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EXISTENCE FOR NONCONVEX INTEGRAL INCLUSIONS VIA FIXED POINTS

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Abstract. We consider a nonconvex integral inclusion and we prove a Filippov type existence theorem by using an appropiate norm on the space of selections of the multifunction and a contraction principle for set-valued maps.

1. Introduction

This paper is concerned with the following integral inclusion

$$x(t) = \lambda(t) + \int_0^t f\big(t, s, u(s)\big)\, ds\,, \tag{1.1}$$

$$u(t) \in F\big(t, V(x)(t)\big)\,, \quad \text{a.e.} \quad (I := [0, T])\,, \tag{1.2}$$

where $\lambda(.) : I \to R^n$, $F(.,.) : I \times X \to \mathcal{P}(X)$, $f(.,.,.) : I \times I \times X \to X$, $V : C(I,X) \to C(I,X)$ are given mappings and X is a separable Banach space.

The aim of this paper is to obtain a version of Filippov's theorem concerning the existence of solutions for problem (1.1)-(1.2). Such kind of results have been proved by Zhu ([8]). The approach proposed in the present paper is different to the ones in [6], [8] and it is based on an idea of Tallos ([7]), applying the contraction principle in the space of selections of the multifunction instead of the space of solutions.

Our estimate is different from the usual form of the Filippov's estimate ([8]). This is a consequence of our method of deriving a "pointwise" inequality from a norm inequality.

We note that similar results are obtained in the case of differential inclusions ([4], [7]), in the case of mild solutions of semilinear differential inclusions in Banach spaces ([2]), and for hyperbolic differential inclusions ([3]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main result.

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2. Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Consider X a real separable Banach space with the norm $\|.\|$ and denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, by $\mathcal{B}(X)$ the family of all Borel subsets of X. The unit ball in X will be denoted by B.

In what follows, as usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $\|x(.)\|_C = \sup_{t\in I} \|x(t)\|$.

In order to study problem (1.1)-(1.2) we introduce the following assumption.

Hypothesis 2.1. Let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a set-valued map with nonempty closed values that verify:

i) The set-valued map $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $L(.) \in L^1(I, R_+)$ such that, for almost all $t \in I$, $F(t,.)$ is L(t)-Lipschitz in the sense that

$$d_H\big(F(t,x), F(t,y)\big) \le L(t)\|x - y\| \quad \forall\, x, y \in X\,,$$

where d_H is the Hausdorff generalized metric on $\mathcal{P}(X)$ defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},\ d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

iii) The mapping $f : I \times I \times X \to X$ is continuous, $V : C(I, X) \to C(I, X)$ and there exist the constants $M_1, M_2 > 0$ such that

$$\|f(t, s, u_1) - f(t, s, u_2)\| \le M_1\|u_1 - u_2\|\,, \qquad \forall u_1, u_2 \in X\,,$$

$$\|V(x_1)(t) - V(x_2)(t)\| \le M_2\|x_1(t) - x_2(t)\|\,, \quad \forall t \in I, \forall x_1, x_2 \in C(I, X)\,.$$

System (1.1)-(1.2) encompasses a large variety of differential inclusions and control systems and, in particular, those defined by partial differential equations.

Example 2.2. Set $f(t, \tau, u) = G(t-\tau)u$, $V(x) = x$, $\lambda(t) = G(t)x_0$ where $\{G(t)\}_{t\ge 0}$ is a C^0-semigroup with an infinitesimal generator A. Then a solution of system (1.1)-(1.2) represents a mild solution of

$$x'(t) \in Ax(t) + F\big(t, x(t)\big)\,, \quad x(0) = x_0\,. \tag{2.1}$$

In particular, this problem includes control systems governed by parabolic partial differential equations as a special case. When $A = 0$, relation (2.1) reduces to classical differential inclusions.

To simplify the notations, we set

$$\Phi(u)(t) = \int_0^t f\big(t, \tau, u(\tau)\big)\, d\tau\,, \quad t \in I\,. \tag{2.2}$$

Then the integral inclusion system (1.1)-(1.2) becames

$$x(t) = \lambda(t) + \Phi(u)(t)\,, \quad u(t) \in F\big(t, V(x)(t)\big) \quad \text{a.e. } (I)\,, \tag{2.3}$$

which may be written in the more “compact” form

$$u(t) \in F\big(t, V(\lambda + \Phi(u))(t)\big) \quad \text{a.e. } (I)\,,$$

but the integral operator $\Phi(.)$ in (2.2) plays a certain role in the proofs of our main results.

Denote $m(t)=\int_0^t L(s)\,ds$, $t\in I$.

Given $\alpha\in R$ we denote by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $u(.):I\to X$ endowed with the norm

$$\|u(.)\|_1=\int_0^T e^{-\alpha M_1M_2m(t)}\|u(t)\|\,dt\,.$$

Definition 2.3. A pair of functions (x,u) is called a *solution pair* of (2.3), if $x(.)\in C(I,X)$, $u(.)\in L^1(I,X)$ and relation (2.3) holds.

We denote by $S(\lambda)$ the solution set of (1.1)-(1.2).

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a l-contraction if there exists $0<l<1$ such that:

$$d\big(T(x),T(y)\big)\le l d(x,y)\,,\quad \forall x,y\in Z\,.$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z\in Z$ such that $z\in T(z)$ (see, for instance, [5]).

We denote by $\mathrm{Fix}\,(T)$ the set of all fixed point of the multifunction T. Obviously, $\mathrm{Fix}\,(T)$ is closed.

Proposition 2.4 ([5])**.** *Let Z be a complete metric space and suppose that T_1,T_2 are l-contractions with closed values in Y. Then*

$$d_H\big(\,\mathrm{Fix}\,(T_1),\mathrm{Fix}\,(T_2)\big)\le\frac{1}{1-l}\sup_{z\in Z}d\big(T_1(z),T_2(z)\big)\,.$$

3. The main result

We are able now to prove a Filippov type existence theorem concerning the existence of solutions of problem (1.1)-(1.2).

Theorem 3.1. *Let Hypothesis 2.1 be satisfied, let $\lambda(.)$, $\mu(.)\in C(I,X)$ and let $v(.)\in L^1(I,X)$ be such that*

$$d\big(v(t),F(t,V(y)(t)\big)\le p(t)\quad a.e.\ (I),$$

where $p(.)\in L^1(I,R_+)$ and $y(t)=\mu(t)+\Phi(v)(t)$, $\forall t\in I$.

Then for every $\alpha>1$ and for every $\epsilon>0$ there exists $x(.)\in S(\lambda)$ such that for every $t\in I$

$$\|x(t)-y(t)\|\le\frac{\alpha}{\alpha-1}e^{\alpha M_1M_2m(T)}\Big[\|\lambda-\mu\|_C+M_1\int_0^T e^{-\alpha M_1M_2m(t)}p(t)\,dt\Big]+\epsilon\,.$$

Proof. For $\lambda\in C(I,X)$ and $u\in L^1(I,X)$ define

$$x_{u,\lambda}(t)=\lambda(t)+\int_0^t f(t,s,u(s))\,ds\,.$$

Consider $\lambda \in C(I,X), \sigma \in L^1(I,X)$ and define the set valued maps:

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)), \quad t \in I, \tag{3.1}$$

$$T_\lambda(\sigma) := \{\psi(.) \in L^1(I,X); \psi(t) \in M_{\lambda,\sigma}(t) \quad \text{a.e.} \quad (I)\}. \tag{3.2}$$

We shall prove first that $T_\lambda(\sigma)$ is nonempty and closed for every $\sigma \in L^1$.

The fact that that the set valued map $M_{\lambda,\sigma}(.)$ is measurable is well known. For example the map $t \to F(t, V(x_{\sigma,\lambda})(t))$ can be approximated by step functions and we can apply Theorem III. 40 in [1]. Since the values of F are closed, with the measurable selection theorem (e.g. Theorem III.6 in [1]) we infer that $M_{\lambda,\sigma}(.)$ admits a measurable selection and $T_\lambda(\sigma)$ is nonempty.

The set $T_\lambda(\sigma)$ is closed. Indeed, if $\psi_n \in T_\lambda(\sigma)$ and $\|\psi_n - \psi\|_1 \to 0$, then we can pass to a subsequence ψ_{n_k} such that $\psi_{n_k}(t) \to \psi(t)$ for a.e. $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

The next step of the proof will show that $T_\lambda(.)$ is a contraction on $L^1(I,X)$.

Let $\sigma_1, \sigma_2 \in L^1(I,X)$ be given, $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following set valued map:

$$G(t) := M_{\lambda,\sigma_2}(t) \cap \Big\{ z \in X; \|\psi_1(t) - z\| \le M_1 M_2 L(t) \int_0^t \|\sigma_1(s) - \sigma_2(s)\|\, ds + \delta \Big\}$$

Since

$$\begin{aligned} d\big(\psi_1(t),\, M_{\lambda,\sigma_2}(t)\big) &\le d_H\big(F(t, V(x_{\sigma_1,\lambda})(t)), F(t, V(x_{\sigma_2,\lambda})(t))\big) \\ &\le L(t)\|V(x_{\sigma_1,\lambda})(t) - V(x_{\sigma_2,\lambda})(t)\| \le L(t) M_2 \|x_{\sigma_1,\lambda}(t) - x_{\sigma_2,\lambda}(t)\| \\ &\le M_2 L(t) \int_0^t \|f\big(t,s,\sigma_1(s)\big) - f\big(t,s,\sigma_2(s)\big)\|\, ds \\ &\le M_1 M_2 L(t) \int_0^t \|\sigma_1(s) - \sigma_2(s)\|\, ds \end{aligned}$$

we deduce that $G(.)$ has nonempty closed values.

Moreover, according to Proposition III.4 in [1], $G(.)$ is measurable.

Let $\psi_2(.)$ be a measurable selection of $G(.)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned} \|\psi_1 - \psi_2\|_1 = &\int_0^T e^{-\alpha M_1 M_2 m(t)} \|\psi_1(t) - \psi_2(t)\|\, dt \\ \le &\int_0^T e^{-\alpha M_1 M_2 m(t)} (M_1 M_2 L(t) \int_0^t \|\sigma_1(s) - \sigma_2(s)\|\, ds)\, dt \\ &+ \delta \int_0^T e^{-\alpha M_1 M_2 m(t)}\, dt \\ \le &\frac{1}{\alpha} \|\sigma_1 - \sigma_2\|_1 + \delta \int_0^T e^{-\alpha M_1 M_2 m(t)}\, dt. \end{aligned}$$

Since δ is arbitrarly, we deduce that

$$d(\psi_1, T_\lambda(\sigma_2)) \le \frac{1}{\alpha} \|\sigma_1 - \sigma_2\|_1.$$

Replacing $\sigma_1(.)$ with $\sigma_2(.)$, we obtain

$$d_H(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \leq \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1 .$$

Hence $T_\lambda(.)$ is a contraction on $L^1(I,X)$.

We consider the following set-valued maps

$$\begin{aligned} \tilde{F}(t,x) &:= F(t,x) + p(t) , \\ \tilde{M}_{\lambda,\sigma}(t) &= \tilde{F}(t, V(x_{\sigma,\lambda})(t)) , \\ \tilde{T}_\mu(\sigma) &= \{\psi \in L^1(I,X) ; \quad \psi(t) \in \tilde{M}_{\mu,\sigma}(t) \quad \text{a.e. } (I)\} . \end{aligned}$$

Obviously, $\tilde{F}(.,.)$ satisfies Hypothesis 2.1.

Repeating the previous step of the proof we obtain that $\tilde{T}_\mu$ is also a $\frac{1}{\alpha}$-contraction on $L^1(I,X)$ with closed nonempty values.

We prove next the following estimate:

$$d_H(T_\lambda(\sigma), \tilde{T}_\mu(\sigma)) \leq \frac{1}{\alpha M_1}\|\lambda - \mu\|_C + \int_0^T e^{-\alpha M_1 M_2 m(t)} p(t)\, dt . \tag{3.3}$$

Let $\phi \in T_\lambda(\sigma), \delta > 0$ and define

$$G_1(t) = \tilde{M}_{\lambda,\sigma}(t) \cap \{z \in X;\ \|\phi(t) - z\| \leq M_2 L(t)\|\lambda - \mu\|_C + p(t) + \delta\} .$$

With the same arguments used for the set valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.) \in \tilde{T}_\mu(\sigma)$. One has:

$$\begin{aligned} \|\phi - \psi\|_1 \leq & \int_0^T e^{-\alpha M_1 M_2 m(t)} \|\phi(t) - \psi(t)\|\, dt \\ \leq & \int_0^T e^{-\alpha M_1 M_2 m(t)} [M_2 L(t)\|\lambda - \mu\|_C + p(t) + \delta]\, dt \\ = & \|\lambda - \mu\|_C \int_0^T e^{-\alpha M_1 M_2 m(t)} M_2 L(t)\, dt \\ & + \int_0^T e^{-\alpha M_1 M_2 m(t)} p(t)\, dt + \delta \int_0^T e^{-\alpha M_1 M_2 m(t)}\, dt . \end{aligned}$$

Since δ is arbitrarly, as above we obtain (3.3).

Applying Proposition 2.4 we obtain:

$$d_H(\mathrm{Fix}\,(T_\lambda), \mathrm{Fix}\,(\tilde{T}_\mu)) \leq \frac{1}{M_1(\alpha - 1)}\|\lambda - \mu\|_C + \frac{\alpha}{\alpha - 1}\int_0^T e^{-\alpha M_1 M_2 m(t)} p(t)\, dt .$$

Since $v(.) \in \mathrm{Fix}\,(\tilde{T}_\mu)$, it follows that there exists $u(.) \in \mathrm{Fix}\,(T_\lambda)$ such that:

(3.4)

$$\|v - u\|_1 \leq \frac{1}{M_1(\alpha - 1)}\|\lambda - \mu\|_C + \frac{\alpha}{\alpha - 1}\int_0^T e^{-\alpha M_1 M_2 m(t)} dt + \frac{\epsilon}{M_1 e^{\alpha M_1 M_2 m(T)}} .$$

We define

$$x(t) = \lambda(t) + \int_0^t f\big(t, s, u(s)\big)\, ds .$$

One has

$$\|x(t)-y(t)\| \leq \|\lambda(t)-\mu(t)\| + M_1\int_0^t \|u(s)-v(s)\|\,ds$$
$$\leq \|\lambda-\mu\|_C + M_1 e^{\alpha M_1 M_2 m(T)}\|u-v\|_1$$

Combining the last inequality with (3.4) we obtain

$$\|x(t)-y(t)\| \leq \|\lambda-\mu\|_C\Big[1+\frac{e^{\alpha M_1 M_2 m(T)}}{\alpha-1}\Big]$$
$$+\frac{M_1\alpha}{\alpha-1}e^{\alpha M_1 M_2 m(T)}\int_0^T e^{-\alpha M_1 M_2 m(t)}p(t)\,dt+\epsilon$$
$$\leq \frac{\alpha}{\alpha-1}e^{\alpha M_1 M_2 m(T)}\Big[\|\lambda-\mu\|_C + M_1\int_0^T e^{-\alpha M_1 M_2 m(t)}p(t)\,dt\Big]+\epsilon$$

and the proof is complete. □

Remark 1. If $f(t,\tau,u)=G(t-\tau)u,\ V(x)=x,\ \lambda(t)=G(t)x_0$ where $\{G(t)\}_{t\geq 0}$ is a C^0-semigroup with an infinitesimal generator A, Theorem 3.1 yields the result in [2] obtained for mild solutions of the semilinear differential inclusion (2.1).

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