ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 103 – 114

LOCAL INTERPOLATION BY A QUADRATIC LAGRANGE FINITE ELEMENT IN 1D

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Abstract. We analyse the error of interpolation of functions from the space $H^3(a,c)$ in the nodes $a<b<c$ of a regular quadratic Lagrange finite element in 1D by interpolants from the local function space of this finite element. We show that the order of the error depends on the way in which the mutual positions of nodes a,b,c change as the length of interval $[a,c]$ approaches zero.

1. Introduction

We motivate and define the notion of a regular quadratic Lagrange finite element in 1D. Then we explain the main results of this article.

A *reference quadratic Lagrange finite element* $\hat{\mathcal{K}}$ is determined by

a) the *interval* $\hat{K} = [-1,1]$,
b) the *local space* $\hat{\mathcal{L}}$ of restrictions of polynomials of degree two or less to the interval $\hat{K}$,
c) the "set of parameters" relating values $\hat{p}(-1), \hat{p}(0), \hat{p}(1)$ to each $\hat{p} \in \hat{\mathcal{L}}$. (These *parameters* determine $\hat{p}$ in $\hat{\mathcal{L}}$ uniquely.)

We denote by $\hat{\Pi}\hat{v}$ a (unique) interpolant of a function $\hat{v} : \hat{K} \longrightarrow \Re$ in the nodes $-1,0,1$ from the space $\hat{\mathcal{L}}$.

To $a<b<c$ real, we relate a *discretisation step* $h = \max(b-a, c-b)$, a *center* $\tilde{b} = (a+c)/2$ and a (unique) function F_h from $\hat{\mathcal{L}}$ with parameters a,b,c. Of course, F_h is an injection if and only if F_h is increasing and, by putting $\nu = 0$ in (3), we can see that this is equivalent to

$$\frac{3a+c}{4} \le b \le \frac{a+3c}{4}\,. \tag{1}$$

In this case we say that F_h is a *transform* and we denote by G_h the transform inverse to F_h.

2000 *Mathematics Subject Classification*: 65D05, 65L60.

Key words and phrases: quadratic Lagrange finite elements in 1D, local interpolation of functions in one variable.

Received February 9, 2004, revised June 2005.

A *quadratic Lagrange finite element* $\mathcal{K}_h$ *in* 1D (briefly a *finite element* $\mathcal{K}_h$) is related to a transform F_h with parameters $a < b < c$. It is determined by

a) the *interval* $K_h = [a, c]$,

b) the *local space* $\mathcal{L}_h$ of functions

$$p_h(x) = \hat{p}\big(G_h(x)\big) \tag{2}$$

for all $\hat{p} \in \hat{\mathcal{L}}$,

c) the "set of parameters" relating the values $p_h(a)$, $p_h(b)$, $p_h(c)$ to each $p_h \in \mathcal{L}_h$. (These *parameters* determine p_h in $\mathcal{L}_h$ uniquely.)

We denote by $\Pi_h v$ a (unique) interpolant of a function $v : K_h \longrightarrow \Re$ in the nodes a, b, c from the space $\mathcal{L}_h$.

In Fig. 1, there is a graph of the transform $x = F_h(\xi)$ with parameters $a < b < c$ such that the node b attains the maximal value $(a+3c)/4$ satisfying condition (1). It is easy to see (by means of (7) for example) that we have

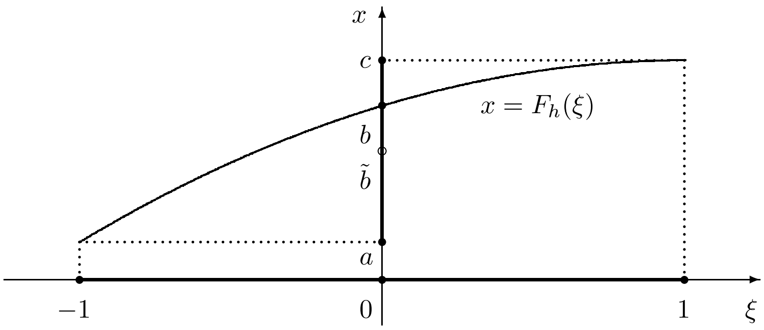

Figure 1

$\frac{d}{d\xi}F_h(\xi) > 0$ for $\xi \in (-1, 1)$, but $\lim_{\xi \longrightarrow 1^-} \frac{d}{d\xi}F_h(\xi) = 0$. As $\frac{d}{dx}G_h(x) = \frac{1}{\frac{d}{d\xi}F_h(\xi)}$ for $x = F_h(\xi) \in (a, c)$, we obtain

$$\lim_{x \longrightarrow c^-} \frac{d}{dx}G_h(x) = +\infty\,.$$

Hence the transform $\xi = G_h(x)$ has an unbounded derivative on the interval (a, c) and, due to definition (2), derivatives of functions $p_h \in \mathcal{L}_h$ are generally unbounded on (a, c), too. We study estimates of the error $v - \Pi_h v$ on finite elements $\mathcal{K}_h$ in which the derivatives $\frac{d}{dx}G_h(x)$ are bounded in the following way.

Let ν be a fixed constant from the open interval $(0, 1)$. We say that a finite element $\mathcal{K}_h$ is *regular* whenever

$$\frac{d}{d\xi}F_h(\xi) \ge \nu h \quad \text{in} \quad [-1, 1]\,.$$

Our Lemma 3 says that a finite element $\mathcal{K}_h$ is regular if and only if

$$\frac{3a + (1+2\nu)c}{4+2\nu} \le b \le \frac{(1+2\nu)a + 3c}{4+2\nu}\,. \tag{3}$$

For regular finite elements $\mathcal{K}_h$ and for functions $v \in H^3(a,c)$, we prove the following relation between the size of error $v - \Pi_h v$ and the "speed" of point b as it approaches the center $\tilde{b}$ for discretisation step h approaching zero:

Let $p \geq 1$ be arbitrary. We have $|b - \tilde{b}| = O(h^p)$ if and only if there exists a constant C satisfying

$$|v - \Pi_h v|_{m,K_h} \leq C\, h^{1-m} \big(h^2 |v|_{3,K_h} + h^p |v|_{2,K_h}\big) \tag{4}$$

for $m = 0, 1, 2$.

As $|b - \tilde{b}| \leq (c-a)/4 \leq h/2$ due to (1), condition $p \geq 1$ is always fulfilled. The accuracies of the error estimates (4) are the worst in the case $p = 1$:

$$|v - \Pi_h v|_{m,K_h} \leq C\, h^{2-m} \left(h |v|_{3,K_h} + |v|_{2,K_h}\right) \tag{5}$$

and the least value for which these accuracies are the best is $p = 2$:

$$|v - \Pi_h v|_{m,K_h} \leq C\, h^{3-m} \left(|v|_{3,K_h} + |v|_{2,K_h}\right) . \tag{6}$$

Our Example 1 illustrates that the estimates (5), (6) are optimal.

In this paper, for a given bounded interval K, the symbols $\|\cdot\|_{0,K}$, $|\cdot|_{0,K}$ mean the norm in the space $L_2(K)$ which we denote by $H^0(K)$, too. For $m = 1, 2, 3$, symbol $\|\cdot\|_{m,K}$ means the norm in the space $H^m(K)$ and $|\cdot|_{m,K}$, $|\cdot|_{m,\infty,K}$ is a notation of the seminorm in $H^m(K)$, $W^{m,\infty}(K)$, respectively.

2. Explicit formulas for some derivatives

We find derivatives of the transform G_h in Lemma 1 and of the interpolants $\Pi_h v$ in Lemma 2.

Definition. We put

$$D_1 = (c-a)/2 \quad \text{and} \quad D_2 = c - 2b + a$$

for a finite element $\mathcal{K}_h$.

As $a < b < c$ are parameters of the transform $F_h \in \hat{\mathcal{L}}$, we obtain

$$x = F_h(\xi) = b + \xi D_1 + \frac{1}{2}\,\xi^2 D_2 \,. \tag{7}$$

Lemma 1. *Let us consider a finite element $\mathcal{K}_h$ and put $F = F_h$, $G = G_h$. Then the following statements a) – c) are valid for all $x \in (a,c)$.*

a) $\frac{d}{dx} G(x) = \frac{1}{F'(G(x))} = \frac{1}{D_1 + G(x) D_2}$,
b) $\frac{d^2}{dx^2} G(x) = -\frac{F''(G(x)) G'(x)^2}{F'(G(x))} = -\frac{D_2}{(D_1 + G(x) D_2)^3}$,
c) $\frac{d^3}{dx^3} G(x) = -\frac{3F''(G(x)) G'(x) G''(x)}{F'(G(x))} = \frac{3 D_2^2}{(D_1 + G(x) D_2)^5}$.

Proof. Insert $\xi = G(x)$ into (7) and compute the first, second and third derivatives of both sides of $x = F(G(x))$. □

Definition. Let $\mathcal{K}_h$ be a finite element. To every function $\hat{v} : \hat{K} \longrightarrow \Re$ we relate a function $v : K_h \longrightarrow \Re$ by the formula

$$v(x) = \hat{v}\big(G_h(x)\big) \tag{8}$$

and we put

$$D_1(v) = \big(v(c) - v(a)\big)/2\,, \qquad D_2(v) = v(a) - 2v(b) + v(c)\,.$$

It is easy to see that

$$v\big(F_h(\xi)\big) = \hat{v}(\xi)$$

for all pairs of functions $v, \hat{v}$ satisfying (8) and that

$$v \in \mathcal{L}_h \Longleftrightarrow \hat{v} \in \hat{\mathcal{L}}\,.$$

Especially,

$$\Pi_h v(F_h(\xi)) = \widehat{\Pi_h v}(\xi) \in \hat{\mathcal{L}}\,.$$

This fact together with $\widehat{\Pi_h v}(-1) = v(a) = \hat{\Pi}\hat{v}(-1)$, $\widehat{\Pi_h v}(0) = v(b) = \hat{\Pi}\hat{v}(0)$, $\widehat{\Pi_h v}(1) = v(c) = \hat{\Pi}\hat{v}(1)$ give us

$$\widehat{\Pi_h v} = \hat{\Pi}\hat{v}, \tag{9}$$

$$\Pi_h v\big(F_h(\xi)\big) = v(b) + \xi\, D_1(v) + \frac{1}{2}\,\xi^2 D_2(v)\,. \tag{10}$$

Lemma 2. *Let be $\mathcal{K}_h$ a finite element and $v : K_h \longrightarrow \Re$ a function. If we put $F = F_h$, $G = G_h$ then the following statements a), b) are valid for all $x \in (a, c)$.*

a) $\frac{d}{dx}\Pi_h v(x) = \frac{D_1(v)+G(x)D_2(v)}{D_1+G(x)D_2}$,

b) $\frac{d^2}{dx^2}\Pi_h v(x) = \frac{v(a)(c-b)+v(b)(a-c)+v(c)(b-a)}{(D_1+G(x)D_2)^3}$.

Proof. If we insert $\xi = G(x)$ into (10) and use Lemma 1, we obtain a):

$$\frac{d}{dx}\Pi_h v(x) = \frac{d}{dx}\big[\Pi_h v\big(F(G(x))\big)\big] = \frac{d}{d\xi}\Pi_h v(F(\xi))\frac{dG}{dx} = \frac{D_1(v) + G(x)D_2(v)}{D_1 + G(x)D_2}$$

and b):

$$\begin{aligned}\frac{d^2}{dx^2}\Pi_h v(x) &= \frac{d}{dx}\Big[\frac{d}{d\xi}\Pi_h v(F(\xi))\frac{dG}{dx}\Big] = \frac{d^2}{d\xi^2}\Pi_h v(F(\xi))\Big(\frac{dG}{dx}\Big)^2 \\ &\quad + \frac{d}{d\xi}\Pi_h v(F(\xi))\frac{d^2G}{dx^2} = \frac{v(a)(c-b) + v(b)(a-c) + v(c)(b-a)}{(D_1 + G(x)D_2)^3}\,.\end{aligned}$$

□

3. Estimates on regular finite elements

We present two characterizations of regular finite elements $\mathcal{K}_h$ in Lemma 3. Under the assumption of regularity, we obtain estimates of certain seminorms of the transforms F_h and G_h in Lemmas 5 and 6 by means of a technical Lemma 4. Estimates of the norm $\|v\|_{0,K_h}$ and of the seminorms $|v|_{m,K_h}$ for $m = 1, 2, 3$ appear in Proposition 1. Corollary 1 gives us an estimate of the seminorm $|\hat{v}|_{3,\hat{K}}$.

Lemma 3. *The following statements a) – c) are equivalent for an arbitrary finite element $\mathcal{K}_h$.*

a) *The finite element $\mathcal{K}_h$ is regular,*
b) $\nu h \le \min(D_1 - D_2, D_1 + D_2)$,
c) *the inequalities* (3) *are satisfied.*

Proof. a) $\Longleftrightarrow$ b): a) $\Longleftrightarrow$ $\nu\, h \le D_1 + \xi\, D_2$ for $\xi \in [-1,1]$ due to (7) $\Longleftrightarrow$ $\nu\, h \le D_1 - D_2$ and $\nu\, h \le D_1 + D_2$.

b) $\Longleftrightarrow$ c): Let us assume that $D_2 \ge 0$. As $D_2 \ge 0 \Longleftrightarrow b \le \tilde{b} \Longleftrightarrow h = c - b$, we have b) $\Longleftrightarrow \nu\, h \le D_1 - D_2 \Longleftrightarrow \nu(c-b) \le \frac{c-a}{2} - c + 2b - a \Longleftrightarrow \frac{3a+(1+2\nu)c}{4+2\nu} \le b$. In the case $D_2 \le 0$, equivalent to $\tilde{b} \le b$, we prove b) $\Longleftrightarrow b \le \frac{(1+2\nu)a+3c}{4+2\nu}$ analogically. □

Lemma 4. *The following statements a) – e) are valid for an arbitrary regular finite element $\mathcal{K}_h$.*

a) $\frac{h}{3}\,(2+\nu) \le D_1 \le h$,
b) $\nu\, h \le D_1 - D_2 \le \frac{h}{3}\,(4-\nu)$,
c) $\nu\, h \le D_1 + D_2 \le \frac{h}{3}\,(4-\nu)$,
d) $-\frac{2}{3}\, h(1-\nu) \le D_2 \le \frac{2}{3}\, h(1-\nu)$,
e) $\frac{h^2}{3}\,\nu(4-\nu) \le D_1^2 - D_2^2 \le h^2$.

Proof of a). As $h = \max(b-a, c-b)$, we have $D_1 = \frac{1}{2}(c - b + b - a) \le h$. If $h = b - a$ then $D_2 \le 0$ and

$$\frac{d}{d\xi} F_h(\xi) = D_1 + \xi\, D_2 \ge \nu\, h \quad \forall \xi \in [-1,1]$$
$$\Longleftrightarrow D_1 + D_2 \ge \nu\, h \Longleftrightarrow \frac{3}{2}\, c + \frac{1}{2}\, a - 2b \ge \nu\, h$$
$$\Longleftrightarrow 3D_1 \ge \nu\, h + 2(b-a) = h(2+\nu)\,.$$

If $h = c - b$ then $D_1 \ge \frac{h}{3}\,(2+\nu)$ can be proved analogically.

Proof of b) and c). The first inequalities in b), c) are valid by Lemma 3 b). If $D_2 \ge 0$ then $h = c - b$ and $D_1 - D_2 \le D_1 + D_2 = \frac{c-a}{2} + c - 2b + a = 2h - D_1 \le 2h - \frac{h}{3}(2+\nu) = \frac{h}{3}(4-\nu)$ due to a). If $D_2 \le 0$ then we can prove $D_1 + D_2 \le D_1 - D_2 \le \frac{h}{3}(4-\nu)$ in the same way.

Proof of d). We obtain d) by multiplying the inequalities b) by -1 and adding them to the inequalities c).

Proof of e). This is a direct consequence of a) and of d) in the form $0 \le |D_2| \le \frac{2}{3} h(1-\nu)$. □

Lemma 5. *If $\mathcal{K}_h$ is a regular finite element then the following statements a) – c) are valid.*

a) $\nu h \le \frac{d}{d\xi} F_h(\xi) \le \frac{h}{3}(4-\nu) \quad \forall \xi \in [-1,1]$,
b) $|F_h|_{2,\infty,\hat{K}} = 2\,|b - \tilde{b}| \le \frac{2}{3}\, h(1-\nu)$,

c) $|F_h|_{3,\infty,\hat{K}} = 0$.

Proof. These statements follow by $F_h'(\xi) = D_1 + \xi D_2$, $F_h'' = D_2 = 2(\tilde{b} - b)$, $F_h''' = 0$ and by Lemma 4 b), c), d). $\square$

Lemma 6. *Let $\mathcal{K}_h$ be a regular finite element with the property $|F_h|_{2,\infty,\hat{K}} \leq C\, h^p$ for some constant C and for some $p \geq 1$. Then*

a) $|G_h|_{1,\infty,K_h} \leq \frac{1}{\nu}\, h^{-1}$,

b) $|G_h|_{2,\infty,K_h} \leq \frac{C}{\nu^3}\, h^{p-3}$,

c) $|G_h|_{3,\infty,K_h} \leq \frac{3C^2}{\nu^5}\, h^{2p-5}$.

Proof. The statement a) is a consequence of Lemmas 1 a) and 5 a). Statement b) follows by Lemmas 1 b), 5 a) and statement c) is valid due to $|F_h|_{2,\infty,\hat{K}} = |D_2|$ and Lemmas 1 c), 5 a). $\square$

Proposition 1. *Let $\mathcal{K}_h$ be a regular finite element. Then the following assertions a) – d) hold true for all pairs of functions $v : K_h \longrightarrow \Re$, $\hat{v} : \hat{K} \longrightarrow \Re$ satisfying condition* (8)*.*

a) *If $\hat{v} \in L_2(\hat{K})$ then $v \in L_2(K_h)$ and*

$$\|v\|_{0,K_h} \leq |F_h|_{1,\infty,\hat{K}}^{\frac{1}{2}} \|\hat{v}\|_{0,\hat{K}}\,,$$

b) *If $\hat{v} \in H^1(\hat{K})$ then $v \in H^1(K_h)$ and*

$$|v|_{1,K_h} \leq |F_h|_{1,\infty,\hat{K}}^{\frac{1}{2}} |G_h|_{1,\infty,K_h} |\hat{v}|_{1,\hat{K}}\,,$$

c) *If $\hat{v} \in H^2(\hat{K})$ then $v \in H^2(K_h)$ and*

$$|v|_{2,K_h} \leq |F_h|_{1,\infty,\hat{K}}^{\frac{1}{2}} \left(|G_h|_{1,\infty,K_h}^2 |\hat{v}|_{2,\hat{K}} + |G_h|_{2,\infty,K_h} |\hat{v}|_{1,\hat{K}} \right)\,,$$

d) *If $\hat{v} \in H^3(\hat{K})$ then $v \in H^3(K_h)$ and $|v|_{3,K_h} \leq |F_h|_{1,\infty,\hat{K}}^{\frac{1}{2}} \cdot$*

$$\left(|G_h|_{1,\infty,K_h}^3 |\hat{v}|_{3,\hat{K}} + 3\, |G_h|_{1,\infty,K_h} |G_h|_{2,\infty,K_h} |\hat{v}|_{2,\hat{K}} + |G_h|_{3,\infty,K_h} |\hat{v}|_{1,\hat{K}} \right)\,.$$

Proof of a). Let us consider $v(x) = \hat{v}(G_h(x))$ for some $\hat{v} \in L_2(\hat{K})$. As $\hat{v}(\xi)^2 F_h'(\xi)$ is non-negative, measurable due to [4], Theorem 10.18 and bounded by the integrable function $\hat{v}(\xi)^2 |F_h|_{1,\infty,\hat{K}}$, we conclude that $\hat{v}(\xi)^2 F_h'(\xi)$ is integrable by [4], Theorem 10.27. Then the change of variables, see [6], Theorem $\mathcal{P}.24$, gives us

$$\|v\|_{0,K_h}^2 = \int_{K_h} v(x)^2 dx = \int_{\hat{K}} \hat{v}(\xi)^2 F_h'(\xi)\, d\xi \leq |F_h|_{1,\infty,\hat{K}} \|\hat{v}\|_{0,\hat{K}}^2$$

as well as $v \in L_2(K_h)$.

Now, we prove

$\tilde{b}$) $\hat{v} \in C^1(\hat{K}) \Longrightarrow v \in C^1(K_h)$ and

$$|v|_{1,K_h} \leq |F_h|_{1,\infty,\hat{K}}^{\frac{1}{2}} |G_h|_{1,\infty,K_h} |\hat{v}|_{1,\hat{K}} :$$

If $\hat{v} \in C^1(\hat{K})$ then $v \in C^1(K_h)$ due to Lemma 1 a) and to the regularity of $\mathcal{K}_h$. Moreover,

$$|v|^2_{1,K_h} = \int_{K_h} \left[\frac{d\hat{v}}{d\xi}(G_h(x))\frac{dG_h}{dx}\right]^2 dx \le |G_h|^2_{1,\infty,K_h} \int_{\hat{K}} \left(\frac{d\hat{v}}{d\xi}\right)^2 F'_h(\xi)\, d\xi$$
$$\le |F_h|_{1,\infty,\hat{K}} |G_h|^2_{1,\infty,K_h} |\hat{v}|^2_{1,\hat{K}} .$$

In the same way, the following assertions $\tilde{c}$), $\tilde{d}$) can be proved.

$\tilde{c}$) $\hat{v} \in C^2(\hat{K}) \Longrightarrow v \in C^2(K_h)$ and

$$|v|_{2,K_h} \le |F_h|^{\frac{1}{2}}_{1,\infty,\hat{K}} \left(|G_h|_{1,\infty,K_h}|\hat{v}|_{2,\hat{K}} + |G_h|_{2,\infty,K_h}|\hat{v}|_{1,\hat{K}}\right) ,$$

$\tilde{d}$) $\hat{v} \in C^3(\hat{K}) \Longrightarrow v \in C^3(K_h)$ and $|v|_{3,K_h} \le |F_h|^{\frac{1}{2}}_{1,\infty,\hat{K}}\cdot$

$$\left(|G_h|^3_{1,\infty,K_h}|\hat{v}|_{3,\hat{K}} + 3\,|G_h|_{1,\infty,K_h}|G_h|_{2,\infty,K_h}|\hat{v}|_{2,\hat{K}} + |G_h|_{3,\infty,K_h}|\hat{v}|_{1,\hat{K}}\right) .$$

The operator $j : L_2(\hat{K}) \longrightarrow L_2(K_h)$, $j(\hat{v})(x) = v(x) \equiv \hat{v}(G_h(x))$, is linear obviously and continuous due to a). The statements a), $\tilde{b}$), $\tilde{c}$), $\tilde{d}$) say that for $m = 1, 2, 3$, the operator $j_m = j|_{C^m(\hat{K})}$ is continuous from $H^m(\hat{K})$ to $H^m(K_h)$. As $C^m(\hat{K})$ is dense in $H^m(\hat{K})$, there exists a unique linear continuous extension J_m of j_m to the space $H^m(\hat{K})$ such that the operator norm $\|J_m\|$ is equal to $\|j_m\|$ by [1], Theorem 3.4.4. The values of J_m have been defined in the following way. For an arbitrary $\hat{v} \in H^m(\hat{K})$, there exists a sequence $(\hat{v}_n)_1^\infty \subseteq C^m(\hat{K})$ such that $\|\hat{v} - \hat{v}_n\|_{m,\hat{K}} \longrightarrow 0$ as $n \longrightarrow \infty$ and $J_m(\hat{v})$ is the limit of the sequence (v_n) in $H^m(K_h)$. But then also $\|\hat{v} - \hat{v}_n\|_{0,\hat{K}} \longrightarrow 0$ as $n \longrightarrow \infty$ and $J_m(\hat{v})$ is the limit of (v_n) in $L_2(K_h)$. This, $J_m(\hat{v}_n) = j(\hat{v}_n)$ for all n and continuity of j in $L_2(\hat{K})$ give us $J_m(\hat{v}) = j(\hat{v})$, so that $J_m = j|_{H^m(\hat{K})}$.

Proof of b). Let us put $\kappa = |F_h|^{\frac{1}{2}}_{1,\infty,\hat{K}}|G_h|_{1,\infty,K_h}$. As $\hat{v} \in H^1(\hat{K}) \Longrightarrow v = J_1(\hat{v}) \in H^1(K_h)$ is valid, it remains to prove that $|v|_{1,K_h} \le \kappa|\hat{v}|_{1,\hat{K}}$: For every $\hat{v} \in H^1(\hat{K})$ there exists a sequence $(\hat{v}_n)_1^\infty \subseteq C^1(\hat{K})$ such that $\|\hat{v} - \hat{v}_n\|_{1,\hat{K}} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence for every $\varepsilon > 0$ there exists $\hat{v}_n$ such that $\|\hat{v} - \hat{v}_n\|_{1,\hat{K}} \le \varepsilon$ and we have $\|v - v_n\|_{1,K_h} = \|J_1(\hat{v} - \hat{v}_n)\|_{1,K_h} \le \|J_1\|\varepsilon$. By this inequality and by $\tilde{b}$) we obtain

$$\begin{aligned} |v|_{1,K_h} &\le |v - v_n|_{1,K_h} + |v_n|_{1,K_h} \le \|J_1\|\varepsilon + \kappa|\hat{v}_n|_{1,\hat{K}} \\ &\le \|J_1\|\varepsilon + \kappa\left(|\hat{v}_n - \hat{v}|_{1,\hat{K}} + |\hat{v}|_{1,\hat{K}}\right) \le \kappa|\hat{v}|_{1,\hat{K}} + (\|J_1\| + \kappa)\,\varepsilon. \end{aligned}$$

As this estimate is valid for all $\varepsilon > 0$, $|v|_{1,K_h} \le \kappa|\hat{v}|_{1,\hat{K}}$ is necessary.

The statements c), d) can be proved in the same way. □

Corollary 1. *There exists a constant C such that if $v \in H^3(K_h)$ then $\hat{v} \in H^3(\hat{K})$ and*

$$|\hat{v}|_{3,\hat{K}} \le C\, h^{\frac{1}{2}} \left(h^2|v|_{3,K_h} + |F_h|_{2,\infty,\hat{K}}|v|_{2,K_h}\right)$$

for all regular finite elements $\mathcal{K}_h$ and for all pairs of functions $v : K_h \longrightarrow \Re$, $\hat{v} : \hat{K} \longrightarrow \Re$ satisfying condition (8).

Proof. If we mutually exchange the intervals $\hat{K}$, K_h, the transforms $x = F_h(\xi)$, $\xi = G_h(x)$ and the functions $\hat{v}(\xi)$, $v(x)$ in Proposition 1 d) and use Lemma 5 c) then we obtain

$$|\hat{v}|_{3,\hat{K}} \leq |G_h|^{\frac{1}{2}}_{1,\infty,K_h} |F_h|_{1,\infty,\hat{K}} \left(|F_h|^2_{1,\infty,\hat{K}} |v|_{3,K_h} + 3|F_h|_{2,\infty,\hat{K}} |v|_{2,K_h} \right) .$$

This inequality and Lemmas 6 a), 5 a) give us the statement. □

4. Main results

An estimate of the interpolation error $\hat{v} - \hat{\Pi}\hat{v}$ on the reference finite element is presented in Lemma 7. Lemmas 8, 9 give us estimates of the seminorms $|v - \Pi_h v|_{m,K_h}$, $m = 0, 1, 2$, on regular finite elements. Theorems 1, 2 formulate an equivalence between the orders of estimates of the above seminorms and the order of the seminorm $|F_h|_{2,\infty,\hat{K}}$ for regular finite elements $\mathcal{K}_h$.

Lemma 7. *There exists a constant C such that*

$$\|\hat{v} - \hat{\Pi}\hat{v}\|_{2,\hat{K}} \leq C\, |\hat{v}|_{3,\hat{K}}$$

for all $\hat{v} \in H^3(\hat{K})$.

Proof. As all norms in the three–dimensional space $\hat{\mathcal{L}}$ are mutually equivalent and the imbedding from $H^3(\hat{K})$ into $C(\hat{K})$ is continuous due to the Sobolev Imbedding Theorem 3.8 from [3], Chap. 2, there exist constants C_1, C_2 such that

$$\|\hat{\Pi}\hat{v}\|_{2,\hat{K}} \leq C_1 \max(|\hat{v}(-1)|, |\hat{v}(0)|, |\hat{v}(1)|) \leq C_1 \|\hat{v}\|_{C(\hat{K})} \leq C_2 \|\hat{v}\|_{3,\hat{K}} \tag{11}$$

for all $\hat{v} \in H^3(\hat{K})$.

Let us take a fixed $\hat{\psi} \in H^2(\hat{K})$ such that $\|\hat{\psi}\|_{2,\hat{K}} = 1$ and consider the scalar product

$$\psi(\hat{v}) = (\hat{v} - \hat{\Pi}\hat{v}, \hat{\psi})_{2,\hat{K}} .$$

Then, due to (11),

$$|\psi(\hat{v})| \leq \|\hat{v} - \hat{\Pi}\hat{v}\|_{2,\hat{K}} \leq \|\hat{v}\|_{2,\hat{K}} + \|\hat{\Pi}\hat{v}\|_{2,\hat{K}} \leq (1 + C_2)\|\hat{v}\|_{3,\hat{K}}$$

for all $\hat{v} \in H^3(\hat{K})$ and, at the same time, $\psi(\hat{p}) = 0$ for all $\hat{p} \in \hat{\mathcal{L}}$. These facts and the Bramble-Hilbert Lemma 4.5 from [2] say that there exists a constant C satisfying

$$\|\hat{v} - \hat{\Pi}\hat{v}\|_{2,\hat{K}} = \sup_{\|\hat{\psi}\|_{2,\hat{K}}=1} |(\hat{v} - \hat{\Pi}\hat{v}, \hat{\psi})_{2,\hat{K}}| \leq C\, |\hat{v}|_{3,\hat{K}}$$

for all $\hat{v} \in H^3(\hat{K})$. □

Lemma 8. *There exists a constant C such that*

$$|v - \Pi_h v|_{m,K_h} \leq C\, h^{\frac{1}{2}-m} |\hat{v}|_{3,\hat{K}}$$

for $m = 0, 1, 2$, all regular finite elements $\mathcal{K}_h$, and for all pairs $v \in H^3(K_h)$, $\hat{v} \in H^3(\hat{K})$ satisfying condition (8).

Proof. Let a regular finite element $\mathcal{K}_h$ and $v \in H^3(K_h)$, $\hat{v} \in H^3(\hat{K})$ satisfying (8) be arbitrary. The interpolants $\Pi_h v \in \mathcal{L}_h$, $\hat{\Pi}\hat{v} \in \hat{\mathcal{L}}$ satisfy $\widehat{\Pi_h v} = \hat{\Pi}\hat{v}$ by (9) and, consequently, $\widehat{v - \Pi_h v} = \hat{v} - \widehat{\Pi_h v} = \hat{v} - \hat{\Pi}\hat{v}$. Then we obtain

$$|v - \Pi_h v|_{m,K_h} \le C\, h^{\frac{1}{2}-m} \|\hat{v} - \hat{\Pi}\hat{v}\|_{m,\hat{K}}$$

by Proposition 1 a) and Lemma 5 a) in the case $m = 0$, by Proposition 1 b) and Lemmas 5 a), 6 a) in the case $m = 1$, and by Proposition 1 c) and Lemmas 5 a), b), 6 a), b) in the case $m = 2$. An application of Lemma 7 concludes the proof. □

Lemma 9. *There exists a constant C such that*

$$|v - \Pi_h v|_{m,K_h} \le C\, h^{1-m} \left(h^2 |v|_{3,K_h} + |F_h|_{2,\infty,\hat{K}} |v|_{2,K_h} \right)$$

for $m = 0, 1, 2$, all regular finite elements $\mathcal{K}_h$ and for all $v \in H^3(K_h)$.

Proof. We obtain this statement by Lemma 8 and Corollary 1. □

The following basic Theorems 1, 2 present an exact formulation and a proof of the fact that for every regular finite element $\mathcal{K}_h$, the property

$$|F_h|_{2,\infty,\hat{K}} \le C_1 h^p \tag{12}$$

is equivalent to the estimate

$$|v - \Pi_h v|_{m,K_h} \le C_2 h^{1-m} \left(h^2 |v|_{3,K_h} + h^p |v|_{2,K_h} \right) \tag{13}$$

for $m = 0, 1, 2$ and for all $v \in H^3(K_h)$.

Theorem 1. *For every constant C_1 there exists a constant C_2 such that the estimate* (13) *is valid on all regular finite elements $\mathcal{K}_h$ with the property* (12).

Proof. This is a direct consequence of Lemma 9. □

Theorem 2. *For every constant C_2 there exists a constant C_1 such that all regular finite elements $\mathcal{K}_h$ on which the estimate* (13) *is satisfied, have the property* (12).

Proof. Let us assume that the estimate (13) is valid on a regular finite element $\mathcal{K}_h$. If we put $m = 2$ and $v = x^2$ in (13) then we can see by means of $|v|_{3,K_h} = 0$, $|v|^2_{2,K_h} = 4\int_a^c dx \le 8h$ that the second power of the right-hand side has an upper estimate $8C_2^2 h^{2p-1}$. Then (13), Lemma 2 b) and substitution $x = F_h(\xi)$ give us

$$\begin{aligned}
8C_2^2 h^{2p-1} &\ge \int_a^c \left(2 - \frac{d^2}{dx^2} \Pi_h x^2 \right)^2 dx \\
&= \int_a^c \left(2 - \frac{a^2(c-b) + b^2(a-c) + c^2(b-a)}{(D_1 + G_h(x) D_2)^3} \right)^2 dx \\
&= \int_{-1}^1 \left(2 - \frac{a^2(c-b) + b^2(a-c) + c^2(b-a)}{(D_1 + \xi D_2)^3} \right)^2 (D_1 + \xi D_2) d\xi\,.
\end{aligned}$$

By means of the identity $a^2(c-b) + b^2(a-c) + c^2(b-a) = D_1(2D_1^2 - \frac{1}{2}D_2^2)$ and by routine computation, we can find the value

$$\frac{D_1 D_2^2}{2(D_1^2 - D_2^2)^4} \left(48 D_1^6 - 31 D_1^4 D_2^2 - 7 D_1^2 D_2^4 + 8 D_2^6 \right)$$

of the last integral. Then we have

$$8C_2^2 h^{2p-1} \geq \frac{D_1 D_2^2}{2(D_1^2 - D_2^2)^4} \big[10D_1^6 + 38D_1^4(D_1^2 - D_2^2) + 7D_1^2 D_2^2 (D_1^2 - D_2^2) + 8D_2^6\big] > 5\frac{D_1^7 D_2^2}{(D_1^2 - D_2^2)^4} > 5\left(\frac{2+\nu}{3}\right)^7 \frac{D_2^2}{h}$$

by Lemma 4 a), e). Hence $|F_h|_{2,\infty,\hat{K}} = |D_2| < C_1 h^p$ for $C_1 = C_2 \left(\frac{8\cdot 3^7}{5(2+\nu)^7}\right)^{\frac{1}{2}}$ due to (7). $\square$

The following example shows us that the error estimate (13) is optimal in the cases $p = 1$ and $p = 2$, formulated in (5) and (6) explicitly.

Example 1. We examine two collections of regular finite elements related to the nodes $a_n < b_n < c_n$ for $n = 1, \dots, 10$.

In both cases, we use the function

$$v(x) = x^4 - e^x$$

and we denote $h_n = \max(c_n - b_n, b_n - a_n)$, $K_n = [a_n, c_n]$,

$$D_1(n) = \frac{1}{2}(c_n - a_n), \quad D_2(n) = c_n - 2b_n + a_n\,,$$

$$F_n(\xi) = b_n + \xi\, D_1(n) + \frac{1}{2}\xi^2\, D_2(n)\,,$$

$$D_1(v,n) = \frac{1}{2}(v(c_n) - v(a_n))\,, \quad D_2(v,n) = v(c_n) - 2v(b_n) + v(a_n)\,,$$

$$\Pi_n v(F_n(\xi)) = v(b_n) + \xi\, D_1(v,n) + \frac{1}{2}\xi^2\, D_2(v,n)\,.$$

Due to Lemma 2 a), b), we compute the seminorms of interpolation error in L_2, H^1, H^2 by the following formulas.

$$|v - \Pi_n v|_{0,K_n}^2 = \int_{-1}^{1} [v(F_n(\xi)) - \Pi_n v(F_n(\xi))]^2\, F_n'(\xi)\, d\xi\,,$$

$$|v - \Pi_n v|_{1,K_n}^2 = \int_{-1}^{1} \Big[\frac{dv}{dx}(F_n(\xi)) - \frac{D_1(v,n) + \xi\, D_2(v,n)}{D_1(n) + \xi\, D_2(n)}\Big]^2 F_n'(\xi)\, d\xi\,,$$

$$|v - \Pi_n v|_{2,K_n}^2 = \int_{-1}^{1} \Big[\frac{d^2 v}{dx^2}(F_n(\xi)) - \frac{v(a_n)(c_n - b_n) + v(b_n)(a_n - c_n) + v(c_n)(b_n - a_n)}{(D_1(n) + \xi\, D_2(n))^3}\Big]^2 F_n'(\xi)\, d\xi\,.$$

In the first case, we put

$$a_n = -2^{1-n}, \quad b_n = 0, \quad c_n = 2^{2-n}\,.$$

Then $h_n = 2^{2-n}$ and $D_2(n) = 2^{2-n} - 2^{1-n} = \frac{1}{2}h_n$, so that $p = 1$. In Table 1, we present values of the lower estimates

$$f_m(n) = \frac{|v - \Pi_n v|_{m,K_n}}{h_n^{2-m}\,(h_n\,|v|_{3,K_n} + |v|_{2,K_n})}$$

of the generic constant C from the formula (5) for $m = 0, 1, 2$, $n = 1, \ldots, 10$.

n	$f_0(n)$	$f_1(n)$	$f_2(n)$
1	0.01326605	0.04911164	0.32236026
2	0.01015252	0.03844066	0.22579268
3	0.00720223	0.04966095	0.72000078
4	0.03227066	0.15416260	1.98841239
5	0.04775567	0.21923563	2.74714388
6	0.05199045	0.23769159	2.97680427
7	0.05298483	0.24254836	3.04760990
8	0.05339632	0.24479767	3.08374039
9	0.05361941	0.24605864	3.10431198
10	0.05373998	0.24674534	3.11551222

Table 1

In the second case, we put

$$a_n = -2^{2-n}(1 - 2^{-n}), \quad b_n = 0, \quad c_n = 2^{2-n}\,.$$

Then $h_n = 2^{2-n}$ and $D_2(n) = 2^{2-2n} = \frac{1}{4}h_n^2$, so that $p = 2$. In Table 2, lower estimates

$$g_m(n) = \frac{|v - \Pi_n v|_{m,K_n}}{h_n^{3-m}\,(|v|_{3,K_n} + |v|_{2,K_n})}$$

of the constant C from (6) for $m = 0, 1, 2$ and $n = 1, \ldots, 10$ are summarized.

n	$g_0(n)$	$g_1(n)$	$g_2(n)$
1	0.01053705	0.03900872	0.25604647
2	0.00803949	0.03232572	0.20823404
3	0.00918685	0.04001027	0.26050865
4	0.01731354	0.06254085	0.31047510
5	0.02703486	0.09081996	0.38322695
6	0.03470397	0.11393049	0.45332569
7	0.03842156	0.12515299	0.48902544
8	0.03968640	0.12889298	0.50080214
9	0.04006793	0.12997263	0.50403188
10	0.04018614	0.13028430	0.50487509

Table 2

Remark. Lemma 9 says that there exists a constant C satisfying

$$|v - \Pi_h v|_{m,K_h} \leq C\,h^{3-m}|v|_{3,K_h} \tag{14}$$

for $m = 0, 1, 2$ and for all $v \in H^3(a, c)$ if and only if

$$|F_h|_{2,\infty,\hat{K}} = 0\,.$$

This condition is equivalent to the linearity of the transform $x = F_h(\xi)$ and also to the fact that $\mathcal{L}_h$ is the space of polynomials of total degree two or less. Estimate (14) is a special case of classical results concerning polynomial interpolation. See the estimate (44) in [5], Section 1.7 for example.

Acknowledgement. This outcome has been achieved with the financial support of the Ministry of Education, Youth and Sports, project No. 1M680470001, within activities of the CIDEAS research centre.

References

[1] Hutson, V. C. L., Pym, J. S. *Applications of Functional Analysis and Operator Theory*, Academic Press, London, 1980.

[2] Křížek, M., Neittaanmäki, P., *Finite Element Approximation of Variational Problems and Applications*, Longman Scientific & Technical, Essex, 1990.

[3] Nečas, J., *Les méthodes directes en théorie des équations elliptiques*, Masson et C^{ie}, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[4] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1964.

[5] Strang, G., Fix, G. J., *An Analysis of the Finite Element Method*, Prentice Hall, Englewood Clifs, N. J., 1973.

[6] Ženíšek, A., *Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations*. Academic Press, London, 1990.

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