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OSCILLATION OF A CLASS OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate a class of higher order neutral functional differential equations, and obtain some new oscillatory criteria of solutions.

1. Introduction

The problem determining oscillatory criteria for higher order differential equations has attracted a great deal of attention in the last several years, We mention here the literatures of Bainov and Mishev [2], Erbe, Kong and Zhang [3], Ladas and Sficas [4], Grace [5, 6], Agarwal, Grace and O'Regan [1] and references cited therein. In this paper, we consider a class of higher order nonlinear neutral functional differential equations of the form

$$[x(t)+p(t)x(t-\tau)]^{(n)}+q(t)f(x[g_1(t)],\dots,x[g_m(t)])=0\,,\quad t\ge t_0 \tag{1.1}$$

in which $n\ge 2$ is an even number, $\tau>0$ is a constant. $p(t)\in C([t_0,\infty),R)$, $0\le p(t)\le 1$; $q(t)\in C([t_0,\infty),R_+)$ is not identically zero on any ray $[t_1,\infty)$, $t_1>t_0$; $g_i(t)\in C([t_0,\infty),R)$, and $\lim_{t\to\infty} g_i(t)=\infty$; $f(u_1,\dots,u_m)\in C(R^m,R)$ is nondecreasing on u_i, $i\in I_m=\{1,2,\dots,m\}$.

We restrict our attention to proper solutions of equation (1.1), i.e. to nonconstant solutions existing on $[T,\infty)$ for some $T\ge t_0$ and satisfying $\sup_{t\ge T}|x(t)|>0$. A proper solution $x(t)$ of equation (1.1) is called oscillatory if it does not have the largest zero, otherwise, it is called nonoscillatory. The existence of oscillatory solutions for functional differential equations of neutral type can be found in [7].

We note that a special case of equations (1.1) is the following equation

$$[x(t)+p(t)x(t-\tau)]^{(n)}+q(t)x[\sigma(t)]=0\,,\qquad t\ge t_0>0\,. \tag{1.2}$$

Erbe, Kong and Zhang [3] discussed the problem of oscillation of solution, and gave the following theorem

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Theorem A. *Assume that the following conditions hold*

A_1) $0 \le p(t) \le 1$, $q(t) \ge 0$, *and* $q(t)$ *is not identical zero on any ray* $[t_1, \infty)$, $t \ge t_1 \ge t_0$*;*

A_2) $0 < \sigma(t) \le t$, $0 < \sigma'(t) \le 1$, *and* $\lim_{t\to\infty} \sigma(t) = \infty$.

If there exists a function $\rho(t) \in C'([t_0, \infty), (0, \infty))$ *such that*

$$\lim_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{(t-s)^{m-3}(\rho'(s)(t-s) - (m-1)\rho(s))^2}{\sigma'(s)\sigma^{n-2}(s)\rho(s)}\, ds < \infty\,, \tag{1.3}$$

$$\lim_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1}\rho(s)q(s)(1 - p(\sigma(s)))\, ds = \infty\,, \tag{1.4}$$

in which $m \ge 3$ *is a integer, then every solution of equation* (1.2) *is oscillatory.*

The aim of this paper is obtain some new oscillatory criteria for equation (1.1) by introducing parameter functions $H(t,s)$ and $h(t,s)$. The results generalize and improve Theorem A at the same time.

The following Lemmas will be found in [8] which are useful for the proof of main results.

Lemma 1. *Let* $u(t)$ *be a positive and* n *times differentiable function on* R_+. *If* $u^{(n)}(t)$ *is of constant sign and not identically zero on any ray* $[t_1, +\infty)$ *for* $(t_1 > 0)$, *then there exists a* $t_u \ge t_1$ *and an integer* l $(0 \le l \le n)$, *with* $n + l$ *even for* $u(t)u^{(n)}(t) \ge 0$ *or* $n + l$ *odd for* $u(t)u^{(n)}(t) \le 0$*; and for* $t \ge t_u$,

$$u^{(k)}(t) > 0\,, \quad 0 \le k < l\,; \qquad (-1)^{k-l}u^{(k)}(t) > 0\,, \quad l \le k < n\,.$$

Lemma 2. *Suppose that the conditions of Lemma 1 is satisfied, and*

$$u^{(n-1)}(t)u^{(n)}(t) \le 0\,, \qquad t \ge t_u\,,$$

then for any $\theta \in (0,1)$ *and sufficiently large* t, *sufficiently large* t, *there exists a constant* M_θ *satisfying*

$$|u'(\theta t)| \ge M_\theta t^{n-2} |u^{(n-1)}(t)|\,. \tag{1.5}$$

2. Main results

We say that a function $H = H(t,s)$ belong to a function class X, denoted by $H \in X$, if $H \in C'(D, R_+)$, where $D = \{(t,s) | t \ge s \ge t_0\}$ which satisfies

$$H(t,t) = 0\,, \quad t \ge t_0\,, \qquad H(t,s) > 0\,, \quad t > s \ge t_0\,;$$

moreover, there exists a function $h(t,s) \in C(D,R)$ such that

$$H_s{}'(t,s) \le 0\,, \quad \text{and} \quad -H_s{}'(t,s) = h(t,s)\sqrt{H(t,s)}\,, \quad (t,s) \in D\,.$$

Theorem 1. *Suppose that the following conditions hold*

(H_1) *there exists a function* $\sigma(t) \in C'([t_0, \infty), (0, \infty))$ *such that* $\sigma(t) \le g_i(t) \le t$, *and* $\lim_{t\to\infty} \sigma(t) = \infty$, $i \in I_m$*;*

(H_2) *$f(u_1,\dots,u_m)$ have same sign with $u_1,\dots,u_m$ when $u_1,\dots,u_m$ have same sign, $i\in I_m=1,2,\dots,m$, and there exist constants $N>0$ and $\lambda>0$ such that*

$$\liminf_{|u_1|\to\infty}\Big|\frac{f(u_1,\dots,u_m)}{u_1}\Big|\ge\lambda\,,\quad |u_i|\ge N\,,\ (i\ne 1)\,. \tag{2.1}$$

If there exists a function $\rho(t)\in C'([t_0,\infty),(0,\infty))$, such that for any $H\in X$

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg\{&\lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\}\\ &-\frac{\Big[\rho'(s)\sqrt{H(t,s)}-h(t,s)\rho(s)\Big]^2}{2M_\theta\rho(s)\sigma'(s)\sigma^{n-2}(s)}\Bigg\}ds=\infty\,,\end{aligned} \tag{2.2}$$

in which M_θ is a constant, then every solution of equation (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1), and $x(t)$ be an eventually positive solution. From the assumption of $g_i(t)$ and $f(u_1,\dots,u_m)$, there exists a $t_1\ge t_0$, such that

$$x(t-\tau)>0\,,\quad x[g_i(t)]>0\,,\quad f(x[g_1(t)],\dots,x[g_m(t)])>0\,,\qquad i\in I_m\,.$$

Let

$$y(t)=x(t)+p(t)x(t-\tau)\,, \tag{2.3}$$

then $y(t)>0$, $t\ge t_1$, and from (1.1), $y^{(n)}(t)\le 0$, and $y^{(n)}(t)$ is not identical zero on any $[t_1\infty)$. Thus, from Lemma 1, there are exists a $t_2\ge t_1$ and a odd $l(0<l<n)$ such that

$$y^{(k)}(t)>0\,,\quad 0\le k<l\,,\qquad (-1)^{k-l}y^{(k)}(t)>0\,,\quad l\le k<n\,,\qquad t\ge t_2\,.$$

Choosing $k=1$ and $k=n-1$, then

$$y'(t)>0\,,\quad y^{(n-1)}(t)>0\,,\qquad t\ge t_2\,. \tag{2.4}$$

Furthermore, from Lemma 2, and noting that $y^{(n)}(t)\le 0$, there exists $M_\theta>0$ and $t_3\ge t_2$, such that

$$y'[\frac{\sigma(t)}{2}]\ge M_\theta\sigma^{n-2}(t)y^{(n-1)}[\sigma(t)]\ge M_\theta\sigma^{n-2}(t)y^{(n-1)}(t)\,,\qquad t\ge t_3\,. \tag{2.5}$$

Let

$$z(t)=\frac{y^{(n-1)}(t)}{y[\frac{\sigma(t)}{2}]}\,, \tag{2.6}$$

then

$$\begin{aligned}z'(t)&=-q(t)\frac{f(x[g_1(t)],\dots,x[g_m(t)])}{y[\frac{\sigma(t)}{2}]}-\frac{1}{2}\sigma'(t)z(t)\frac{y'[\frac{\sigma(t)}{2}]}{y[\frac{\sigma(t)}{2}]}\\ &=I_1+I_2\,,\qquad t\ge t_3\,,\end{aligned} \tag{2.7}$$

for I_1, from (2.3), $(H_1)-(H_2)$, $y(t)\geq x(t)$ and $y'(t)>0$, we have

$$I_1 \leq -\lambda q(t)\frac{x[g_1(t)]}{y[\frac{\sigma(t)}{2}]} \leq -\lambda q(t)\frac{\{1-p[g_1(t)]\}y[g_1(t)]}{y[\frac{\sigma(t)}{2}]} \leq -\lambda q(t)\{1-p[g_1(t)]\}\,,$$

for I_2, from (2.5) and (2.6), we have

$$\frac{1}{2}\sigma'(t)z(t)\frac{y'[\frac{\sigma(t)}{2}]}{y[\frac{\sigma(t)}{2}]} \geq \frac{M_\theta}{2}\sigma'(t)\sigma^{n-2}(t)z^2(t)\,,$$

then

$$z'(t) \leq -\lambda q(t)\{1-p[g_1(t)]\} - \frac{M_\theta}{2}\sigma'(t)\sigma^{n-2}(t)z^2(t)\,, \qquad t\geq t_3\,. \tag{2.8}$$

For any $H\in X$, from (2.8), for any $t>T\geq t_3$, we have

$$\begin{aligned}
&\int_T^t \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\}\,ds \leq -\int_T^t H(t,s)\rho(s)z'(s)\,ds \\
&\quad -\frac{M_\theta}{2}\int_T^t H(t,s)\rho(s)\sigma'(s)\sigma^{n-2}(s)z^2(s)\,ds \\
&= H(t,T)\rho(T)z(T) + \int_T^t \left(\rho'(s)H(t,s) - h(t,s)\sqrt{H(t,s)}\rho(s)\right) z(s)\,ds \\
&\quad -\frac{M_\theta}{2}\int_T^t H(t,s)\rho(s)\sigma'(s)\sigma^{n-2}(s)z^2(s)\,ds \\
&= H(t,T)\rho(T)z(T) \\
&\quad -\frac{M_\theta}{2}\int_T^t \left[\sqrt{H(t,s)\rho(s)\sigma'(s)\sigma^{n-2}(s)}z(s) - \frac{\rho'(s)\sqrt{H(t,s)}-h(t,s)\rho(s)}{M_\theta\sqrt{\rho(s)\sigma'(s)\sigma^{n-2}(s)}}\right]^2 ds \\
&\quad +\int_T^t \frac{\left[\rho'(s)\sqrt{H(t,s)}-h(t,s)\rho(s)\right]^2}{2M_\theta\rho(s)\sigma'(s)\sigma^{n-2}(s)}\,ds \\
&\leq H(t,T)\rho(T)z(T) + \int_T^t \frac{\left[\rho'(s)\sqrt{H(t,s)}-h(t,s)\rho(s)\right]^2}{2M_\theta\rho(s)\sigma'(s)\sigma^{n-2}(s)}\,ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_T^t \left\{\lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)}-h(t,s)\rho(s)\right]^2}{2M_\theta\rho(s)\sigma'(s)\sigma^{n-2}(s)}\right\} ds \\
&\qquad \leq H(t,T)\rho(T)z(T)\,.
\end{aligned}$$

From $H'_s(t,s) \le 0$, for $t_3 \ge t_0$, we have $H(t,t_3) \le H(t,t_0)$, thus

$$\int_{t_3}^{t} \left\{ \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds$$
$$\le H(t,t_3)\rho(t_3)z(t_3) \le H(t,t_0)\rho(t_3)z(t_3)\,.$$

Furthermore, we have

$$\begin{aligned}
&\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds \\
&= \frac{1}{H(t,t_0)} \Bigg(\int_{t_0}^{t_3} \left\{ \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds \\
&\quad + \int_{t_3}^{t} \left\{ \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds \Bigg) \\
&\le \rho(t_3)z(t_3) + \lambda \int_{t_0}^{t_3} \frac{H(t,s)}{H(t,t_0)} \rho(s)q(s)\{1-p[g_1(s)]\}\, ds \\
&\le z(t_3) + \lambda \int_{t_0}^{t_3} \rho(s)q(s)\{1-p[g_1(s)]\}\, ds\,.
\end{aligned} \tag{2.9}$$

Let $t \to \infty$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_3}^{t} \left\{ \lambda H(t,s)\rho(s)q(s)\{1-p[g_1(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds \\
&\le \rho(t_3)z(t_3) + \lambda \int_{t_0}^{t_3} \rho(s)q(s)\{1-p[g_1(s)]\}\, ds\,,
\end{aligned} \tag{2.10}$$

which contradicts (2.2).

If $x(t)$ is an eventually negative solution of equation (1.1), let $x^*(t) = -x(t)$, then equation (1.1) transfer to

$$(1.1^*) \quad [x^*(t) + p(t)x^*(t-\tau)]^{(n)} + q(t)f^*(x^*[g_1(t)], \ldots, x^*[g_m(t)]) = 0\,, \quad t \geq t_0\,,$$

in which $f^*(x^*[g_1(t)], \ldots, x^*[g_m(t)]) = -f(-x^*[g_1(t)], \ldots, -x^*[g_m(t)])$, and $x^*(t)$ is an eventually positive solution of equation (1.1*), f^* satisfies $(H_1) - (H_2)$, then similar to the case of $x(t) > 0$, we can also obtain a contradiction. This completes the proof of Theorem 1. □

From Theorem 1, we have the following corollary.

Corollary 1. *Suppose that the conditions* (H_1) – (H_2) *hold, and there exists a function* $\rho(t) \in C'([t_0, \infty), (0, \infty))$, *such that for any* $H \in X$

$$(2.11) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)\{1 - p[g_1(s)]\}\, ds = \infty\,,$$

$$(2.12) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{\rho(s)\sigma'(s)\sigma^{n-2}(s)}\, ds < \infty\,,$$

then every solution of equation (1.1) *is oscillatory.*

Remark 1. When $f(x[g_1(t)], \ldots, x[g_m(t)]) \equiv x[\sigma(t)]$, choosing $H(t,s) = (t - s)^{m-1}$, then it is clear that Corollary 1 reduce to Theorem A.

Example 1. Consider the high-order equation

$$(*) \quad [x(t) + (1 - \frac{1}{t})x(t-\tau)]^{(4)} + \frac{1}{t}x(t)(1 + x^2(t - \frac{1}{2}))(1 + x^2(t-1)) = 0\,, \quad t \geq 1\,,$$

where $p(t) = 1 - \frac{1}{t}$, $q(t) = \frac{1}{t}$, $g_1(t) = x(t)$, $g_2(t) = t - \frac{1}{2}$, $g_3(t) = t - 1$, $m = 3$, $f(x[g_1(t)], x[g_2(t)], x[g_3(t)]) = f(x,y,z) = x(1+y^2)(1+z^2)$.

Taking $H(t,s) = (t-s)^{k-1}$, $h(t,s) = (k-1)(t-s)^{\frac{k-3}{2}}$, $\rho(s) = s$, in which $(t,s) \in D$ and $k > 2$ is a constant. It is easy known that the condition of Corollary 1 are satisfied, then all solutions of equation (*) are oscillatory.

When $f(x[g_1(t)], \ldots, x[g_m(t)]) = \sum_{i=1}^{m} f_i(x[g_i(t)])$, in which $f_i(x) \in C(R,R)$, $xf_i(x) > 0$ for $x \neq 0$, $i \in I_m$. We have the following theorem.

Theorem 2. *Suppose that the following conditions hold*

(H_3) *there exists a function* $\sigma(t) \in C'([t_0, \infty), (0, \infty))$ *such that* $\sigma(t) = \inf_{s\geq t}\{s, \min_{i\in I_m}\{g_i(s)\}\}$, *and* $\lim_{t\to\infty} \sigma(t) = \infty$, $i \in I_m$;

(H_4) $\frac{f_i(x)}{x} \geq \lambda_i > 0$, $x \neq 0$, *where* $\lambda_i > 0$ *are some constants,* $i \in I_m$. *If there exists a function* $\rho(t) \in C'([t_0, \infty), (0, \infty))$, *such that for any* $H \in X$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)q(s) \sum_{i=1}^{m} \lambda_i \{1 - p[g_i(s)]\} - \frac{\left[\rho'(s)\sqrt{H(t,s)} - h(t,s)\rho(s)\right]^2}{2M_\theta \rho(s)\sigma'(s)\sigma^{n-2}(s)} \right\} ds = \infty \,, \tag{2.13}$$

in which M_θ *is a constant, then every solution of equation* (1.1) *is oscillatory.*

Proof. Suppose that equation (1.1) has a nonoscillatory solution $x(t) > 0$. By using the same arguments as in the proof of Theorem 1, there exists a $t_1 \geq t_0$ such that $y(t) > 0$, $y'(t) > 0$, $y^{(n-1)}(t) > 0$ and $y^{(n)}(t) \leq 0$ for $t \geq t_1$, and there exists $M_\theta > 0$ and $t_2 \geq t_1$, such that

$$y'[\frac{\sigma(t)}{2}] \geq M_\theta \sigma^{n-2}(t) y^{(n-1)}[\sigma(t)] \geq M_\theta \sigma^{n-2}(t) y^{(n-1)}(t) \,, \quad t \geq t_2 \,. \tag{2.5}$$

Let

$$z(t) = \frac{y^{(n-1)}(t)}{y[\frac{\sigma(t)}{2}]} \,, \tag{2.6}$$

then

$$z'(t) = -q(t) \sum_{i=1}^{m} \frac{f_i(x[g_i(t)])}{y[\frac{\sigma(t)}{2}]} - \frac{1}{2}\sigma'(t) z(t) \frac{y'[\frac{\sigma(t)}{2}]}{y[\frac{\sigma(t)}{2}]} \,, \quad t \geq t_2 \,. \tag{2.14}$$

From (2.3), $(H_3) - (H_4)$, $y(t) \geq x(t)$ and $y'(t) > 0$, we have

$$\begin{aligned} -q(t) \sum_{i=1}^{m} \frac{f_i(x[g_i(t)])}{y[\frac{\sigma(t)}{2}]} &\leq -q(t) \sum_{i=1}^{m} \lambda_i \frac{x[g_i(t)]}{y[\frac{\sigma(t)}{2}]} \\ &= -q(t) \sum_{i=1}^{m} \lambda_i \frac{y[g_i(t)] - p[g_i(t)] x[g_i(t) - \tau]}{y[\frac{\sigma(t)}{2}]} \\ &\leq -q(t) \sum_{i=1}^{m} \lambda_i \frac{\{1 - p[g_i(t)]\} y[g_i(t)]}{y[\frac{\sigma(t)}{2}]} \\ &\leq -q(t) \sum_{i=1}^{m} \lambda_i \{1 - p[g_i(t)]\}, \end{aligned}$$

then

$$z'(t) \leq -q(t) \sum_{i=1}^{m} \lambda_i \{1 - p[g_1(t)]\} - \frac{M_\theta}{2} \sigma'(t) \sigma^{n-2}(t) z^2(t) \,, \quad t \geq t_2 \,. \tag{2.15}$$

The remainder proof is as same as proof of Theorem 1, we omit it. This completes the proof of Theorem 2. □

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