ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 195 – 203

GEOMETRIC STRUCTURES ON THE TANGENT BUNDLE OF THE EINSTEIN SPACETIME

JOSEF JANYŠKA

Abstract. We describe conditions under which a spacetime connection and a scaled Lorentzian metric define natural symplectic and Poisson structures on the tangent bundle of the Einstein spacetime.

Introduction

Geometrical structures induced on the tangent bundle of the Einstein spacetime play a fundamental role in the covariant classical and quantum mechanics. The covariant classical and quantum mechanics over the Einstein spacetime proposed in [3, 4] is natural in the sense of [7, 8, 10] and independent of the base of scales, so the "spaces of scales" are systematically used. Roughly speaking, a space of scales has the algebraic structure of $I\!R^+$ but has no distinguished 'basis'. The basic objects of the theory (metric, 2-forms, 2-vectors, etc.) are valued into *scaled* vector bundles, that is into vector bundles multiplied tensorially with spaces of scales. In this way, each tensor field carries explicit information on its "scale dimension". Actually, in this paper, we assume the space of *lengths* $\mathbb{L}$. Moreover, $\mathbb{L}^p$ denotes $\otimes^p\mathbb{L}$.

In [1, 5] the classification of symplectic and Poisson structures on the tangent bundle of a pseudo-Riemannian manifold was given for a non-scaled metric g and a torsion free linear connection K. In this case the metric g and the connection K admit a family of symplectic 2-forms $\Upsilon[g,K]$ or Poisson 2-vectors $\Lambda[g,K]$ on the tangent bundle parametrized by a function $\mu(g(u,u))$ satisfying certain conditions. Moreover, g and K are related by the condition that ∇g is a symmetric (0,3)-tensor field. For a scaled metric g and a general spacetime connections the constructions of the 2-form $\Upsilon[g,K]$ and the 2-vector $\Lambda[g,K]$ are the same but

2000 *Mathematics Subject Classification*: 53B15, 53B30, 53D05, 53D17, 58A10, 58A20, 58A32.

Key words and phrases: spacetime, spacetime connection, Schouten bracket, Frölicher–Nijenhuis bracket, symplectic structure, Poisson structure.

This research has been supported by the Ministry of Education of the Czech Republic under the project MSM0021622409 and by the Grant agency of the Czech Republic under the project GA 201/05/0523.

Received August 24, 2005.

from the independence on the base of scales it follows that $\Upsilon[g,K]$ and $\Lambda[g,K]$ are unique, up to a multiplicative real constant. In this paper we generalize the results of [1, 5] for a scaled metric and a general spacetime connection. Namely, we shall describe the condition under which the 2-form $\Upsilon[g,K]$ and the 2-vector $\Lambda[g,K]$ give symplectic and Poisson structures, respectively.

If $\boldsymbol{E}$ is a manifold, then the tangent bundle will be denoted by $\tau[\boldsymbol{E}] : T\boldsymbol{E} \to \boldsymbol{E}$ and local coordinates (x^λ) on $\boldsymbol{E}$ induce the fibered local coordinates $(x^\lambda, \dot{x}^\lambda)$ on $T\boldsymbol{E}$. By $\mathrm{map}(\boldsymbol{E}, \boldsymbol{E}')$ we denote the sheaf of smooth maps.

1. Geometry of the spacetime

We recall basic properties of the Einstein spacetime and its tangent bundle.

1.1. Spacetime. We assume *spacetime* to be an oriented and time oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a scaled Lorentzian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ with signature $(-+++)$. The dual metric will be denoted by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{*2} \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$. Let us note that the dimension is not relevant. Our results are valid for any dimension $n \geq 3$ and a pseudo-Riemannian metric of the signature $(1, n-1)$.

A *spacetime chart* is defined to be an ordered chart $(x^0, x^i) \in \mathrm{map}(\boldsymbol{E}, I\!R \times I\!R^3)$ of $\boldsymbol{E}$, which fits the orientation of spacetime and such that the vector ∂_0 is timelike and time oriented and the vectors $\partial_1, \partial_2, \partial_3$ are spacelike. In the following we shall always refer to spacetime charts. Latin indices $i, j, \dots$ will span spacelike coordinates, while Greek indices $\lambda, \mu, \dots$ will span spacetime coordinates.

We have the coordinate expressions

$$g = g_{\lambda\mu}\, d^\lambda \otimes d^\mu\,, \qquad \text{with} \qquad g_{\lambda\mu} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes I\!R)$$
$$\bar{g} = g^{\lambda\mu}\, \partial_\lambda \otimes \partial_\mu\,, \qquad \text{with} \qquad g^{\lambda\mu} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{*2} \otimes I\!R)\,.$$

1.2. Spacetime connections. We define a *(general) spacetime connection* to be a connection K of the bundle $\tau[\boldsymbol{E}] : T\boldsymbol{E} \to \boldsymbol{E}$. We recall that a connection K of the bundle $T\boldsymbol{E} \to \boldsymbol{E}$ can be expressed, equivalently, by a tangent valued form $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, which is projectable over $\mathbf{1} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, or by the vertical valued form $\nu[K] : T\boldsymbol{E} \to T^*T\boldsymbol{E} \otimes VT\boldsymbol{E}$. Their coordinate expressions are of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^\nu\, \dot{\partial}_\nu)\,, \qquad \nu[K] = (\dot{d}^\nu - K_\lambda{}^\nu\, d^\lambda) \otimes \dot{\partial}_\nu\,, \tag{1.1}$$

where $K_\lambda{}^\nu \in \mathrm{map}(T\boldsymbol{E}, I\!R)$ and $(\partial_\lambda, \dot{\partial}_\lambda)$ or $(d^\lambda, \dot{d}^\lambda)$ are the induced bases of local sections of $TT\boldsymbol{E} \to T\boldsymbol{E}$ or $T^*T\boldsymbol{E} \to T\boldsymbol{E}$, respectively.

The connection K is said to be *linear* if it is a linear fibred morphism over $\mathbf{1} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$. Moreover, the connection K is linear if and only if its coordinate expression is of the type

$$K_\lambda{}^\nu = K_\lambda{}^\nu{}_\mu\, \dot{x}^\mu\,, \qquad \text{with} \qquad K_\lambda{}^\nu{}_\mu \in \mathrm{map}(\boldsymbol{E}, I\!R)\,.$$

The *torsion* of the connection K is defined to be the vertical valued 2–form

$$\tau[K] =: -[\vartheta, K] : T\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes VT\boldsymbol{E}\,,$$

where $[\,,]$ is the Frölicher-Nijenhuis bracket and $\vartheta : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes VT\boldsymbol{E}$ is the natural vertical valued 1–form with the coordinate expression $\vartheta = d^\lambda \otimes \dot\partial_\lambda$. We have the coordinate expression

$$\tau[K] = \dot\partial_\mu K_\lambda{}^\nu \, d^\lambda \wedge d^\mu \otimes \dot\partial_\nu \,. \tag{1.2}$$

In the linear case, the torsion can be identified with a section $\tau[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes T\boldsymbol{E}$ and its coordinate expression turns out to be the usual formula $\tau[K] = K_\lambda{}^\nu{}_\mu \, d^\lambda \wedge d^\mu \otimes \partial_\nu$. Thus, the connection K is linear and torsion free if and only if its coordinate expression is of the type

$$K_\lambda{}^\nu = K_\lambda{}^\nu{}_\mu \, \dot x^\mu \,, \qquad \text{with} \qquad K_\lambda{}^\nu{}_\mu = K_\mu{}^\nu{}_\lambda \in \operatorname{map}(\boldsymbol{E}, I\!R) \,.$$

We shall denote by $K[g]$ the canonical torsion free linear spacetime metric connection given by $\nabla g = 0$. We have

$$K[g]_\mu{}^\lambda{}_\nu = -\frac{1}{2} \, g^{\lambda\rho} \, (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \,. \tag{1.3}$$

The *curvature* of the connection K is defined to be the vertical valued 2–form

$$R[K] =: -[K, K] : T\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes VT\boldsymbol{E} \,, \tag{1.4}$$

where $[\,,]$ is the Frölicher-Nijenhuis bracket. We have the coordinate expression

$$\begin{aligned} R[K] &= R[K]_{\lambda\mu}{}^\nu \, d^\lambda \wedge d^\mu \otimes \dot\partial_\nu \\ &= -2 \, (\partial_\lambda K_\mu{}^\nu + K_\lambda{}^\rho \, \dot\partial_\rho K_\mu{}^\nu) \, d^\lambda \wedge d^\mu \otimes \dot\partial_\nu \,. \end{aligned} \tag{1.5}$$

In the linear case, the coordinate expression turns out to be the usual formula

$$\begin{aligned} R[K] &= R[K]_{\lambda\mu}{}^\nu{}_\sigma \, \dot x^\sigma d^\lambda \wedge d^\mu \otimes \dot\partial_\nu \\ &= -2 \, (\partial_\lambda K_\mu{}^\nu{}_\sigma + K_\lambda{}^\rho{}_\sigma \, K_\mu{}^\nu{}_\rho) \, \dot x^\sigma d^\lambda \wedge d^\mu \otimes \dot\partial_\nu \,. \end{aligned} \tag{1.6}$$

Hence, in the linear case, the curvature can be identified with a section

$$R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes T\boldsymbol{E} \otimes T^*\boldsymbol{E} \,,$$

with the usual coordinate expression

$$\begin{aligned} R[K] &= R[K]_{\lambda\mu}{}^\nu{}_\sigma \, d^\lambda \wedge d^\mu \otimes \partial_\nu \otimes d^\sigma \\ &= -2 \, (\partial_\lambda K_\mu{}^\nu{}_\sigma + K_\lambda{}^\rho{}_\sigma \, K_\mu{}^\nu{}_\rho) \, d^\lambda \wedge d^\mu \otimes \partial_\nu \otimes d^\sigma \,. \end{aligned} \tag{1.7}$$

1.3. **The Lie derivative and the exterior covariant differential with respect to a spacetime connection.** A (general) spacetime connection K considered as a tangent valued 1-form on $T\boldsymbol{E}$ admits as usual, [8], the Lie derivative of forms on $T\boldsymbol{E}$. Namely,

$$L[K] \, \phi = \big(i(K) \, d - d \, i(K)\big) \, \phi : T\boldsymbol{E} \to \Lambda^{r+1} T^* T\boldsymbol{E}$$

for any r-form $\phi : T\boldsymbol{E} \to \Lambda^r T^* T\boldsymbol{E}$. Similarly we can define the Lie derivative

$$L\big[R[K]\big] \, \phi = \big(i(R[K]) \, d + d \, i(R[K])\big) \, \phi : T\boldsymbol{E} \to \Lambda^{r+2} T^* T\boldsymbol{E} \,.$$

On the other hand a linear spacetime connection K admits covariant exterior differential, [8], of vector-valued forms on $\boldsymbol{E}$. We apply this operation on $T^*\boldsymbol{E}$-valued forms on $\boldsymbol{E}$ and compare it with the Lie derivative.

Let ϕ be an $T^*\boldsymbol{E}$-valued r-form on $\boldsymbol{E}$, or equivalently $\phi : \boldsymbol{E} \to \Lambda^r T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T^*\boldsymbol{E}$ be a section. The *covariant exterior differential* of ϕ with respect to K is then defined to be the $T^*\boldsymbol{E}$-valued $(r+1)$-form $d_K\phi$ on $\boldsymbol{E}$ given by

$$d_K\phi(X_1, \cdots, X_{r+1})(Y) = \sum_{i=1}^{r+1} (-1)^{i+1} \nabla_{X_i}(\phi(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}))(Y) + \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1})(Y), \tag{1.8}$$

for any vector fields $Y, X_1, \cdots, X_{r+1}$ on $\boldsymbol{E}$, the vector fields $\hat{X}_i$ being omitted.

Any $T^*\boldsymbol{E}$-valued r-form on $\boldsymbol{E}$ can be considered to be a linear horizontal r-form on $T\boldsymbol{E}$. Then we have

Lemma 1.1. *Let ϕ be a linear horizontal r-form on $T\boldsymbol{E}$ and K be a spacetime connection. Then the Lie derivative $L[K]\,\phi$ is a linear horizontal $(r+1)$-form on $T\boldsymbol{E}$ if and only if K is linear. Moreover, $(r+1)\,L[K]\,\phi$ and $d_K\phi$ coincides.*

Proof. Let $\phi = \phi_{\rho\lambda_1\ldots\lambda_r}\, \dot{x}^\rho\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r}$, $\phi_{\rho\lambda_1\ldots\lambda_r} \in \operatorname{map}(\boldsymbol{E}, I\!R)$, be a linear horizontal r-form on $T\boldsymbol{E}$. Then we have

$$L[K]\,\phi = (\partial_\mu \phi_{\rho\lambda_1\ldots\lambda_r}\, \dot{x}^\rho + \phi_{\sigma\lambda_1\ldots\lambda_r}\, K_\mu{}^\sigma)\, d^\mu \wedge d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r}\,,$$

i.e., in the linear spacetime connection case,

$$L[K]\,\phi = (\partial_\mu \phi_{\rho\lambda_1\ldots\lambda_r} + \phi_{\sigma\lambda_1\ldots\lambda_r}\, K_\mu{}^\sigma{}_\rho)\, \dot{x}^\rho\, d^\mu \wedge d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r}\,,$$

which implies that $L[K]\,\phi$ is a linear horizontal $(r+1)$-form.

On the other hand ϕ can be considered to be a $T^*\boldsymbol{E}$-valued r-form on $\boldsymbol{E}$ with coordinate expression $\phi = \phi_{\rho\lambda_1\ldots\lambda_r}\, d^\rho \otimes (d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r})$. Then

$$d_K\phi = (r+1)\,(\partial_\mu \phi_{\rho\lambda_1\ldots\lambda_r} + \phi_{\sigma\lambda_1\ldots\lambda_r}\, K_\mu{}^\sigma{}_\rho)\, d^\rho \otimes (d^\mu \wedge d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r})\,. \qquad \square$$

Remark 1.2. Now we shall apply $L[K]$ and d_K on specific situation of the scaled metric g. The metric g can be considered to be a $\mathbb{L}^2 \otimes T^*\boldsymbol{E}$-valued 1-form on $\boldsymbol{E}$. Then the covariant exterior differential $d_K g$ is a $\mathbb{L}^2 \otimes T^*\boldsymbol{E}$-valued 2-form defined for any vector fields X, Y, Z by

$$(d_K g)(X, Y)(Z) = \big(\nabla_X(Y^\flat) - \nabla_Y(X^\flat) - ([X, Y]^\flat)\big)(Z)\,,$$

where $^\flat$ denotes the musical mapping $g^\flat : T\boldsymbol{E} \to \mathbb{L}^2 \otimes T^*\boldsymbol{E}$. We have the coordinate expression

$$d_K g = 2\,(\partial_\lambda g_{\rho\mu} + g_{\sigma\mu}\, K_\lambda{}^\sigma{}_\rho)\, d^\rho \otimes (d^\lambda \wedge d^\mu)\,. \tag{1.9}$$

On the other hand the musical mapping $g^\flat$ can be considered as a linear horizontal 1-form on $T\boldsymbol{E}$ with the coordinate expression $g^\flat = g_{\lambda\mu}\, \dot{x}^\lambda\, d^\mu$. Then we have the coordinate expression

$$L[K]\, g^\flat = (\partial_\lambda g_{\rho\mu}\, \dot{x}^\rho + g_{\rho\mu}\, K_\lambda{}^\rho)\, d^\lambda \wedge d^\mu \tag{1.10}$$

and, if K is linear,

$$L[K]\, g^\flat = (\partial_\lambda g_{\rho\mu} + g_{\sigma\mu}\, K_\lambda{}^\sigma{}_\rho)\, \dot{x}^\rho\, d^\lambda \wedge d^\mu\,, \tag{1.11}$$

i.e., in the linear case, $L[K]\,g^\flat$ is a linear horizontal 2-form on $\boldsymbol{TE}$ which can be considered to be a $\mathbb{L}^2 \otimes T^*\boldsymbol{E}$ valued 2-form on $\boldsymbol{E}$ which coincides with $\frac{1}{2}\, d_K\, g$. □

1.4. **Spacetime 2–forms and 2–vectors.** The map $T\tau[\boldsymbol{E}] : TT\boldsymbol{E} \to T\boldsymbol{E}$, can be regarded as a vector valued 1-form $\upsilon : T\boldsymbol{E} \to T^*T\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} T\boldsymbol{E}$, with coordinate expression $\upsilon = d^\lambda \otimes \partial_\lambda$.

We define the *spacetime 2–form* of $T\boldsymbol{E}$ associated with g and a spacetime connection K to be the scaled 2–form

$$\Upsilon[g,K] =: g \lrcorner \big(\nu[K] \wedge \upsilon\big) : T\boldsymbol{E} \to \mathbb{L}^2 \otimes \Lambda^2 T^*T\boldsymbol{E}.$$

We have the coordinate expression

$$\Upsilon[g,K] = g_{\lambda\mu}\,(\dot{d}^\lambda - K_\nu{}^\lambda\, d^\nu) \wedge d^\mu \tag{1.12}$$

and, if K is linear,

$$\Upsilon[g,K] = g_{\lambda\mu}\,(\dot{d}^\lambda - K_\nu{}^\lambda{}_\rho\, \dot{x}^\rho\, d^\nu) \wedge d^\mu \tag{1.13}$$

We define the *spacetime 2–vector* of $T\boldsymbol{E}$ associated with g and a spacetime connection K to be the scaled 2–vector

$$\Lambda[g,K] =: \bar{g} \lrcorner \big(K \wedge \vartheta\big) : T\boldsymbol{E} \to \mathbb{L}^{*2} \otimes \Lambda^2 TT\boldsymbol{E}.$$

We have the coordinate expression

$$\Lambda[g,K] = g^{\lambda\mu}\,(\partial_\lambda + K_\lambda{}^\nu\, \dot{\partial}_\nu) \wedge \dot{\partial}_\mu \tag{1.14}$$

and, if K is linear,

$$\Lambda[g,K] = g^{\lambda\mu}\,(\partial_\lambda + K_\lambda{}^\nu{}_\rho\, \dot{x}^\rho\, \dot{\partial}_\nu) \wedge \dot{\partial}_\mu. \tag{1.15}$$

Lemma 1.3. *We have*

$$i(\Lambda[g,K])\Upsilon[g,K] = -4.$$

Proof. We have

$$i(\Lambda[g,K])\Upsilon[g,K] = -g^{\lambda\mu}\, g_{\lambda\mu} = -4.$$

□

2. Induced structures on the tangent bundle of the spacetime

We study symplectic and Poisson structures induced on the tangent bundle of the spacetime by the metric g and a spacetime connection K.

2.1. **General spacetime connection case.** Let us assume a spacetime connection K given by (1.1) , the spacetime 2–form $\Upsilon[g,K]$ given by (1.12) and the spacetime 2–vector $\Lambda[g,K]$ given by (1.14).

Lemma 2.1. $\Upsilon[g,K]$ *is closed if and only if the following two conditions are satisfied*

$$\partial_\nu g_{\lambda\mu} + g_{\rho\mu}\, \dot{\partial}_\lambda K_\nu{}^\rho - \partial_\mu g_{\lambda\nu} - g_{\rho\nu}\, \dot{\partial}_\lambda K_\mu{}^\rho = 0 \tag{2.1}$$

$$R[K]_{\lambda\mu\nu} + R[K]_{\mu\nu\lambda} + R[K]_{\nu\lambda\mu} = 0, \tag{2.2}$$

where we have set $R[K]_{\lambda\mu\nu} = g_{\rho\nu}\, R[K]_{\lambda\mu}{}^\rho$.

Proof. It follows immediately from the coordinate expression

$$\begin{aligned} d\Upsilon[g,K] &= -(\partial_\lambda g_{\rho\nu}\, K_\mu{}^\rho + g_{\rho\nu}\, \partial_\lambda K_\mu{}^\rho)\, d^\lambda \wedge d^\mu \wedge d^\nu \\ &\quad - (\partial_\mu g_{\lambda\nu} \;+ g_{\rho\nu}\, \dot\partial_\lambda K_\mu{}^\rho)\, \dot d^\lambda \wedge d^\mu \wedge d^\nu \\ &= \frac{1}{2}\; R[K]_{\lambda\mu\nu}\, d^\lambda \wedge d^\mu \wedge d^\nu \\ &\quad - (\partial_\mu g_{\rho\nu} + g_{\sigma\nu}\, \dot\partial_\rho K_\mu{}^\sigma)\, (\dot d^\rho - K_\lambda{}^\rho\, d^\lambda) \wedge d^\mu \wedge d^\nu\,. \end{aligned}$$ □

Now we shall describe the geometrical interpretation of the equations (2.1) and (2.2). Let us consider the Liouville vector field $I = \dot x^\lambda\, \dot\partial_\lambda$.

Lemma 2.2. *The conditions* (2.1) *or* (2.2) *are equivalent with*

$$L[I]\, L[K]\, g^\flat = 0 \tag{2.3}$$

or

$$L[K]\, L[K]\, g^\flat = 0\,, \tag{2.4}$$

respectively.

Proof. We have

$$\begin{aligned} L[I]\, L[K]\, g^\flat &= \big(i(I)\, d + d\, i(I)\big)\, \big((\partial_\lambda g_{\rho\mu}\, \dot x^\rho + g_{\rho\mu}\, K_\lambda{}^\rho)\, d^\lambda \wedge d^\mu\big) \\ &= \dot x^\rho\, (\partial_\lambda g_{\rho\mu} + g_{\sigma\mu}\, \dot\partial_\rho K_\lambda{}^\sigma)\, d^\lambda \wedge d^\mu\,. \end{aligned}$$

It is easy to see that $L[I]\, L[K]\, g^\flat = 0$ if and only if the condition (2.1) is satisfied.

Further from (1.5) we have

$$L\big[R[K]\big]\, g^\flat = g_{\rho\nu}\, R[K]_{\lambda\mu}{}^\rho\, d^\lambda \wedge d^\mu \wedge d^\nu$$

i.e., the condition (2.2) is equivalent with $L\big[R[K]\big]\, g^\flat = 0$. But, from (1.4),

$$L\big[R[K]\big]\, g^\flat = -L\big[[K,K]\big]\, g^\flat = -2\, L[K]\, L[K]\, g^\flat.$$

Hence (2.2) is equivalent with $L[K]\, L[K]\, g^\flat = 0$. □

Lemma 2.3. *The Schouten bracket*

$$\big[\Lambda[g,K], \Lambda[g,K]\big] : T\boldsymbol{E} \to \mathbb{L}^{*4} \otimes \Lambda^3 TT\boldsymbol{E}$$

has the coordinate expression

$$\begin{aligned} \big[\Lambda[g,K], \Lambda[g,K]\big] &= 2\, g^{\rho\nu}\, (\partial_\rho g^{\lambda\mu} - g^{\sigma\lambda}\, \dot\partial_\sigma K_\rho{}^\mu)\, (\partial_\lambda + K_\lambda{}^\kappa\, \dot\partial_\kappa) \wedge \dot\partial_\mu \wedge \dot\partial_\nu \\ &\quad + R[K]^{\kappa\mu\nu}\, \dot\partial_\kappa \wedge \dot\partial_\mu \wedge \dot\partial_\nu\,, \end{aligned}$$

where we have set $R[K]^{\lambda\mu\nu} = g^{\lambda\rho}\, g^{\mu\sigma}\, R[K]_{\rho\sigma}{}^\nu$.

Proof. We have

$$i(\big[\Lambda[g,K], \Lambda[g,K]\big])\, \beta = 2\, i(\Lambda[g,K])\, di(\Lambda[g,K])\, \beta$$

for any closed 3-form β.

Then

$$\begin{aligned}\big[\Lambda[g,K],\Lambda[g,K]\big] &= 2\,g^{\rho\nu}\,(\partial_\rho g^{\lambda\mu} - g^{\sigma\lambda}\,\dot\partial_\sigma K_\rho{}^\mu)\,\partial_\lambda\wedge\dot\partial_\mu\wedge\dot\partial_\nu \\ &\quad + \Big(g^{\omega\nu}\,g^{\rho\mu}\,R_{\rho\omega}{}^\kappa + 2\,g^{\sigma\nu}\,K_\rho{}^\kappa\,(\partial_\sigma g^{\rho\mu} - g^{\rho\omega}\,\dot\partial_\omega K_\sigma{}^\mu)\Big)\,\dot\partial_\kappa\wedge\dot\partial_\mu\wedge\dot\partial_\nu \\ &= 2\,g^{\rho\nu}\,(\partial_\rho g^{\lambda\mu} - g^{\sigma\lambda}\,\dot\partial_\sigma K_\rho{}^\mu)\,(\partial_\lambda + K_\lambda{}^\kappa\,\dot\partial_\kappa)\wedge\dot\partial_\mu\wedge\dot\partial_\nu \\ &\quad + R[K]^{\kappa\mu\nu}\,\dot\partial_\kappa\wedge\dot\partial_\mu\wedge\dot\partial_\nu\,. \end{aligned}$$

□

Lemma 2.4. $\big[\Lambda[g,K],\Lambda[g,K]\big]=0$ *if and only if the conditions* (2.1) *and* (2.2) *are satisfied, i.e., if and only if the conditions* (2.3) *and* (2.4) *are satisfied.*

Proof. From Proposition 2.3 it follows that $\big[\Lambda[g,K],\Lambda[g,K]\big]=0$ if and only if

$$g^{\rho\nu}\,(\partial_\rho g^{\lambda\mu} - g^{\sigma\lambda}\,\dot\partial_\sigma K_\rho{}^\mu) - g^{\rho\mu}\,(\partial_\rho g^{\lambda\nu} + g^{\sigma\lambda}\,\dot\partial_\sigma K_\rho{}^\nu) = 0 \tag{2.5}$$

$$R[K]^{\kappa\mu\nu} + R[K]^{\mu\nu\kappa} + R[K]^{\nu\kappa\mu} = 0\,. \tag{2.6}$$

But by lowering the indices in (2.5) we get from $\partial_\rho g^{\lambda\mu} = -g^{\lambda\tau}\,g^{\mu\omega}\,\partial_\rho g_{\tau\omega}$ just (2.1) and by lowering indices in (2.6) we get just (2.2). □

Theorem 2.5. *The metric g and a general spacetime connection K induce on $T\boldsymbol{E}$ natural symplectic and natural Poisson structures if and only if the conditions* (2.3) *and* (2.4) *are satisfied.*

Proof. The regularity of g implies that $\Upsilon[g,K]$ and $\Lambda[g,K]$ are non degenerate. Lemmas 2.1, 2.2 and 2.4 then imply that $\Upsilon[g,K]$ and $\Lambda[g,K]$ define symplectic and Poisson structures, respectively, if and only if (2.3) and (2.4) are satisfied. □

2.2. **Linear spacetime connection case.** We assume a linear spacetime connection K.

Lemma 2.6. *Let K be a linear spacetime connection. $\Upsilon[g,K]$ is closed if and only if $L[K]\,g^\flat = 0$.*

Proof. By Lemmas 2.1 and 2.2 $\Upsilon[g,K]$ is closed if and only if (2.3) and (2.4) are satisfied. But for a linear spacetime connection K the horizontal 2-form $L[K]\,g^\flat$ is linear. Moreover, for any linear horizontal r-form ϕ, we have $L[I]\phi=\phi$, i.e.,

$$L[I]\,L[K]\,g^\flat = L[K]\,g^\flat\,.$$

Then the condition (2.3) is equivalent with $L[K]\,g^\flat = 0$ which implies (2.4). □

Remark 2.7. In [1] we have proved that the spacetime 2-form $\Upsilon[g,K]$ is closed if and only if $d_K g = 0$. By Remark 1.2 it coincides with Lemma 2.6. □

Lemma 2.8. *Let K be a linear spacetime connection then $\big[\Lambda[g,K],\Lambda[g,K]\big]=0$ if and only if $L[K]\,g^\flat = 0$.*

Proof. This follows immediately. □

Theorem 2.9. *Let K be a linear spacetime connection. Then the following identities are equivalent:*

(1) $L[K]\, g^\flat = 0\,.$

(2) $d_K g = 0\,.$

(3) $d\Upsilon[g,K] = 0\,.$

(4) $\big[\Lambda[g,K], \Lambda[g,K]\big] = 0\,.$

Proof. This follows from Remark 1.2 and Lemmas 2.6 and 2.8. □

Corollary 2.10. *A linear spacetime connection K and the metric g induce on $T\boldsymbol{E}$ natural symplectic and Poisson structures if and only if $d_K g = 0 = L[K]\, g^\flat$.* □

Lemma 2.11. *Let K be a linear spacetime connection then the following three identities are equivalent:*

(1) $d_K g = 0.$

(2) $L[K]\; g^\flat = 0.$

(3) $(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = 2\, g(\tau[K](X,Y),Z)\,.$

Proof. (1) $\Leftrightarrow$ (2). This follows immediately from Remark 1.2.

(1) $\Leftrightarrow$ (3). Let us recall that for a linear connection K we have

$$2\,\tau[K](X,Y) = \nabla_Y X - \nabla_X Y + [X,Y]\,. \tag{2.7}$$

Then, by Remark 1.2,

$$\begin{aligned}(d_K g)(X,Y)(Z) &= (\nabla_X g)(Y,Z) + g(\nabla_X Y,Z) - (\nabla_Y g)(X,Z)\\ &\quad - g(\nabla_Y X,Z) - g([X,Y],Z)\\ &= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(\nabla_X Y - \nabla_Y X - [X,Y],Z)\\ &= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) - 2\, g(\tau[K](X,Y),Z)\,.\end{aligned}$$

□

Corollary 2.12. *If K is a torsion free connection then $d_K g = 0 = L[K]\, g^\flat$ is equivalent to $(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$, i.e., for a torsion free linear connection, $d_K g = 0 = L[K]\, g^\flat$ is equivalent with the symmetry of the (0,3)-tensor field ∇g.* □

Theorem 2.13. *Let K be a linear torsion free spacetime connection. Then the following identities are equivalent:*

(1) *∇g is a symmetric (0,3)-tensor field.*

(2) $d\Upsilon[g,K] = 0\,.$

(3) $\big[\Lambda[g,K], \Lambda[g,K]\big] = 0\,.$

Proof. By Corollary 2.12 for a linear torsion free connection the identity $d_K g = 0 = L[K]\, g^\flat$ is equivalent with ∇g to be fully symmetric. □

Corollary 2.14. *A linear torsion free spacetime connection K and the metric g induce on $T\boldsymbol{E}$ natural symplectic and Poisson structures if and only if the covariant differential ∇g is a symmetric (0,3)-tensor field.* □

Remark 2.15. Let us assume the torsion free spacetime metric connection $K[g]$ given by the Christtoffel symbols (1.3). Then we have $\nabla g = 0$, i.e., ∇g is symmetric in the canonical way, and we have the canonical natural symplectic and Poisson structures on $T\boldsymbol{E}$ given by $\Upsilon[g] = \Upsilon\big[g, K[g]\big]$ and $\Lambda[g] = \Lambda\big[g, K[g]\big]$. Moreover, in the metric case, $\Upsilon[g] = dg^\flat$. $\square$

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Department of Mathematics, Masaryk University
Janáčkovo nám 2a, 602 00 Brno, Czech Republic
E-mail: janyska@math.muni.cz