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ON HEREDITY OF STRONGLY PROXIMAL ACTIONS

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Abstract. We prove that action of a semigroup T on compact metric space X by continuous selfmaps is strongly proximal if and only if T action on $\mathcal{P}(X)$ is strongly proximal. As a consequence we prove that affine actions on certain compact convex subsets of finite-dimensional vector spaces are strongly proximal if and only if the action is proximal.

Let X be a complete separable metric space. Let T be a semigroup acting on X by continuous selfmaps. A system (X, T) is a pair consisting of a complete separable metric space X and a semigroup T acting on X by continuous selfmaps. In such a situation X is a called a T-space.

Two points x and y in a T-space X are said to be *proximal* if there exists a sequence (t_n) in T such that $\lim t_n x = \lim t_n y$.

We say that a system (X, T) is *proximal* or the action of T on X is *proximal* if any two points x and y in X are proximal.

It is easy to see that group of special linear automorphisms on $\mathbb{R}^n$ action on $\mathbb{R}^n$ is proximal and the compact group actions are not proximal.

Let $\mathcal{P}(X)$ be the space of all regular Borel probability measures on X, equipped with the weak* topology with respect to all continuous bounded functions. It may be seen that $\mathcal{P}(X)$ equipped with the weak* topology is a complete separable metric space (see [P]). The map $x \mapsto \delta_x$, maps X homeomorphically onto a closed subset δ_X, say of $\mathcal{P}(X)$ (see [P]) where δ_x is the measure concentrated at the point x. Suppose a semigroup T acts on X by continuous selfmaps. Then the action of T on X extends to an action on $\mathcal{P}(X)$ in the following natural way, for any $\lambda \in \mathcal{P}(X)$ and any $t \in T$ $t\lambda(E) = \lambda(t^{-1}E)$ for any Borel subset E of X.

We say that a system (X, T) is *strongly proximal* or the action of T on X is *strongly proximal* if for any $\lambda \in \mathcal{P}(X)$, there exists a sequence $(t_n) \subset T$ such that $t_n\lambda \to \delta_x$ for some $x \in X$.

By considering $\frac{1}{2}(\delta_x + \delta_y)$ for any $x, y \in X$, it is easy to see that any strongly proximal system is proximal; see [G] for more details on proximal and strongly proximal systems. But not all proximal systems are strongly proximal. The action of the special Linear group $SL(V)$ on V is proximal but it is not strongly proximal.

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We now prove the following interesting result which is needed in the sequel.

Proposition 1. *Let T be a semigroup acting on a complete separable metric space X by continuous selfmaps. Then the action of T on X is strongly proximal if and only if the action of T on $\mathcal{P}(X)$ is proximal.*

Proof. Suppose the action of T on X is strongly proximal. Let λ_1 and λ_2 be in $\mathcal{P}(X)$. Then there exists a sequence (t_n) in T such that $t_n(\frac{1}{2}(\lambda_1 + \lambda_2)) \to \delta_x$ for some $x \in X$. Then for given $1 > \epsilon > 0$ there exists a compact subset K of X such that

$$t_n\lambda_1(K) + t_n\lambda_2(K) > 2 - \epsilon \tag{i}$$

for all $n \geq 1$. Suppose for some $i = 1, 2$ and for some $m \geq 1$, $t_m\lambda_i(K) \leq 1 - \epsilon$. Then since λ_1 and λ_2 are probability measures, we get that

$$t_m\lambda_1(K) + t_m\lambda_2(K) \leq 2 - \epsilon$$

for some $m \geq 1$ which is a contradiction to (i). Thus,

$$t_n\lambda_i(K) > 1 - \epsilon$$

for $i = 1, 2$ and for all $n \geq 1$. By Prohorov's theorem (see [B] or [P]), the sequences $(t_n\lambda_1)$ and $(t_n\lambda_2)$ are relatively compact in $\mathcal{P}(X)$. Let μ_1 be a limit point of $(t_n\lambda_1)$. Then there exists a $\mu_2 \in \mathcal{P}(X)$ such that

$$\frac{1}{2}(\mu_1 + \mu_2) = \delta_x$$

and hence $\mu_1 = \delta_x$. This implies that

$$\lim t_n\lambda_1 = \delta_x = \lim t_n\lambda_2 \,.$$

Thus, the action of T on $\mathcal{P}(X)$ is proximal.

Suppose the action of T on $\mathcal{P}(X)$ is proximal. Let $\lambda \in \mathcal{P}(X)$. Now for any $x \in X$, there exists a sequence (t_n) in T such that

$$\lim t_n\lambda = \lim t_n\delta_x \,. \tag{ii}$$

For any $n \geq 1$, $t_n x \in \delta_X$ which is a closed T-invariant set and hence $\lim t_n x \in \delta_X$. Thus, (ii) implies that

$$t_n\lambda \to \delta_y$$

for some $y \in X$. □

Let (X, T) be a dynamical system where X is a complete separable metric space and T be topological semigroup. Let us now consider the map $\Psi\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ defined as

$$\Psi(\rho) = \int_{\mathcal{P}(X)} y \, d\rho(y) \in \mathcal{P}(X)$$

for any $\rho \in \mathcal{P}(\mathcal{P}(X))$.

We first establish the following properties of Ψ.

Proposition 2. *Let X, T and Ψ be as above. Then*

1. *Ψ is a continuous T-equivariant map,*
2. *$\Psi(\delta_y) = y$ for all $y \in \mathcal{P}(X)$,*
3. *for $\rho \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\rho) = \delta_x$ for some $x \in X$ implies ρ is a point mass concentrated at the point x.*
4. *suppose X is a semigroup, then Ψ is a semigroup homomorphism.*

Proof. Since X, $\mathcal{P}(X)$ and $\mathcal{P}(\mathcal{P}(X))$ are all metrizable, it is enough to prove sequential continuity of Ψ. Let (ρ_n) be a sequence in $\mathcal{P}(\mathcal{P}(X))$ such that $\rho_n \to \rho \in \mathcal{P}(\mathcal{P}(X))$. Let $\nu_n = \Psi(\rho_n)$ for all n and $\Psi(\rho) = \nu$. Let f be a bounded continuous function on X. Then the function $y \mapsto y(f)$ is a continuous bounded function on $\mathcal{P}(X)$ and hence since $\rho_n \to \rho$ in $\mathcal{P}(\mathcal{P}(X))$, we have

$$\nu_n(f) = \int y(f)\, d\rho_n(y) \to \int y(f)\, d\rho(y) = \nu(f)\,.$$

Thus, $\nu_n \to \nu$ in $\mathcal{P}(X)$. This proves that Ψ is continuous. Since the action of T on X is by continuous maps, we have

$$t\nu = \int ty\, d\rho(y) = \int y\, d(t\rho)(y)\,,$$

that is $t\Psi(\rho) = \Psi(t\rho)$. Thus, verifying property (1) of the map Ψ. It is easy to verify property (2) of the map Ψ.

Suppose for $\rho \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\rho) = \delta_x = \nu$, say for some $x \in X$. Then for any $\epsilon > 0$ and any bounded continuous function f on X such that $f \geq 0$ and $f(x) = 0$, let

$$\sigma(f, \epsilon) = \{y \in \mathcal{P}(X) \mid y(f) > \epsilon\}\,.$$

Then

$$0 = \nu(f) \geq \int_{y \in \sigma(f,\epsilon)} y(f)\, d\rho(y) \geq \epsilon\rho(\sigma(f,\epsilon))$$

and hence $\rho(\sigma(f,\epsilon)) = 0$. It is easy to see that $\sigma(f,\epsilon)$ is an open set for all continuous bounded f and $\epsilon > 0$. Now let W be the set of all nonnegative bounded continuous functions f on X which vanish at x and

$$B = \cup_{f \in W} \cup_{n=1}^{\infty} \sigma\left(f, \frac{1}{n}\right)\,.$$

Then B is an open set.

We now claim that $B \cup x = \mathcal{P}(X)$ and $x \notin B$. Let $\lambda (\neq \delta_x) \in \mathcal{P}(X)$. Then choose a compact set K such that $\lambda(K) > \frac{1}{n}$ for some n and $x \notin K$. Since $x \notin K$, there exists a continuous function f on X such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(y) = 1$ for all $y \in K$. Then $\lambda(f) > \frac{1}{n}$ and hence $\lambda \in \sigma(f, \frac{1}{n}) \subset B$. Thus, $\mathcal{P}(X) = B \cup x$ and it is easy to see that $x \notin B$.

We now claim that $\rho(B) = 0$. Let K be any compact set contained B. Then there exists a finite number nonnegative continuous function $f_1, f_2, \cdots, f_k$ and a finite set of integers $n_1, n_2, \cdots, n_k$ such that $K \subset \cup\sigma(f_i, \frac{1}{n_i})$. Since $\rho(\sigma(f,\epsilon)) = 0$

for all $f \in W$ and $\epsilon > 0$, we have $\rho(K) = 0$ and hence since K is any arbitrary compact subset contained in B, we have $\rho(B) = 0$. Thus, ρ is the mass concentrated at the point $x \in X$. Thus, verifying property (3) of the map Ψ.

Suppose X is a semigroup. The Ψ is a semigroup homomorphism follows from the facts

1. Ψ is affine, that is for any $0 \leq \alpha_1, \alpha_2, \cdots, \alpha_n \leq 1$ with $\sum \alpha_i = 1$ and for $\rho_1, \rho_2, \cdots, \rho_n \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\sum \alpha_i \rho_i) = \sum \alpha_i \Psi(\rho_i)$,
2. the set of measures with finite supports in $\mathcal{P}(X)$ is dense in $\mathcal{P}(\mathcal{P}(X))$ and
3. Ψ is continuous. □

We now prove that strongly proximal actions on compact metric spaces is hereditary in the following sense.

Theorem 1. *Let X be a compact metric space and T be a topological semigroup acting on X. Then the following are equivalent:*

1. *T action on X is strongly proximal;*
2. *the action of T on $\mathcal{P}(X)$ is strongly proximal.*

Proof. Suppose (X, T) is strongly proximal. Let $\rho \in \mathcal{P}(\mathcal{P}(X))$, and let $\nu = \Psi(\rho) \in \mathcal{P}(X)$. Then since the action of T on X is strongly proximal, there exists a sequence (t_n) in T such that $t_n \nu \to \delta_x$, for some $x \in X$. Since for each n, $t_n \rho \in \mathcal{P}(\mathcal{P}(X))$ which is a compact metrizable space, the sequence $(t_n \rho)$ is a relatively compact sequence. Now let $\rho_o \in \mathcal{P}(\mathcal{P}(X))$ be a limit point of $(t_n \rho)$. Since $\mathcal{P}(\mathcal{P}(X))$ is a metrizable space, there exists a subsequence (t_{k_n}) of (t_n) such that $t_{k_n} \rho \to \rho_0$ in $\mathcal{P}(\mathcal{P}(X))$. Let $\nu_0 = \Psi(\rho_0) \in \mathcal{P}(X)$. Then by Proposition 2,

$$t_{k_n} \nu = t_{k_n} \Psi(\rho) = \Psi(t_{k_n} \rho) \to \Psi(\rho_0) = \nu_0$$

in $\mathcal{P}(X)$ and hence since $t_n \nu \to \delta_x$ in $\mathcal{P}(X)$ which is a metric space, we get that $\nu_0 = \delta_x$. Again by Proposition 2, ρ_0 is the mass concentrated at $x \in X$. Thus, the relatively compact sequence $(t_n \rho)$ has a unique limit point and hence it converges to the point mass concentrated at $x \in X$. This proves that (1) implies (2). That (2) implies (1) follows from Proposition 1 and from the remark that strongly proximal actions are proximal. □

As a consequence we have the following corollary for certain affine actions: *affine action* of a semigroup T on a closed convex subset X of a locally convex vector space is an action of T on X such that $t(ax + (1-a)y) = at(x) + (1-a)t(y)$ for all $x, y \in X$ and all $0 \leq a \leq 1$.

Corollary 1. *Let $\{v_1, v_2, \cdots, v_n\}$ be a set of linearly independent vectors on a locally convex vector space V over reals. Let X be the convex hull of $\{v_1, v_2, \cdots, v_n\}$. Let T be a topological semigroup acting on X by affine surjective maps. Then the T action on X is proximal if and only if the T action on X is strongly proximal.*

Proof. Let $F = \{v_i \mid 1 \leq i \leq n\}$. Since F is compact, it is easy to see that X is compact. Since T is an affine action of X by surjective maps and F is the set of extreme points of X, F is a T-invariant set. Thus, T acts on F also.

Let $f\colon \mathcal{P}(F) \to X$ be defined by

$$f(\lambda) = \sum \lambda(v_i)v_i$$

for all $\lambda \in \mathcal{P}(F)$. Suppose (λ_n) is a sequence in $\mathcal{P}(F)$ such that $\lambda_n \to \lambda$ in $\mathcal{P}(F)$. Then $\lambda_n(v_i) \to \lambda(v_i)$ for all i. This implies that $f(\lambda_n) = \sum \lambda_n(v_i)v_i \to \sum \lambda(v_i)v_i$. Thus, f is continuous.

Given any point x in X there exist $0 \leq \lambda_i \leq 1$ for $i = 1, 2, \cdots n$ such that $x = \sum \lambda_i v_i$ and $\sum \lambda_i = 1$. Since $\{v_i \mid 1 \leq i \leq n\}$ is a linearly independent set λ_i's are unique. This implies that f is a bijection. Since $\mathcal{P}(F)$ is compact, f is a homeomorphism. Since the action is affine it is easy to verify that $f(t\lambda) = tf(\lambda)$ for all $t \in T$. Suppose the T action on X is proximal. Then the T action on $\mathcal{P}(F)$ is proximal. Now by Proposition 1, T action on F is strongly proximal and hence by Theorem 1, T action on $\mathcal{P}(F)$ is strongly proximal. Since f is a homeomorphism preserving the T-action, T action on X is strongly proximal. □

Remark 1. It should be noted that the conclusion of Theorem 1 is valid for any Polish space if the map Ψ is proper.

In general for a complete separable metric space X, it is not clear that $(\mathcal{P}(X), T)$ is strongly proximal if (X, T) is strongly proximal. However the system $(\mathcal{P}(X), T)$ does not admit a non-trivial T-invariant measure in the following sense:

Corollary 2. *Let T be a topological semigroup acting strongly proximally on a complete separable metric space X. Suppose $\rho \in \mathcal{P}(\mathcal{P}(X))$ is T-invariant. Then ρ is the mass concentrated at a point $x \in X$.*

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