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Unstable hyperplanes for Steiner bundles and multidimensional matrices

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Abstract. We study some properties of the natural action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ on non-degenerate multidimensional complex matrices $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ of boundary format (in the sense of Gelfand, Kapranov and Zelevinsky); in particular we characterize the non-stable ones as the matrices which are in the orbit of a "triangular" matrix, and the matrices with a stabilizer containing $\mathbb{C}^*$ as those which are in the orbit of a "diagonal" matrix. For $p = 2$ it turns out that a non-degenerate matrix $A \in \mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ detects a Steiner bundle S_A (in the sense of Dolgachev and Kapranov) on the projective space $\mathbb{P}^n$, $n = \dim(V_2) - 1$. As a consequence we prove that the symmetry group of a Steiner bundle is contained in SL(2) and that the SL(2)-invariant Steiner bundles are exactly the bundles introduced by Schwarzenberger [Schw], which correspond to "identity" matrices. We can characterize the points of the moduli space of Steiner bundles which are stable for the action of $\mathrm{Aut}(\mathbb{P}^n)$, answering in the first nontrivial case a question posed by Simpson. In the opposite direction we obtain some results about Steiner bundles which imply properties of matrices. For example the number of unstable hyperplanes of S_A (counting multiplicities) produces an interesting discrete invariant of A, which can take the values $0, 1, 2, \ldots, \dim V_0 + 1$ or ∞; the ∞ case occurs if and only if S_A is Schwarzenberger (and A is an identity). Finally, the Gale transform for Steiner bundles introduced by Dolgachev and Kapranov under the classical name of association can be understood in this setting as the transposition operator on multidimensional matrices.

1 Introduction

A multidimensional matrix of boundary format is an element $A \in V_0 \otimes \cdots \otimes V_p$ where V_i is a complex vector space of dimension $k_i + 1$ for $i = 0, \ldots, p$ and

$$k_0 = \sum_{i=1}^{p} k_i.$$

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We denote by $\operatorname{Det} A$ the hyperdeterminant of A (see [GKZ]). Let $e_0^{(j)}, \dots, e_{k_j}^{(j)}$ be a basis in V_j so that every $A \in V_0 \otimes \cdots \otimes V_p$ has a coordinate form

$$A = \sum a_{i_0,\dots,i_p} e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}.$$

Let $x_0^{(j)}, \dots, x_{k_j}^{(j)}$ be the coordinates in V_j. Then A has the following different descriptions:

1) A multilinear form

$$\sum_{(i_0,\dots,i_p)} a_{i_0,\dots,i_p} x_{i_0}^{(0)} \otimes \cdots \otimes x_{i_p}^{(p)}.$$

2) An ordinary matrix $M_A = (m_{i_1 i_0})$ of size $(k_1+1) \times (k_0+1)$ whose entries are multilinear forms

$$m_{i_1 i_0} = \sum_{(i_2,\dots,i_p)} a_{i_0,\dots,i_p} x_{i_2}^{(0)} \otimes \cdots \otimes x_{i_p}^{(p)}. \tag{1.1}$$

3) A sheaf morphism f_A on the product $X = \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_p}$:

$$\mathcal{O}_X^{k_0+1} \xrightarrow{f_A} \mathcal{O}_X(1,\dots,1)^{k_1+1}. \tag{1.2}$$

Theorem 3.1 of chapter 14 of [GKZ] easily translates into:

Theorem. *The following properties are equivalent*:

i) $\operatorname{Det} A \neq 0$;

ii) *the matrix M_A has constant rank k_1+1 on $X = \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_p}$*;

iii) *the morphism f_A is surjective so that $S_A^* = \ker f_A$ is a vector bundle of rank $k_0 - k_1$.*

The above remarks set up a basic link between non-degenerate multidimensional matrices of boundary format and vector bundles on a product of projective spaces. In the particular case $p = 2$ the (dual) vector bundle S_A lives on the projective space $\mathbb{P}^n$, $n = k_2$, and is a Steiner bundle as defined by Dolgachev and Kapranov in [DK]. We can keep for S_A the name Steiner also in the case $p \geqslant 3$.

The action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ on $V_0 \otimes \cdots \otimes V_p$ translates to an action on the corresponding bundle in two steps: first the action of $\mathrm{SL}(V_0) \times \mathrm{SL}(V_1)$ leaves the bundle in the same isomorphism class; then $\mathrm{SL}(V_2) \times \cdots \times \mathrm{SL}(V_p)$ acts on the classes, i.e. on the moduli space of Steiner bundles. It follows that the invariants of matrices for the action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ coincide with the invariants of the action of $\mathrm{SL}(V_2) \times \cdots \times \mathrm{SL}(V_p)$ on the moduli space of the corresponding bundles. Moreover the stable points of both actions correspond to each other.

The aim of this paper is to investigate the properties and the invariants of both the above actions. When we look at the vector bundles, we restrict ourselves to the case $p = 2$, that is Steiner bundles on projective spaces. This is probably the first case where Simpson's question ([Simp], p. 11) about the natural $\mathrm{SL}(n+1)$-action on the moduli spaces of bundles on $\mathbb{P}^n$ has been investigated.

Section 2 is devoted to the study of multidimensional matrices. We denote by the same letter matrices in $V_0 \otimes \cdots \otimes V_p$ and their projections in $\mathbb{P}(V_0 \otimes \cdots \otimes V_p)$. In Theorem 2.4 we prove that a matrix $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ of boundary format with $\operatorname{Det} A \neq 0$ is not stable for the action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ if and only if there is a coordinate system such that $a_{i_0 \dots i_p} = 0$ for $i_0 > \sum_{t=1}^{p} i_t$. A matrix satisfying this condition is called triangulable. The other main results of this section are Theorems 2.5 and 2.6 which describe the behaviour of the stabilizer subgroup $\mathrm{Stab}(A)$. In Remark 5.14 we introduce a discrete $\mathrm{SL}(V_0) \times \mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$-invariant of non-degenerate matrices in $\mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ and we show that it can assume only the values $0, \dots, k_0 + 2, \infty$.

The second part of the paper, consisting of Sections 3 to 6, can be read independently of Section 2, except that we will use Theorem 2.4 in two crucial points (Theorem 5.9 and Section 6). In this part we study the Steiner bundles on $\mathbb{P}^n = \mathbb{P}(V)$. As we mentioned above, they are rank-n vector bundles S whose dual S^* appears in an exact sequence

$$0 \to S^* \to W \otimes \mathcal{O} \xrightarrow{f_A} I \otimes \mathcal{O}(1) \to 0 \tag{1.3}$$

where W and I are complex vector spaces of dimension $n + k$ and k, respectively. The map f_A corresponds to $A \in W^* \otimes V \otimes I$ (which is of boundary format) and f_A is surjective if and only if $\operatorname{Det} A \neq 0$. We denote by $\mathcal{S}_{n,k}$ the family of Steiner bundles described by a sequence as (1.3). $\mathcal{S}_{n,1}$ contains only the quotient bundle. Important examples of Steiner bundles are the Schwarzenberger bundles, whose construction goes back to the pioneering work of Schwarzenberger [Schw]. Other examples are the logarithmic bundles $\Omega(\log \mathcal{H})$ of meromorphic forms on $\mathbb{P}^n$ having at most logarithmic poles on a finite union $\mathcal{H}$ of hyperplanes with normal crossing; Dolgachev and Kapranov showed in [DK] that they are Steiner. The Schwarzenberger bundles are a special case of logarithmic bundles, when all the hyperplanes osculate the same rational normal curve. Dolgachev and Kapranov proved a Torelli type theorem, namely that the logarithmic bundles are uniquely determined up to isomorphism by the above union of hyperplanes, with a weak additional assumption. This assumption was recently removed by Vallès [V2], who shares with us the idea of looking at the scheme $W(S) = \{H \in \mathbb{P}^{n\vee} \mid h^0(S_H^*) \neq 0\} \subset \mathbb{P}^{n\vee}$ of unstable hyperplanes of a Steiner bundle S. Vallès proves that any $S \in \mathcal{S}_{n,k}$ with at least $n + k + 2$ unstable hyperplanes with normal crossing is a Schwarzenberger bundle and $W(S)$ is a rational normal curve. We strengthen this result by showing the following: for any $S \in \mathcal{S}_{n,k}$ any subset of closed points in $W(S)$ has always normal crossing (see Theorem 3.10). Moreover $S \in \mathcal{S}_{n,k}$ is logarithmic if and only if $W(S)$ contains at least $n + k + 1$ closed points (Corollaries 5.11 and 5.10). In particular if $W(S)$ contains exactly $n + k + 1$ closed points then $S \simeq \Omega(\log W(S))$. The Torelli Theorem follows.

It turns out that the length of $W(S)$ defines an interesting filtration into irreducible subschemes of $\mathscr{S}_{n,k}$ which gives also the discrete invariant of multidimensional matrices of boundary format mentioned above. This filtration is well behaved with respect to the $\mathrm{PGL}(n+1)$-action on $\mathbb{P}^n$ and also with respect to the classical notion of association reviewed in [DK]. Eisenbud and Popescu realized in [EP] that the association is exactly what nowadays is called Gale transform. For Steiner bundles corresponding to $A \in W^* \otimes V \otimes I$ this operation amounts to exchanging the role of V with I, so that it corresponds to the transposition operator on multidimensional matrices.

The Gale transform for Steiner bundles can be decribed by the natural isomorphism

$$\mathscr{S}_{n,k}/\mathrm{SL}(n+1) \to \mathscr{S}_{k-1,n+1}/\mathrm{SL}(k).$$

Both quotients in the previous formula are isomorphic to the GIT-quotient

$$\mathbb{P}(W^* \otimes V \otimes I)/\mathrm{SL}(W) \times \mathrm{SL}(V) \times \mathrm{SL}(I)$$

which is a basic object in linear algebra.

As an application of the tools developed in the first section we show that all the points of $\mathscr{S}_{n,k}$ are semistable for the action of $\mathrm{SL}(n+1)$ and we compute the stable points. Moreover we characterize the Steiner bundles $S \in \mathscr{S}_{n,k}$ whose symmetry group (i.e. the group of linear projective transformations preserving S) contains $\mathrm{SL}(2)$ or contains $\mathbb{C}^*$.

Finally we mention that $W(S)$ has a geometrical construction by means of the Segre variety. From this construction $W(S)$ can be easily computed by means of current software systems.

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2 Multidimensional matrices of boundary format and geometric invariant theory

It is well known that all one dimensional subgroups of the complex Lie group $\mathrm{SL}(2)$ either are conjugated to the maximal torus consisting of diagonal matrices (which is isomorphic to $\mathbb{C}^*$) or are conjugated to the subgroup $\mathbb{C} \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| \, b \in \mathbb{C} \right\}$.

Definition 2.1. A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is called triangulable if one of the following equivalent conditions holds:

i) there exist bases in V_j such that $a_{i_0,\dots,i_p} = 0$ for $i_0 > \sum_{t=1}^{p} i_t$;

ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^* \subset \mathrm{SL}(U)$ and isomorphisms $V_j \simeq S^{k_j} U$ such that if $V_0 \otimes \cdots \otimes V_p = \bigoplus_{n \in \mathbb{Z}} W_n$ is the decomposition into a direct sum of eigenspaces of the induced representation then we have $A \in \bigoplus_{n \geqslant 0} W_n$.

Proof of the equivalence between i) *and* ii). Let x, y be a basis of U such that $t \in \mathbb{C}^*$ acts on x and y as tx and $t^{-1}y$. Set $e_k^{(j)} := x^k y^{k_j - k}\binom{k_j}{k} \in S^{k_j}U$ for $j > 0$ and $e_k^{(0)} := x^{k_0 - k} y^k \binom{k_0}{k} \in S^{k_0}U$ so that $e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}$ is a basis of $S^{k_0}U \otimes \cdots \otimes S^{k_p}U$ which diagonalizes the action of $\mathbb{C}^*$. The weight of $e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}$ is $2(\sum_{t=1}^p i_t - i_0)$, hence ii) implies i). The converse is trivial.

The following definition agrees with the one in [WZ], p. 639.

Definition 2.2. A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is called diagonalizable if one of the following equivalent conditions holds:

i) there exist bases in V_j such that $a_{i_0,\dots,i_p} = 0$ for $i_0 \neq \sum_{t=1}^p i_t$;

ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^* \subset \mathrm{SL}(U)$ and isomorphisms $V_j \simeq S^{k_j}U$ such that A is a fixed point of the induced action of $\mathbb{C}^*$.

The following definition agrees with the one in [WZ], p. 639.

Definition 2.3. A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is an identity if one of the following equivalent conditions holds:

i) there exist bases in V_j such that

$$a_{i_0,\dots,i_p} = \begin{cases} 0 & \text{for } i_0 \neq \sum_{t=1}^p i_t \\ 1 & \text{for } i_0 = \sum_{t=1}^p i_t; \end{cases}$$

ii) there exist a vector space U of dimension 2 and isomorphisms $V_j \simeq S^{k_j}U$ such that A belongs to the unique one-dimensional $\mathrm{SL}(U)$-invariant subspace of $S^{k_0}U \otimes S^{k_1}U \otimes \cdots \otimes S^{k_p}U$.

The equivalence between i) and ii) follows easily from the following remark: the matrix A satisfies the condition ii) if and only if it corresponds to the natural multiplication map $S^{k_1}U \otimes \cdots \otimes S^{k_p}U \to S^{k_0}U$ (after a suitable isomorphism $U \simeq U^*$ has been fixed).

From now on, we consider the natural action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ on $\mathbb{P}(V_0 \otimes \cdots \otimes V_p)$. We may suppose $p \geqslant 2$. The definitions of triangulable, diagonalizable and identity apply to elements of $\mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ as well. In particular all identity matrices fill a distinguished orbit in $\mathbb{P}(V_0 \otimes \cdots \otimes V_p)$. The hyperdeterminant of elements of $V_0 \otimes \cdots \otimes V_p$ was introduced by Gelfand, Kapranov and Zelevinsky in [GKZ]. They proved that the dual variety of the Segre product $\mathbb{P}(V_0) \times \cdots \times \mathbb{P}(V_p)$ is a hypersurface if and only if $k_j \leqslant \sum_{i \neq j} k_i$ for $j = 0, \dots, p$ (which is obviously true for a matrix of boundary format). When the dual variety is a hypersurface, its equation is called the hyperdeterminant of format $(k_0+1) \times \cdots \times (k_p+1)$ and denoted by Det. The hyperdeterminant is a homogeneous polynomial function over $V_0 \otimes \cdots \otimes V_p$ so that the condition $\mathrm{Det}\, A \neq 0$ is meaningful for $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$. The function Det is $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$-invariant, in particular if $\operatorname{Det} A \neq 0$ then A is semistable for the action of $\mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$. We denote by $\mathrm{Stab}(A) \subset \mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ the stabilizer subgroup of A and by $\mathrm{Stab}(A)^0$ its connected component containing the identity. The main results of this section are the following.

Theorem 2.4. *Let* $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ *of boundary format such that* $\operatorname{Det} A \neq 0$. *Then*

$$A \text{ is triangulable } \Leftrightarrow A \text{ is not stable for the action of } \mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p).$$

Theorem 2.5. *Let* $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ *be of boundary format such that* $\operatorname{Det} A \neq 0$. *Then*

$$A \text{ is diagonalizable } \Leftrightarrow \mathrm{Stab}(A) \text{ contains a subgroup isomorphic to } \mathbb{C}^*.$$

We state the following theorem only in the case $p = 2$, although we believe that it is true for all $p \geqslant 2$. We point out that in particular $\dim \mathrm{Stab}(A) \leqslant 3$ which is a bound independent of k_0, k_1, k_2.

Theorem 2.6. *Let* $A \in \mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ *of boundary format such that* $\operatorname{Det} A \neq 0$. *Then there exists a* 2*-dimensional vector space* U *such that* $\mathrm{SL}(U)$ *acts over* $V_i \simeq S^{k_i} U$ *and according to this action on* $V_0 \otimes V_1 \otimes V_2$ *we have* $\mathrm{Stab}(A)^0 \subset \mathrm{SL}(U)$. *Moreover the following cases are possible*:

$$\mathrm{Stab}(A)^0 \simeq \begin{cases} 0 & (\textit{trivial subgroup}) \\ \mathbb{C} & \\ \mathbb{C}^* & \\ \mathrm{SL}(2) & (\textit{this case occurs if and only if } A \textit{ is an identity}). \end{cases}$$

Remark. When A is an identity then $\mathrm{Stab}(A) \simeq \mathrm{SL}(2)$.

Let X_j be the finite set $\{0, \ldots, j\}$. We set $\mathscr{B} := X_{k_1} \times \cdots \times X_{k_p}$. A *slice* (in the q-direction) is the subset $\{(\alpha_1, \ldots, \alpha_p) \in \mathscr{B} : \alpha_q = k\}$ for some $k \in X_q$. Two slices in the same direction are called *parallel*. An *admissible path* is a finite sequence of elements $(\alpha_1, \ldots, \alpha_p) \in \mathscr{B}$ starting from $(0, \ldots, 0)$, ending with $(k_1, \ldots, k_p)$, such that at each step exactly one α_i increases by 1 and all other remain unchanged. Note that each admissible path consists exactly of $k_0 + 1$ elements.

Tom Thumb's Lemma 2.7. *Put a mark* (*or a piece of bread*) *on every element of every admissible path. Then two parallel slices contain the same number of marks.*

Proof. Any admissible path P corresponds to a sequence of k_0 integers between 1 and p such that the integer i occurs exactly k_i times. We call this sequence the code of the path P. More precisely the j-th element of the code is the integer i such that α_i

increases by 1 from the j-th element of the path to the $(j+1)$-th element. The occurrences of the integer i in the code divide all other integers different from i appearing in the code into k_i+1 strings (possibly empty); each string encodes the part of the path contained in one of the k_i+1 parallel slices. The symmetric group Σ_{k_i+1} acts on the set $\mathscr{A}$ of all the admissible paths by permuting the strings. Let P^i_j the number of elements (marks) of the path $P \in \mathscr{A}$ on the slice $\alpha_i = j$. In particular for all $\sigma \in \Sigma_{k_i+1}$ we have

$$\sum_{P\in\mathscr{A}} P^i_j = \sum_{P\in\mathscr{A}} (\sigma\cdot P)^i_j = \sum_{P\in\mathscr{A}} P^i_{\sigma^{-1}(j)},$$

which proves our lemma.

We will often use the following well-known lemma.

Lemma 2.8. *If $\mathcal{O}^k_X \xrightarrow{\phi} F$ is a morphism of vector bundles on a variety X with $k \leqslant \operatorname{rank} F = f$ and $c_j(F) \neq 0$ for some $j \geqslant f-k+1$, then the degeneracy locus $D_k(\phi) = \{x \in X \mid \operatorname{rank}(\phi_x) \leqslant k-1\}$ is nonempty of codimension $\leqslant f-k+1$.*

Proof. Suppose that $D_k(\phi) = \emptyset$. Then consider the projection $X \times \mathbb{P}^{k-1} \xrightarrow{\pi} X$ and let H be the pullback of the hyperplane divisor according to the second projection. The natural composition

$$\mathcal{O} \to \pi^*\mathcal{O}^k \otimes H \to \pi^*F \otimes H$$

gives a section of $\pi^*F \otimes H$ without zeroes, hence $\pi^*F \otimes H$ has a trivial line subbundle. It follows

$$0 = c_f(\pi^*F \otimes H) = \pi^*c_f(F) + \cdots + \pi^*c_{f-k+1}(F)\cdot H^{k-1},$$

which is a contradiction because $1, \dots, H^{k-1}$ are independent modulo $\pi^*H^*(X, \mathbb{C})$. We get $D_k(\phi) \neq \emptyset$ and the result follows from the Theorem 14.4 (b) of [Fu].

A square matrix with a zero left-lower submatrix with the NE-corner on the diagonal has zero determinant. The following lemma generalizes this remark to multidimensional matrices of boundary format.

Lemma 2.9. *Let $A \in V_0 \otimes \cdots \otimes V_p$. Suppose that in a suitable coordinate system there is $(\beta_1, \dots, \beta_p) \in \mathscr{B}$ such that $a_{i_0\dots i_p} = 0$ for $i_k \leqslant \beta_k$ $(k \geqslant 1)$ and $i_0 \geqslant \beta_0 := \sum_{t=1}^p \beta_t$. Then* $\operatorname{Det} A = 0$.

Proof. The submatrix of A given by elements $a_{i_0\dots i_p}$ satisfying $i_k \leqslant \beta_k$ $(k \geqslant 1)$ gives on $X = \mathbb{P}^{\beta_2} \times \cdots \times \mathbb{P}^{\beta_p}$ the sheaf morphism

$$\mathcal{O}_X^{\beta_1+1} \to \mathcal{O}_X(1, \dots, 1)^{\beta_0}$$

whose rank by Lemma 2.8 drops on a subvariety of codimension $\leqslant \beta_0 - \beta_1 = \sum_{t=2}^{p} \beta_t = \dim \mathbb{P}^{\beta_2} \times \cdots \times \mathbb{P}^{\beta_p}$; hence there are nonzero vectors $v_i \in V_i^*$ for $1 \leqslant i \leqslant p$ such that $A(v_1 \otimes \cdots \otimes v_p) = 0$ and then $\operatorname{Det} A = 0$ by Theorem 3.1 of Chapter 14 of [GKZ].

Lemma 2.10. *Let $p \geqslant 2$ and a_j^i be integers with $0 \leqslant i \leqslant p$, $0 \leqslant j \leqslant k_i$ satisfying the inequalities $a_j^0 \geqslant a_{j+1}^0$ for $0 \leqslant j \leqslant k_0 - 1$, $a_j^i \leqslant a_{j+1}^i$ for $i > 0$, $0 \leqslant j \leqslant k_i - 1$ and the linear equations*

$$\sum_{j=0}^{k_i} a_j^i = 0 \quad \text{for } 0 \leqslant i \leqslant p$$

$$a^0_{\sum_{t=1}^{p} \beta_i} + a^1_{\beta_1} + \cdots + a^p_{\beta_p} = 0 \quad \text{for all } (\beta_1, \ldots, \beta_p) \in \mathscr{B}.$$

Then there is $N \in \mathbb{Q}$ such that

$$a_i^0 = N(k_0 - 2i), \quad a_i^j = N(-k_j + 2i) \quad j > 0.$$

Moreover $N \in \mathbb{Z}$ if at least one k_j is not even, and $2N \in \mathbb{Z}$ if all the k_j are even.

Proof. If $1 \leqslant s \leqslant p$ and $\beta_s \geqslant 1$ we have the two equations

$$a^0_{\sum_{t=1}^{p} \beta_t} + a^1_{\beta_1} + \cdots + a^s_{\beta_s} + \cdots + a^p_{\beta_p} = 0,$$
$$a^0_{\sum_{t=1}^{p} \beta_t - 1} + a^1_{\beta_1} + \cdots + a^s_{\beta_s - 1} + \cdots + a^p_{\beta_p} = 0.$$

Subtracting we obtain

$$a^0_{\sum_{t=1}^{p} \beta_t} - a^0_{\sum_{t=1}^{p} \beta_t - 1} = -(a^s_{\beta_s} - a^s_{\beta_s - 1}),$$

so that the right-hand side does not depend on s.

Moreover for $p \geqslant 2$ from the equations

$$a^0_{\sum_{t=1}^{p} \beta_t} + a^1_{\beta_1} + \cdots + a^q_{\beta_q + 1} + \cdots + a^s_{\beta_s - 1} + \cdots + a^p_{\beta_p} = 0,$$
$$a^0_{\sum_{t=1}^{p} \beta_t - 1} + a^1_{\beta_1} + \cdots + a^q_{\beta_q} + \cdots + a^s_{\beta_s} + \cdots + a^p_{\beta_p} = 0$$

we get

$$a^q_{\beta_q + 1} - a^q_{\beta_q} = a^s_{\beta_s} - a^s_{\beta_s - 1},$$

which implies that the right-hand side does not depend on β_s either. Let $a^s_{\beta_s} - a^s_{\beta_s - 1} = 2N \in \mathbb{Z}$. Then $a_t^s = a_0^s + 2Nt$ for $t > 0$, $s > 0$. By the assumption $\sum_{t=0}^{k_s} a_t^s = 0$ we get

$$(k_s + 1)a_0^s + 2N \sum_{t=1}^{k_s} t = 0,$$

that is

$$a_0^s = -k_s N.$$

The formulas for a_i^s and a_i^0 follow immediately. If some k_s is odd we have $2N \in \mathbb{Z}$ and $k_s N \in \mathbb{Z}$ so that $N \in \mathbb{Z}$.

Proof of Theorem 2.4. If A is triangulable it is not stable. Conversely suppose A not stable and denote by A again a representative of A in $V_0 \otimes \cdots \otimes V_p$. By the Hilbert–Mumford criterion there exists a 1-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ such that $\lim_{t \to 0} \lambda(t) A$ exists. Let

$$a_0^s \leqslant \cdots \leqslant a_{k_s}^s, \quad 0 \leqslant s \leqslant p$$

be the weights of the 1-parameter subgroup of $\mathrm{SL}(V_s)$ induced by λ; with respect to a basis consisting of eigenvectors the coordinate $a_{i_0 \dots i_p}$ describes the eigenspace of λ whose weight is $a_{i_0}^0 + a_{i_1}^1 + \cdots + a_{i_p}^p$. Recall that

$$\sum_{j=0}^{k_i} a_s^i = 0, \quad 0 \leqslant s \leqslant p.$$

We note that for all $(\beta_1, \dots, \beta_k) \in \mathscr{B}$ we have

$$a^0_{\sum_{t=1}^p \beta_t} + a^1_{\beta_1} + \cdots + a^p_{\beta_p} \geqslant 0; \tag{2.1}$$

otherwise the coefficient $a_{i_0 \dots i_p}$ is zero for $i_k \leqslant \beta_k$, $1 \leqslant k \leqslant p$ and $i_0 \geqslant \sum_{t=1}^p \beta_t$ and Lemma 2.9 implies $\operatorname{Det} A = 0$. The sum on all $(\beta_1, \dots, \beta_k) \in \mathscr{B}$ for any admissible path of the left-hand side of (2.1) is nonnegative. The contribution of a^t's in this sum is zero by Lemma 2.7. Also the contribution of a^0's is zero because it is zero on any admissible path. It follows that

$$a^0_{\sum_{t=1}^p \beta_t} + a^1_{\beta_1} + \cdots + a^p_{\beta_p} = 0 \quad \text{for all } (\beta_1, \dots, \beta_k) \in \mathscr{B},$$

and by Lemma 2.10 we get explicit expressions for the weights which imply that A is triangulable.

Proof of Theorem 2.5. Again we denote by A any representative of A in $V_0 \otimes \cdots \otimes V_p$. If A is diagonal in a suitable basis $e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}$, we construct a 1-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{SL}(V_0) \times \cdots \times \mathrm{SL}(V_p)$ by the equation $\lambda(t) e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)} := t^{i_0 - \sum_{t=1}^p i_t} e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}$, so that $\mathbb{C}^* \subset \mathrm{Stab}(A)$. Conversely let $\mathbb{C}^* \subset \mathrm{Stab}(A)$. By Theorem 2.4, A is triangulable and by Lemma 2.9 all diagonal elements $a_{i_0 \dots i_p}$ with $i_0 = \sum_{t=1}^p i_t$ are nonzero. We can arrange the action on the representative in order that the diagonal corresponds to the zero eigenspace. Then the assumption

$\mathbb{C}^* \subset \mathrm{Stab}(A)$ and the explicit expressions of the weights as in the proof of Theorem 2.4 show that A is diagonal.

We will prove Theorem 2.6 by geometric arguments at the end of Section 6.

3 Preliminaries about Steiner bundles

Definition 3.1. A Steiner bundle over $\mathbb{P}^n = \mathbb{P}(V)$ is a vector bundle S whose dual S^* appears in an exact sequence

$$0 \to S^* \to W \otimes \mathcal{O} \xrightarrow{f_A} I \otimes \mathcal{O}(1) \to 0 \tag{3.1}$$

where W and I are complex vector spaces of dimension $n+k$ and k respectively.

A Steiner bundle is stable ([BS], Theorem 2.7 or [AO], Theorem 2.8) and is invariant by small deformations ([DK], Corollary 3.3). Hence the moduli space $\mathscr{S}_{n,k}$ of Steiner bundles defined by (3.1) is isomorphic to an open subset of the Maruyama moduli scheme of stable bundles. On the other hand $\mathscr{S}_{n,k}$ is also isomorphic to the GIT-quotient of a suitable open subset of $\mathbb{P}(\mathrm{Hom}(W, I \otimes V))$ for the action of $\mathrm{SL}(W) \times \mathrm{SL}(I)$ (see Section 6). It is interesting to remark that these two approaches give two different compactifications of $\mathscr{S}_{n,k}$, but we do not pursue this direction in this paper. For other results about $\mathbb{P}(\mathrm{Hom}(W, I \otimes V))$, see [EH] and [C].

Definition 3.2. Let $S \in \mathscr{S}_{n,k}$ be a Steiner bundle. A hyperplane $H \in \mathbb{P}(V^*)$ is an unstable hyperplane of S if $h^0(S^*_{|H}) \neq 0$. The set $W(S)$ of the unstable hyperplanes is the degeneracy locus over $\mathbb{P}(V^*)$ of the natural map $H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1)$, hence it has a natural structure of scheme. $W(S)$ is called the scheme of the unstable hyperplanes of S. Note that since $h^0(S^*_{|H}) \leqslant 1$ ([V2]) the rank of the previous map drops at most by one.

3.3. Let us describe more explicitly the map $H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1)$. From (3.1) it follows that $H^1(S^*(-1)) \simeq I$ and $H^1(S^*) \simeq (V \otimes I)/W$. The projection $V \otimes I \xrightarrow{B} (V \otimes I)/W$ can be interpreted as a map $V \otimes H^1(S^*(-1)) \to H^1(S^*)$ which induces on $\mathbb{P}(V^*)$ the required morphism $H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1)$.

For a generic S, $W(S) = \emptyset$. Examples show that $W(S)$ can have a nonreduced structure.

We recall that if D is a divisor with normal crossing then $\Omega(\log D)$ is the bundle of meromorphic forms having at most logarithmic poles over D. If $\mathscr{H}$ is the union of m hyperplanes H_i with normal crossing, it is shown in [DK] that for $m \leqslant n+1$, $\Omega(\log \mathscr{H})$ splits while for $m \geqslant n+2$ we have $S = \Omega(\log \mathscr{H}) \in \mathscr{S}_{n,k}$ where $k = m - n - 1$.

The following is a simple consequence of [BS], Theorem 2.5.

Proposition 3.4. *Let $S \in \mathcal{S}_{n,k}$, then*

$$h^0(S^*(t)) = 0 \Leftrightarrow t \leqslant k - 1.$$

Proof. $S^*(t) \simeq \bigwedge^{n-1} S(-k+t)$. The $\bigwedge^{n-1}$-power of the sequence dual to (3.1) is

$$0 \to S^{n-1} I^* \otimes \mathcal{O}(-n+1-k+t) \to S^{n-2} I^* \otimes W^* \otimes \mathcal{O}(-n+2-k+t) \to \cdots$$
$$\cdots \to \bigwedge^{n-1} W^* \otimes \mathcal{O}(-k+t) \to \bigwedge^{n-1} S(-k+t) \to 0,$$

and from this sequence the result follows.

Let us fix a basis in each of the vector spaces W and I. Then the morphism f_A in (3.1) can be represented by a $k \times (n+k)$ matrix A (it was called M_A in the introduction, see (1.1)) with entries in V. In order to simplify the notations we will use the same letter A to denote also its class in $\mathbb{P}(\mathrm{Hom}(W, I \otimes V))$. A has rank k at every point of $\mathbb{P}(V)$. Two such matrices represent isomorphic bundles if and only if they lie in the same orbit of the action of $\mathrm{GL}(W) \times \mathrm{GL}(I)$.

3.5. In particular $H^0(S^*(t))$ identifies with the space of $(n+k) \times 1$-column vectors v with entries in $S^t V$ such that

$$Av = 0. \tag{3.2}$$

Moreover $H \in W(S)$ (as closed point) if and only if there are nonzero vectors w_1 of size $(n+k) \times 1$ and i_1 of size $k \times 1$ both with constant coefficients such that

$$Aw_1 = i_1 H. \tag{3.3}$$

This means that w_1 is in the kernel of the map $W \simeq H^0(W \otimes \mathcal{O}_H) \to H^0(I \otimes \mathcal{O}_H(1))$.

3.6. According to the theorem stated in the introduction $A \in \mathrm{Hom}(W, V \otimes I)$ has nonzero hyperdeterminant if and only if it corresponds to a vector bundle. The locus in $\mathbb{P}(\mathrm{Hom}(W, V \otimes I))$ where the hyperdeterminant vanishes is an irreducible hypersurface of degree $k \cdot \binom{n+k}{k}$ ([GKZ], Chapter 14, Corollary 2.6). It is interesting to remark that Proposition 3.4 can be proved also as a consequence of [GKZ], Chapter 14, Theorem 3.3.

3.7. The above description has a geometrical counterpart. Here $P(V)$ is the projective space of lines in V, dual to the usual projective space $\mathbb{P}$ of hyperplanes in V. Consider in $P(V \otimes I)$ the variety X_r corresponding to elements of $V \otimes I$ of rank $\leqslant r$. In particular X_1 is the Segre variety $P(V) \times P(I)$. Let $m = \min(n, k-1)$ so that X_m is the variety of non maximum rank elements. Then $A \in \mathrm{Hom}(W, V \otimes I)$ defines a vector bundle if and only if it induces an embedding $P(W) \subset P(V \otimes I)$ such that at every smooth point of $X_m \cap P(W)$, $P(W)$ and X_m meet transversally. This follows from [GKZ], Chapter 14, Propostion 3.14 and Chapter 1, Proposition 4.11.

3.8. $W(S)$ has the following geometrical description. Let p_V be the projection of the Segre variety $P(V) \times P(I)$ on the $P(V)$. Then

$$W(S)_{\mathrm{red}} = p_V[P(W) \cap (P(V) \times P(I))]_{\mathrm{red}}$$

(according to the natural isomorphism $P(V) = \mathbb{P}(V^*)$). In fact $i_1 H$ in formula (3.3) is a decomposable tensor in $V \otimes I$.

3.9. About the scheme structure we remark that $W(S)$ is the degeneration locus of the morphism $I \otimes \mathcal{O}_{\mathbb{P}(V^*)} \to \dfrac{V \otimes I}{W} \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$. The following construction is standard. The projective bundle $\mathbb{P} = \mathbb{P}(I^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}) \xrightarrow{\pi} \mathbb{P}(V^*)$ is isomorphic to the Segre variety $T = \mathbb{P}(V^*) \times \mathbb{P}(I^*) = P(V) \times P(I)$ and $\mathcal{O}_{\mathbb{P}}(1) \simeq \mathcal{O}_T(0,1)$. The morphism

$$\mathbb{C} \to \frac{V \otimes I}{W} \otimes V^* \otimes I^*$$

defines a section of $O_T(1,1) \otimes \dfrac{V \otimes I}{W}$ with zero locus $Z = T \cap P(W)$. Now assume that dim $W(S) = 0$, hence dim $T = 0$. By applying π_* to the exact sequence

$$\mathcal{O}_T \otimes \left(\frac{V \otimes I}{W}\right)^* \to \mathcal{O}_T(1,1) \to \mathcal{O}_Z \to 0$$

we get that the structure sheaf of $W(S)$ is contained in $\pi_* \mathcal{O}_Z$. We do not know if the equality always holds. In particular if Z is reduced also $W(S)$ is reduced. We will show in Proposition 6.5 that a multiple point occurs in Z iff it occurs in $W(S)$.

Theorem 3.10. *Let $S \in \mathcal{S}_{n,k}$ be a Steiner bundle. Then any set of distinct unstable hyperplanes of S has normal crossing.*

Proof. We fix a coordinate system $x_0, \ldots, x_n$ on $\mathbb{P}^n$ and a basis $e^1, \ldots, e^{n+k}$ of W. Let A be a matrix representing S. If the assertion is not true, we may suppose that $W(S)$ contains the hyperplanes $x_0 = 0, \ldots, x_j = 0$, $\sum_{i=0}^{j} x_i = 0$ for some j such that $1 \leqslant j \leqslant n-1$. By (3.3) there are $c^0 \in W$, $b^0 \in I$ such that $Ac^0 = b^0 x_0$. We may suppose that the first coordinate of c^0 is nonzero, hence $A \cdot [c^0, e^2, \ldots, e^{n+k}] = [b^0 x_0, \ldots] = A'$.

The matrix A' still represents S, hence by (3.3) there are $c^1 \in W$, $b^1 \in I$ such that $A'c^1 = b^1 x_1$. At least one coordinate of c^1 after the first is nonzero, say the second. It follows that $A' \cdot [e^1, c^1, e^3, \ldots, e^{n+k}] = [b^0 x_0, b^1 x_1, \ldots] = A''$ and again A'' represents S. Proceeding in this way we get in the end that

$$[b^0 x_0, \ldots, b^j x_j, \ldots]$$

is a matrix representing S, which we denote again by A.

By (3.3) there are $c = (c_1, \dots, c_{n+k})^t \in W$, $b \in I$ such that $A \cdot c = [b^0 x_0, \dots b^j x_j, \dots] \cdot c = b \sum_{i=0}^{j} x_i$.

Now we distinguish two cases. If $c_i = 0$ for $i \geqslant j+2$ we get $b = c_1 b^0 = c_2 b^1 = \dots = c_{j+1} b^j$, that is the submatrix of A given by the first $j+1$ columns has generically rank one. If we take the $k \times (n+k-j)$ matrix which has b^j as first column and the last $n+k-j-1$ columns of A in the remaining places, we obtain a morphism

$$\mathcal{O}^k \to \mathcal{O} \oplus \mathcal{O}(1)^{n+k-j-1},$$

which by Lemma 2.8 has rank $\leqslant k-1$ on a nonempty subscheme Z of $\mathbb{P}^n$. It follows that also A has rank $\leqslant k-1$ on Z, contradicting the assumption that S is a bundle. So this case cannot occur.

In the second case there exists a nonzero c_i for some $i \geqslant j+2$, we may suppose $c_{j+2} \neq 0$. Then the matrix

$$A' = A \cdot [e^1, \dots, e^{j+1}, c, e^{j+3}, \dots, e^{n+k}] = \left[b^0 x_0, \dots b^j x_j, b \sum_{i=0}^{j} x_i \dots \right]$$

represents S.

The last $n+k-j-2$ columns of A' define a sheaf morphism $\mathcal{O}^k \to \mathcal{O}(1)^{n+k-j-2}$ on the subspace $\mathbb{P}^{n-j-1} = \{x_0 = \dots = x_j = 0\}$ and again by Lemma 2.8 we find a point where the rank of A is $\leqslant k-1$. So neither case can occur.

Proposition 3.11. *Let $S \in \mathscr{S}_{n,k}$ and let $\xi_1, \dots, \xi_s \in W(S)$, $s \leqslant n+k$. There exists a matrix representing S whose first s columns are $[b^1 \xi_1, \dots, b^s \xi_s]$, where the b^i are vectors with constant coefficients of size $k \times 1$. Moreover any p columns among $b^1, \dots, b^s$ with $p \leqslant k$ are independent. Conversely if the first s columns of a matrix representing S have the form $[b^1 \xi_1, \dots, b^s \xi_s]$ then $\xi_1, \dots, \xi_s \in W(S)$.*

Proof. The last assertion is obvious. The proof of the existence of a matrix A representing S having the required form is analogous to that of Theorem 3.10. Then it is sufficient to prove that $b^1, \dots, b^p$ are independent. Suppose $\sum_{i=1}^{p} b^i \lambda_i = 0$. Let $\xi = \prod_{i=1}^{p} \xi_i$. Let c be the $(n+k) \times 1$ vector (whith coefficients in $S^{p-1} V$) whose i-th entry is $\lambda_i \xi / \xi_i$ for $i = 1, \dots, p$ and zero otherwise. It follows that $A \cdot c = \xi \sum_{i=1}^{p} b^i \lambda_i = 0$ and by (3.2) we get a nonzero section of $S^*(p-1)$, which contradicts Proposition 3.4.

3.12 Elementary transformations. Consider $H = \{\xi = 0\} \in W(S)$. The map $\mathcal{O}_H \to S^*_{|H}$ induces a surjective map $S \to \mathcal{O}_H$ and an exact sequence

$$0 \to S' \to S \to \mathcal{O}_H \to 0 \tag{3.4}$$

(see also [V2], Theorem 2.1); it is easy to check (e.g. by Beilinson's theorem) that $S' \in \mathscr{S}_{n,k-1}$. According to [M] we say that S' has been obtained from S by an elementary transformation. By Proposition 3.11 there exists a matrix A representing S of the following form

$$A = \begin{bmatrix} \xi & * & \cdots & * \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix} \tag{3.5}$$

where A' is a matrix representing S'. Since $h^0(S^*_{|H}) \leqslant 1$, S' is uniquely determined by S and H.

Theorem 3.13. *With the above notations we have the inclusion of schemes* $W(S) \subset W(S') \cup H$. *In particular we have*:

i) length $W(S') \geqslant$ length $W(S) - 1$;

ii) *if* $\dim W(S') = 0$ *then* $\mathrm{mult}_H\, W(S') \geqslant \mathrm{mult}_H\, W(S) - 1$, *so that if* H *is a multiple point of* $W(S)$, *then* $H \in W(S')$;

iii) *if* $\dim W(S') = 0$ *then for any hyperplane* $K \neq H$,

$$\mathrm{mult}_K\, W(S') \geqslant \mathrm{mult}_K\, W(S).$$

Proof. The sequence dual to (3.4)

$$0 \to S^* \to S'^* \to \mathcal{O}_H(1) \to 0$$

gives the commutative diagram on $\mathbb{P}(V^*)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & H^1(S^*(-1))\otimes\mathcal{O} & \longrightarrow & H^1(S'^*(-1))\otimes\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_H(1))\otimes\mathcal{O}(1) & \longrightarrow & H^1(S^*)\otimes\mathcal{O}(1) & \longrightarrow & H^1(S'^*)\otimes\mathcal{O}(1) & \longrightarrow & 0 \end{array}$$

It follows that the matrix B' of the map

$$H^1(S'^*(-1)) \otimes \mathcal{O} \to H^1(S'^*) \otimes \mathcal{O}(1)$$

can be seen as a submatrix of the matrix B of the map

$$H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1).$$

In a suitable system of coordinates:

$$B = \begin{bmatrix} y_1 & * \\ \vdots & * \\ y_n & * \\ 0 & B' \end{bmatrix} \tag{3.6}$$

where $(y_1, \dots, y_n)$ is the ideal of H (in the dual space). It follows that

$$I(W(S')) \cdot (y_1, \dots, y_n) \subset I(W(S)),$$

which concludes the proof.

4 The Schwarzenberger bundles

Let U be a complex vector space of dimension 2. The natural multiplication map $S^{k-1}U^* \otimes S^nU^* \to S^{n+k-1}U^*$ induces the $\mathrm{SL}(U)$-equivariant injective map $S^{n+k-1}U \to S^{k-1}U \otimes S^nU$ and defines a Steiner bundle on $\mathbb{P}(S^nU) \simeq \mathbb{P}^n$ as the dual of the kernel of the surjective morphism

$$\mathcal{O}_{\mathbb{P}(S^nU)} \otimes S^{n+k-1}U \to \mathcal{O}_{\mathbb{P}(S^nU)}(1) \otimes S^{k-1}U.$$

It is called a Schwarzenberger bundle (see [ST], [Schw]). Let us remark that in the correspondence between Steiner bundles and multidimensional matrices mentioned in the introduction, the Schwarzenberger bundles correspond exactly to the identity matrices (see Definition 2.3).

By interchanging the role of $S^{k-1}U$ and S^nU we obtain also a Schwarzenberger bundle on $\mathbb{P}(S^{k-1}U) \simeq \mathbb{P}^{k-1}$ as the dual of the kernel of the surjective morphism

$$\mathcal{O}_{\mathbb{P}(S^{k-1}U)} \otimes S^{n+k-1}U \to \mathcal{O}_{\mathbb{P}(S^{k-1}U)}(1) \otimes S^nU.$$

Both the above bundles are $\mathrm{SL}(U)$-invariant. We sketch the original Schwarzenberger construction for the first one. The diagonal map $u \mapsto u^n$ and the isomorphism $\mathbb{P}(S^nU) \simeq \mathbb{P}^n$ detect a rational normal curve $\mathbb{P}(U) = C_n \subset \mathbb{P}^n$. In the same way a second rational normal curve $\mathbb{P}(U) = C_{n+k-1}$ arises in $\mathbb{P}(S^{n+k-1}U)$. We define a morphism

$$\begin{aligned} \mathbb{P}(S^nU) = S^n\mathbb{P}(U) &\to \mathrm{Gr}(\mathbb{P}^{n-1}, \mathbb{P}(S^{n+k-1}U)) \\ n \text{ points in } \mathbb{P}(U) &\mapsto \text{Span of } n \text{ points in } C_{n+k-1}. \end{aligned}$$

The pullback of the dual of the universal bundle on the Grassmannian is a Schwarzenberger bundle.

It is easy to check that if S is a Schwarzenberger bundle then $W(S) = C_n^* \subset \mathbb{P}(S^nU^*)$ (the dual rational normal curve). See e.g. [ST], [V1].

This can be explicitly seen from the matrix form given by [Schw], Proposition 2

$$M_A = \begin{bmatrix} x_0 & \dots & x_n & & \\ & \ddots & & \ddots & \\ & & x_0 & \dots & x_n \end{bmatrix}. \tag{4.1}$$

Let $t_1, \dots, t_{n+k}$ be any distinct complex numbers. Let w be the $(n+k) \times (n+k)$ Vandermonde matrix whose (i, j) entry is $t_j^{(i-1)}$; the (i, j)-entry of the product $M_A w$ is $t_j^{(i-1)} \cdot (\sum_{k=0}^{n} x_k t_j^k)$; hence $\{\sum_{k=0}^{n} x_k t^k = 0\} \in W(S)$ for all $t \in \mathbb{C}$ by Proposition 3.11. On the other hand $W(S)$ is $\mathrm{SL}(U)$-invariant; if it were strictly bigger than C_n^* then it would contain the hyperplane $H = \{x_0 + x_1 = 0\}$, which lies in the next $\mathrm{SL}(U)$-orbit; now equation (3.3) implies immediately that $w_1 = i_1 = 0$.

In Theorem 5.13 we will need the following result.

Lemma 4.1. *Let S be a Schwarzenberger bundle and let $(x_0, \dots, x_n)$ be coordinates in $\mathbb{P}(V)$ such that S is represented (with respect to suitable basis of I and W) by the matrix M_A in* (4.1). *Let $(y_0, \dots, y_n)$ be dual coordinates in $\mathbb{P}(V^*)$. Then the morphism $H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1)$ (with respect to the obvious basis) is represented by the matrix*

$$B = \begin{bmatrix} y_1 & -y_0 & & & \\ & y_1 & -y_0 & & \\ & & \ddots & \ddots & \\ & & & y_1 & -y_0 \\ y_2 & 0 & -y_0 & & \\ & \ddots & \ddots & \ddots & \\ & & y_2 & 0 & -y_0 \\ & & & y_2 & -y_1 \\ y_3 & 0 & 0 & -y_0 & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proof. By (3.3) it is enough to check that the composition

$$W \xrightarrow{A} V \otimes I \xrightarrow{B} (V \otimes I)/W$$

is zero, which is straightforward.

Theorem 4.2 ([Schw], Theorem 1, see also [DK], Proposition 6.6). *The moduli space of Schwarzenberger bundles is* $\mathrm{PGL}(n+1)/\mathrm{SL}(2)$, *which is the open subscheme of the Hilbert scheme parametrizing rational normal curves.*

In particular $W(S)$ uniquely determines S in the class of Schwarzenberger bundles.

5 A filtration of $\mathcal{S}_{n,k}$ and the Gale transform of Steiner bundles

Definition 5.1.

$$\mathcal{S}_{n,k}^i := \{S \in \mathcal{S}_{n,k} \mid \text{length } W(S) \geqslant i\}.$$

In particular

$$\mathscr{S}_{n,k} = \mathscr{S}^0_{n,k} \supset \mathscr{S}^1_{n,k} \supset \cdots.$$

We will see (Corollary 5.5) that $\mathscr{S}^\infty_{n,k}$ corresponds to Schwarzenberger bundles.

Each $\mathscr{S}^i_{n,k}$ is invariant for the action of $\mathrm{SL}(V)$ on $\mathscr{S}_{n,k}$. We will see in Section 6 that all the points of $\mathscr{S}_{n,k}$ are semistable (in the sense of Mumford's GIT) for the action of $\mathrm{SL}(V)$.

Let $\mathscr{S}$ be the open subset of $\mathbb{P}(\mathrm{Hom}(W, V \otimes I))$ representing Steiner bundles. The quotient $\mathscr{S}_{n,k}/\mathrm{SL}(V)$ is isomorphic to $\mathscr{S}/\mathrm{SL}(W) \times \mathrm{SL}(I) \times \mathrm{SL}(V)$.

By interchanging the role of V and I, also $\mathscr{S}_{k-1,n+1}/\mathrm{SL}(I)$ turns out to be isomorphic to $\mathscr{S}/\mathrm{SL}(W) \times \mathrm{SL}(I) \times \mathrm{SL}(V)$, so that we obtain an isomorphism

$$\mathscr{S}_{n,k}/\mathrm{SL}(n+1) \simeq \mathscr{S}_{k-1,n+1}/\mathrm{SL}(k).$$

For any $E \in \mathscr{S}_{n,k}/\mathrm{SL}(n+1)$ we will call the *Gale transform* of E the corresponding class in $\mathscr{S}_{k-1,n+1}/\mathrm{SL}(k)$ and we denote it by E^G. In [DK] the above construction is called association. Here we follow [EP]. Our Gale transform is a generalization of the one in [EP]. In fact in the case $i = n+k+1$ Eisenbud and Popescu in [EP] review the classical association between $\mathrm{PGL}(n+1)$-classes of $n+k+1$ points of $\mathbb{P}^n$ in general position and $\mathrm{PGL}(k)$-classes of $n+k+1$ points of $\mathbb{P}^{k-1}$ in general position and call it Gale transform. If we take the union $\mathscr{H}$ of $n+k+1$ hyperplanes with normal crossing in $\mathbb{P}^n$ (as points in the dual projective space) the Gale transform (as points in the dual projective space) $\mathscr{H}^G$ consists of a $\mathrm{PGL}(k)$-class of $n+k+1$ hyperplanes with normal crossing in $\mathbb{P}^{k-1}$. As remarked in [DK], $[\Omega(\log \mathscr{H})]^G \simeq [\Omega(\log \mathscr{H}^G)]$. That is, the Gale transform in our sense reduces to that in [EP] when the Steiner bundles are logarithmic. It is also clear that the PGL-class of Schwarzenberger bundles over $\mathbb{P}(V)$ corresponds under the Gale transform to the PGL-class of Schwarzenberger bundles over $\mathbb{P}(I)$.

We point out that one can define the Gale transform of a PGL-class of Steiner bundles but it is not possible to define the Gale transform of a single Steiner bundle. This was implicit (but not properly written) in [DK]. Nevertheless by a slight abuse we will also speak about the Gale transform of a Steiner bundle S, which will be any Steiner bundle in the class of the Gale transform of $S \bmod \mathrm{SL}(n+1)$.

The following elegant theorem due to Dolgachev and Kapranov is a first beautiful application of the Gale transform.

Theorem 5.2 ([DK], Theorem 6.8). *Any $S \in \mathscr{S}_{n,2}$ is a Schwarzenberger bundle.*

Proof.

$$\mathscr{S}_{n,2}/\mathrm{SL}(n+1) \simeq \mathscr{S}_{1,n+1}/\mathrm{SL}(2),$$

and it is obvious that a Steiner bundle on the line $\mathbb{P}^1$ is Schwarzenberger.

Theorem 5.3. *Two Steiner bundles having in common $n+k+1$ distinct unstable hyperplanes are isomorphic.*

Proof. We prove that if S is a Steiner bundle such that the hyperplanes $\{\xi_i = 0\}$ for $i = 1, \dots, n+k+1$ belong to $W(S)$, then S is uniquely determined. By Proposition 3.11 there exist column vectors $a_i \in \mathbb{C}^k$ such that S is represented by the matrix $[a^1\xi_1, \dots, a^{n+k}\xi_{n+k}]$. Moreover by (3.3) there are $b \in \mathbb{C}^{n+k}$ and $c \in \mathbb{C}^k$ such that

$$[a^1\xi_1, \dots, a^{n+k}\xi_{n+k}]b = c\xi_{n+k+1}.$$

We claim that *all* the components of b are nonzero. The last formula can be written

$$[a^1b^1, \dots, a^{n+k}b^{n+k}, -c] \cdot [\xi_1, \dots, \xi_{n+k+1}]^t = 0$$

where in the right matrix we identify ξ_i with the $(n+1) \times 1$ vector given by the coordinates of the corresponding hyperplane. We may suppose that there exists s with $1 \leqslant s \leqslant n+k-1$ such that $b_i = 0$ for $1 \leqslant i \leqslant s$ and $b_i \neq 0$ for $s+1 \leqslant i \leqslant n+k$. If $s \geqslant k$, it follows that $n+1$ hyperplanes among the ξ_i have a nonzero syzygy, which contradicts Proposition 3.11. Hence $s \leqslant k-1$ and we have

$$[a^{s+1}b^{s+1}, \dots, a^{n+k}b^{n+k}, -c] \cdot [\xi_{s+1}, \dots, \xi_{n+k+1}]^t = 0.$$

The rank of the right matrix is $n+1$, hence the rank of the left matrix is $\leqslant k-s$, in particular the first $k-s+1$ columns are dependent and this contradicts Proposition 3.11. This proves the claim.

In particular $[a^1, \dots, a^{n+k}, -c] \cdot B = 0$ where

$$B = \mathrm{Diag}(b_1, \dots, b_{n+k}, 1) \cdot [\xi_1, \dots, \xi_{n+k+1}]^t$$

is a $(n+k+1) \times (n+1)$ matrix with constant entries of rank $(n+1)$. Therefore the matrix $[a^1, \dots a^{n+k}, -c]$ is uniquely determined up to the (left) $GL(k)$-action, which implies that S is uniquely determined up to isomorphism.

Corollary 5.4. *A Steiner bundle is logarithmic if and only if it admits at least $(n+k+1)$ unstable hyperplanes.*

Proof. In fact $\mathscr{H} \subset W(\Omega(\log \mathscr{H}))$ by formula (3.5) of [DK] and Proposition 3.11.

Corollary 5.5 ([V2], Theorem 3.1]). *A Steiner bundle is Schwarzenberger if and only if it admits at least $(n+k+2)$ unstable hyperplanes. In particular $\mathscr{S}^{\infty}_{n,k}$ coincides with the moduli space of Schwarzenberger bundles.*

Proof. Let S be a Steiner bundle, and $H \in W(S)$. Let us consider the elementary transformation (3.12)

$$0 \to S' \to S \to \mathcal{O}_H \to 0$$

where $S' \in \mathscr{S}_{n,k-1}$; by Theorem 3.13, S' has $n+k+1$ unstable hyperplanes. Picking $H' \in W(S')$ and repeating the above procedure after $(k-2)$ steps we reach a $S^{(k-2)} \in \mathscr{S}_{n,2}$; by Theorem 5.2, $S^{(k-2)}$ is a Schwarzenberger bundle. In particular the remaining $n+4$ unstable hyperplanes lie on a rational normal curve. It is then clear that any subset of $n+4$ hyperplanes in $W(S)$ lies on a rational normal curve. Since there is a unique rational normal curve through $n+3$ points in general position, it follows that $W(S)$ is contained in a rational normal curve, so that S is a Schwarzenberger bundle by Theorem 5.3.

Theorem 5.6. *Let* $n \geqslant 2$, $k \geqslant 3$.

i) $\mathscr{S}^i_{n,k}$ *for* $0 \leqslant i \leqslant n+k+1$ *is an irreducible unirational closed subvariety of* $\mathscr{S}_{n,k}$ *of dimension* $(k-1)(n-1)(k+n+1) - i[(n-1)(k-2)-1]$.

ii) $\mathscr{S}^{n+k+1}_{n,k}$ *contains as an open dense subset the variety of Steiner logarithmic bundles which coincides with the open subvariety of* $\mathrm{Sym}^{n+k+1}\,\mathbb{P}^{n\vee}$ *consisting of hyperplanes in* $\mathbb{P}^n$ *with normal crossing.*

Proof. (ii) follows from Theorem 5.3.

The irreducibility in (i) follows from the geometric construction 3.8. The numerical computation in (i) is performed (for $i \leqslant n+k$) by adding $i(n+k-1)$ (moduli of i points in $\mathbb{P}(V) \otimes \mathbb{P}(I)$) to $n(k-1)(n+k-i)$ (dimension of Grassmannian of linear $\mathbb{P}^{n+k-1}$ in $\mathbb{P}(V \otimes I)$ containing the span of the above i points) and subtracting k^2-1 ($\dim \mathrm{SL}(I)$).

Remark 5.7. In the case $(n,k) = (2,3)$ the generic Steiner bundle is logarithmic (this was remarked in [DK], 3.18). In fact the generic $\mathbb{P}^4$ linearly embedded in $\mathbb{P}^8$ meets the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$ in $\deg \mathbb{P}^2 \times \mathbb{P}^2 = 6 = n+k+1$ points.

Remark. The dimension of $\mathscr{S}^i_{n,k}/\mathrm{SL}(n+1)$ is equal to $(n+k+1-i)[(k-2)(n-1)-1] + n(k-1)$ for $k \geqslant 3$, $n \geqslant 2$, $0 \leqslant i \leqslant n+k+1$ and it is 0 for $i \geqslant n+k+2$.

5.8. Corollary 5.5 implies the following property of the Segre variety: if a generic linear $P(W)$ meets $P(V) \times P(I)$ in $n+k+2$ points, then $P(W)$ meets it in infinitely many points.

Theorem 5.9. *Consider a nontrivial* (*linear*) *action of* $\mathrm{SL}(2) = \mathrm{SL}(U)$ *over* $\mathbb{P}^n$. *If a Steiner bundle is* $\mathrm{SL}(2)$-*invariant then it is a Schwarzenberger bundle and* $\mathrm{SL}(U)$ *acts over* $\mathbb{P}^n = \mathbb{P}(S^n U)$. *Hence* $\mathscr{S}^\infty_{n,k}$ *is the subset of the fixed points of the action of* $\mathrm{SL}(2)$ *on* $\mathscr{S}_{n,k}$.

Proof. By Theorem 2.4 there exists a coordinate system such that all the entries (except the first) of the first column of the matrix representing the Steiner bundle S are zero. By Proposition 3.11, $W(S)$ is nonempty. By the assumption $W(S)$ is

SL(2)-invariant and closed; it follows that $W(S)$ is a union of rational curves and of simple points. If $W(S)$ is infinite we can apply Corollary 5.5. If $W(S)$ is finite we argue by induction on k. We pick up $H \in W(S)$ and we consider the elementary transformation $0 \to S' \to S \to \mathcal{O}_H \to 0$. We get for all $g \in \mathrm{SL}(U)$ the diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & \mathcal{O}_H \\ \Big\downarrow{\scriptstyle i} & & \\ g^*S & \xrightarrow{g^*\phi} & \mathcal{O}_H. \end{array}$$

Since $h^0(S^*_{|H}) \leqslant 1$ we get that ϕ and $g^*\phi \circ i$ coincide up to a scalar multiple. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & g^*S' & \longrightarrow & g^*S & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0. \end{array}$$

It follows that $S' \simeq g^*S'$, hence $\mathrm{SL}(U) \subset \mathrm{Sym}(S')$ and by the inductive assumption S' is Schwarzenberger and $\mathrm{SL}(U)$ acts over $\mathbb{P}^n = \mathbb{P}(S^n U)$. Hence $W(S)$ is infinite and we apply again Corollary 5.5.

Corollary 5.10. *If $\mathcal{H}$ is the union of $n+k+1$ hyperplanes with normal crossing then*

$$W(\Omega(\log \mathcal{H})) = \begin{cases} \mathcal{H} & \text{when } \mathcal{H} \text{ does not osculate a rational normal curve,} \\ C_n & \text{when } \mathcal{H} \text{ osculates the rational normal curve } C_n, \\ & (\text{this case occurs iff } \Omega(\log \mathcal{H}) \text{ is Schwarzenberger}). \end{cases}$$

Proof. $\mathcal{H} \subset \Omega(\log \mathcal{H})$ by Proposition 3.11. The result follows by Theorem 5.3 and Corollary 5.5.

Corollary 5.11. *Let $S \in \mathcal{S}_{n,k}$ be a Steiner bundle. If $W(S)$ contains at least $n+k+1$ hyperplanes then for every subset $\mathcal{H} \subset W(S)$ consisting of $n+k+1$ hyperplanes $S \simeq \Omega(\log \mathcal{H})$, in particular S is logarithmic.*

Corollary 5.12 (Torelli theorem, see [DK] for $k \geqslant n+2$ or [V2] in general). *Let $\mathcal{H}$ and $\mathcal{H}'$ be two finite unions of $n+k+1$ hyperplanes with normal crossing in $\mathbb{P}(V)$ with $k \geqslant 3$ not osculating any rational normal curve. Then*

$$\mathcal{H} = \mathcal{H}' \Leftrightarrow \Omega(\log \mathcal{H}) \simeq \Omega(\log \mathcal{H}').$$

Theorem 5.13. *Let* $S \in \mathcal{S}_{n,k}$ *be a Steiner bundle. If* length $W(S) \geqslant n+k+2$ *then* length $W(S) = \infty$ *and* S *is Schwarzenberger.*

Proof. We proceed by induction on k. If $k = 2$ the result follows from Theorem 5.2, so we can suppose $k \geqslant 3$. Let us pick any $H \in W(S)$ and perform the elementary transformation (3.4). Then $S' \in \mathcal{S}_{n,k-1}$ and by Theorem 3.13 i), length $W(S') \geqslant n+k+1$, so that by induction S' is Schwarzenberger, in particular $W(S')$ is a rational normal curve C_n.

It follows that S is represented by the matrix

$$M_A = \begin{bmatrix} x_0 & f_1 & f_2 & \cdots & & & f_{n+k-1} \\ & x_0 & x_1 & \cdots & x_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & x_0 & x_1 & \cdots & x_n \end{bmatrix}$$

where $f_i = -\sum_{j=1}^{n} c_j^i x_j$. It is easy to check by Lemma 4.1 (and the proof of Theorem 3.13) that the morphism $H^1(S^*(-1)) \otimes \mathcal{O} \to H^1(S^*) \otimes \mathcal{O}(1)$ is represented by the matrix

$$B = \begin{bmatrix} y_1 & c_1^1 y_0 & c_1^2 y_0 & \cdots & c_1^{k-2} y_0 & \sum_{h=0}^{n} c_1^{k+h-1} y_h \\ y_2 & c_2^1 y_0 & c_2^2 y_0 & \cdots & c_2^{k-2} y_0 & \sum_{h=0}^{n} c_2^{k+h-1} y_h \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_n & c_n^1 y_0 & c_n^2 y_0 & \cdots & c_n^{k-2} y_0 & \sum_{h=0}^{n} c_n^{k+h-1} y_h \\ & y_1 & -y_0 & & & \\ & & \ddots & \ddots & & \\ & & & & y_1 & -y_0 \\ & y_2 & 0 & -y_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & y_2 & 0 & -y_0 \\ & & & & y_2 & -y_1 \\ & y_3 & 0 & 0 & -y_0 & \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

By Theorem 3.13 we have that

$$\text{length}(W(S) \cap C_n) \geqslant n+k+1. \tag{5.1}$$

The points of C_n are parametrized by $y_i = t^i$ and $W(S) \cap C_n$ is given by the $k \times k$ minors of B where we substitute $y_i = t^i$. It is sufficient to look at the first $n+k-2$

rows because the others are linear combination of these. The first two rows and the last $k-2$ give the submatrix

$$\begin{bmatrix} t & c_1^1 & c_1^2 & \dots & c_1^{k-2} & \sum_{h=0}^n c_1^{k+h-1} t^h \\ t^2 & c_2^1 & c_2^2 & \dots & c_2^{k-2} & \sum_{h=0}^n c_2^{k+h-1} t^h \\ & t & -1 & & & \\ & & t & -1 & & \\ & & & \ddots & \ddots & \\ & & & & t & -1 \end{bmatrix}$$

whose determinant is given up to sign by

$$t^{n+k} c_1^{n+k-1} + t^{n+k-1}(c_1^{n+k-2} - c_2^{n+k-1}) + \dots + t^2(c_1^1 - c_2^2) - tc_2^1; \tag{5.2}$$

by (5.1) all the coefficients of this polynomial are zero. When $n=2$ this is enough to conclude that M_A represents a Schwarzenberger bundle because the matrix M_A reduces to (4.1) after a Gaussian elimination on the rows. If $n \geqslant 3$ we have to look also at other minors. For example the minor given by the first, third and the last $k-2$ rows is

$$\begin{bmatrix} t & c_1^1 & c_1^2 & \dots & c_1^{k-2} & \sum_{h=0}^n c_1^{k+h-1} t^h \\ t^3 & c_3^1 & c_3^2 & \dots & c_3^{k-2} & \sum_{h=0}^n c_3^{k+h-1} t^h \\ & t & -1 & & & \\ & & t & -1 & & \\ & & & \ddots & \ddots & \\ & & & & t & -1 \end{bmatrix}$$

whose determinant is equal up to sign to

$$t^{n+k+1} c_1^{n+k-1} + t^{n+k} c_1^{n+k-2} + t^{n+k-1}(c_1^{n+k-3} - c_3^{n+k-1}) + \dots + t^3(c_1^1 - c_3^3) - t^2 c_3^2 - tc_3^1.$$

By (5.2) the leading term c_1^{n+k-1} vanishes and the degree drops so that by (5.1) also the coefficients of this last polynomial vanish. The reader can convince himself that the same argument of the case $n=2$ works also in this case.

We remark that the above proof does not use the Corollary 5.5 and gives a second proof of this corollary.

Remark. There are examples of Steiner bundles $S \in \mathcal{S}_{n,k}$ such that length $W(S) = n+k+1$ and $W(S)$, as a set, consists of only one point.

Remark 5.14. The above theorem shows that the only possible values for length $W(S)$ are $0, 1, \dots, n+k+1, \infty$. With the notation of Section 2, every multidimensional matrix $A \in V_0 \otimes V_1 \otimes V_2$ of boundary format such that $\operatorname{Det} A \neq 0$ has a $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-invariant

$$w(A) := \text{length } W(\ker f_A)^*$$

which can assume only the values $0, 1, \dots, \dim V_0 + 1, \infty$.

6 Moduli spaces of Steiner bundles and geometric invariant theory

Let $\mathscr{S} \subset \mathbb{P}(\mathrm{Hom}(W, V \otimes I))$ be the open subset consisting of all $\phi : W \to V \times I$ such that for every nonzero $v^* \in V^*$ the composite $v^* \circ \phi : W \to I$ has maximum rank. By (3.6), $\mathscr{S}$ is the complement of a hypersurface, and it is invariant for the natural action of $\mathrm{SL}(W) \times \mathrm{SL}(I)$. By interchanging the roles of all V and I (or, in the language of the previous section, by performing the Gale transform) it is easy to check that $\mathscr{S}$ coincides with the open subset of all $\phi : W \to V \times I$ such that for every nonzero $i^* \in I^*$ the composite $i^* \circ \phi : W \to V$ has maximum rank.

Lemma 6.1. *Every point of $\mathscr{S}$ is stable for the action of* $\mathrm{SL}(W) \times \mathrm{SL}(I)$.

Proof. Suppose that $A \in \mathscr{S}$ is not stable. Then by the Hilbert–Mumford criterion there exists a one-parameter subgroup $\lambda(t) : \mathbb{C}^* \to \mathrm{SL}(W) \times \mathrm{SL}(I)$ such that $\lim_{t \to 0} \lambda(t) A$ exists. We may suppose that the two projections of $\lambda(t)$ on the factors act diagonally with weights $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k$ and $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_{n+k}$ such that $\sum_i \beta_i = \sum_j \gamma_j = 0$.

We claim that there exists p such that $1 \leqslant p \leqslant k$ and $\beta_p + \gamma_{k+1-p} < 0$. Otherwise we get $0 \leqslant \sum_{i=1}^k (\beta_i + \gamma_{k+1-i}) = \sum_{i=1}^k \beta_i + \sum_{j=1}^k \gamma_j = \sum_{j=1}^k \gamma_j \leqslant k\gamma_k$, hence $0 \leqslant \gamma_k$. If $\gamma_{n+k} > 0$ we have $0 \leqslant \sum_{j=1}^k \gamma_j < \sum_{j=1}^{n+k} \gamma_j = 0$ which is a contradiction. If $\gamma_{n+k} = 0$ then $\gamma_j = 0$ for all j and the claim is obvious. It follows that $\beta_i + \gamma_j < 0$ for $i \leqslant p$ and $j \leqslant k+1-p$. Hence the first $p \times (k+1-p)$ block of the matrix corresponding to A is zero. The first p rows of A have nonzero elements only in the last $n+p-1$ columns and define a morphism $\mathcal{O}^p \to \mathcal{O}(1)^{n+p-1}$ that by Lemma 2.8 drops rank on a nonempty set contradicting the fact that A has maximum rank at every point.

Theorem 6.2. *Every point of $\mathscr{S}$ is semistable for the action of* $\mathrm{SL}(W) \times \mathrm{SL}(V) \times \mathrm{SL}(I)$.

Proof. By (3.6), $\mathscr{S}$ is the complement of a $\mathrm{SL}(W) \times \mathrm{SL}(V) \times \mathrm{SL}(I)$-invariant hypersurface ([GKZ], Chapter 14, Proposition 1.4).

Corollary 6.3. *Every point of $\mathscr{S}_{n,k}$ is semistable for the action of* $\mathrm{SL}(V)$ (*with respect to the natural polarization of $\mathscr{S}_{n,k}$ as GIT-quotient*).

Proof. We look at the hyperdeterminant as a polynomial in the coordinate ring of the GIT quotient $\mathbb{P}(\mathrm{Hom}(W, V \otimes I))/\mathrm{SL}(W) \times \mathrm{SL}(I) \supset \mathscr{S}_{n,k}$ which is invariant by the action of $\mathrm{SL}(V)$.

Theorem 6.4. *An element $A \in \mathscr{S}_{n,k}$ is not stable for the action of $\mathrm{SL}(n+1) = \mathrm{SL}(V)$ if and only if there is a coordinate system such that the ordinary matrix M_A (with entries in V) associated to A (see (1.1)) has the triangular form $M_A = \sum_{j=0}^{n} A^m x_m$, where the (i, j)-entry a_{ij}^m of A^m is zero for $j < i + m$.*

Proof. It is a reformulation of Theorem 2.4 in the case $p = 2$.

Proposition 6.5. *Let S be a Steiner bundle. The following properties are equivalent*:

i) *there is a hyperplane H which is a multiple point for $W(S)$, or S is a Schwarzenberger bundle*;

ii) *there is a coordinate system such that $H = \{x_0 = 0\}$ and the matrix $M_A = \sum_{j=0}^{n} A^m x_m$ satisfies $a_{ij}^0 = 0$ for $j < i$, $j = 1, 2$ and $a_{ij}^m = 0$ for $m \geqslant 1$, $j \leqslant i$, $j = 1, 2$.*

Proof. By (3.7), (3.8) and (3.9) (with the same notation) if the condition i) occurs then S is Schwarzenberger or Z has a multiple point. In both cases there is some point of $P(V) \times P(I)$ whose tangent space intersects $P(W)$ in a subspace of positive dimension. The tangent space at a point $[v_0 \otimes i_0] \in P(V) \times P(I)$ is the span of the two linear subspaces $P(V \otimes \langle i_0 \rangle)$ and $P(\langle v_0 \rangle \otimes I)$, so that any point of the tangent space has the form $[v_1 \otimes i_0 + v_0 \otimes i_1]$. If the point $[v_1 \otimes i_0 + v_0 \otimes i_1]$ with $v_0 \neq v_1$, $i_0 \neq i_1$ belongs to $P(W)$ it is easy to check that the matrix of S satisfies ii). Conversely if the matrix of S satisfies ii) then according to (3.5) we can perform twice the elementary transformation at the hyperplane H corresponding to v_0. Let $y_0, \ldots, y_n$ be coordinates in $\mathbb{P}(V^*)$ such that the ideal of $\{H\}$ is defined by $y_1, \ldots, y_n$. The matrix B in (3.6) has the form

$$B = \begin{bmatrix} y_1 & g_1(y_0, \ldots, y_n) & * \\ \vdots & \vdots & * \\ y_n & g_n(y_0, \ldots, y_n) & * \\ 0 & y_1 & * \\ \vdots & \vdots & * \\ 0 & y_n & * \\ 0 & 0 & B' \end{bmatrix}$$

where g_i are linear forms. It is straightforward to check that the maximal minors of the restriction of B to the line parametrized by $y_0 = 1$, $y_i = tg_i(1, 0, \ldots, 0)$ for $i = 1, \ldots, n$ have a multiple root for $t = 0$, hence either H is a multiple point of $W(S)$ or $W(S)$ is a curve and S is Schwarzenberger by Corollary 5.5.

Corollary 6.6. *With the notation of* (3.4) *if* H *is a multiple point of* $W(S)$ *then* $H \in W(S')$.

Proof. By Theorem 6.5 the matrix A representing S has the form (3.5) where A' has the same form.

Corollary 6.7. *If* $S \in \mathscr{S}_{n,k}$ *is not stable for the action of* $\mathrm{SL}(V)$ *then* $S \in \mathscr{S}^2_{n,k}$.

Proof. This follows from Theorem 6.4 and Proposition 6.5.

Remark. We conjecture that if $S \in \mathscr{S}_{n,k}$ $(k \geqslant 3, (n,k) \neq (2,3))$ is not stable for the action of $\mathrm{SL}(V)$ then $S \in S^3_{n,k}$ and moreover S is Schwarzenberger or $W(S)$ has a point of multiplicity at least 3. We can prove that S is Schwarzenberger or, in the notation of (3.8), $Z = P(W) \cap (P(V) \times P(I))$ has a point of multiplicity at least 3.

Theorem 6.8. *Let* $S \in \mathscr{S}_{n,k}$ *be a Steiner bundle. The following two conditions are equivalent*:

i) $\mathrm{Sym}(S) \supset \mathbb{C}^*$;

ii) *there is a coordinate system such that the matrix of S has the diagonal form*

$$\begin{bmatrix} a_{0,1}x_0 & \dots & a_{n,1}x_n & & \\ & \ddots & & \ddots & \\ & & a_{0,k}x_0 & \dots & a_{n,k}x_n \end{bmatrix}.$$

Proof. This is a reformulation of Theorem 2.5 in the case $p = 2$.

Corollary 6.9. *Let* $S \in \mathscr{S}_{n,k}$ *be a Steiner bundle such that* $\mathrm{Sym}(S) \supset \mathbb{C}^*$. *Then the* $\mathbb{C}^*$-*action on* $\mathbb{P}^n$ *has exactly* $n+1$ *fixed points whose weights are proportional to* $-n, -n+2, \dots, n-2, n$.

Proof. The statement follows from Definition 2.2.

Corollary 6.10. *Let* $S \in \mathscr{S}_{n,k}$ *be a Steiner bundle such that* $\mathrm{Sym}(S) \supset \mathbb{C}^*$. *Then either* $W(S)$ *is a rational normal curve and S is a Schwarzenberger bundle, or* $W(S)$ *has only two closed points, namely the two fixed points of the dual* $\mathbb{C}^*$-*action on* $\mathbb{P}^{n\vee}$ *having minimum and maximum weights.*

Proof. If S is not Schwarzenberger, $W(S)$ is finite (by Corollary 5.5); since it is $\mathrm{Sym}(S)$-invariant, it must be contained in the $n+1$ fixed points of the $\mathbb{C}^*$-action on $\mathbb{P}^{n\vee}$. It is now easy to check, with the notations of (3.3), that the equation

$$\begin{bmatrix} a_{0,1}x_0 & \dots & a_{n,1}x_n & & \\ & \ddots & & \ddots & \\ & & a_{0,k}x_0 & \dots & a_{n,k}x_n \end{bmatrix} \cdot w_1 = i_1 \cdot x_j$$

has nonzero solutions only for $j = 0, n$.

Proposition 6.11. *A logarithmic bundle in $\mathcal{S}_{n,k}$ which is not stable for the action of* $\mathrm{SL}(n+1)$ *is Schwarzenberger.*

Proof. The proof is by induction on k. For $k = 2$ the result is true by Theorem 5.2. By Theorem 6.4 there exists a triangular matrix corresponding to S. Then $H = \{x_0 = 0\}$ is an unstable hyperplane of S. By (3.12) there is an elementary transformation

$$0 \to S' \to S \to \mathcal{O}_H \to 0$$

where also S' is logarithmic (by Theorem 3.13 and Corollary 5.10). Moreover the matrix representing S' is again triangular by (3.5). S' is not stable by Theorem 6.4. By induction S' is Schwarzenberger and $W(S') = C_n$ is a rational normal curve. For every $K \in W(S)$, $K \neq H$, we have $K \in W(S') = C_n$ by Theorem 3.13. The crucial point is that in this case also $H \in W(S') = C_n$; this can be checked by looking at the matrix of S'. Hence every closed point of $W(S)$ lies in C_n and by Theorem 5.3, S is isomorphic to the Schwarzenberger bundle determined by C_n.

Lemma 6.12. *Let U be a 2-dimensional vector space, and $C_n \simeq \mathbb{P}(U) \to \mathbb{P}(S^n U)$ be the* $\mathrm{SL}(U)$*-equivariant embedding (whose image is a rational normal curve). Let $\mathbb{C}^* \subset \mathrm{SL}(U)$ act on $\mathbb{P}(S^n U)$. We label the $n+1$ fixed points $P_i, i = -n + 2j, j = 0, \dots, n$ of the $\mathbb{C}^*$-action with an index proportional to their weights. Then P_{-n}, P_n lie on C_n and $P_{-n+2j} = T^j P_{-n} \cap T^{n-j} P_n$, where T^j denotes the j-dimensional osculating space to C_n.*

Proof. We choose a coordinate system which diagonalizes the $\mathbb{C}^*$-action. Then the result follows by a straightforward computation.

Lemma 6.13. *Let $S \in \mathcal{S}_{n,k}$ be a Steiner bundle. Let* $\mathrm{Sym}(S)^0$ *be the connected component containing the identity of* $\mathrm{Sym}(S)$*. If there are two different one-parameter subgroups $\lambda_1, \lambda_2 : \mathbb{C}^* \to$* $\mathrm{Sym}(S)$ *then S is Schwarzenberger.*

Proof. The proof is by induction on k. If $k = 2$ the theorem is true by Theorem 5.2. By applying Theorem 6.8 to λ_1 we may suppose that the matrix representing S is diagonal, and that $H = \{x_0 = 0\}$ is the fixed point with minimum weight of the dual action λ_1^* on $\mathbb{P}^{n\vee}$. By (3.12) there is an elementary transformation

$$0 \to S' \to S \to \mathcal{O}_H \to 0$$

where the matrix of S' is also diagonal ((3.5)), so that λ_1 is a one-parameter subgroup of $\mathrm{Sym}(S')$. Let us suppose by contradiction that S is not Schwarzenberger; by Corollary 6.10 we find that H is also the fixed point with minimum weight of the dual λ_2^* (replacing λ_2 with λ_2^{-1} if necessary). Hence by the same argument also λ_2 is a one-parameter subgroup of $\mathrm{Sym}(S')$, so that S' is Schwarzenberger by the inductive assumption. It follows that λ_1 and λ_2 are contained in the same $\mathrm{SL}(2) = \mathrm{Sym}(S')$ and have the same two fixed points with minimum and maximum weight. By Lemma

6.12, λ_1 and λ_2 have the same fixed points and have also the same image in $\mathrm{SL}(n+1)$. This is a contradiction.

Proof of Theorem 2.6. In view of Theorem 5.9, Theorem 2.6 is equivalent to the following (the equivalence will be clear from the proof).

Theorem 6.14. *Let $S \in \mathscr{S}_{n,k}$ be a Steiner bundle. Let* $\mathrm{Sym}(S)^0$ *be the connected component containing the identity of* $\mathrm{Sym}(S)$. *Then there is a* 2-*dimensional vector space U such that* $\mathrm{SL}(U)$ *acts over* $\mathbb{P}^n = \mathbb{P}(S^n U)$ *and according to this action* $\mathrm{Sym}(S)^0 \subset \mathrm{SL}(U)$. *Moreover*

$$\mathrm{Sym}(S)^0 \simeq \begin{cases} 0 \\ \mathbb{C} \\ \mathbb{C}^* \\ \mathrm{SL}(2) \quad (\textit{this case occurs if and only if } S \textit{ is Schwarzenberger}). \end{cases}$$

We prove this theorem. The proof is by induction on k. If $k=2$ the theorem is true by Theorem 5.2. We may suppose that $G = \mathrm{Sym}(S)^0$ has dimension $\geqslant 1$. By Theorem 2.4 the matrix A representing S is triangulable. By the Proposition 3.11, $W(S)$ is not empty and we pick up $H \in W(S)$. By Corollary 5.5 we may suppose that $W(S)$ is finite, hence H is G-invariant. We repeat the argument of the proof of Theorem 5.9. We get for all $g \in G$ the diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & \mathcal{O}_H \\ \downarrow{\scriptstyle i} & & \\ g^*S & \xrightarrow{g^*\phi} & \mathcal{O}_H. \end{array}$$

Since $h^0(S^*_{|H}) \leqslant 1$ we obtain that ϕ and $g^*\phi \circ i$ coincide up to a scalar multiple. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & g^*S' & \longrightarrow & g^*S & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0. \end{array}$$

It follows that $S' \simeq g^*S'$, hence $G \subset \mathrm{Sym}(S')$ and by the inductive assumption $G \subset \mathrm{SL}(U)$ and $\mathrm{SL}(U)$ acts over $\mathbb{P}^n = \mathbb{P}(S^n U)$. We remark that the above considered elementary transformation gives the decompositions $W = W' \oplus \mathbb{C}$, $I = I' \oplus \mathbb{C}$ such that the inclusion $\mathrm{Hom}(W', V \otimes I') \subset \mathrm{Hom}(W, V \otimes I)$ identifies with the $\mathrm{SL}(U)$-invariant inclusion $S^{n+k-2}U \otimes S^n U \otimes S^{k-2}U \subset S^{n+k-1}U \otimes S^n U \otimes S^{k-1}U$ according to the natural actions. In fact no other morphism of $\mathrm{SL}(U)$ in $\mathrm{SL}(W) \times \mathrm{SL}(S^n U) \times \mathrm{SL}(I)$ can give $S^{n+k-2}U \otimes S^n U \otimes S^{k-2}U$ as an invariant summand of $W \otimes S^n U \otimes I$. Now consider the Levi decomposition $G = R \cdot M$ where R is the radical and M is maximal semisimple. If S is not Schwarzenbeger we have $M = 0$ and G is solvable. By the Lie theorem G is contained (after a convenient basis has been fixed) in the subgroup $T = \left\{ \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} \middle| a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$. If there is a subgroup $\mathbb{C}^*$ properly contained in T then there is a conjugate of $\mathbb{C}^*$ different from itself and this is a contradiction by Lemma 6.13. If there is no subgroup $\mathbb{C}^*$ contained in T then G is isomorphic to $\mathbb{C} \simeq \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| b \in \mathbb{C} \right\}$.

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