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LIFTING OF PROJECTABLE VECTOR FIELDS TO VERTICAL FIBER PRODUCT PRESERVING VECTOR BUNDLES

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Abstract. We classify all natural operators lifting projectable vector fields from fibered manifolds to vector fields on vertical fiber product preserving vector bundles. We explain this result for some more known such bundles.

Introduction

The category of vector bundles and vector bundle maps will be denoted by $\mathcal{VB}$. The category of fibered manifolds and their fibered maps will be denoted by $\mathcal{FM}$. The category of fibered manifolds with m-dimensional bases and their fibered maps covering local diffeomorphisms will be denoted by $\mathcal{FM}_m$. The category of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local fibered diffeomorphisms will be denoted by $\mathcal{FM}_{m,n}$.

A bundle functor F on $\mathcal{FM}_m$ is fiber product preserving if for any fiber product projections $Y_1 \xleftarrow{pr_1} Y_1 \times_M Y_2 \xrightarrow{pr_2} Y_2$ in the category $\mathcal{FM}_m$,

$$FY_1 \xleftarrow{Fpr_1} F(Y_1 \times_M Y_2) \xrightarrow{Fpr_2} FY_2$$

are fiber product projections in the category $\mathcal{FM}_m$. In other words we have $F(Y_1 \times_M Y_2) = F(Y_1) \times_M F(Y_2)$.

A bundle functor F on $\mathcal{FM}_m$ is called vertical if for any $\mathcal{FM}_m$-objects $Y \to M$ and $Y_1 \to M$ with the same basis, any $x \in M$ and any $\mathcal{FM}_m$-map $f : Y \to Y_1$ covering the identity of M the fiber restriction $F_x f : F_x Y \to F_x Y_1$ depends only on $f_x : Y_x \to (Y_1)_x$.

A bundle functor F on $\mathcal{FM}_m$ is vector if it has values in the category $\mathcal{VB}$ of vector bundles. This means that for every fibered manifold Y, $\pi : FY \to Y$ is a vector bundle, and for every $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$, $Ff : FY_1 \to FY_2$ is a vector bundle map covering f.

From now on we are interested in vertical fiber product preserving vector bundle functors on $\mathcal{FM}_m$.

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A well-know example of a vertical fiber product preserving vector bundle functor on $\mathcal{FM}_m$ is the usual vertical bundle functor $V : \mathcal{FM}_m \to \mathcal{VB}$. We recall that $VY \to Y$ is the vertical bundle of Y.

Another example of vertical fiber product preserving vector bundle functor F on $\mathcal{FM}_m$ is the functor $F^{\langle r\rangle} : \mathcal{FM}_m \to \mathcal{VB}$, see [8]. More precisely, for a $\mathcal{FM}_m$-object $p : Y \to M$ we have vector bundle

$$F^{\langle r\rangle}Y = \bigcup_{y\in Y}\{j^r_{p(y)}\sigma \mid \sigma : M \to T_yY_{p(y)}\}$$

over Y. For a $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\underline{f} : M_1 \to M_2$ we have vector bundle map $F^{\langle r\rangle}f : F^{\langle r\rangle}Y_1 \to F^{\langle r\rangle}Y_2$ covering f by

$$F^{\langle r\rangle}f(j^r_{p(y)}\sigma) = j^r_{\underline{f}(p(y))}(Tf \circ \sigma \circ \underline{f}^{-1})\,.$$

In [4], we classified all fiber product preserving bundle functors F on $\mathcal{FM}_m$ of finite order r in terms of triples (A, H, t), where A is a Weil algebra, $H : G^r_m \to \mathrm{Aut}(A)$ is a smooth group homomorphism from $G^r_m = \mathrm{inv}\, J^r_0(\mathbf{R}^m, \mathbf{R}^m)_0$ into the group $\mathrm{Aut}(A)$ of algebra automorphisms of A, and $t : \mathcal{D}^r_m \to A$ is a G^r_m-equivariant algebra homomorphism from $\mathcal{D}^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A. Moreover, we proved that all fiber product preserving bundle functors F on $\mathcal{FM}_m$ are of finite order. Analyzing the construction on (A, H, t) one can easily seen that the triple (A, H, t) corresponding to a vertical F in question has trivial t. Moreover, one can easily seen that the triple (A, H, t) corresponding to a vector fiber product preserving bundle functor F in question has "vector Weil algebra" A which is of the form $A = \mathbf{R} \times W$ for some vector space W, see [3; Lemma 37.2]. It implies that all vertical fiber product preserving vector bundle functors on $\mathcal{FM}_m$ can be constructed (up to $\mathcal{FM}_m$-equivalence) as follows.

Let $G : \mathcal{M}f_m \to \mathcal{VB}$ be a vector natural bundle. For any $\mathcal{FM}_m$-object $p : Y \to M$ we put

$$F^GY = GM \otimes VY\,,$$

the tensor product over Y of the pull-back of GM with respect $p : Y \to M$ and the vertical bundle $VY \to Y$ of Y, and for any $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\underline{f} : M_1 \to M_2$ we put

$$F^Gf = G\underline{f} \otimes Vf : F^GY_1 \to F^GY_2\,.$$

The correspondence $F^G : \mathcal{FM}_m \to \mathcal{VB}$ is a vertical fiber product preserving vector bundle functor on $\mathcal{FM}_m$.

In the present short note we study the problem how a projectable vector field X on a fibered mainfold Y from $\mathcal{FM}_{m,n}$ can induce canonically a vector field $A(X)$ on FY for some vertical fiber product preserving vector bundle functor $F : \mathcal{FM}_m \to \mathcal{VB}$. This problem is reflected in the concept of $\mathcal{FM}_{m,n}$-natural operators $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF$ in the sense of [3]. We can assume that $F = F^G$

for some vector natural bundle $G : \mathcal{M}f_m \to \mathcal{VB}$. We deduce (see Theorem 1) that any such operator A is of the form

$$A(X) = \lambda \mathcal{F}^G X + B(\underline{X}) \otimes L$$

for some real number λ and some $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TG$ of vertical type transforming vector fields $\underline{X} \in \mathcal{X}(M)$ from m-dimensional manifolds M into linear vertical vector fields $B(\underline{X})$ on GM, where X denote a projectable vector field on a $\mathcal{FM}_{m,n}$-object Y, $\underline{X}$ is the underlying vector field of X, $\mathcal{F}^G X$ is the flow lifting of X and L is the Liouville vector field on the vertical vector bundle $VY \to Y$ of Y.

In Section 3, we explain this main result for some more known vertical fiber product preserving vector bundle functors F on $\mathcal{FM}_m$. In particular, for the vertical functor V we reobtain the result saying that any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TV$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on V.

The trivial bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$ with standard fiber $\mathbf{R}^n$ will be denoted by $\mathbf{R}^{m,n}$. The coordinates on $\mathbf{R}^m$ will be denoted by $x^1, \dots, x^m$. The fiber coordinates on $\mathbf{R}^{m,n}$ will be denoted by $y^1, \dots, y^n$.

All manifolds are assumed to be finite dimensional and smooth. Maps are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$.

1. The main result

From now on, let F be a vertical fiber product preserving vector bundle functor on $\mathcal{FM}_m$. We can assume that $F = F^G$ for some natural vector bundle $G : \mathcal{M}f_m \to \mathcal{VB}$, where F^G is described in Introduction.

A $\mathcal{FM}_{m,n}$-natural operator $T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^G$ is a $\mathcal{FM}_{m,n}$-invariant family of regular operators $A : \mathcal{X}_{\mathrm{proj}}(Y) \to \mathcal{X}(F^G Y)$ for $\mathcal{FM}_{m,n}$-objects Y. The invariance means that if projectable vector fields X' and X" on $\mathcal{FM}_{m,n}$-objects are f related then $A(X')$ and $A(X")$ are $F^G f$-related. The regularity means that A transforms smoothly parametrized family of projectable vector fields into smoothly parametrized family of vector fields.

We present examples of $\mathcal{FM}_{m,n}$-natural operators $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^G$.

Example 1 (*The flow operator*)**.** Let X be a projectable vector field on a $\mathcal{FM}_{m,n}$-object Y. The flow Fl^X_t of X is formed by $\mathcal{FM}_{m,n}$-morphisms. Applying F^G we obtain a flow $F^G(Fl^X_t)$ on $F^G Y$. The last flow generates a vector field $\mathcal{F}^G X$ on $F^G Y$. The corresponding $\mathcal{FM}_{m,n}$-operator $\mathcal{F}^G : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^G$ is called the flow operator.

Example 2. Suppose we have a vertical type $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TG$ lifting vector fields $\underline{X}$ from m-manifolds M into linear vector fields $B(\underline{X})$ on GM. Let X be a projectable vector field on a $\mathcal{FM}_{m,n}$-object $Y \to M$ with the underlying vector field $\underline{X}$ on M. Applying B to $\underline{X}$ we produce the linear vertical vector field $B(\underline{X})$ on GM. On VY we have the Liouville vector field L generated by fiber homotheties of VY. Clearly, L is vertical and linear. Then we have the tensor product $B(\underline{X}) \otimes L \in \mathcal{X}_{\mathrm{lin}}(F^G Y)$ (generated by the tensor product of the flows of $B(\underline{X})$ and L). Thus we have the corresponding $\mathcal{FM}_{m,n}$-natural operator $B \otimes L : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^G$ of vertical type.

Theorem 1. *Let $G : \mathcal{M}f_m \to \mathcal{VB}$ be a natural vector bundle. Any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^G$ is of the form*

$$A(X) = \lambda \mathcal{F}^G X + B(\underline{X}) \otimes L$$

for some real number λ and some $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TG$ lifting vector fields $\underline{X}$ from m-manifolds M into vertical linear vector fields $B(\underline{X})$ on GM, where X denotes a projectable vector field on a $\mathcal{FM}_{m,n}$-object Y, $\underline{X}$ is the underlying vector field of X, $\mathcal{F}^G X$ is the flow lifting of X and L is the Liouville vector field on $VY \to Y$.

2. Proof of Theorem 1

We will modify the proof of Theorem 1 in [5].

Since any projectable vector field X with the non-vanishing underlying vector field is locally $\frac{\partial}{\partial x^1}$ in some $\mathcal{FM}_{m,n}$-trivialization, A is uniquely determined by $A\big(\frac{\partial}{\partial x^1}\big)$ over $(0,0) \in \mathbf{R}^{m,n}$.

Consider $T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(u) \in T_{(0,0)}\mathbf{R}^{m,n}$ for $u \in F^G_{(0,0)}\mathbf{R}^{m,n}$, where $\pi : F^G Y \to Y$ is the bundle projection. Using the invariance of $A\big(\frac{\partial}{\partial x^1}\big)$ with respect to the fiber homotheties we deduce that $T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(u) = T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(tu)$ for any $u \in F^G_{(0,0)}\mathbf{R}^{m,n}$, $t \neq 0$. Then $T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(u) = T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(0)$ for u as above. Then using the invariance of $A\big(\frac{\partial}{\partial x^1}\big)$ with respect to the $\mathcal{FM}_{m,n}$-maps $(x^1, tx^2, \ldots, tx^m, ty^1, \ldots, ty^n)$ we deduce that $T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(u) = \lambda \frac{\partial}{\partial x^1}(0,0)$ for some real number λ. Then replacing A by $A - \lambda \mathcal{F}^G$ we have $T\pi \circ A\big(\frac{\partial}{\partial x^1}\big)(u) = 0$ for any u as above. Then A is of vertical type because of the first sentence of the proof.

We define an $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TG$ as follows.

Let $\underline{X} \in \mathcal{X}(M)$ be a vector field on an m-manifold M. We consider $\underline{X}$ as the projectable vector field $\underline{X}$ on the trivial vector bundle $M \times \mathbf{R}^n$ over M. Applying A we obtain the vertical vector field $A(\underline{X})$ on $F^G(M \times \mathbf{R}^n)$. Using the invariance of $A(\underline{X})$ with respect to the fiber homotheties of $M \times \mathbf{R}^n$ we easily see that $A(\underline{X})$ is linear. Define a fiber linear map $\tilde{A}(\underline{X}) : GM \to GM$ covering id_M by

$$\tilde{A}(\underline{X})(y) = \langle pr_2 \circ A(\underline{X})\big(y \otimes e_1(x)\big), e_1^*(x)\rangle\,,$$

$y \in G_x M$, $x \in M$, where $pr_2 : V\big(F^G(M \times \mathbf{R}^n)\big) \tilde{=} F^G(M \times \mathbf{R}^n) \times_{M \times \mathbf{R}^n} F^G(M \times \mathbf{R}^n) \to F^G(M \times \mathbf{R}^n)$ is the projection on the second (essential) factor, $e_1(x), \ldots, e_n(x)$ is the usual basis of vector space $V_{(x,0)}(M \times \mathbf{R}^n)$ (i.e. $e_j(x) = \frac{\partial}{\partial y^j}(x,0)$) and

$e_1^*(x),\dots,e_n^*(x)$ is the dual basis. Since $A(\underline{X})$ is linear, $\tilde{A}(\underline{X})$ is a vector bundle map. Consequently we have a linear vertical vector field $B(\underline{X})$ on GM such that

$$B(\underline{X})(y)=\frac{d}{dt}_{|0}\big(y+t(\tilde{A}(\underline{X})(y)-y)\big)\,,$$

$y\in G_xM$, $x\in M$. Let $B:T_{\mathcal{M}f_m}\rightsquigarrow TG$ be the corresponding $\mathcal{M}f_m$-natural operator.

It remains to show that $A=B\otimes L$.

It is immediately seen that

$$\Big\langle pr_2\circ\Big(B\Big(\frac{\partial}{\partial x^1}\Big)\otimes L\Big)\big(y\otimes e_1(0)\big),e_1^*(0)\Big\rangle=\Big\langle pr_2\circ A\Big(\frac{\partial}{\partial x^1}\Big)\big(y\otimes e_1(0)\big),e_1^*(0)\Big\rangle$$

for all $y\in G_0\mathbf{R}^m$. Let $i=2,\dots,n$. Then by the invariance of $A\big(\frac{\partial}{\partial x^1}\big)$ and $B\big(\frac{\partial}{\partial x^1}\big)\otimes L$ with respect to a $\mathcal{FM}_{m,n}$-map of the form $\mathrm{id}_{\mathbf{R}^m}\times\psi$ for a linear isomorphism $\psi:\mathbf{R}^n\to\mathbf{R}^n$ preserving $e_1(0)$ and sending $e_1^*(0)$ into $e_1^*(0)+e_i^*(0)$ we obtain that

$$\Big\langle pr_2\circ\Big(B\Big(\frac{\partial}{\partial x^1}\Big)\otimes L\Big)\big(y\otimes e_1(0)\big),e_i^*(0)\Big\rangle=\Big\langle pr_2\circ A\Big(\frac{\partial}{\partial x^1}\Big)\big(y\otimes e_1(0)\big),e_i^*(0)\Big\rangle$$

for all $i=1,\dots,n$ and all $y\in G_0\mathbf{R}^m$. Then

$$pr_2\circ\Big(B\Big(\frac{\partial}{\partial x^1}\Big)\otimes L\Big)\big(y\otimes e_1(0)\big)=pr_2\circ A\Big(\frac{\partial}{\partial x^1}\Big)\big(y\otimes e_1(0)\big)$$

for all $y\in G_0\mathbf{R}^m$. Then by the invariance of $B\big(\frac{\partial}{\partial x^1}\big)\otimes L$ and $A\big(\frac{\partial}{\partial x^1}\big)$ with respect to $\mathcal{FM}_{m,n}$-maps of the form $\mathrm{id}_{\mathbf{R}^m}\times\psi$ for linear isomorphisms $\psi:\mathbf{R}^n\to\mathbf{R}^n$ we deduce that

$$pr_2\circ\Big(B\Big(\frac{\partial}{\partial x^1}\Big)\otimes L\Big)(y\otimes e)=pr_2\circ A\Big(\frac{\partial}{\partial x^1}\Big)(y\otimes e)$$

for all not-zero $e\in V_{(0,0)}(\mathbf{R}^{m,n})$ and all $y\in G_0\mathbf{R}^m$. Then

$$pr_2\circ A\Big(\frac{\partial}{\partial x^1}\Big)(u)=pr_2\circ\Big(B\Big(\frac{\partial}{\partial x^1}\Big)\otimes L\Big)(u)$$

for all $u\in F^G_{(0,0)}\mathbf{R}^{m,n}$. (For, all $y\otimes e$ as above generate $F^G_{(0,0)}\mathbf{R}^{m,n}$, and both sides of the last equality are linear in u because of the invariance of $A\big(\frac{\partial}{\partial x^1}\big)$ and $B\big(\frac{\partial}{\partial x^1}\big)\otimes L$ with respect to the fiber homotheties of $\mathbf{R}^{m,n}$ and the homogeneous function theorem.) Then $A(\frac{\partial}{\partial x^1})=B(\frac{\partial}{\partial x^1})\otimes L$ over $(0,0)\in\mathbf{R}^{m,n}$. Then $A=B\otimes L$ because of the first sentence of the proof. $\square$

3. Applications

Let p be a non-negative integer. Let $T^{(p,0)} = \otimes^p T : \mathcal{M}f_m \to \mathcal{VM}$ be the natural vector bundle of tensor fields of type $(p,0)$ over m-manifolds. Let $F^{T^{(p,0)}} : \mathcal{FM}_m \to \mathcal{VB}$ be the corresponding vertical fiber product preserving vector bundle functor (see Introduction). Clearly, $F^{T^{(0,0)}}$ is isomorphic to the vertical functor V. In [2], I. Kolář showed that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{(p,0)}$ lifting vector fields $\underline{X}$ from m-manifolds M into linear vertical vector fields $B(\underline{X})$ on $T^{(p,0)}M$ is a constant multiple of the Liouville vector field on $T^{(p,0)}$. Thus we have the following corollary.

Corollary 1. *Any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^{T^{(p,0)}}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on $F^{T^{(p,0)}}$.*

Let $T^{(r)} = (J^r(\cdot,\mathbf{R})_0)^* : \mathcal{M}f_m \to \mathcal{VB}$ be the linear r-tangent bundle functor. Let $F^{T^{(r)}} : \mathcal{FM}_m \to \mathcal{VB}$ be the corresponding vertical fiber product preserving gauge bundle functor. In [7], we showed that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{(r)}$ lifting vector fields $\underline{X}$ from m-manifolds M into linear vertical vector fields $B(\underline{X})$ on $T^{(r)}M$ is a constant multiple of the Liouville vector field on $T^{(r)}$ (see also [1] in the case $r = 2$). Thus we obtain the following result.

Corollary 2. *Any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^{T^{(r)}}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on $F^{T^{(r)}}$.*

Let $T^{r*} = J^r(\cdot,\mathbf{R})_0 : \mathcal{M}f_m \to \mathcal{VB}$ be the r-cotangent bundle functor. Let $F^{T^{r*}} : \mathcal{FM}_m \to \mathcal{VB}$ be the corresponding vertical fiber product preserving gauge bundle functor. From [6] it follows that any $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TT^{r*}$ is a constant multiple of the Liouville vector field on T^{r*}. Thus we obtain the following result.

Corollary 3. *Any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^{T^{r*}}$ is a linear combination with real coefficients of the flow operator and the Liouville vector field on $F^{T^{r*}}$.*

Let $E^{r*} = T^{r*} \times \mathbf{R} = J^r(\cdot,\mathbf{R}) : \mathcal{M}f_m \to \mathcal{VB}$ be the extended r-cotangent bundle functor. The corresponding vertical fiber product preserving vector bundle functor on $\mathcal{FM}_m$ is equivalent to the mentioned in Introduction functor $F^{\langle r\rangle} : \mathcal{FM}_m \to \mathcal{VB}$.

Lemma 1 ([5]). *The vector space of all $\mathcal{M}f_m$-natural operator $B : T_{\mathcal{M}f_m} \rightsquigarrow TE^{r*}$ transforming vector fields $\underline{X}$ from m-manifolds M into linear vertical vector fields $B(\underline{X})$ on $E^{r*}M$ is of dimension less or equal to $r+2$.*

Thus using the dimension argument we obtain the following result.

Corollary 4. *Any $\mathcal{FM}_{m,n}$-natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TF^{\langle r\rangle}$ is a linear combination with real coefficients of the flow operator, the Liouville vector field on $F^{\langle r\rangle}$ and the described in Example 3 in [8] operators $V^{\langle s\rangle}$ for $s = 0,\dots,r$.*

We recall that given a projectable vector field X on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$ covering a vector field $\underline{X}$ on M the vector field $V^{\langle s\rangle}(X)$ on $F^{\langle r\rangle}Y$ is defined as follows. Let $j^r_{p(y)}\sigma \in F^{\langle r\rangle}_y Y$, $\sigma : M \to T_y Y_{p(y)}$, $y \in Y$. We put

$$V^{\langle s\rangle}(X)(y) = \big(y, j^r_{p(y)}(\underline{X}^{(s)}\sigma(p(y)))\big) \in \{y\} \times F^{\langle r\rangle}_y Y = V_y F^{\langle r\rangle} Y\,,$$

where $\underline{X}^{(s)} = X \circ \dots \circ X$ (s-times) and $\underline{X}^{(s)}\sigma\big(p(y)\big) : M \to T_y Y_{p(y)}$ is the constant map.

Remark 1. In [8], we introduced the concept of admissible fiber product preserving bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$. Let (A, H, t) be the triple corresponding to F. Then F is called admissible if for every derivation $D \in \operatorname{Der}(A)$,

$$\text{if}\quad H\big(j^r_0(\tau\, \mathrm{id}_{\mathbf{R}^m})\big) \circ D \circ H\big(j^r_0(\tau^{-1}\, \mathrm{id}_{\mathbf{R}^m})\big) \to 0 \quad\text{as}\quad \tau \to 0 \quad\text{then}\quad D = 0\,.$$

In [8], we also studied the problem how a projectable vector field X on a fibered manifold Y from $\mathcal{FM}_{m,n}$ can induce a vector field $A(X)$ on FY. We proved that for admissible F we have

$$A(X) = \lambda\mathcal{F}(X) + \text{ a canonical vector field}\,,$$

where $\lambda \in \mathbf{R}$ depends only on A (not on X). From this characterization and Theorem 1 it follows directly that F^G is not admissible if there exists an essentially dependent on $\underline{X}$ natural operator $B : T_{\mathcal{M}f_m} \to \mathcal{VB}$ transforming vector fields $\underline{X}$ from m-manifolds M into vertical linear vector fields on GM. For example, $F^{E^{r*}} : \mathcal{FM}_m \to \mathcal{VB}$ is not admissible.

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