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A NONLINEAR DIFFERENTIAL EQUATION INVOLVING REFLECTION OF THE ARGUMENT

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Abstract. We study the nonlinear boundary value problem involving reflection of the argument

$$-M\Big(\int_{-1}^{1}|u'(s)|^2\,ds\Big)\,u''(x)=f\big(x,u(x),u(-x)\big)\qquad x\in[-1,1]\,,$$

where M and f are continuous functions with $M>0$. Using Galerkin approximations combined with the Brouwer's fixed point theorem we obtain existence and uniqueness results. A numerical algorithm is also presented.

1. Introduction

In this note we are concerned with the nonlinear differential equation

$$-M\Big(\int_{-1}^{1}|u'(s)|^2 ds\Big)\,u''(x)=f\big(x,u(x),u(-x)\big)\quad x\in[-1,1] \tag{1.1}$$

subject to the boundary condition

$$u(-1)=u(1)=0\,, \tag{1.2}$$

where $M:[0,\infty)\to\mathbb{R}$ and $f:[-1,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions with M satisfying:

$$\exists\,\delta>0\quad\text{such that}\quad M(s)\ge\delta\quad\text{for all}\quad s\ge0\,. \tag{1.3}$$

The equation (1.1) is related to the stationary states of the Kirchhoff equation

$$u_{tt}-\left[c_0+c_1\int_0^L|u_x|^2\,dx\right]u_{xx}=0\,,$$

which is a classical nonlinear model for the study of the free vibrations in elastic strings. The Kirchhoff equation was studied by several authors and we refer the reader to the paper by A. Arosio and S. Panizzi [1] for a short survey of its mathematical aspects and references. We also mention the papers [2] and [6] for

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others stationary problems of Kirchhoff type. On the other hand, the equation (1.1) involves a reflection of the argument x in the nonlinearity $f\big(x, u(x), u(-x)\big)$. Boundary value problem involving reflection of the argument was firstly considered by J. Wiener and A. R. Aftabizadeh in [9], where (1.1) was studied with $M = 1$. Using Schauder's fixed point theorem, they proved existence and uniqueness results. Later, their results were extended or improved by several authors, for example, Gupta [3], Hai [4], O'Regan [7] and Sharma [8].

We note that equation (1.1) has a nonlocal nonlinearity given by the function M. Then instead using a direct application of the Degree arguments, our analysis is based on the Galerkin approximations and a well known variation of the Brouwer's fixed point theorem whose statement is the following: Any continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying $\langle F(u), u\rangle \geq 0$ on the boundary $\partial B(0,\rho)$, for some $\rho > 0$, has a zero in the closed ball $\overline{B}(0,\rho)$. This result can be found, for example, in the book by S. Kesavan ([5], Theorem 5.2.5). Our results are the following two theorems.

Theorem 1. *Let us suppose that condition* (1.3) *holds. Let us suppose in addition that there exist positive constants* a, b *such that*

$$a + b < \frac{\delta\pi^2}{4} \tag{1.4}$$

and satisfying

$$f(x,u,v)u \leq a|u|^2 + b|u|\,|v| + c|u|\,, \tag{1.5}$$

for all $x \in [-1,1]$, $u, v \in \mathbb{R}$ *and any fixed constant* $c > 0$. *Then problem* (1.1)–(1.2) *has at least one solution* $u \in C^2([-1,1])$.

Theorem 2. *Let us assume the assumptions of Theorem* 1 *with* (1.5) *replaced by*

$$[f(x,u_1,v_1) - f(x,u_2,v_2)]\,(u_1 - u_2) \leq a|u_1 - u_2|^2 + b|u_1 - u_2|\,|v_1 - v_2|\,, \tag{1.6}$$

for all $x \in [-1,1]$ *and* u_1, u_2, v_1, $v_2 \in \mathbb{R}$. *Then if* M *is continuously differentiable and* $\|M'\|_\infty$ *is sufficiently small, problem* (1.1)–(1.2) *has exactly one solution.*

The proofs of the theorems are given in Section 2. In Section 3 we consider a numerical example using the finite-difference method.

2. Existence and uniqueness

We begin with some notations. Let $H^k(-1,1)$ be the Sobolev space of the functions $u : [-1,1] \to \mathbb{R}$ with the derivative u^{k-1} absolutely continuous and u^k in $L^2(-1,1)$ and let $H_0^1(-1,1) = \{u \in H^1(-1,1) : u(-1) = u(1) = 0\}$. In $H_0^1(-1,1)$ we consider the norm $\|u\|_{H_0^1} = \|u'\|_2$, where $\|\cdot\|_p$ denotes standard L^p norm. Then it is known that both embeddings $H^2(-1,1)\cap H_0^1(-1,1) \hookrightarrow H_0^1(-1,1)$ and $H_0^1(-1,1) \hookrightarrow C^0([-1,1])$ are compacts. Besides, the following Wirtinger type inequalities

$$\|u\|_2 \leq \frac{2}{\pi}\|u'\|_2 \quad \text{and} \quad \|u\|_1 \leq \frac{2\sqrt{2}}{\pi}\|u'\|_2 \tag{2.1}$$

hold for every $u \in H_0^1(-1,1)$.

Proof of Theorem 1. The proof is given in three steps.

Step 1 – Approximate Problem: Let $\{\omega_k\}$ be the complete orthonormal system for $H_0^1(-1,1)$ given by the eigenfunctions of $-\omega'' = \lambda\omega$ with $\omega(-1) = \omega(1) = 0$. Let us put

$$V_n = \mathrm{Span}\{\omega_1, \cdots, \omega_n\}\,.$$

Then V_n is isometric to $\mathbb{R}^n$ in the following way: Each $v \in V_n$ is uniquely associated to $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ through the relation $v = \sum \xi_k \omega_k$. Since $\{\omega_k\}$ is orthonormal in $H_0^1(-1,1)$, we see that

$$\|v\|_{V_n}^2 = \|v'\|_2^2 = \sum_{k=1}^{n} \xi_k^2 = \|\xi\|_{\mathbb{R}^n}^2\,.$$

We search for a function $u_n \in V_n$ such that for $k = 1, 2, \cdots, n$.

$$\int_{-1}^{1} \left[M(\|u_n'\|_2^2)u_n''(x) + f(x, u_n(x), u_n(-x))\right] \omega_k(x)\, dx = 0\,. \tag{2.2}$$

The equations in (2.2) define a nonlinear algebraic system in $\mathbb{R}^n$. In fact, system (2.2) can be written as $F_n(v) = 0$, where F_n is the operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose k-component is defined by

$$\langle F_n(v), \omega_k\rangle = \int_{-1}^{1} \left[-M(\|v'\|_2^2)v''(x) - f(x, v(x), v(-x))\right] \omega_k(x)\, dx\,,$$

which is continuous because of the continuity of the functions M and f. To solve (2.2) we apply the Brouwer fixed point theorem. From (1.3), (1.5), (2.1) and integration by parts, we have for $v \in V_n$

$$\begin{aligned}
\langle F_n(v), v\rangle = &\int_{-1}^{1} \left[-M(\|v'\|_2^2)v''(x) - f(x, v(x), v(-x))\right] v(x)\, dx \\
&\geq \delta\, \|v'\|_2^2 - (a+b)\, \|v\|_2^2 - c\|v\|_1 \\
&\geq \left[\delta - (a+b)\frac{4}{\pi^2}\right] \|v'\|_2^2 - \frac{c2\sqrt{2}}{\pi}\|v'\|_2\,.
\end{aligned}$$

This shows the existence of $R_1 > 0$, depending only on δ, a, b and c, such that $\langle F_n(v), v\rangle \geq 0$ if $\|v\|_{V_n} = R_1$. Then from the Brouwer fixed point theorem, system (2.2) has a solution $u_n \in V_n$ satisfying

$$\|u_n'\|_2 \leq R_1 \qquad \forall\, n \in \mathbb{N}\,. \tag{2.3}$$

Step 2 – A Priori Estimates: Now we obtain an additional estimate in order to have strong convergence of the approximate solutions u_n in $H_0^1(-1,1)$. Since $\omega_k'' = -\lambda_k\omega_k$, we see that (2.2) holds with ω_k replaced by ω_k'' and then

$$\delta\|u_n''\|_2^2 \leq \int_{-1}^{1} |f(x, u_n(x), u_n(-x))|\, |u_n''(x)|\, dx\,. \tag{2.4}$$

But from (2.3) we have that (u_n) is a bounded sequence in $C^0([-1,1])$ and therefore $f(x, u_n(x), u_n(-x))$ is uniformly bounded. This combined with (2.4) yields a constant $R_2 > 0$ such that

$$\|u_n''\|_2 \leq R_2 \qquad \forall\, n \in \mathbb{N}\,. \tag{2.5}$$

Step 3 – Passage to the Limit: From the estimates (2.3) and (2.5) and the Sobolev embeddings, there exists $u \in H^2(-1,1) \cap H_0^1(-1,1)$ such that, going to a subsequence if necessary,

$$u_n \to u \quad \text{strongly in} \quad H_0^1(-1,1)$$

and

$$u_n'' \rightharpoonup u'' \quad \text{weakly in} \quad L^2(-1,1)\,. \tag{2.6}$$

Then passing to the limit in (2.2) we conclude that

$$\int_{-1}^{1} \left[-M(\|u'\|_2^2)u''(x) - f(x,u(x),u(-x))\right] v(x)\,dx = 0$$

for all $v \in H_0^1(-1,1)$. Therefore u is a weak solution of (1.1)-(1.2) and, from the regularity of f, we get that u is in fact a solution in $C^2([-1,1])$. $\square$

Proof of Theorem 2. The existence part follows from Theorem 1 since (1.6) implies (1.5). In fact, taking $u_2 = v_2 = 0$ we see that

$$f(x,u_1,v_1)\,u_1 \leq a|u_1|^2 + b|u_1|\,|v_1| + c|u_1|\,,$$

with $c = \max\{|f(x,0,0)|;\ x \in [-1,1]\}$.
Now let u_1 and u_2 be two solutions of problem (1.1)–(1.2). Putting $w = u_1 - u_2$ we have

$$\begin{aligned} M(\|u_1'\|_2^2)w''(x) = \; & -\left[M(\|u_1'\|_2^2) - M(\|u_2'\|_2^2)\right] u_2''(x) \\ & - \left[f(x,u_1(x),u_1(-x)) - f(x,u_2(x),u_2(-x))\right]. \end{aligned}$$

Then, multiplying this identity by $w(x)$ and integrating by parts we have, after some re arrangements,

$$M(\|u_1'\|_2^2)\,\|w'\|_2^2 = -\left[M(\|u_1'\|_2^2) - M(\|u_2'\|_2^2)\right] \int_{-1}^{1} u_2'(x)\,w'(x)\,dx$$

$$+ \int_{-1}^{1} \left[f(x,u_1(x),u_1(-x)) - f(x,u_2(x),u_2(-x))\right]\left[u_1(x) - u_2(x)\right] dx\,. \tag{2.7}$$

Next we note that the arguments used to obtain (2.3) also imply that every solution u of (1.1)-(1.2) satisfies

$$\|u'\|_2 \leq R_3 = \frac{c2\sqrt{2}}{\pi}\left(\delta - (a+b)\frac{4}{\pi^2}\right)^{-1}.$$

Then, since this estimate is independent of $\|M'\|_\infty$, using the inequality $|\|p\|^2 - \|q\|^2| \le (\|p\| + \|q\|)\,\|p-q\|$, we infer that

$$\begin{aligned}
&\left|\left[M(\|u_1'\|_2^2) - M(\|u_2'\|_2^2)\right]\int_{-1}^{1} u_2'(x)\,w'(x)\,dx\right| \\
&\qquad \le \|M'\|_\infty \left|\|u_1'\|_2^2 - \|u_2'\|_2^2\right| \|u_2'\|_2 \|w'\|_2 \\
&\qquad \le \|M'\|_\infty\, 2R_3^2\, \|w'\|_2^2.
\end{aligned}$$

Hence from (1.6) and (2.7) it follows that

$$\left[\delta - (a+b)\frac{4}{\pi^2}\right]\|w'\|_2^2 \le \|M'\|_\infty\, 2R_3^2\, \|w'\|_2^2\,.$$

If $\|M'\|_\infty$ is sufficiently small, we conclude that $\|w'\|_2 = 0$ and hence $w \equiv 0$. $\square$

3. Numerical solutions

In this section we consider a numerical algorithm for the problem (1.1)-(1.2) based on the finite-differences method. Let $-1 = x_0 < x_1 < \cdots < x_n = 1$ be a discretization of the interval $[-1, 1]$ with mesh size $h = 2/n$. Then putting $u_i = u(x_i)$, $f_i = f(x_i, u_i, u_{n-i})$ and using central differences formula, the equation (1.1) becomes

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i K^{-1}, \quad 1 \le i \le n-1\,, \tag{3.1}$$

where K is the finite-difference approximation of $M(\int_{-1}^{1} u'^2\,dx)$. From the boundary conditions we know that $u_0 = u_n = 0$, and therefore the trapezoidal method gives

$$K \approx M\left(\frac{1}{2h}(u_1^2 + u_{n-1}^2) + \frac{1}{4h}\sum_{i=1}^{n-1}(u_{i+1} - u_{i-1})^2\right)\,.$$

Now we can compute $u_1, \cdots, u_{n-1}$ by solving the nonlinear system (3.1) through successive linearization combined with the Gauss–Seidel method. The basic algorithm is the following.

1 - Choose initial guess u^0
2 - For $N = 0, 1, 2, 3, \ldots$
- compute K and $f_i(x_i, u_i^N, u_{n-i}^N)$, $1 \le i \le n-1$
- solve linear system (3.1)
- test convergence

3 - End iteration.

Next we give a numerical example. Let us consider problem (1.1)–(1.2) with

$$M(s) = 1 + \frac{25}{64}s^2 \quad \text{and} \quad f(x,p,q) = -x^6 + 2x^4 + 2x^3 - x^2 + 10x + p^2 - 2q\,.$$

The exact solution is $u(x) = x - x^3$. Using $u \equiv 0$ as initial approximation and mesh size $h = 0.1$, we have obtained the following error table, where $E^N = \|u^N - u\|_\infty$ and $\varepsilon^N = \|u^N - u^{N-1}\|_\infty$.

Iteration N	E^N	ε^N
1	0.33342	$0.91629 \cdot 10^{-1}$
50	$0.14407 \cdot 10^{-1}$	$0.41263 \cdot 10^{-3}$
100	$0.70612 \cdot 10^{-2}$	$0.72186 \cdot 10^{-4}$
200	$0.63687 \cdot 10^{-2}$	$0.22164 \cdot 10^{-5}$
300	$0.64039 \cdot 10^{-2}$	$0.67990 \cdot 10^{-7}$
400	$0.64049 \cdot 10^{-2}$	$0.21000 \cdot 10^{-8}$

$\square$

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