

FIXED POINTS THEOREMS OF NON-EXPANDING FUZZY MULTIFUNCTIONS

ABDELKADER STOUTI

ABSTRACT. We prove the existence of a fixed point of non-expanding fuzzy multifunctions in α -fuzzy preordered sets. Furthermore, we establish the existence of least and minimal fixed points of non-expanding fuzzy multifunctions in α -fuzzy ordered sets.

1. INTRODUCTION

In [19], Zadeh introduced the notion of fuzzy order and similarity, which was investigated by several authors (see [1, 3, 7, 13]). During the last few decades many authors have established the existence of a lots of fixed point theorems in fuzzy setting: Beg [2, 4], Bose and Sahani [6], Fang [8], Hadzic [9], Heilpern [10], Kaleva [11] and the present author [13, 14, 15, 16]. The aim of this paper is to study the existence of fixed points of non-expanding fuzzy multifunctions in α -fuzzy setting.

Let X be a nonempty crisp set, with generic element of X denoted by x . A fuzzy subset A of X is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy subset A for each $x \in X$. Let A and B be two fuzzy subsets of X . We say that A is included in B and we write $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$. In particular, if $x \in X$ and $\mu_A(x) = 1$, then $\{x\} \subseteq A$.

Let X be a nonempty crisp set and $\alpha \in]0, 1]$. An α -fuzzy preorder relation on X is a fuzzy subset r_α of $X \times X$ satisfying the following two properties:

- (i) for all $x \in X$, $r_\alpha(x, x) = \alpha$,
- (ii) for all $x, y \in X$, $r_\alpha(x, y) + r_\alpha(y, x) > \alpha$ implies $x = y$.

A nonempty set X with an α -fuzzy preorder r_α defined on it, is called an α -fuzzy preorder and we denote it by (X, r_α) .

An α -fuzzy preordered set (X, r_α) is called an α -fuzzy ordered set (see [14]) if

- (iii) for all $x, z \in X$, $r_\alpha(x, z) \geq \sup_{y \in X} [\inf\{r_\alpha(x, y), r_\alpha(y, z)\}]$.

2000 *Mathematics Subject Classification*: 04A72, 03E72, 06A06, 47H10.

Key words and phrases: fuzzy set, α -fuzzy preorder relation, α -fuzzy order relation, non-expanding fuzzy multifunction, fixed point.

Received May 19, 2003.

Let (X, r_α) be a nonempty α -fuzzy preordered set. A fuzzy multifunction $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ is called non-expanding if for every $x \in X$ there exists $y \in X$ such that $\{y\} \subseteq T(x)$ and $r_\alpha(y, x) > \frac{\alpha}{2}$.

In the third section of this paper, we first prove the following result (Theorem 3.1): if (X, r_α) is a nonempty α -fuzzy preordered complete set and $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ is a non-expanding fuzzy multifunction, then T has a fixed point.

Secondly, we establish the existence of least and minimal fixed points of non-expanding fuzzy multifunctions in α -fuzzy ordered sets (Theorems 3.3 and 3.5). As consequences we obtain some fixed point theorems for non-expanding maps.

2. PRELIMINARIES

In order to establish our main results, we give some concepts and results.

Definition 2.1. Let (X, r_α) be an α -fuzzy preordered set. Then

(a) The α -fuzzy preorder r_α is said to be total if for all $x \neq y$ we have either $r_\alpha(x, y) > \frac{\alpha}{2}$ or $r_\alpha(y, x) > \frac{\alpha}{2}$. An α -fuzzy ordered set on which fuzzy order is total is called r_α -fuzzy chain.

(b) Let A be a subset of X . An element $l \in X$ is a r_α -lower bound of A if $r_\alpha(l, y) > \frac{\alpha}{2}$ for all $y \in A$. If l is a r_α -lower bound of A and $l \in A$, then l is called a least element of A . Similarly, we can define r_α -upper bounds and greatest elements of A .

(c) An element m of A is called a minimal element of A if $r_\alpha(y, m) > \frac{\alpha}{2}$ for some $y \in A$, then $y = m$. Maximal elements are defined analogously.

Let A be a nonempty subset of X . Then,

$$\sup_{r_\alpha}(A) = \text{the least element of } r_\alpha\text{-upper bounds of } A \text{ (if it exists),}$$

and

$$\inf_{r_\alpha}(A) = \text{the greatest element of } r_\alpha\text{-lower bounds of } A \text{ (if it exists).}$$

Definition 2.2. Let (X, r_α) be a nonempty α -fuzzy preordered set. A map $f : X \rightarrow X$ is called non-expanding if for every $x \in X$, $r_\alpha(f(x), x) > \frac{\alpha}{2}$.

An element x of X is called a fixed point of a map $f : X \rightarrow X$ if $f(x) = x$. We denote by $\text{Fix}(f)$ the set of all fixed points of f .

Definition 2.3. Let (X, r_α) be a nonempty α -fuzzy preordered set and let (x_β) be a family of X . We say that (x_β) is an α -fuzzy decreasing family if $r_\alpha(x_{\beta+1}, x_\beta) > \frac{\alpha}{2}$.

Definition 2.4. A nonempty α -fuzzy preordered set (X, r_α) is said to be an α -fuzzy ordered complete set if r_α is total and for every decreasing family (x_β) of X , $\inf_{r_\alpha}(x_\beta)$ exists in X .

Let X be a nonempty crisp set. A fuzzy multifunction is any map $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ such that for every $x \in X$, $T(x)$ is a nonempty fuzzy subset of X .

An element x of X is called a fixed point of a fuzzy multifunction $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ if $\{x\} \subseteq T(x)$. We denote by \mathcal{F}_T the set of all fixed points of T .

Definition 2.5 ([13]). Let (X, r_α) be an α -fuzzy ordered set. The inverse fuzzy relation s_α of r_α is defined by $s_\alpha(x, y) = r_\alpha(y, x)$, for all $x, y \in X$.

In [13], we established the following results.

Lemma 2.6 ([13, Lem. 3.6]). *Let (X, r_α) be a nonempty α -fuzzy ordered set. If every nonempty r_α -fuzzy chain has a r_α -upper bound, then X has a maximal element.*

Lemma 2.7 ([13, Prop. 3.6]). *Let (X, r_α) be a nonempty α -fuzzy ordered set and let s_α be the inverse α -fuzzy relation of r_α . Then,*

- (i) *The α -fuzzy relation s_α is an α -fuzzy order on X .*
- (ii) *If every nonempty r_α -fuzzy chain has a r_α -infimum, then every nonempty s_α -fuzzy chain has a r_α -supremum.*

3. MAIN RESULTS

We begin this section by proving the existence of fixed point of non-expanding fuzzy multifunctions. More precisely, we shall show the following:

Theorem 3.1. *Let (X, r_α) be a nonempty α -fuzzy preordered complete set and let $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Then, T has a fixed point.*

Proof. Let (X, r_α) be a nonempty α -fuzzy preordered complete set and let $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be an expanding fuzzy multifunction. Assume that T has no fixed point and let x_0 be a given element of X .

Next, we shall produce an α -fuzzy decreasing family (x_β) of X where β is an ordinal as follows:

- (i) First case: if $\beta = 0$, then the element x_0 is given by our hypothesis.
- (ii) Second case: β is a nonzero non limit ordinal. Since T is an expanding fuzzy multifunction and r_α is total, then for $x_{\beta-1}$ there is $x_\beta \in X$ such that

$$\begin{cases} \{x_\beta\} \subseteq T(x_{\beta-1}) \\ \alpha > r_\alpha(x_\beta, x_{\beta-1}) > \frac{\alpha}{2}. \end{cases}$$

- (iii) Third case: β is a limit ordinal. As (X, r_α) is an α -fuzzy ordered complete set, hence we have

$$x_\beta = \inf_{r_\alpha} \{x_\gamma : \gamma < \beta\}.$$

It follows that if β and γ are two ordinals with $\beta \neq \gamma$, then we have $x_\beta \neq x_\gamma$.

Now, we shall defining an ordinal valued function G by assign to every $x \in X$, an ordinal $G(x)$ as follows:

$$G(x) = \begin{cases} \beta & \text{if } x = x_\beta \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the range of G is the set of all ordinals. From ZF Axioms of substitution [12, page 261], we conclude that the range of G is a set. That is a contradiction. Therefore, T has a fixed point. □

As an application of Theorem 3.1, we obtain the following:

Corollary 3.2. *Let (X, r_α) be a nonempty α -fuzzy preordered complete set and let $f : X \rightarrow X$ be a non-expanding map. Then, f has a fixed point.*

For the existence of the least fixed point of non-expanding fuzzy multifunctions, we shall show the following:

Theorem 3.3. *Let (X, r_α) be a nonempty α -fuzzy ordered set with a least element l and let $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Then the set \mathcal{F}_T of all fixed points of T is nonempty and l is the least element of \mathcal{F}_T .*

Proof. Let (X, r_α) be a nonempty α -fuzzy ordered set with a least element l and let $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. Since T is a non-expanding fuzzy multifunction, there exists an element x of X such that $\{x\} \subseteq T(l)$ and $r_\alpha(x, l) > \frac{\alpha}{2}$. As $l = \inf_{r_\alpha}(X)$, then $r_\alpha(l, x) > \frac{\alpha}{2}$. Hence, $r_\alpha(x, l) + r_\alpha(l, x) > \alpha$. Therefore, $x = l$. So l is fixed point of T . On the other hand, l is the least element of X . Therefore, we deduce that l is the least fixed point of T . \square

As a consequence of Theorem 3.3, we have:

Corollary 3.4. *Let (X, r_α) be a nonempty α -fuzzy ordered set with a least element l and $f : X \rightarrow X$ be a non-expanding map. Then, the set $\text{Fix}(f)$ of all fixed points of f is nonempty and l is the least element of $\text{Fix}(f)$.*

Next, we shall establish the existence of a minimal fixed point of non-expanding fuzzy multifunctions.

Theorem 3.5. *Let (X, r_α) be a nonempty α -fuzzy ordered set with the property that every nonempty r_α -fuzzy chain has a r_α -infimum. Let $T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding r_α -fuzzy multifunction. Then, the set \mathcal{F}_T of all fixed points of T is nonempty and has a minimal element.*

To prove Theorem 3.5, we shall need the following lemma.

Lemma 3.6. *Let (X, r_α) be a nonempty α -fuzzy ordered set with the property that every nonempty r_α -fuzzy chain has a r_α -infimum. Then, X has a minimal element.*

Proof. Let (X, r_α) be a nonempty α -fuzzy ordered set with the property that every nonempty r_α -fuzzy chain has a r_α -infimum. Let s_α be the α -fuzzy inverse order relation of r_α . From Lemma 2.7, every nonempty r_α -fuzzy chain has a s_α -supremum. Then, by Lemma 2.6, X has a maximal element m (say) in (X, s_α) . Let x be an element of X such that $r_\alpha(x, m) > \frac{\alpha}{2}$. Then, $s_\alpha(m, x) > \frac{\alpha}{2}$. Since m is a maximal element in (X, s_α) , hence $x = m$. Therefore, m is a minimal element in (X, r_α) . \square

Now we are ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. Let (X, r_α) be a nonempty α -fuzzy ordered set with the property that every nonempty r_α -fuzzy chain has a r_α -infimum and let

$T : X \rightarrow [0, 1]^X \setminus \{\emptyset\}$ be a non-expanding fuzzy multifunction. By using Lemma 3.6, we deduce that X has a minimal element m (say). As T is a non-expanding fuzzy multifunction, so there is an element x of X such that $\{x\} \subseteq T(m)$ and $r_\alpha(x, m) > \frac{\alpha}{2}$. Since m is a minimal element of X , then $x = m$. Thus, m is a fixed point of T . Using the fact that m is a minimal element of X , we conclude that m is a minimal fixed point of T . \square

By using Theorem 3.5, we get:

Corollary 3.7. *Let (X, r_α) be a nonempty α -fuzzy ordered set with the property that every nonempty r_α -fuzzy chain has a r_α -infimum and let $f : X \rightarrow X$ be a non-expanding map. Then, the set of all fixed points of f is nonempty and has a minimal element.*

REFERENCES

- [1] Beg, I. and Islam, M., *Fuzzy order linear spaces*, J. Fuzzy Math. **3** (3) (1995), 659–670.
- [2] Beg, I., *Fixed points of fuzzy multivalued mappings with values in fuzzy ordered sets*, J. Fuzzy Math. **6** (1) (1998), 127–131.
- [3] Beg, I., *On fuzzy Zorn's lemma*, Fuzzy Sets and Systems **101** (1999), 181–183.
- [4] Beg, I., *Fixed points of expansive mappings on fuzzy preordered sets*, J. Fuzzy Math., 7(2)(1999), 397–400.
- [5] Beg, I., *Fixed points of fuzzy monotone maps*, Arch. Math. (Brno) **35** (1999), 141–144.
- [6] Bose, B. K. and Sahani, D., *Fuzzy mappings and fixed point theorems*, Fuzzy Sets and Systems **21** (1987), 53–58.
- [7] Billot, A., *Economic theory of fuzzy equilibria*, Lecture Notes in Econom. and Math. Systems **373** Springer-Verlag, Berlin 1992.
- [8] Fang, J. X., *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **46** (1992), 107–113.
- [9] Hadzic, O., *Fixed point theorems for multivalued mapping in some classes of fuzzy metric spaces*, Fuzzy Sets and Systems **29** (1989), 115–125.
- [10] Heilpern, S., *Fuzzy mappings and fixed point theorems*, J. Math. Anal. Appl. **83** (1981), 566–569.
- [11] Kaleva, O., *A note on fixed points for fuzzy mappings*, Fuzzy Sets and Systems **15** (1985), 99–100.
- [12] Negoita, C. V. and Ralescu, D. A., *Applications of fuzzy sets to systems analysis*, Fuzzy Sets and Systems **1** (1978), 155–167.
- [13] Stouti, A., *Fixed point theory for fuzzy monotone multifunctions*, J. Fuzzy Math. **11** (2) (2003), 455–466.
- [14] Stouti, A., *Fixed points of fuzzy monotone multifunctions*, Arch. Math. (Brno) **39** (2003), 209–212.
- [15] Stouti, A., *A fuzzy version of Tarski's fixpoint Theorem*, Arch. Math. (Brno) **40** (2004), 273–279.
- [16] Stouti, A., *Fixed points of expanding fuzzy multifunctions*, J. Fuzzy Math. **12**(4) (2004), 1542–1546.
- [17] Venugopalan, P., *Fuzzy ordered sets*, Fuzzy Sets and Systems **46** (1992), 221–226.
- [18] Zadeh, L. A., *Fuzzy sets*, Information and Control **8** (1965), 338–353.
- [19] Zadeh, L. A., *Similarity and fuzzy ordering*, Information Sciences **3** (1971), 177–200.

UNITÉ DE RECHERCHE: MATHÉMATIQUES ET APPLICATIONS
FACULTÉ DES SCIENCES ET TECHNIQUES BENI-MELLAL
B. P. 523, 23000 BENI-MELLAL, MOROCCO
E-mail: stouti@yahoo.com