$\mathrm{det}B^{\alpha_2}=\frac{\mathrm{det}M_2}{\mathrm{det}M_1}\mathrm{det}B^{\alpha_1}\,.$$

Note that $\dfrac{\det M_2}{\det M_1}$ is a determinant quotient computed in (4.12).
From (4.10), we get a relation between B_{12} and B_{33}:

$$\begin{aligned}\det\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix} &= B_{11}B_{22} - B_{12}B_{22} \\ &= -\frac{1}{A_2E^-}B_{33}\left(B_{33}A_2E^+ - (E^-E^+ + A_1A_2)B_{21}\right) .\end{aligned}$$

We can also compare $(2,1)$ entries of both sides in (4.11). Applying (4.10) and (4.13) to the both relations, we can finally write B_{21} in terms of B_{33} with a "big" help from computer algebra package.

2×2 block on

$$\mathcal{V}_\Xi(f; j, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2})$$

in terms of $(3,3)$

$$\mathcal{V}_\Xi(f+1; j, \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{\varepsilon}{2})$$

is:

$$\begin{pmatrix} \dfrac{4C_1C_2}{(n-1)C_3C_4} - 1 & \dfrac{-2(n-2)\Xi C_5C_2}{(n-1)^2C_3C_4} \\ \dfrac{8n\Xi C_2}{C_3C_4} & \dfrac{-4C_5C_2}{(n-1)C_1C_3C_4} + \dfrac{C_6}{C_1}\end{pmatrix} \bullet Z(r; f+1, J, \Xi\varepsilon)\,, \tag{5.15}$$

where

$$\begin{aligned} C_1 &= 2fn - 2f - 2n + 1 + n^2 + 2rn - 2r - 2\Xi J_a\,, \\ C_2 &= 2fr + \Xi J_a\,, \\ C_3 &= n - 1 + 2r\,, \\ C_4 &= (2f + 2r - \Xi + 2J_a)(2f + 2r + \Xi - 2J_a)\,, \\ C_5 &= (n - 1 + 2J_a)(n - 1 - 2J_a)\,, \text{ and} \\ C_6 &= 2fn - 2f - 2n + 1 + n^2 - 2rn + 2r + 2\Xi J_a\,. \end{aligned}$$

Remark 2. In particular, if $r = \frac{1}{2}$ and $(3,3)$ entry

$$\sqrt{-1}f - \sqrt{-1}\Xi\varepsilon J$$

of the exchanged Rarita-Schwinger operator is put into the above formula, we recover the other 2×2 entries

$$\begin{pmatrix} -\dfrac{n-2}{n}\sqrt{-1}\left(f + \dfrac{n+1}{n-1}\Xi\varepsilon J\right) & -\dfrac{2\sqrt{-1}\Xi}{n(n-1)}\left(\dfrac{(n-1)(n-2)}{4} - \dfrac{n-2}{n-1}J^2\right) \\ 2\sqrt{-1}\Xi & \sqrt{-1}f - \dfrac{n-3}{n-1}\sqrt{-1}\Xi\varepsilon J \end{pmatrix}$$

of the operator ([7]).

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