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Symplectic translation planes and line ovals

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Abstract. A symplectic spread of a $2n$-dimensional vector space V over $GF(q)$ is a set of $q^n + 1$ totally isotropic n-subspaces inducing a partition of the points of the underlying projective space. The corresponding translation plane is called symplectic. We prove that a translation plane of even order is symplectic if and only if it admits a completely regular line oval. Also, a geometric characterization of completely regular line ovals, related to certain symmetric designs $\mathscr{S}^1(2d)$, is given. These results give a complete solution to a problem set by W. M. Kantor in apparently different situations.

Key words. Translation plane, symplectic spread, line oval, regular triple, Lüneburg plane, symmetric design.

1 Introduction

Let Π_q be a finite projective plane of order q. An *oval* is a set of $q + 1$ points, no three of which are collinear. Dually, a *line oval* is a set of $q + 1$ lines no three of which are concurrent. Any line of the plane meets the oval $\mathcal{O}$ at either 0, 1 or 2 points and is called exterior, tangent or secant, respectively. For an account on ovals the reader is referred to [1], [2] and [7]. If the the order of the plane is even all the tangent lines to the oval $\mathcal{O}$ concur at the same point N, called the *nucleus* (or the knot) of $\mathcal{O}$. The set $\mathcal{O} \cup \{N\}$ becomes a *hyperoval*, that is a set of $q + 2$ points, no three of which are collinear. A *regular hyperoval* is a conic plus its nucleus in a desarguesian plane. If $\mathcal{O}$ is a line oval, then there is exactly one line n such that on each of its points there is only one line of $\mathcal{O}$. This line n is called the (dual) *nucleus* of $\mathcal{O}$. The $(q + 2)$-set $\mathcal{O} \cup \{n\}$ is a *line hyperoval* or *dual hyperoval*.

Let $\mathscr{A}_q$ be a translation plane of even order $q = 2^d$ and $\mathcal{O}$ a line oval whose nucleus is the line at infinity. Let T be the translation group of $\mathscr{A}_q$ and $\boldsymbol{A}$ its set of points. Identifying the elements of $\boldsymbol{A}$ with those of T and using addition as the operation on $\boldsymbol{A}$, define

$$B(\mathcal{O}) = \{P \in \boldsymbol{A} \mid P \text{ is on a line of } \mathcal{O}\}.$$

In [3], Theorem 7, it is proved that $B(\mathcal{O})$ is a difference set in the abelian group $\boldsymbol{A}$. The corresponding symmetric design $\mathscr{D}(\mathcal{O})$ has parameters

$$v = q^2, \quad k = \frac{q^2}{2} + \frac{q}{2}, \quad \lambda = \frac{q^2}{4} + \frac{q}{2}.$$

This design has the same parameters as certain designs $\mathscr{S}^1(2d)$, see [3] and also Section 2. In two cases Kantor proved, see [3], Theorems 8 and 9, that $\mathscr{D}(\mathcal{O})$ is isomorphic to $\mathscr{S}^1(2d)$, namely

1. $\mathscr{A}_q$ is desarguesian and $\mathcal{O}$ is a line conic (i.e. $\mathcal{O}$ becomes a conic in the dual of the projectivization of $\mathscr{A}_q$);
2. $\mathscr{A}_q$ is the Lüneburg plane of order q, where $q = 2^{2d}$ with $d > 1$ odd and $\mathcal{O}$ is a suitable line oval.

Such a line oval in the Lüneburg plane has the property of being stabilised by a collineation group isomorphic to the Suzuki group $\mathrm{Sz}(2^d)$ acting 2-transitively on its lines. Its existence was first proved in [3] by methods related to the symmetric design $\mathscr{S}^1(2d)$. There is also a direct construction, based on analytical methods, see [6].

Quite naturally W. M. Kantor raised the problem of finding out which translation planes were related to $\mathscr{S}^1(2d)$ and which geometric conditions on a line oval of a translation plane of order 2^d were necessary and sufficient in order that $\mathscr{D}(\mathcal{O})$ be isomorphic to $\mathscr{S}^1(2d)$.

The aim of this paper is to give a complete solution to the above problem. To get such a solution results about P-regular line ovals are used. In [9] and [10] ovals admitting a strongly regular tangent line are investigated. Here we need analogous results in a dual setting. So, we recall some basic definitions.

Definition 1. Let $\mathcal{O}$ be an oval with nucleus N in Π_q, where $q \geqslant 8$ is even. A tangent line s to $\mathcal{O}$ is *strongly regular* if for every pair of distinct points $X, Y \in s \setminus ((s \cap \mathcal{O}) \cup \{N\})$ there is a third point $Z \in s \setminus ((s \cap \mathcal{O}) \cup \{N\})$ such that for every point $P \neq N$ of Π_q at least one of the lines PX, PY, PZ is a secant line. Each non-ordered triple of points with the above property is called *s-regular*.

The dual definition is as follows. Let $\mathcal{O}$ be a line oval of Π_q, q even, and n its nucleus. Denote by $\Pi_q^n = \mathscr{A}_q$ the affine plane deduced by Π_q by deleting the line n and by A the set of points of $\mathscr{A}_q$. As above, set

$$B(\mathcal{O}) = \{P \in A \mid P \text{ is on a line of } \mathcal{O}\}.$$

Let $\mathscr{F}_P$ denote the pencil of lines on P, where P is a point of Π_q.

Definition 2. Let $\mathcal{O}$ be a line oval with nucleus n and P a point on n. $\mathcal{O}$ is called *P-regular* if for any pair of distinct affine lines $x, y \in \mathscr{F}_P \setminus (\mathscr{F}_P \cap \mathcal{O})$ there is a third affine line $z \in \mathscr{F}_P \setminus (\mathscr{F}_P \cap \mathcal{O})$ such that for every affine line ℓ not on P at least one of the points $\ell \cap x$, $\ell \cap y$ or $\ell \cap z$ belongs to $B(\mathcal{O})$. Each non-ordered triple of lines sharing the above property is called *P-regular*.

In [9], Theorem 3, it is proved that if the oval $\mathcal{O}$ has a strongly regular tangent line, then the order q of the plane is a power of 2. By duality the same result holds in the case of a P-regular line oval.

Known examples of ovals with a strongly regular tangent line are the translation ovals, see [9] and [10]. By duality we obtain examples of P-regular line ovals.

Non-degenerate conics are characterized by the following result, see [10], Corollary 1.

Theorem 1. *In* $\mathrm{PG}(2, 2^d)$, *where* $d \geqslant 3$, *an oval* $\mathcal{O}$ *is a non-degenerate conic if and only if* $\mathcal{O}$ *admits two distinct strongly regular tangent lines.*

This shows that a non-degenerate conic admits $q + 1$ strongly regular tangent lines.

Definition 3. An oval $\mathcal{O}$ with nucleus N is called *completely N-regular* if every line on $N \in \mathcal{O}$ is strongly regular.

We need the dual definition.

Definition 4. A line oval $\mathcal{O}$ is called *completely regular* with respect to its nucleus n if $\mathcal{O}$ is P-regular for every point P on n.

Our main results are summarized in the following theorems.

Theorem 2. *Let* $\mathscr{A}_q$ *be a translation plane of even order* $q = 2^d$, *where* $d \geqslant 3$, *and* $\mathcal{O}$ *a line oval whose nucleus is the line at infinity* n. *Then* $\mathscr{D}(\mathcal{O})$ *is isomorphic to* $\mathscr{S}^1(2d)$ *if and only if* $\mathcal{O}$ *is completely regular with respect to the line* n.

In a $2n$-dimensional vector space over $\mathrm{GF}(q)$, equipped with a non-singular alternating bilinear form, a *symplectic spread* is a family of $q^n + 1$ totally isotropic n-subspaces which induces a partition of the points of the underlying projective space.

Theorem 3. *Let* $\mathscr{A}_q$ *be a translation plane of even order* $q = 2^d$, *where* $d \geqslant 3$. *Then* $\mathscr{A}_q$ *admits a completely regular line oval with respect to the line at infinity if and only if* $\mathscr{A}_q$ *is defined by a symplectic spread of a* $2d$*-dimensional vector space over* $\mathrm{GF}(2)$.

In particular, the above theorem states that any symplectic translation plane of even order admits a line oval, a well known result, see [12]. There are many examples of symplectic translation planes, see [4] and [5]. So there are many examples of completely regular line ovals. Note that the above theorem answers the question of finding an internal criterion for a translation plane to be symplectic, see [5], page 318.

The paper is organized as follows. In Section 2 we fix some notation and introduce the designs $\mathscr{D}(\mathcal{O})$ and $\mathscr{S}^1(2d)$. Section 3 is devoted to prove that the only translation planes admitting a completely regular line oval are the symplectic ones. This is the content of Theorem 3 above. Also, a method to determine explicitly the regular triples of a completely regular line oval is described.

Finally in Section 4 we prove that, for a completely regular line oval $\mathcal{O}$ in a translation plane, $\mathcal{D}(\mathcal{O}) \cong \mathcal{S}^1(2d)$ holds. This result and those of Section 3 will provide a proof of Theorem 2.

2 Preliminary results

We will use fairly standard notation. In particular, dealing with planes or symmetric designs, points will be denoted by $P, Q, \ldots, X, Y, Z$, lines by $\ell, m, \ldots, r, s, \ldots, x, y, z$ and blocks by $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{x}$. The symbol $\mathcal{F}_P$ will denote the pencil of lines of a projective plane through the point P. Sometimes the line through two distinct points P and Q will be denoted by PQ.

If T is a finite set, then $|T|$ denotes the size of T, $\mathcal{C}T$ the complement of T and $T \backslash S$ the set of elements of T not in S. Finally, if $h : A \to B$ is a map between the sets A and B, then P^h is the image under h of the element $P \in A$ (in some cases also the symbol $h(P)$ is used).

Let Π_q be a projective plane of even order q, $\mathcal{O}$ a line oval with nucleus n and $\mathcal{A}_q = \Pi_q^n$ the affine plane deduced from Π_q by deleting the line n. Let $\mathcal{L}$ be its set of affine lines. Denote by $B(\mathcal{O})$ the set of affine points which are on the lines of $\mathcal{O}$ and by $\mathcal{C}B(\mathcal{O})$ its affine complement. It is easy to prove that

$$|B(\mathcal{O})| = \frac{q(q+1)}{2}, \quad |\mathcal{C}B(\mathcal{O})| = \frac{q(q-1)}{2},$$

and if $R \in B(\mathcal{O})$ then there are two lines of $\mathcal{O}$ through R. Moreover, if $\ell \notin \mathcal{O}$ is any affine line then

$$|\ell \cap B(\mathcal{O})| = |\ell \cap \mathcal{C}B(\mathcal{O})| = \frac{q}{2}.$$

The following proposition is a useful criterion to decide if a set of $q+2$ lines of Π_q is a line hyperoval.

Proposition 1. *Let Ω be a set of $q+2$ lines of Π_q. Then Ω is a line hyperoval if and only if the number of points which are not on the lines of Ω is greater than or equal to $q(q-1)/2$.*

Proof. (See also [11], Theorem 3) Let $k \geqslant 2$ be the maximum number of concurrent lines of Ω and t_s the number of points which are on s lines of Ω, $s = 0, 1, \ldots, k$. By a standard counting argument

$$\sum_{s=0}^{k} t_s = q^2 + q + 1 \tag{1}$$

$$\sum_{s=1}^{k} s t_s = (q+1)(q+2) \tag{2}$$

$$\sum_{s=2}^{k} s(s-1)t_s = (q+2)(q+1). \tag{3}$$

Subtracting Equation (3) from (2)

$$t_1 - \sum_{s=3}^{k} s(s-2)t_s = 0. \tag{4}$$

Since $t_0 \geqslant q(q-1)/2$, elimination of t_1 from (1) and (4) gives

$$t_2 + \sum_{s=3}^{k} (s^2 - 2s + 1)t_s \leqslant q^2 + q + 1 - \frac{q(q-1)}{2}. \tag{5}$$

From (3)

$$2t_2 = (q+2)(q+1) - \sum_{s=3}^{k} s(s-1)t_s. \tag{6}$$

From (5) and (6)

$$\sum_{s=3}^{k} (s^2 - 3s + 2)t_s \leqslant 0.$$

As $s^2 - 3s + 2 > 0$ for any $s \geqslant 3$, we infer $t_s = 0$ for any $s \geqslant 3$. Therefore $k \leqslant 2$, that is Ω is a line hyperoval. The converse is trivial. □

For the theory of translation planes we refer to [8]. Let $\mathscr{A}_q$ be a translation plane of even order $q = 2^d$, where $d \geqslant 3$, T its translation group and $\mathcal{O}$ a line oval with nucleus the line at infinity n. Note that $\mathcal{O}^g$ is a line oval with nucleus n for every $g \in T$. Also, if $\mathcal{O}^g$ and $\mathcal{O}^h$, $g, h \in T$, are distinct line ovals, then they have exactly one line in common.

For every $g \in T$, let $B(\mathcal{O}^g)$ be the set of affine points which are on the lines of $\mathcal{O}^g$. Denote by $\mathscr{D}(\mathcal{O})$ the incidence structure whose *points* are the points of $\mathscr{A}_q$ and whose *blocks* are the sets $B(\mathcal{O}^g)$, $g \in T$.

Theorem 4. *$\mathscr{D}(\mathcal{O})$ is a symmetric design with parameters*

$$v = q^2, \quad k = \frac{q(q+1)}{2}, \quad \lambda = \frac{q^2}{4} + \frac{q}{2}.$$

Proof. (see also [3], Theorem 7 (i)) The number of points is q^2 and equals the number of blocks. Each block contains $q(q+1)/2$ points, which is the total number of points which are on the lines of a line oval. It remains to prove that any two distinct blocks have $q^2/4 + q/2$ common points. Consider any two distinct line ovals $\mathcal{O}^g$ and $\mathcal{O}^h$. Let s be the unique line they have in common and S_m the point $n \cap s$. For any line ℓ of the plane not in $\mathcal{O}^h$ there are $q/2$ points of $B(\mathcal{O}^h)$ which belong to ℓ, one of which is $\ell \cap s$. Let ℓ vary on $\mathcal{O}^g \backslash \{s\}$. Since a point on $\ell \cap B(\mathcal{O}^g)$ not on s is also determined by another line of $\mathcal{O}^g$, we have $q/2(q/2 - 1)$ common points. To these we add the q points on s (excluding S_m) to obtain $q^2/4 + q/2$ common points. □

We introduce now another symmetric design, having the same parameters as $\mathcal{D}(\mathcal{O})$ and investigated in [3]. So our reference is [3], with only some minor change in notation. We use only one type of orthogonal group of a $2d$-dimensional vector space over GF(2), namely $O^+(2d, 2)$, which is the linear group preserving a non-degenerate quadratic form with index d. The symplectic group of a $2d$-dimensional vector space over GF(2) will be denoted by $\mathrm{Sp}(2d, 2)$.

If S and T are sets of points of a design, then $S \triangle T$ is the symmetric difference $(S \cup T) \backslash (S \cap T)$.

Set

$$H(2) = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

For each positive integer d, let $H(2d)$ be the tensor product of d copies of $H(2)$. Rows and columns of $H(2d)$ can be regarded as the points and blocks of a symmetric design $\mathcal{S}^1(2d)$, a point being on a block if and only if the corresponding entry is 1. $\mathcal{S}^1(2d)$ has parameters

$$v = 2^{2d}, \quad k = 2^{2d-1} + 2^{d-1}, \quad \lambda = 2^{2d-2} + 2^{d-1}.$$

Theorem 5. *Let $\mathcal{D}$ be a symmetric design admitting a sharply point-transitive automorphism group T. Define addition of points so that T is the set of right translations of the group G of the points. Then the following statements are equivalent.*

1. *$\mathcal{D}$ is isomorphic to $\mathcal{S}^1(2d)$ for some d.*
2. *$\boldsymbol{b} \triangle \boldsymbol{c}$ is a left coset of a subgroup of G whenever $\boldsymbol{b}$ and $\boldsymbol{c}$ are distinct blocks.*
3. *$\mathcal{C}(\boldsymbol{b} \triangle \boldsymbol{c})$ is a left coset of a subgroup of G whenever $\boldsymbol{b}$ and $\boldsymbol{c}$ are distinct blocks.*

Proof. See [3], Theorem 2. □

In [3], Section 4, the full automorphism group $\mathcal{G}$ of $\mathcal{S}^1(2d)$ is completely determined: it is a semidirect product of the translation group T of the $2d$-dimensional affine geometry over GF(2), AG$(2d, 2)$, with $\mathrm{Sp}(2d, 2)$. Moreover, if $\boldsymbol{x}$ is a block, then $\mathcal{G}_{\boldsymbol{x}}$ is isomorphic to $\mathrm{Sp}(2d, 2)$ and is 2-transitive on $\boldsymbol{x}$ and $\mathcal{C}\boldsymbol{x}$. Finally, if $P \in \boldsymbol{x}$ then $\mathcal{G}_{P\boldsymbol{x}}$ is $O^+(2d, 2)$.

It follows that identifying the points of $\mathcal{S}^1(2d)$ with the vectors of a $2d$-dimensional vector space V over GF(2) there exists a quadratic form Q with group $O^+(2d,2)$ such that $\boldsymbol{x}$ is the set of singular vectors of Q (a vector v is a singular vector of Q if $Q(v) = 0$). Therefore $\mathcal{S}^1(2d)$ can be constructed as follows.

Proposition 2. *Let V be a $2d$-dimensional vector space over* GF(2) *and Q a non-degenerate quadratic form on V whose group is $O^+(2d,2)$. Let $S(Q)$ be the set of singular vectors of Q. Then the points and blocks of $\mathcal{S}^1(2d)$ are the vectors of V and the translates $S(Q) + v$, $v \in V$.*

Proof. See [3], Corollary 3. □

3 Symplectic translation planes

Let $V = V(2n,q)$ be a $2n$-dimensional vector space over $\mathbb{F}_q = \mathrm{GF}(q)$. Vectors will be denoted by $v, w, \ldots, z$, subspaces by $S, T, U, \ldots, X, Y$. A *spread* of V is a family Σ of $q^n + 1$ n-dimensional subspaces of V any two of which have in common the zero vector only. A *symplectic spread* of V is a spread which consists of totally isotropic subspaces with respect to a non-degenerate alternating bilinear form f.

Let $\Sigma = \{S_0, S_1, \ldots, S_{q^n}\}$ be a spread of V and $\mathcal{A}(\Sigma)$ the corresponding translation plane of order q^n, see [8]. If T is its translation group, then the points of $\mathcal{A}(\Sigma)$ are the vectors of V and the lines are the translates of the components of Σ. A translation plane defined by a symplectic spread is said to be *symplectic*.

Fix two distinct component of Σ, say S_0 and S_1. Then $V = S_0 \oplus S_1$. Choose bases $\{v_1, v_2, \ldots, v_n\}$ in S_0 and $\{w_1, w_2, \ldots, w_n\}$ in S_1, so that $\mathcal{B} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ is a basis of V. The subspaces S_0 and S_1 are identified with $\mathbb{F}_q^n$ and V with $\mathbb{F}_q^n \times \mathbb{F}_q^n$. Vectors of $\mathbb{F}_q^n$ are identified with $n \times 1$ matrices, represented by symbols like $\boldsymbol{x}, \boldsymbol{y}, \ldots$.

With respect to the basis $\mathcal{B}$, the spread Σ determines a set $\mathcal{M}$ of $n \times n$ matrices over $\mathbb{F}_q$ such that (see [8])

1. $|\mathcal{M}| = q^n$ and $O \in \mathcal{M}$
2. if $A, B \in \mathcal{M}$ and $A \neq B$ then $A - B$ is non-singular
3. $\mathcal{M} \backslash \{O\}$ acts sharply transitively on $\mathbb{F}_q^n \backslash \{\mathbf{0}\}$.

The set $\mathcal{M}$ is called the *spread-set* associated with Σ. With respect to $\mathcal{M}$ and the basis $\mathcal{B}$

$$\Sigma = \{\boldsymbol{x} = \mathbf{0}\} \cup \{\boldsymbol{y} = M\boldsymbol{x} \mid M \in \mathcal{M}\}.$$

Note that we write $\boldsymbol{y} = M\boldsymbol{x}$ to denote the subspace $\{(\boldsymbol{x}, M\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{F}_q^n\}$.

From now on we assume that q is a power of 2, $q = 2^d$. Let $\Sigma = \{S_0, S_1, \ldots, S_{q^n}\}$ be a symplectic spread with respect to a non-degenerate alternating bilinear form f. Then the bases in S_0 and S_1 can be chosen so that $f(v_i, w_j) = \delta_{ij}$, where δ_{ij} is the symbol of Kronecker and $i, j = 1, \ldots, n$. Such bases are called *dual*. Therefore in the basis $\mathcal{B} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ of V, f is represented by the matrix

$$\begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

where O and I denote the $n \times n$ zero and identity matrices. Then

$$f((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) = \boldsymbol{x}^\top \boldsymbol{y}' + \boldsymbol{y}^\top \boldsymbol{x}',$$

where $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'$ are vectors of $\mathbb{F}_2^n$, the symbol $\top$ denotes transposition and the product is the ordinary product between matrices. A quadratic form Q which polarises to f (i.e. $Q(v + w) = Q(v) + Q(w) + f(v, w)$ for $v, w \in V$) is

$$Q((\boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{x}^\top \boldsymbol{y}.$$

With respect to this basis the associated spread-set $\mathscr{M}$ consists of symmetric matrices. For, if $\boldsymbol{y} = M\boldsymbol{x}$ is a component of Σ, then

$$f((\boldsymbol{x}, M\boldsymbol{x}), (\boldsymbol{x}', M\boldsymbol{x}')) = 0$$

for every $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{F}_q^n$ if and only if $M = M^\top$.

The vector space $\mathbb{F}_q^n \times \mathbb{F}_q^n$ can be viewed as a $2nd$-dimensional vector space over $\mathbb{F}_2$. Let $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_2$ be the trace map: $\mathrm{Tr}(x) = \sum_{i=0}^{d-1} x^{2^i}$. Then the bilinear map $f' = \mathrm{Tr} \circ f$ is a non-degenerate alternating bilinear form on $\mathbb{F}_2^{nd} \times \mathbb{F}_2^{nd}$ and $Q' = \mathrm{Tr} \circ Q$ is a quadratic form which polarises to f'. The symplectic spread Σ gives rises to a symplectic spread Σ' of $\mathbb{F}_2^{nd} \times \mathbb{F}_2^{nd}$, such that the plane $\mathscr{A}(\Sigma)$ is identical to the plane $\mathscr{A}(\Sigma')$, see also [5].

The definition of completely regular line oval is in the Introduction, Definition 4.

Theorem 6. *Let $\mathscr{A}_q$ be a translation plane of even order $q = 2^d$ with $d \geqslant 3$ and $\mathscr{O}$ a line oval with nucleus the line at infinity such that $\mathscr{D}(\mathscr{O}) \cong \mathscr{S}^1(2d)$. Then*

1. *$\mathscr{A}_q$ is a symplectic translation plane*
2. *$\mathscr{O}$ is a completely regular line oval.*

Proof. Let $\Sigma = \{S_0, S_1, \ldots, S_q\}$ be a spread of a $2d$-dimensional vector space V over GF(2) which defines $\mathscr{A}_q$. We can assume that the lines of $\mathscr{O}$ are $\{S_0, S_1, S_2 + v_2, \ldots, S_q + v_q\}$, where $v_2, \ldots, v_q$ are, in some ordering, the non-zero vectors of S_0. As $\mathscr{D}(\mathscr{O}) \cong \mathscr{S}^1(2d)$, so, because of Proposition 2, there is a quadratic form on V with group $O^+(2d, 2)$ such that $B(\mathscr{O})$ is the set of singular vectors of Q. Let f be the non-degenerate alternating bilinear on V form polarised by Q, that is

$$f(v, w) = Q(v + w) + Q(v) + Q(w) \quad \text{for } v, w \in V.$$

Let $S(Q) = B(\mathscr{O})$ be the set of singular vectors of Q. Then for every $v \in S(Q)$ the quadratic form Q_v defined by $Q_v(w) = Q(v + w)$, $w \in V$, also polarises to f and its set of singular vectors is $S(Q) + v$. As $S(Q) = B(\mathscr{O})$, so $S(Q_v) = S(Q) + v = B(\mathscr{O}^{\tau_v})$, where τ_v is the translation $w \mapsto w + v$. Therefore the subspaces S_i, $i = 0, 1, \ldots, q$, are totally isotropic with respect to f. For S_0 and S_1 are totally singular, since they are contained in $S(Q)$, and S_i is contained in $S(Q_{v_i}) = S(Q) + v_i$, $i = 2, \ldots, q$. The spread Σ is then symplectic and $\mathscr{A}_q$ is a symplectic translation plane. This proves item 1 of the theorem.

To prove that $\mathcal{O}$ is completely regular we make use of coordinates to write explicitly its regular triples. Referring back to the construction at the beginning of the section, write $V = S_0 \oplus S_1$ and choose dual bases $\{v_1, \ldots, v_d\}$ in S_0 and $\{w_1, \ldots, w_d\}$ in S_1 so that, in the basis $\mathscr{B} = \{v_1, \ldots, v_d, w_1, \ldots, w_d\}$,

$$f((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) = \boldsymbol{x}^\top \boldsymbol{y}' + \boldsymbol{y}^\top \boldsymbol{x}',$$

where $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'$ are vectors of $\mathbb{F}_2^d$. Thus the quadratic form Q is

$$Q((\boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{x}^\top \boldsymbol{y}$$

and the points which are on the lines of $\mathcal{O}$ are the vectors $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{F}_2^d \times \mathbb{F}_2^d$ such that

$$Q(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^\top \boldsymbol{y} = 0.$$

The above equation represents the set of points $B(\mathcal{O})$.

Let $\mathscr{M}$ be the spread-set relative to Σ and $\mathscr{B}$. Then

$$\Sigma = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M\boldsymbol{x} \mid M \in \mathscr{M}\}.$$

Recall that $\mathscr{M}$ is a set of 2^d symmetric matrices. The line oval $\mathcal{O}$ is

$$\mathcal{O} = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{x}_M \mid M \in \mathscr{M}\},$$

where the vector $\boldsymbol{x}_M$ is determined by the condition

$$Q(\boldsymbol{x}, M\boldsymbol{x} + \boldsymbol{x}_M) = \boldsymbol{x}^\top M\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{x}_M = 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{F}_2^d.$$

If $\boldsymbol{x} = (x_1, \ldots, x_d)^\top$, $\boldsymbol{x}_M = (\alpha_1, \ldots, \alpha_d)^\top$ and the symmetric matrix M has entries a_{ij}, $i, j = 1, \ldots, d$, a calculation proves that

$$\sum_{i=1}^{d} (a_{ii} x_i^2 + \alpha_i x_i) = 0, \quad \text{for all } x_i \in \mathbb{F}_2.$$

Hence $\alpha_i = a_{ii}$, $i = 1, \ldots, d$. We reserve the symbol $\boldsymbol{x}_M$ to denote the vector $(a_{11}, \ldots, a_{dd})^\top$, where $(a_{11}, \ldots, a_{dd})$ is the main diagonal of the matrix M.

Now we can write the regular triples of $\mathcal{O}$. Denote by (∞) and (M), $M \in \mathscr{M}$, the points on the line at infinity, corresponding to the subspaces $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{y} = M\boldsymbol{x}$, respectively. We claim:

for every triple $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$ *of distinct vectors of* $\mathbb{F}_2^d \backslash \{\boldsymbol{x}_M\}$ *such that* $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 = \boldsymbol{x}_M$, *the triple of lines of* $\mathscr{A}_q$

$$\{\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{r}_1, \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{r}_2, \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{r}_3\}$$

is (M)*-regular.*

Also, for every triple $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$ *of distinct vectors of* $\mathbb{F}_2^d \backslash \{\boldsymbol{0}\}$ *such that* $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 = \boldsymbol{0}$, *the triple*

$$\boldsymbol{x} = \boldsymbol{r}_1, \boldsymbol{x} = \boldsymbol{r}_2, \boldsymbol{x} = \boldsymbol{r}_3$$

is (∞)*-regular.*

To prove the claim, consider the intersection between the line $\boldsymbol{y} = N\boldsymbol{x} + \boldsymbol{h}$ and the lines of the first triple, where $N \neq M$. We find the vectors

$$\boldsymbol{v}_k = ((N+M)^{-1}(\boldsymbol{r}_k + \boldsymbol{h}), M(N+M)^{-1}(\boldsymbol{r}_k + \boldsymbol{h}) + \boldsymbol{r}_k), \quad k = 1, 2, 3.$$

Since M and N are symmetric matrices and $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 = \boldsymbol{x}_M$, we have

$$\begin{aligned} &Q(\boldsymbol{v}_1) + Q(\boldsymbol{v}_2) + Q(\boldsymbol{v}_3) \\ &\quad = \boldsymbol{h}^\top (N+M)^{-1} M (N+M)^{-1} \boldsymbol{h} + \boldsymbol{h}^\top (N+M)^{-1} \boldsymbol{x}_M \\ &\qquad + \boldsymbol{x}_M^\top (N+M)^{-1} M (N+M)^{-1} \boldsymbol{x}_M + \boldsymbol{x}_M^\top (N+M)^{-1} \boldsymbol{x}_M. \end{aligned}$$

As $\boldsymbol{x}^\top M\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{x}_M = 0$ for all $\boldsymbol{x} \in \mathbb{F}_2^d$, putting in the above equation $(N+M)^{-1}\boldsymbol{h} = \boldsymbol{x}$ and $(N+M)^{-1}\boldsymbol{x}_M = \boldsymbol{y}$, we get

$$Q(\boldsymbol{v}_1) + Q(\boldsymbol{v}_2) + Q(\boldsymbol{v}_3) = (\boldsymbol{x}^\top M\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{x}_M) + (\boldsymbol{y}^\top M\boldsymbol{y} + \boldsymbol{y}^\top \boldsymbol{x}_M) = 0.$$

Consequently, if two of the vectors $\boldsymbol{v}_k$, $k = 1, 2, 3$, are not in $B(\mathcal{O}) = S(Q)$ then the third is in $B(\mathcal{O})$.

The other case is similar. □

Now we will prove that any symplectic translation plane admits a completely regular line oval.

Let $\mathscr{A}_q$ be a symplectic translation plane of even order $q = 2^d$, defined by the symplectic spread $\Sigma = \{S_0, S_1, \ldots, S_q\}$ of a $2d$-dimensional vector space V over GF(2), equipped with a non-degenerate alternating bilinear form f. Let Q be a quadratic form which polarises to f and whose group is $O^+(2d, 2)$. Let $S(Q) = \{v \in V \mid Q(v) = 0\}$ be the set of singular vectors of Q. Then

$$|S(Q)| = 2^{2d-1} + 2^{d-1} = \frac{q(q+1)}{2}.$$

Lemma 1. *Any maximal totally isotropic subspace U not lying on $S(Q)$ meets $S(Q)$ in a $(d-1)$-dimensional subspace.*

Proof. U has dimension d and the restriction of Q to U gives rise to a linear form on U which is not the zero form, since U is not contained in $S(Q)$. Therefore $S(Q) \cap U = \{v \in U \mid Q(v) = 0\}$ is a hyperplane of U and so its dimension is $d-1$. □

Lemma 2. *$S(Q)$ contains exactly two distinct components of Σ.*

Proof. Let k be the number of components of Σ which are contained in $S(Q)$. Since Σ is a spread, then

$$S(Q) = (S_0 \cap S(Q)) \cup \cdots \cup (S_q \cap S(Q))$$

where $(S_i \cap S(Q)) \cap (S_j \cap S(Q)) = (0)$, for $i \neq j$. By the previous lemma

$$|S(Q)| = 1 + k(2^d - 1) + (2^d + 1 - k)(2^{d-1} - 1).$$

Since $|S(Q)| = 2^{2d-1} + 2^{d-1}$, $k = 2$ follows. □

For any $v \in S(Q)$, the function $Q_v : V \to \mathbb{F}_2$, defined by $Q_v(w) = Q(v+w)$, is a quadratic form on V which polarises to f. Also, the set of singular vectors of Q_v is $S(Q) + v$. Therefore Lemma 2 applies to each $S(Q) + v$, with $v \in S(Q)$.

Lemma 3. *Let S and T be two distinct components of Σ contained in $S(Q)$. Then for every $v \in S \backslash \{0\}$ there is exactly one component $S_v \in \Sigma$ such that $S_v + v \subset S(Q)$ and $S_v \neq S$, $S_v \neq T$. Moreover, if $v, w \in S \backslash \{0\}$ and $v \neq w$ then $S_v \neq S_w$.*

Proof. Let $v \in S \backslash \{0\}$. Then $S(Q) + v$ contains two distinct components of Σ, one of which is S. Let S_v be the other. As $S_v \subset S(Q) + v$, so $S_v + v \subset S(Q)$. Clearly $S_v \neq S$. We claim that $S_v \neq T$. If $S_v = T$, then $T \subset S(Q) \cap (S(Q) + v))$. Therefore for all $z \in T$

$$0 = Q(z) = Q_v(z) = Q(v+z) = Q(v) + Q(z) + f(v,z).$$

Since $Q(v) = Q(z) = 0$, then $f(v,z) = 0$ for all $z \in T$, which is absurd.

In a similar way we can prove the last assertion of the lemma. By way of contradiction, let $S_v = S_w$ for some $v \neq w$ in $S \backslash \{0\}$. Then $S_v \subset (S(Q) + v) \cap (S(Q) + w)$. Therefore for all $z \in S_v$

$$0 = Q_v(z) = Q_w(z).$$

Then $Q(v+z) = Q(w+z)$ implies

$$Q(v) + Q(z) + f(v,z) = Q(w) + Q(z) + f(w,z).$$

As $Q(v) = Q(w) = 0$, so $f(v + w, z) = 0$ for all $z \in S_v$, which is absurd, as $v + w \in S \setminus \{0\}$. □

Theorem 7. *Let V be a $2d$-dimensional vector space over* GF(2), *f a non-degenerate alternating bilinear form and $\Sigma = \{S_0, S_1, \ldots, S_q\}$ a symplectic spread of V, where $q = 2^d$ and $d \geqslant 3$. Then the following statements hold.*

1. *The set $\mathcal{O} = \{S_0, S_1, S_2 + v_2, \ldots, S_q + v_q\}$, where $v_2, \ldots, v_q$ are, in a suitable ordering, the non-zero vectors of S_0, is a line oval in the translation plane $\mathscr{A}(\Sigma)$ of order $q = 2^d$ defined by Σ.*
2. *The vector set $S_0 \cup S_1 \cup (S_2 + v_2) \cup \cdots \cup (S_q + v_q)$ is the set of singular vectors of a quadratic form Q which polarises to f.*
3. *$\mathcal{O}$ is completely regular.*

Proof. Let Q be a quadratic form which polarises to f and whose group is $O^+(2d, 2)$. Further, let $S(Q)$ be the set of singular vectors of Q. By Lemma 2, $S(Q)$ contains two components of Σ; let them be S_0 and S_1. So $S_0 \cup S_1 \subseteq S(Q)$. By Lemma 3, $S(Q)$ contains the subset

$$S_0 \cup S_1 \cup (S_2 + v_2) \cup \cdots \cup (S_q + v_q),$$

where $v_2, \ldots, v_q$ are, in a suitable ordering, the non-zero vectors of S_0 and the sets $S_k + v_k$, $k = 2, \ldots, q$, are pairwise distinct. Then in the translation plane $\mathscr{A}(\Sigma)$ the vector set $S(Q)$ contains $q + 1$ distinct lines. Denote by $\mathcal{O}$ this set of lines. Since $|S(Q)| = q(q + 1)/2$, then the number of points which are not on the lines of $\mathcal{O}$ is $t_0 \geqslant q(q - 1)/2$, since the vector space has q^2 vectors. Because of Proposition 1, $\mathcal{O}$ is a line oval whose nucleus is the line at infinity. Also, since the number of points which are on its lines is $q(q + 1)/2$, then

$$S(Q) = S_0 \cup S_1 \cup (S_2 + v_2) \cup \cdots \cup (S_q + v_q).$$

This proves statements 1 and 2. Statement 3 follows from Proposition 2 and Theorem 6. □

Theorems 7 and 6 prove Theorem 3 stated in the Introduction.

At this point we have a computational tool to describe line ovals when the translation plane is defined by a spread of a vector space over GF(2). However, a translation plane of order q^n is usually constructed from spreads of a $2n$-dimensional vector space over $\mathbb{F}_q = \mathrm{GF}(q)$, with $q > 2$. Therefore it is useful to illustrate how the methods developed during the proof of Theorem 6 can be applied to this more common situation. So let $V = V(2n, q)$ be a $2n$-dimensional vector space over $\mathbb{F}_q = \mathrm{GF}(q)$, where $q = 2^d$, equipped with a non-degenerate alternating bilinear form f and $\Sigma = \{S_0, S_1, \ldots, S_{q^n}\}$ a symplectic spread of V. Denote by $\mathscr{A}(\Sigma)$ the corresponding translation plane of order q^n. Fix two components of Σ, say S_0 and S_1 and choose dual bases $\{v_1, v_2, \ldots, v_n\}$ in S_0 and $\{w_1, w_2, \ldots, w_n\}$ in S_1 so that

$$f((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{x}',\boldsymbol{y}')) = \boldsymbol{x}^\top \boldsymbol{y}' + \boldsymbol{y}^\top \boldsymbol{x}',$$

where $\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y},\boldsymbol{y}'$ are vectors of $\mathbb{F}_q^n$. A quadratic form Q which polarises to f and whose group is $O^+(2n,q)$ is

$$Q((\boldsymbol{x},\boldsymbol{y})) = \boldsymbol{x}^\top \boldsymbol{y}.$$

View the vector space $\mathbb{F}_q^n \times \mathbb{F}_q^n$ as a $2nd$-dimensional vector space over $\mathbb{F}_2$. Let $\mathrm{Tr}: \mathbb{F}_q \to \mathbb{F}_2$ be the trace map. As explained at the beginning of the section, using the bilinear map $f' = \mathrm{Tr} \circ f$, the symplectic spread Σ gives rises to a symplectic spread Σ' of $\mathbb{F}_2^{nd} \times \mathbb{F}_2^{nd}$, such that the plane $\mathscr{A}(\Sigma)$ is identical to the plane $\mathscr{A}(\Sigma')$. Because of Theorem 7, the plane $\mathscr{A}(\Sigma')$ admits a completely regular line oval $\mathscr{O}'$, such that $B(\mathscr{O}')$ is the set of singular vectors of the quadratic form $Q' = \mathrm{Tr} \circ Q$ which polarises to f'. Regard the line oval $\mathscr{O}'$ as a line oval $\mathscr{O}$ of $\mathscr{A}(\Sigma)$. Consequently, the points which are on the lines of $\mathscr{O}$ are the vectors $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ such that

$$\mathrm{Tr}(Q(\boldsymbol{x},\boldsymbol{y})) = \mathrm{Tr}(\boldsymbol{x}^\top \boldsymbol{y}) = 0.$$

The above equation represents the set of points $B(\mathscr{O})$.

Let $\mathscr{M}$ be the spread-set associated to Σ and $\mathscr{B}$. Then

$$\Sigma = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M\boldsymbol{x} \mid M \in \mathscr{M}\},$$

the matrices of $\mathscr{M}$ are symmetric and the line oval $\mathscr{O}$ is represented as

$$\mathscr{O} = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{x}_M \mid M \in \mathscr{M}\},$$

where the vector $\boldsymbol{x}_M$ is determined by the condition

$$\mathrm{Tr}[Q(\boldsymbol{x}, M\boldsymbol{x} + \boldsymbol{x}_M)] = \mathrm{Tr}[\boldsymbol{x}^\top M\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{x}_M] = 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{F}_q^n.$$

If $\boldsymbol{x} = (x_1,\ldots,x_n)^\top$, $\boldsymbol{x}_M = (\alpha_1,\ldots,\alpha_n)^\top$ and the symmetric matrix M has entries a_{ij}, $i,j = 1,\ldots,n$, a calculation proves that

$$\mathrm{Tr}\left[\sum_{i=1}^n (a_{ii}x_i^2 + \alpha_i x_i)\right] = 0, \quad \text{for all } x_i \in \mathbb{F}_q.$$

Hence $\alpha_i = a_{ii}^{2^{d-1}}$, $i = 1,\ldots,n$. We use the symbol $\sqrt{a}$ to denote $a^{2^{d-1}}$ and reserve now the symbol $\boldsymbol{x}_M$ to denote the vector $(\sqrt{a_{11}},\ldots,\sqrt{a_{nn}})^\top$, where $(a_{11},\ldots,a_{nn})$ is the main diagonal of the matrix M.

With a similar construction as that in the proof of Theorem 6, we can prove that the line oval $\mathscr{O}$ is completely regular writing explicitly its regular triples.

Denote by (∞) and (M), $M \in \mathscr{M}$, the points on the line at infinity of $\mathscr{A}(\Sigma)$, which correspond to the subspaces $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{y} = M\boldsymbol{x}$, respectively.

Theorem 8. *The triple* $\{\boldsymbol{y}=M\boldsymbol{x}+\boldsymbol{r}_1, \boldsymbol{y}=M\boldsymbol{x}+\boldsymbol{r}_2, \boldsymbol{y}=M\boldsymbol{x}+\boldsymbol{r}_3\}$ *is* (M)*-regular if and only if* $\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3=\boldsymbol{x}_M$, *where* $\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3$ *are distinct vectors of* $\mathbb{F}_q^n\backslash\{\boldsymbol{x}_M\}$. *Also, the triple* $\{\boldsymbol{x}=\boldsymbol{r}_1,\boldsymbol{x}=\boldsymbol{r}_2,\boldsymbol{x}=\boldsymbol{r}_3\}$ *is* (∞)*-regular if and only if* $\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3=\boldsymbol{0}$, *where* $\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3$ *are distinct vectors of* $\mathbb{F}_q^n\backslash\{\boldsymbol{0}\}$.

Proof. The proof is essentially similar to that in the proof of Theorem 6. For the sake of completeness, we repeat it.

Consider the intersection between the line $\boldsymbol{y}=N\boldsymbol{x}+\boldsymbol{h}$ and the lines of the first triple, where $N\neq M$. We find the vectors

$$\boldsymbol{v}_k=((N+M)^{-1}(\boldsymbol{r}_k+\boldsymbol{h}), M(N+M)^{-1}(\boldsymbol{r}_k+\boldsymbol{h})+\boldsymbol{r}_k), \quad k=1,2,3.$$

Assume $\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3=\boldsymbol{x}_M$. Then

$$\begin{aligned}&\mathrm{Tr}[Q(\boldsymbol{v}_1)+Q(\boldsymbol{v}_2)+Q(\boldsymbol{v}_3)]\\&\quad=\mathrm{Tr}[\boldsymbol{h}^\top(N+M)^{-1}M(N+M)^{-1}\boldsymbol{h}+\boldsymbol{h}^\top(N+M)^{-1}\boldsymbol{x}_M\\&\qquad+\boldsymbol{x}_M^\top(N+M)^{-1}M(N+M)^{-1}\boldsymbol{x}_M+\boldsymbol{x}_M^\top(N+M)^{-1}\boldsymbol{x}_M].\end{aligned}$$

Since $\mathrm{Tr}[\boldsymbol{x}^\top M\boldsymbol{x}+\boldsymbol{x}^\top\boldsymbol{x}_M]=0$ for all $\boldsymbol{x}\in\mathbb{F}_q^n$, putting in the above equation $(N+M)^{-1}\boldsymbol{h}=\boldsymbol{x}$ and $(N+M)^{-1}\boldsymbol{x}_M=\boldsymbol{y}$, we get

$$\begin{aligned}&\mathrm{Tr}[Q(\boldsymbol{v}_1)+Q(\boldsymbol{v}_2)+Q(\boldsymbol{v}_3)]\\&\quad=\mathrm{Tr}(\boldsymbol{x}^\top M\boldsymbol{x}+\boldsymbol{x}^\top\boldsymbol{x}_M)+\mathrm{Tr}(\boldsymbol{y}^\top M\boldsymbol{y}+\boldsymbol{y}^\top\boldsymbol{x}_M)=0.\end{aligned}$$

Consequently, as the trace map is additive, if two of the vectors $\boldsymbol{v}_k$, $k=1,2,3$, are not in $B(\mathcal{O})$ then the third is in $B(\mathcal{O})$.

To prove the only if part of the theorem it suffices to note that the number of (M)-regular triples is $(q^n-1)(q^n-2)/6$, which is also the number of triples $\{\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3\}$, $\boldsymbol{r}_i\in\mathbb{F}_q^n\backslash\{\boldsymbol{x}_M\}$, whose sum is $\boldsymbol{x}_M$.

The other case is similar. □

Examples. 1. *The desarguesian plane* $\mathrm{AG}(2,q)$. Let f be the non-degenerate alternating bilinear form

$$f((x,y),(x',y'))=xy'+yx', \quad x,y,x',y'\in\mathbb{F}_q$$

with associated quadratic form $Q((x,y))=xy$. The symplectic spread is

$$\Sigma=\{x=0\}\cup\{y=mx\,|\,x\in\mathbb{F}_q\}$$

and a completely regular line oval is

$$\mathcal{O}=\{x=0\}\cup\{y=mx+\sqrt{m}\,|\,m\in\mathbb{F}_q\}.$$

Since $m \mapsto m^2$ is an automorphism of $\mathbb{F}_q$, letting $k^2 = m$, we can write

$$\mathcal{O} = \{x = 0\} \cup \{y = k^2 x + k \mid k \in \mathbb{F}_q\}.$$

$\mathcal{O}$ is a line conic.

2. *The Lüneburg plane of order* q^2, $q = 2^{2k+1}$, see [6] and [8]. Let σ be the automorphism of $\mathbb{F}_q$ defined by $a \mapsto a^{2^{k+1}}$. Then $\sigma^2 = 2$ and $\sigma + 1$ and $\sigma + 2$ are automorphisms of the cyclic group $\mathbb{F}_q^*$. Using the standard alternating bilinear form, define the symplectic spread

$$\Sigma = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M_{a,b}\boldsymbol{x} \mid a, b \in \mathbb{F}_q\}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^2$ and

$$M_{a,b} = \begin{pmatrix} a & a^{\sigma^{-1}} + b^{1+\sigma^{-1}} \\ a^{\sigma^{-1}} + b^{1+\sigma^{-1}} & b \end{pmatrix}.$$

A completely regular line oval is

$$\mathcal{O} = \{\boldsymbol{x} = \boldsymbol{0}\} \cup \{\boldsymbol{y} = M_{a,b}\boldsymbol{x} + (\sqrt{a}, \sqrt{b})^\top \mid a, b \in \mathbb{F}_q\}.$$

4 Completely regular line ovals

This last section is devoted to the proof that if $\mathcal{O}$ is a completely regular line oval in a translation plane of even order $q = 2^d$, then $\mathcal{D}(\mathcal{O}) \cong \mathcal{S}^1(2d)$. We will need some known results about P-regular line ovals (see Definition 2 in the Introduction). Notation is as in Section 2.

Let Π_q be a projective plane of even order q, $\mathcal{A}_q = \Pi_q^n$ the affine plane deduced from Π_q by deleting the line n and $\mathcal{L}$ its set of affine lines. Let $\mathcal{O}$ be a line oval with nucleus n. For every line $\ell \in \mathcal{L}\backslash\mathcal{O}$, let L be the point $\ell \cap n$ and define

$$\mathcal{S}_\ell = \{m \in \mathcal{L}\backslash\mathcal{O} \mid m \neq \ell \text{ and } \ell \cap m \in B(\mathcal{O})\} \cup (\mathcal{F}_L \backslash (\mathcal{O} \cap \mathcal{F}_L)).$$

Note that $\ell \in \mathcal{S}_\ell$.

Result 1. *The incidence structure $\mathcal{H}(\mathcal{O})$ whose set of points is $\mathcal{L}\backslash\mathcal{O}$ and whose blocks are the subsets $\mathcal{S}_\ell$, $\ell \in \mathcal{L}\backslash\mathcal{O}$, is a Hadamard symmetric design with parameters*

$$v = q^2 - 1, \quad k = \frac{q^2}{2} - 1, \quad \lambda = \frac{q^2}{4} - 1,$$

admitting the null polarity $\ell \mapsto \mathcal{S}_\ell$.

The proof is straightforward.

Suppose now that the order of the plane is greater than or equal to 8. Let P be any point on n and o the unique line of $\mathcal{O} \cap \mathcal{F}_P$. From now on we assume that $\mathcal{O}$ is a P-regular line oval.

Result 2. *$\mathcal{F}_P^* = \mathcal{F}_P \backslash \{n\}$ is a d-dimensional vector space over* GF(2). *Such a structure is determined as follows. Let $x, y \in \mathcal{F}_P^*$ with $x \neq y$. Then define*:

$$x + x = o, \quad x + o = o + x = x, \quad x + y = z,$$

where z is the third line of $\mathcal{F}_P^*$ such that $\{x, y, z\}$ is P-regular.

The dual proof is in [9], Theorem 3, where associativity of addition is proved. Note that $|\mathcal{F}_P^*| = q$ and that the number of P-regular triples equals the number of 2-dimensional subspaces of $\mathcal{F}_P^*$. This number is $(q-1)(q-2)/6$.

The hyperplanes of $\mathcal{F}_P^*$, which are its additive subgroups of order $q/2$, can be recovered from the sets $\mathcal{S}_\ell$.

Lemma 4. *For every affine line $\ell \notin \mathcal{O} \cup \mathcal{F}_P$, $\mathcal{S}_\ell \cap \mathcal{F}_P$ is the set of points of a hyperplane of $\mathcal{F}_P^*$.*

Proof. As $|\ell \cap B(\mathcal{O})| = q/2$, so $|\mathcal{S}_\ell \cap \mathcal{F}_P| = q/2$. Therefore it suffices to prove that $\mathcal{S}_\ell \cap \mathcal{F}_P$ is a subgroup of $\mathcal{F}_P^*$. Now if $x, y \in \mathcal{S}_\ell \cap \mathcal{F}_P$, then also $z = x + y$ is in $\mathcal{S}_\ell \cap \mathcal{F}_P$, since $\{x, y, z\}$ is a P-regular triple. □

It follows that for every $\ell, m \in \mathcal{L} \backslash (\mathcal{O} \cup \mathcal{F}_P)$ with $\ell \neq m$,

$$|\mathcal{S}_\ell \cap \mathcal{S}_m \cap \mathcal{F}_P| = \begin{cases} \frac{q}{2} & \text{if } \mathcal{S}_\ell \cap \mathcal{F}_P = \mathcal{S}_m \cap \mathcal{F}_P \\ \frac{q}{4} & \text{otherwise.} \end{cases}$$

Lemma 5. *Let $\ell \in \mathcal{L} \backslash (\mathcal{O} \cup \mathcal{F}_P)$. Then on each point R of ℓ with $R \neq \ell \cap o$ there is exactly one line $m \neq \ell$ such that $\mathcal{S}_\ell \cap \mathcal{F}_P = \mathcal{S}_m \cap \mathcal{F}_P$.*

Proof. Let $\mathcal{F}_R^0 = \mathcal{F}_R \backslash \{n, PR, \mathcal{O} \cap \mathcal{F}_R\}$. Note that $|\mathcal{F}_R^0| = q - 1$ if $R \notin B(\mathcal{O})$, while $|\mathcal{F}_R^0| = q - 3$ if $R \in B(\mathcal{O})$.

Let $H_{\ell,r} = |\mathcal{S}_\ell \cap \mathcal{S}_r \cap \mathcal{F}_P|$, $r \in \mathcal{F}_R^0$. Then

$$\sum_{r \in \mathcal{F}_R^0} H_{\ell,r} = \begin{cases} \frac{q^2}{4} & \text{if } R \notin B(\mathcal{O}) \\ \frac{q^2}{4} + 2q - 5\frac{q}{2} & \text{if } R \in B(\mathcal{O}). \end{cases} \tag{$*$}$$

Relation $(*)$ is obtained counting in two different ways the pairs $(z, r) \in (\mathcal{S}_\ell \cap \mathcal{F}_P) \times \mathcal{F}_R^0$ such that $z \cap r \in B(\mathcal{O})$.

Let k be the number of lines $r \in \mathcal{F}_R^0$ such that $\mathcal{S}_\ell \cap \mathcal{F}_P = \mathcal{S}_r \cap \mathcal{F}_P$. If $R \notin B(\mathcal{O})$, from $(*)$

$$\sum_{r \in \mathscr{F}_R^0} H_{\ell,r} = k\frac{q}{2} + (q-1-k)\frac{q}{4} = \frac{q^2}{4}.$$

So $k = 1$. If $R \in B(\mathcal{O})$, again from $(*)$ we get

$$\sum_{r \in \mathscr{F}_R^0} H_{\ell,r} = k\frac{q}{2} + (q-3-k)\frac{q}{4} = \frac{q^2}{4} + 2q - 5\frac{q}{2}.$$

Hence $k = 1$. $\square$

Using Lemma 5 it is easy to prove that the equivalence relation on $\mathscr{L} \backslash (\mathscr{F}_P \cup \mathcal{O})$

$$\ell \sim m \quad \text{if } \mathscr{S}_\ell \cap \mathscr{F}_P = \mathscr{S}_m \cap \mathscr{F}_P$$

has $q-1$ classes, each class has q elements and that the lines of a class plus the line o constitute a line oval. Therefore the P-regular line oval $\mathcal{O}$ determines $q-1$ other line ovals $\mathcal{O}_i$, $i = 1, \ldots, q-1$, all with nucleus n, such that

1. $\mathcal{O} \cap \mathcal{O}_i = \mathcal{O} \cap \mathscr{F}_P = \{o\}$, $i = 1, \ldots, q-1$;
2. $\ell, m \in \mathcal{O}_i \backslash \{o\}$ if and only if $\mathscr{S}_\ell \cap \mathscr{F}_P = \mathscr{S}_m \cap \mathscr{F}_P$.

Definition 5. The set of line ovals $\{\mathcal{O}, \mathcal{O}_i\}_{i=1,\ldots,q-1}$, as above determined, is called the *P-bundle* of $\mathscr{A}_q$.

Note that the $q-1$ hyperplanes of $\mathscr{F}_P^*$ are determined by the sets $\mathscr{S}_{\ell_i} \cap \mathscr{F}_P$, where $\ell_i \in \mathcal{O}_i$, $i = 1, \ldots, q-1$.

A property which characterizes the line ovals $\mathcal{O}_i$, $i = 1, \ldots, q-1$, is given by the following lemma.

Lemma 6. *Let $\mathcal{O}'$ be a line oval such that $\mathcal{O}' \cap \mathcal{O} = \mathcal{O} \cap \mathscr{F}_P = \{o\}$. Then $\mathcal{O}'$ is one of the line ovals $\mathcal{O}_i$ if and only if $x \cap B(\mathcal{O}) = x \cap B(\mathcal{O}')$ for every line $x \in \mathscr{F}_P^* \backslash \{o\}$ such that $x \cap B(\mathcal{O}) \cap B(\mathcal{O}') \neq \emptyset$.*

Proof. Let $\mathcal{O}' = \mathcal{O}_i$, some i. If $x \in \mathscr{F}_P^* \backslash \{o\}$ and $R \in x \cap B(\mathcal{O}) \cap B(\mathcal{O}_i)$, then there is a line ℓ of $\mathcal{O}_i$ such that $\ell \cap x = R$. Therefore $x \in \mathscr{S}_\ell \cap \mathscr{F}_P$. Because of property 2 above, $x \in \mathscr{S}_m \cap \mathscr{F}_P$ for every $m \in \mathcal{O}_i$. Using the null polarity of the 2-design $\mathscr{H}(\mathcal{O})$, $m \in \mathscr{S}_x$ for every $m \in \mathcal{O}_i$. Therefore $x \cap B(\mathcal{O}_i) \subseteq x \cap B(\mathcal{O})$. As $|x \cap B(\mathcal{O}_i)| = |x \cap B(\mathcal{O})|$, so $x \cap B(\mathcal{O}) = x \cap B(\mathcal{O}_i)$.

To prove the converse, it suffices to show that if ℓ and m are in $\mathcal{O}'$ then $\mathscr{S}_\ell \cap \mathscr{F}_P = \mathscr{S}_m \cap \mathscr{F}_P$. By way of contradiction, let $\ell \cap x \in B(\mathcal{O})$, but $m \cap x \notin B(\mathcal{O})$, some $x \in \mathscr{F}_P^* \backslash \{o\}$. Since $\ell \cap x \in B(\mathcal{O}) \cap B(\mathcal{O}')$, then $x \cap B(\mathcal{O}) = x \cap B(\mathcal{O}')$. Therefore $m \cap x \notin B(\mathcal{O}')$, a contradiction. $\square$

From the above lemma we deduce that the line oval $\mathcal{O}_i$, $i = 1, \ldots, q-1$, is P-regular and has the same P-regular triples as $\mathcal{O}$.

Proof. Let $\{x, y, z\}$ be a P-regular triple for $\mathcal{O}$. Let ℓ be a line such that $\ell \cap x \notin B(\mathcal{O}_i)$ and $\ell \cap y \notin B(\mathcal{O}_i)$. We claim that $\ell \cap z \in B(\mathcal{O}_i)$. We treat the cases $\ell \cap x \in B(\mathcal{O})$ and $\ell \cap y \in B(\mathcal{O})$, the others being similar.

Since $\{x, y, z\}$ is P-regular for $\mathcal{O}$ and $\ell \cap x \in B(\mathcal{O})$, $\ell \cap y \in B(\mathcal{O})$, then $\ell \cap z \in B(\mathcal{O})$ and $\mathscr{C}(\mathscr{S}_x) \cap \mathscr{C}(\mathscr{S}_y) \subset \mathscr{S}_z$ Because of Lemma 6, from $\ell \cap x \notin B(\mathcal{O}_i)$ and $\ell \cap y \notin B(\mathcal{O}_i)$

$$\mathcal{O}_i \subset \mathscr{C}(\mathscr{S}_x), \quad \mathcal{O}_i \subset \mathscr{C}(\mathscr{S}_y)$$

follows. Therefore $\mathcal{O}_i \subset \mathscr{C}(\mathscr{S}_x) \cap \mathscr{C}(\mathscr{S}_y) \subset \mathscr{S}_z$. Thus $m \cap z \in B(\mathcal{O})$; whence $\ell \cap z \in B(\mathcal{O}_i)$. □

Let now $\overline{\mathcal{O}}$ be a line oval such that $\overline{\mathcal{O}} \cap \mathcal{O} = \mathcal{O} \cap \mathscr{F}_P = \{o\}$. If $\overline{\mathcal{O}}$ is P-regular and has the same P-regular triples as $\mathcal{O}$, then $\overline{\mathcal{O}}$ is one of the $\mathcal{O}_i$, $i = 1, \ldots, q-1$. To prove the assertion, first note that $\mathcal{O}$ and $\overline{\mathcal{O}}$ induce on $\mathscr{F}_P^*$ the same additive structure. Let $x \in \mathscr{F}_P^*$. Denote by $I_1, \ldots, I_{q/2}$ the $q/2$ hyperplanes of $\mathscr{F}_P^*$ which contain x. As each I_j can be realized as $\mathscr{S}_{\ell_j} \cap \mathscr{F}_P$, where ℓ_j is any affine line of $\mathscr{F}_Q$ with $Q \neq P$, so $\ell_j \cap x \in B(\mathcal{O})$ if and only if $\ell_j \cap x \in B(\overline{\mathcal{O}})$. Therefore $x \cap B(\mathcal{O}) = x \cap B(\overline{\mathcal{O}})$. Because of Lemma 6, $\overline{\mathcal{O}}$ is one of the line ovals $\mathcal{O}_i$, $i = 1, \ldots, q-1$.

We have proved

Result 3. (see also [10], Theorem 1 and Lemma 3) *Let $\mathcal{O}$ be a P-regular line oval. Then there exist $q-1$ other line ovals $\mathcal{O}_i$, $i = 1, \ldots, q-1$, all with nucleus n, such that*

1. $\mathcal{O} \cap \mathcal{O}_i = \mathcal{O} \cap \mathscr{F}_P = \{o\}$, $i = 1, \ldots, q-1$;
2. $\ell, m \in \mathcal{O}_i \backslash \{o\}$ *if and only if* $\mathscr{S}_\ell \cap \mathscr{F}_P = \mathscr{S}_m \cap \mathscr{F}_P$.

Moreover, each line oval $\mathcal{O}_i$, $i = 1, \ldots, q-1$, is P-regular, has the same P-regular triples as $\mathcal{O}$ and any other P-regular line oval having the same P-regular triples as $\mathcal{O}$ is one of the $\mathcal{O}_i$, $i = 1, \ldots, q-1$.

We apply now Result 3 to the case where $\mathscr{A}_q = \Pi_q^n$ is a translation plane of even order $q = 2^d$ with translation group T and $\mathcal{O}$ is a completely regular line oval with respect to the line at infinity n. If P is a point on n, denote by T_P the group of all translations with centre P.

Lemma 7. *Let $g \in T$. Then $\mathcal{O}^g$ is a completely regular line oval and if $g \in T_P$ then $\mathcal{O}$ and $\mathcal{O}^g$ have the same P-regular triples.*

Proof. First of all note that $\mathcal{O}^g$ is a line oval with nucleus n. Also, $B(\mathcal{O})^g = B(\mathcal{O}^g)$. For, if $R \in B(\mathcal{O})$, then there is a line r of $\mathcal{O}$ such that $R \in r$. Therefore $R^g \in B(\mathcal{O}^g)$, and so $B(\mathcal{O})^g \subseteq B(\mathcal{O}^g)$. Since $|B(\mathcal{O})^g| = |B(\mathcal{O}^g)|$, then $B(\mathcal{O})^g = B(\mathcal{O}^g)$ follows.

Let $\{x, y, z\}$ be any P-regular triple for $\mathcal{O}$, where P is any point on n. We prove that $\{x^g, y^g, z^g\}$ is a P-regular triple for $\mathcal{O}^g$. Let ℓ be any line not on P and assume that $\ell \cap x^g$ and $\ell \cap y^g$ are not in $B(\mathcal{O}^g)$. If $\ell \cap z^g \notin B(\mathcal{O}^g)$, then the points $\ell^g \cap x$, $\ell^g \cap y$, $\ell^g \cap z$ are not in $B(\mathcal{O})$, which is absurd, as $\{x, y, z\}$ is a P-regular triple for $\mathcal{O}$.

In particular, if $g \in T_P$, then $\{x^g, y^g, z^g\} = \{x, y, z\}$. □

Because of this lemma and Result 3 above the P-bundle defined by $\mathcal{O}$ is $\{\mathcal{O}^g \mid g \in T_P\}$. We fix the following notation:

$\{S_0, \dots, S_q\}$ is the set of points of n;

T_i is the group of all translations with centre S_i, $i = 0, 1, \dots, q$;

$\mathcal{F}_i$ is the pencil of lines thought S_i, $i = 0, 1, \dots, q$;

o_i is the line $\mathcal{O} \cap \mathcal{F}_i$, $i = 0, 1, \dots, q$;

$\mathcal{F}_i^* = \mathcal{F}_i \setminus \{o_i\}$, $i = 0, 1, \dots, q$.

Recall that $\mathcal{F}_i^*$ is a d-dimensional vector space over GF(2) and that the S_i-bundle is $\{\mathcal{O}^g \mid g \in T_i\}$.

Lemma 8. *Let $I = \{o_j, m_1, \dots, m_{q/2-1}\}$ be any hyperplane of the vector space $\mathcal{F}_j^*$. Then there is a subgroup H of T_i, with $i \neq j$, of order $q/2$ which stabilizes I. Moreover, also the group HT_j of order $q^2/2$ stabilizes I.*

Proof. Let ℓ be a line of $\mathcal{F}_i \setminus \{o_i, n\}$, such that $\mathcal{S}_\ell \cap \mathcal{F}_j = I$. Then the points $\ell \cap m_k$, $k = 1, \dots, q/2 - 1$, are in $B(\mathcal{O})$. Let $\{\mathcal{O}, \mathcal{O}^{h_1}, \dots, \mathcal{O}^{h_{q-1}}\}$ be the S_i-bundle, where $\{1, h_1, \dots, h_{q-1}\} = T_i$. Each of the lines m_k, $k = 1, \dots, q/2 - 1$, is on one of the line ovals of the S_i-bundle. It is not restrictive to assume that $m_k \in \mathcal{O}^{h_k}$, $k = 1, \dots, q/2 - 1$. So let $H = \{1, h_1, \dots, h_{q/2-1}\}$. From Lemma 6, $\ell \cap m_k \in B(\mathcal{O}) \cap B(\mathcal{O}^{h_k})$ implies $\ell \cap B(\mathcal{O}) = \ell \cap B(\mathcal{O}^{h_k})$ for every $h_k \in H$. Therefore for any h_k and h_r in H

$$(\ell \cap B(\mathcal{O}^{h_k}))^{h_r} = \ell \cap B(\mathcal{O}^{h_k h_r}) = \ell \cap B(\mathcal{O}).$$

Hence H is a subgroup of T_i which stabilizes I.

Clearly, also HT_j stabilizes I and has order $q^2/2$, as $H \cap T_j = \{1\}$. □

The elementary abelian 2-group T is sharply transitive on the points of $\mathcal{A}_q$. Fix a point P_0 of $\mathcal{A}_q$. Then for any point $P \neq S_0, \dots, S_q$ there is exactly one $g \in T$ such that $P = P_0^g$. If $P = P_0^g$ and $Q = P_0^h$, then addition of points is meaningful: $P + Q := P_0^{gh}$. In this way the set of points of $\mathcal{A}_q$ becomes an elementary abelian 2-group G of order q^2 isomorphic to T, whose identity element is the point P_0.

The design $\mathcal{D}(\mathcal{O})$ has been defined in Section 2.

Lemma 9. *For any distinct blocks $\boldsymbol{b}$ and $\boldsymbol{c}$ of $\mathcal{D}(\mathcal{O})$, $\boldsymbol{b} \triangle \boldsymbol{c}$ is a left coset of a subgroup of G.*

Proof. First we consider the case $\boldsymbol{b} = B(\mathcal{O})$ and $\boldsymbol{c} = B(\mathcal{O}^g)$, $g \in T_S$, where S is one of the points $S_0, S_1, \dots, S_q$ and T_S is the group of all translations with centre S. Let $\mathcal{F}_S \cap \mathcal{O} = \{o\}$. If $\ell \in \mathcal{O}^g \setminus \{o\}$, then $\mathcal{S}_\ell \cap \mathcal{F}_S$ is a hyperplane of $\mathcal{F}_S^*$ and $\mathcal{S}_\ell \cap \mathcal{F}_S = \mathcal{S}_m \cap \mathcal{F}_S$ for every $\ell, m \in \mathcal{O}^g \setminus \{o\}$. Let $I = \mathcal{S}_\ell \cap \mathcal{F}_S = \{o, z_1, \dots, z_{q/2-1}\}$. The remaining affine lines of $\mathcal{F}_S$, say $\bar{z}_1, \dots, \bar{z}_{q/2}$, share the following property:

any point on $\bar{z}_i$, $i = 1, \dots, q/2$, *is either in* $B(\mathcal{O})$ *or in* $B(\mathcal{O}^g)$.

Thus $B(\mathcal{O}) \triangle B(\mathcal{O}^g)$ is the set of points on the lines $\bar{z}_1, \dots, \bar{z}_{q/2}$. These points are $q^2/2$ in number.

Let $P_0 \in B(\mathcal{O}) \triangle B(\mathcal{O}^g)$ (P_0 is the identity element of G). Then P_0 is on one of the lines $\bar{z}_1, \dots, \bar{z}_{q/2}$, say $\bar{z}_1$. If $h \in T_S$, then P_0^h is a point on $\bar{z}_1$. Therefore $P_0^h \in B(\mathcal{O}) \triangle B(\mathcal{O}^g)$ for any $h \in T_S$. By Lemma 8, let H be a subgroup of T_i, where $S_i \neq S$, of order $q/2$ which stabilizes I and its complement $\{\bar{z}_1, \dots, \bar{z}_{q/2}\}$. Then HT_S stabilizes I and its complement. So $B(\mathcal{O}) \triangle B(\mathcal{O}^g)$ consists of the points $\{P_0^{h_i g_j} \mid h_i \in H, g_j \in T_S\}$, which is a subgroup of G of order $q^2/2$.

Next let us examine the case where $P_0 \notin B(\mathcal{O}) \triangle B(\mathcal{O}^g)$. Then P_0 is in $B(\mathcal{O}) \cap B(\mathcal{O}^g)$. So P_0 is on one of the lines $\{o, z_1, \dots, z_{q/2-1}\}$. Using the subgroup H as determined above, we have that

$$K = \{P_0^{h_i g_j} \mid h_i \in H, g_j \in T_S\} = (B(\mathcal{O}) \cap (B(\mathcal{O}^g)) \cup (\mathscr{C}B(\mathcal{O}) \cap \mathscr{C}B(\mathcal{O}^g))$$

is a subgroup of G. Therefore if P is any point on one of the lines $\bar{z}_1, \dots, \bar{z}_{q/2}$, then $B(\mathcal{O}) \triangle B(\mathcal{O}^g) = P + K$ is a left coset of a subgroup of G.

The general case follows from the above ones. It suffices to note that if $\boldsymbol{b}$ and $\boldsymbol{c} = B(\mathcal{O}^h)$ are two distinct blocks of $\mathscr{D}(\mathcal{O})$, then $\boldsymbol{b} = B(\mathcal{O}^g)$, where $g \in T$. Since $B(\mathcal{O}) \triangle B(\mathcal{O}^{hg})$ is a left coset of a subgroup of G the same holds for $B(\mathcal{O}^g) \triangle B(\mathcal{O}^h)$. □

Theorem 9. *Let* $\mathscr{A}_q$ *be a translation plane of even order* $q = 2^d$ *with* $d \geqslant 3$ *and* $\mathcal{O}$ *a completely regular line oval. Then* $\mathscr{D}(\mathcal{O}) \cong \mathscr{S}^1(2d)$.

Proof. The proof follows from the above lemma and Theorem 4. □

Theorems 9 and 6 prove Theorem 2 stated in the Introduction.

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