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The connected components of the projective line over a ring

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Abstract. The main result of the present paper is that the projective line over a ring R is connected with respect to the relation "distant" if, and only if, R is a GE_2-ring.

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1 Introduction

One of the basic notions for the projective line $\mathbb{P}(R)$ over a ring R is the relation *distant* ($\triangle$) on the point set. Non-distant points are also called *parallel*. This terminology goes back to the projective line over the real dual numbers, where parallel points represent parallel spears of the Euclidean plane [4, 2.4].

We say that $\mathbb{P}(R)$ is *connected* (with respect to $\triangle$) if the following holds: For any two points p and q there is a finite sequence of points starting at p and ending at q such that each point other than p is distant from its predecessor. Otherwise $\mathbb{P}(R)$ is said to be *disconnected*. For each *connected component* a *distance function* and a *diameter* (with respect to $\triangle$) can be defined in a natural way.

One aim of the present paper is to characterize those rings R for which $\mathbb{P}(R)$ is connected. Here we use certain subgroups of the group $\mathrm{GL}_2(R)$ of invertible 2×2-matrices over R, namely its *elementary subgroup* $\mathrm{E}_2(R)$ and the subgroup $\mathrm{GE}_2(R)$ generated by $\mathrm{E}_2(R)$ and the set of all invertible diagonal matrices. It turns out that $\mathbb{P}(R)$ is connected exactly if R is a GE_2-ring, i.e., if $\mathrm{GE}_2(R) = \mathrm{GL}_2(R)$.

Next we turn to the diameter of connected components. We show that all connected components of $\mathbb{P}(R)$ share a common diameter.

It is well known that $\mathbb{P}(R)$ is connected with diameter $\leqslant 2$ if R is a ring of stable rank 2. We give explicit examples of rings R such that $\mathbb{P}(R)$ has one of the following properties: $\mathbb{P}(R)$ is connected with diameter 3, $\mathbb{P}(R)$ is connected with diameter ∞, and $\mathbb{P}(R)$ is disconnected with diameter ∞. In particular, we show that there are *chain geometries* over disconnected projective lines.

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2 Preliminaries

Throughout this paper we shall only consider associative rings with a unit element 1, which is inherited by subrings and acts unitally on modules. The trivial case $1 = 0$ is not excluded. The group of invertible elements of a ring R will be denoted by R^*.

Firstly, we turn to the projective line over a ring: Consider the free left R-module R^2. Its automorphism group is the group $\mathrm{GL}_2(R)$ of invertible 2×2-matrices with entries in R. A pair $(a, b) \in R^2$ is called *admissible*, if there exists a matrix in $\mathrm{GL}_2(R)$ with (a, b) being its first row. Following [14, p. 785], the *projective line over* R is the orbit of the free cyclic submodule $R(1, 0)$ under the action of $\mathrm{GL}_2(R)$. So

$$\mathbb{P}(R) := R(1,0)^{\mathrm{GL}_2(R)}$$

or, in other words, $\mathbb{P}(R)$ is the set of all $p \leqslant R^2$ such that $p = R(a, b)$ for an admissible pair $(a, b) \in R^2$. As has been pointed out in [8, Proposition 2.1], in certain cases $R(x, y) \in \mathbb{P}(R)$ does not imply the admissibility of $(x, y) \in R^2$. However, throughout this paper we adopt the convention that points are represented by admissible pairs only. Two such pairs represent the same point exactly if they are left-proportional by a unit in R.

The point set $\mathbb{P}(R)$ is endowed with the symmetric relation *distant* ($\triangle$) defined by

$$\triangle := (R(1,0), R(0,1))^{\mathrm{GL}_2(R)}. \tag{1}$$

Letting $p = R(a, b)$ and $q = R(c, d)$ gives then

$$p \triangle q \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

In addition, $\triangle$ is anti-reflexive exactly if $1 \neq 0$.

The vertices of the *distant graph* on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, the edges of this graph are the unordered pairs of distant points. Therefore basic graph-theoretical concepts are at hand: $\mathbb{P}(R)$ can be decomposed into *connected components* (maximal connected subsets), for each connected component there is a *distance function* ($\mathrm{dist}(p, q)$ is the minimal number of edges needed to go from vertex p to vertex q), and each connected component has a *diameter* (the supremum of all distances between its points).

Secondly, we recall that the set of all *elementary matrices*

$$B_{12}(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{21}(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{with } t \in R \tag{2}$$

generates the *elementary subgroup* $\mathrm{E}_2(R)$ of $\mathrm{GL}_2(R)$. The group $\mathrm{E}_2(R)$ is also generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} = B_{12}(1) \cdot B_{21}(-1) \cdot B_{12}(1) \cdot B_{21}(t) \quad \text{with } t \in R, \tag{3}$$

since $B_{12}(t) = E(-t) \cdot E(0)^{-1}$ and $B_{21}(t) = E(0)^{-1} \cdot E(t)$. Moreover, we have $E(t)^{-1} = E(0) \cdot E(-t) \cdot E(0)$, which implies that all finite products of matrices $E(t)$ already comprise the group $\mathrm{E}_2(R)$.

The subgroup of $\mathrm{GL}_2(R)$ which is generated by $\mathrm{E}_2(R)$ and the set of all invertible diagonal matrices is denoted by $\mathrm{GE}_2(R)$. By definition, a GE_2-*ring* is characterized by $\mathrm{GL}_2(R) = \mathrm{GE}_2(R)$; see, among others, [10, p. 5] or [18, p. 114].

3 Connected components

We aim at a description of the connected components of the projective line $\mathbb{P}(R)$ over a ring R. The following lemma, although more or less trivial, will turn out useful:

Lemma 3.1. *Let $X' \in \mathrm{GL}_2(R)$ and suppose that the 2×2-matrix X over R has the same first row as X'. Then X is invertible if, and only if, there is a matrix*

$$M = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} \in \mathrm{GE}_2(R) \tag{4}$$

such that $X = MX'$.

Proof. Given X' and X then $XX'^{-1} = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} =: M$ for some $s, u \in R$. Further, $X = MX'$ is invertible exactly if $u \in R^*$. This in turn is equivalent to (4).

Here is our main result, where we use the generating matrices of $\mathrm{E}_2(R)$ introduced in (3).

Theorem 3.2. *Denote by C_∞ the connected component of the point $R(1,0)$ in the projective line $\mathbb{P}(R)$ over a ring R. Then the following holds*:

(a) *The group $\mathrm{GL}_2(R)$ acts transitively on the set of connected components of $\mathbb{P}(R)$.*

(b) *Let $t_1, t_2, \ldots, t_n \in R$, $n \geqslant 0$, and put*

$$(x, y) := (1,0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1). \tag{5}$$

Then $R(x, y) \in C_\infty$ and, conversely, each point $r \in C_\infty$ can be written in this way.

(c) *The stabilizer of C_∞ in $\mathrm{GL}_2(R)$ is the group $\mathrm{GE}_2(R)$.*

(d) *The projective line $\mathbb{P}(R)$ is connected if, and only if, R is a GE_2-ring.*

Proof. (a) This is immediate from the fact that the group $\mathrm{GL}_2(R)$ acts transitively on the point set $\mathbb{P}(R)$ and preserves the relation $\triangle$.

(b) Every matrix $E(t_i)$ appearing in (5) maps C_∞ onto C_∞, since $R(0,1) \in C_\infty$ goes over to $R(1,0) \in C_\infty$. Therefore $R(x, y) \in C_\infty$.

On the other hand let $r \in C_\infty$. Then there exists a sequence of points $p_i = R(a_i, b_i) \in \mathbb{P}(R)$, $i \in \{0, 1, \dots, n\}$, such that

$$R(1,0) = p_0 \mathbin{\triangle} p_1 \mathbin{\triangle} \cdots \mathbin{\triangle} p_n = r. \tag{6}$$

Now the arbitrarily chosen admissible pairs (a_i, b_i) are "normalized" recursively as follows: First define $(x_{-1}, y_{-1}) := (0, -1)$ and $(x_0, y_0) := (1, 0)$. So $p_0 = R(x_0, y_0)$. Next assume that we already are given admissible pairs (x_j, y_j) with $p_j = R(x_j, y_j)$ for all $j \in \{0, 1, \dots, i-1\}$, $1 \leqslant i \leqslant n$. From Lemma 3.1, there are $s_i \in R$ and $u_i \in R^*$ such that

$$\begin{pmatrix} x_{i-1} & y_{i-1} \\ a_i & b_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_i & u_i \end{pmatrix} \begin{pmatrix} x_{i-1} & y_{i-1} \\ -x_{i-2} & -y_{i-2} \end{pmatrix}. \tag{7}$$

By putting $x_i := u_i^{-1} a_i$, $y_i := u_i^{-1} b_i$, and $t_i := u_i^{-1} s_i$ we get

$$\begin{pmatrix} x_i & y_i \\ -x_{i-1} & -y_{i-1} \end{pmatrix} = E(t_i) \cdot \begin{pmatrix} x_{i-1} & y_{i-1} \\ -x_{i-2} & -y_{i-2} \end{pmatrix} \tag{8}$$

and $p_i = R(x_i, y_i)$. Therefore, finally, (x_n, y_n) is the first row of the matrix

$$G' := E(t_n) \cdot E(t_{n-1}) \cdots E(t_1) \in E_2(R), \tag{9}$$

and $r = R(x_n, y_n)$.

(c) As has been noticed at the end of Section 2, the set of all matrices (3) generates $\mathrm{E}_2(R)$. This together with (b) implies that $\mathrm{E}_2(R)$ stabilizes C_∞. Further, $R(1,0)$ remains fixed under each invertible diagonal matrix. Therefore $\mathrm{GE}_2(R)$ is contained in the stabilizer of C_∞.

Conversely, suppose that $G \in \mathrm{GL}_2(R)$ stabilizes C_∞. Then the first row of G, say (a, b), determines a point of C_∞. By (5) and (9), there is a matrix $G' \in \mathrm{E}_2(R)$ and a unit $u \in R^*$ such that $(a, b) = (1, 0) \cdot (uG')$. Now Lemma 3.1 can be applied to G and $uG' \in \mathrm{GE}_2(R)$ in order to establish that $G \in \mathrm{GE}_2(R)$.

(d) This follows from (a) and (c).

From Theorem 3.2 and (9), the connected component of $R(1,0) \in \mathbb{P}(R)$ is given by all pairs of $(1,0) \cdot \mathrm{E}_2(R)$ or, equivalently, by all pairs of $(1,0) \cdot \mathrm{GE}_2(R)$. Each product (5) gives rise to a sequence

$$(x_i, y_i) = (1, 0) \cdot E(t_i) \cdot E(t_{i-1}) \cdots E(t_1), \ i \in \{0, 1, \dots, n\}, \tag{10}$$

which in turn defines a sequence $p_i := R(x_i, y_i)$ of points with $p_0 = R(1, 0)$. By putting $p_n =: r$ and by reversing the arguments in the proof of (b), it follows that (6) is true. So, if the diameter of C_∞ is finite, say $m \geqslant 0$, then in order to reach all points of C_∞ it is sufficient that n ranges from 0 to m in formula (5).

By the action of $\mathrm{GL}_2(R)$, the connected component C_p of any point $p \in \mathbb{P}(R)$ is $\mathrm{GL}_2(R)$-equivalent to the connected component C_∞ of $R(1,0)$ and the stabilizer of C_p in $\mathrm{GL}_2(R)$ is conjugate to $\mathrm{GE}_2(R)$. Observe that in general $\mathrm{GE}_2(R)$ is not normal in $\mathrm{GL}_2(R)$. Cf. the example in 5.7 (c). All connected components are isomorphic subgraphs of the distant graph.

4 Generalized chain geometries

If $K \subset R$ is a (not necessarily commutative) subfield, then the K-sublines of $\mathbb{P}(R)$ give rise to a *generalized chain geometry* $\Sigma(K,R)$; see [7]. In contrast to an ordinary chain geometry (cf. [14]) it is not assumed that K is in the centre of R. Any three mutually distant points are on at least one K-chain. Two distinct points are distant exactly if they are on a common K-chain. Therefore each K-chain is contained in a unique connected component. Each connected component C together with the set of K-chains entirely contained in it defines an incidence structure $\Sigma(C)$. It is straightforward to show that the automorphism group of the incidence structure $\Sigma(K,R)$ is isomorphic to the wreath product of $\operatorname{Aut}\Sigma(C)$ with the symmetric group on the set of all connected components of $\mathbb{P}(R)$.

If $\Sigma(K,R)$ is a chain geometry then the connected components are exactly the *maximal connected subspaces* of $\Sigma(K,R)$ [14, p. 793, p. 821]. Cf. also [15] and [16].

An R-semilinear bijection of R^2 induces an automorphism of $\Sigma(K,R)$ if, and only if, the accompanying automorphism of R takes K to $u^{-1}Ku$ for some $u \in R^*$. On the other hand, if $\mathbb{P}(R)$ is disconnected then we cannot expect all automorphisms of $\Sigma(K,R)$ to be semilinearly induced. More precisely, we have the following:

Theorem 4.1. *Let* $\Sigma(K,R)$ *be a* disconnected *generalized chain geometry, i.e., the projective line* $\mathbb{P}(R)$ *over* R *is disconnected. Then* $\Sigma(K,R)$ *admits automorphisms that cannot be induced by any semilinear bijection of* R^2.

Proof. (a) Suppose that two semilinearly induced bijections γ_1, γ_2 of $\mathbb{P}(R)$ coincide on all points of one connected component C of $\mathbb{P}(R)$. We claim that $\gamma_1 = \gamma_2$. For a proof choose two distant points $R(a,b)$ and $R(c,d)$ in C. Also, write α for that projectivity which is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\beta := \alpha\gamma_1\gamma_2^{-1}\alpha^{-1}$ is a semilinearly induced bijection of $\mathbb{P}(R)$ fixing the connected component C_∞ of $R(1,0)$ pointwise. Hence $R(1,0)$, $R(0,1)$, and $R(1,1)$ are invariant under β, and we get

$$R(x,y)^\beta = R(x^\zeta u, y^\zeta u) \quad \text{for all } (x,y) \in R^2$$

with $\zeta \in \operatorname{Aut}(R)$ and $u \in R^*$, say. For all $x \in R$ the point $R(x,1)$ is distant from $R(1,0)$; so it remains fixed under β. Therefore $x = u^{-1}x^\zeta u$ or, equivalently, $x^\zeta u = ux$ for all $x \in R$. Finally, $R(x,y)^\beta = R(ux,uy) = R(x,y)$ for all $(x,y) \in R^2$, whence $\gamma_1 = \gamma_2$.

(b) Let γ be a non-identical projectivity of $\mathbb{P}(R)$ given by a matrix $G \in \mathrm{GE}_2(R)$, for example, $G = B_{12}(1)$. From Theorem 3.2, the connected component C_∞ of $R(1,0)$ is invariant under γ. Then

$$\delta : \mathbb{P}(R) \to \mathbb{P}(R) : \begin{cases} p \mapsto p^\gamma & \text{for all } p \in C_\infty \\ p \mapsto p & \text{for all } p \in \mathbb{P}(R) \setminus C_\infty \end{cases} \tag{11}$$

is an automorphism of $\Sigma(K, R)$. The projectivity γ and the identity on $\mathbb{P}(R)$ are different and both are linearly induced. The mapping δ coincides with γ on C_∞ and with the identity on every other connected component. There are at least two distinct connected components of $\mathbb{P}(R)$. Hence it follows from (a) that δ cannot be semilinearly induced.

If a cross-ratio in $\mathbb{P}(R)$ is defined according to [14, 1.3.5] then four points with cross-ratio are necessarily in a common connected component. Therefore, the automorphism δ defined in (11) preserves all cross-ratios. However, cross-ratios are not invariant under δ if one adopts the definition in [4, p. 90] or [14, 7.1] which works for commutative rings only. This is due to the fact that here four points with cross-ratio can be in two distinct connected components.

We shall give examples of disconnected (generalized) chain geometries in the next section.

5 Examples

There is a widespread literature on (non-)GE_2-rings. We refer to [1], [9], [10], [11], [12], [13], and [18]. We are particularly interested in rings containing a field and the corresponding generalized chain geometries.

Remark 5.1. Let R be a ring. Then each admissible pair $(x, y) \in R^2$ is *unimodular*, i.e., there exist $x', y' \in R$ with $xx' + yy' = 1$. We remark that

$$(x, y) \in R^2 \text{ unimodular} \Rightarrow (x, y) \text{ admissible} \tag{12}$$

is satisfied, in particular, for all *commutative* rings, since $xx' + yy' = 1$ can be interpreted as the determinant of an invertible matrix with first row (x, y). Also, all rings of *stable rank* 2 [19, p. 293] satisfy (12); cf. [19, 2.11]. For example, local rings, matrix rings over fields, and finite-dimensional algebras over commutative fields are of stable rank 2. See [13, 4.1B], [19, §2], [20], and the references given there.

The following example shows that (12) does not hold for all rings: Let $R := K[X, Y]$ be the polynomial ring over a proper skew field K in independent central indeterminates X and Y. There are $a, b \in K$ with $c := ab - ba \neq 0$. From

$$(X + a)(Y + b)c^{-1} - (Y + b)(X + a)c^{-1} = 1,$$

the pair $(X + a, -(Y + b)) \in R^2$ is unimodular. However, this pair is not admissible: Assume to the contrary that $(X + a, -(Y + b))$ is the first row of a matrix

$M \in \mathrm{GL}_2(R)$ and suppose that the second column of M^{-1} is the transpose of $(v_0, w_0) \in R^2$. Then

$$P := \{(v, w) \in R^2 \mid (X + a)v - (Y + b)w = 0\} = (v_0, w_0)R.$$

On the other hand, by [17, Proposition 1], the right R-module P cannot be generated by a single element, which is a contradiction.

Examples 5.2. (a) If R is a ring of stable rank 2 then $\mathbb{P}(R)$ is connected and its diameter is $\leqslant 2$ [14, Proposition 1.4.2]. In particular, the diameter is 1 exactly if R is a field and it is 0 exactly if $R = \{0\}$.

As has been pointed out in [2, (2.1)], the points of the projective line over a ring R of stable rank 2 are exactly the submodules $R(t_2 t_1 + 1, t_2)$ of R^2 with $t_1, t_2 \in R$. Clearly, this is just a particular case of our more general result in Theorem 3.2 (b).

Conversely, if an arbitrary ring R satisfies (12) and $\mathbb{P}(R)$ is connected with diameter $\leqslant 2$, then R is a ring of stable rank 2 [14, Proposition 1.1.3].

(b) The projective line over a (not necessarily commutative) Euclidean ring R is connected, since every Euclidean ring is a GE_2-ring [13, Theorem 1.2.10].

Our next examples are given in the following theorem:

Theorem 5.3. *Let U be an infinite-dimensional vector space over a field K and put $R :=$* $\mathrm{End}_K(U)$. *Then the projective line $\mathbb{P}(R)$ over R is connected and has diameter* 3.

Proof. We put $V := U \times U$ and denote by $\mathscr{G}$ those subspaces W of V that are isomorphic to V/W. By [5, 2.4], the mapping

$$\Phi \;:\; \mathbb{P}(R) \to \mathscr{G} \;:\; R(\alpha, \beta) \mapsto \{(u^\alpha, u^\beta) \mid u \in U\} \tag{13}$$

is bijective and two points of $\mathbb{P}(R)$ are distant exactly if their Φ-images are complementary. By an abuse of notation, we shall write $\mathrm{dist}(W_1, W_2) = n$, whenever W_1, W_2 are Φ-images of points at distance n, and $W_1 \bigtriangleup W_2$ to denote complementary elements of $\mathscr{G}$. As V is infinite-dimensional, $2 \dim W = \dim V = \dim W$ for all $W \in \mathscr{G}$.

We are going to verify the theorem in terms of $\mathscr{G}$: So let $W_1, W_2 \in \mathscr{G}$. Put $Y_{12} := W_1 \cap W_2$ and choose $Y_{23} \leqslant W_2$ such that $W_2 = Y_{12} \oplus Y_{23}$. Then $W_1 \cap Y_{23} = \{0\}$ so that there is a $W_3 \in \mathscr{G}$ through Y_{23} with $W_1 \bigtriangleup W_3$. By the law of modularity,

$$W_2 \cap W_3 = (Y_{23} + Y_{12}) \cap W_3 = Y_{23} + (Y_{12} \cap W_3) = Y_{23}.$$

Finally, choose $Y_{14} \leqslant W_1$ with $W_1 = Y_{12} \oplus Y_{14}$ and $Y_{34} \leqslant W_3$ with $W_3 = Y_{23} \oplus Y_{34}$. Hence we arrive at the decomposition

$$V = Y_{14} \oplus Y_{12} \oplus Y_{23} \oplus Y_{34}. \tag{14}$$

As $W_2 \in \mathscr{G}$, so is also $W_4 := Y_{14} \oplus Y_{34}$. Now there are two possibilities:

Case 1: There exists a linear bijection $\sigma : Y_{14} \to Y_{23}$. We define $Y := \{v + v^\sigma \mid v \in Y_{14}\}$. Then Y_{14}, Y_{23}, and Y are easily seen to be mutually complementary subspaces of $Y_{14} \oplus Y_{23}$. Therefore, from (14),

$$V = Y_{14} \oplus Y_{12} \oplus Y \oplus Y_{34} = Y \oplus Y_{12} \oplus Y_{23} \oplus Y_{34}, \tag{15}$$

i.e., $W_1 \triangle (Y \oplus Y_{34}) \triangle W_2$. So $\operatorname{dist}(W_1, W_2) \leqslant 2$.

Case 2: Y_{14} and Y_{23} are not isomorphic. Then $\dim Y_{12} = \dim W_1$, since otherwise $\dim Y_{12} < \dim W_1 = \dim W_2$ together with well-known rules for the addition of infinite cardinal numbers would imply

$$\dim W_1 = \max\{\dim Y_{12}, \dim Y_{14}\} = \dim Y_{14},$$

$$\dim W_2 = \max\{\dim Y_{12}, \dim Y_{23}\} = \dim Y_{23},$$

a contradiction to $\dim Y_{14} \neq \dim Y_{23}$.

Likewise, it follows that $\dim Y_{34} = \dim W_3$. But this means that Y_{12} and Y_{34} are isomorphic, whence the proof in case 1 can be modified accordingly to obtain a $Y \leqslant Y_{12} \oplus Y_{34}$ such that $W_1 \triangle W_3 \triangle (Y \oplus Y_{14}) \triangle W_2$. So now $\operatorname{dist}(W_1, W_2) \leqslant 3$.

It remains to establish that in $\mathcal{G}$ there are elements with distance 3: Choose any subspace $W_1 \in \mathcal{G}$ and a subspace $W_2 \leqslant W_1$ such that W_1/W_2 is 1-dimensional. With the previously introduced notations we get $Y_{12} = W_2$, $\dim Y_{14} = 1$, $Y_{23} = \{0\}$, $Y_{34} = W_3 \in \mathcal{G}$, and $W_4 = Y_{14} \oplus W_3$. As before, $V = W_2 \oplus W_4$ and from $\dim W_2 = 1 + \dim W_2 = \dim W_1 = \dim W_3 = 1 + \dim W_3 = \dim W_4$ we obtain $W_2, W_4 \in \mathcal{G}$. By construction, $\operatorname{dist}(W_1, W_2) \neq 0, 1$. Also, this distance cannot be 2, since $W \triangle W_1$ implies $W + W_2 \neq V$ for all $W \in \mathcal{G}$.

This completes the proof.

If K is a proper skew field, then K can be embedded in $\operatorname{End}_K(U)$ in several ways [6, p. 17]; each embedding gives rise to a connected generalized chain geometry. (In [6] this is just called a "chain geometry".) If K is commutative, then $\operatorname{End}_K(U)$ is a K-algebra and $x \mapsto x\,\mathrm{id}_U$ is a distinguished embedding of K into the centre of $\operatorname{End}_K(U)$. In this way an ordinary connected chain geometry arises; cf. [14, 4.5. Example (4)].

Our next goal is to show the existence of chain geometries with connected components of infinite diameter.

Remark 5.4. If R is an arbitrary ring then each matrix $A \in \mathrm{GE}_2(R)$ can be expressed in *standard form*

$$A = \operatorname{diag}(u, v) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1), \tag{16}$$

where $u, v \in R^*$, $t_1, t_n \in R$, $t_2, t_3, \ldots, t_{n-1} \in R \backslash (R^* \cup \{0\})$, and $t_1, t_2 \neq 0$ in case $n = 2$ [10, Theorem (2.2)]. Since $E(0)^2 = \operatorname{diag}(-1, -1)$, each matrix $A \in \mathrm{GE}_2(R)$ can also be written in the form (16) subject to the slightly modified conditions $u, v \in R^*$, $t_1, t_n \in R$, $t_2, t_3, \ldots, t_{n-1} \in R \backslash (R^* \cup \{0\})$, and $n \geqslant 1$. We call this a *modified standard form* of A.

Suppose that there is a unique standard form for $\mathrm{GE}_2(R)$. For all non-diagonal matrices in $\mathrm{GE}_2(R)$ the unique representation in standard form is at the same time the unique representation in modified standard form. Any diagonal matrix $A \in \mathrm{GE}_2(R)$ is already expressed in standard form, but its unique modified standard form reads $A = -A \cdot E(0)^2$. Therefore there is also a unique modified standard form for $\mathrm{GE}_2(R)$.

By reversing these arguments it follows that the existence of a unique modified standard form for $\mathrm{GE}_2(R)$ is equivalent to the existence of a unique standard form for $\mathrm{GE}_2(R)$.

Theorem 5.5. *Let R be a ring with a unique standard form for* $\mathrm{GE}_2(R)$ *and suppose that R is not a field. Then every connected component of the projective line* $\mathbb{P}(R)$ *over R has infinite diameter.*

Proof. Since R is not a field, there exists an element $t \in R \backslash (R^* \cup \{0\})$. We put

$$q_m := R(c_m, d_m) \text{ where } (c_m, d_m) := (1, 0) \cdot E(t)^m \text{ for all } m \in \{0, 1, \dots\}. \tag{17}$$

Next fix one $m \geqslant 1$, and put $n - 1 := \mathrm{dist}(q_0, q_{m-1}) \geqslant 0$. Hence there exists a sequence

$$p_0 \vartriangle p_1 \vartriangle \cdots \vartriangle p_{n-1} \vartriangle p_n \tag{18}$$

such that $p_0 = q_0$, $p_{n-1} = q_{m-1}$, and $p_n = q_m$. Now we proceed as in the proof of Theorem 3.2 (b): First let $p_i = R(a_i, b_i)$ and put $(x_{-1}, y_{-1}) := (0, -1)$, $(x_0, y_0) := (1, 0)$. Then pairs $(x_i, y_i) \in R^2$ and matrices $E(t_i) \in \mathrm{E}_2(R)$ are defined in such a way that $p_i = R(x_i, y_i)$ and that (8) holds for $i \in \{1, 2 \dots, n\}$. It is immediate from (8) that a point p_i, $i \geqslant 2$, is distant from p_{i-2} exactly if $t_i \in R^*$. Also, $p_i = p_{i-2}$ holds if, and only if, $t_i = 0$. We infer from (8) and $\mathrm{dist}(p_i, p_j) = |i - j|$ for all $i, j \in \{0, 1, \dots, n-1\}$ that

$$\begin{pmatrix} x_n & y_n \\ -x_{n-1} & -y_{n-1} \end{pmatrix} = E(t_n) \cdot E(t_{n-1}) \cdots E(t_1), \tag{19}$$

where $t_i \in R \backslash (R^* \cup \{0\})$ for all $i \in \{2, 3, \dots, n-1\}$. On the other hand, by (17) and $(c_{m-1}, d_{m-1}) = (0, -1) \cdot E(t)^m$, there are $v, v' \in R^*$ with

$$\begin{pmatrix} x_n & y_n \\ -x_{n-1} & -y_{n-1} \end{pmatrix} = \mathrm{diag}(v, v') \cdot E(t)^m. \tag{20}$$

From Remark 5.4, the modified standard forms (19) and (20) are identical. Therefore, $n = m$, $\mathrm{dist}(q_0, q_{m-1}) = m - 1$, and the diameter of the connected component of q_0 is infinite.

By Theorem 3.2 (a), all connected components of $\mathbb{P}(R)$ have infinite diameter.

Remark 5.6. Let R be a ring such that $R^* \cup \{0\}$ is a field, say K, and suppose that we have a *degree function*, i.e. a function $\deg : R \to \{-\infty\} \cup \{0, 1, \dots\}$ satisfying

$$\begin{aligned}\deg a = -\infty \quad &\text{if, and only if, } a = 0,\\ \deg a = 0 \quad &\text{if, and only if, } a \in R^*,\\ \deg(a+b) &\leqslant \max\{\deg a, \deg b\},\\ \deg(ab) &= \deg(a) + \deg(b),\end{aligned}$$

for all $a, b \in R$. Then, following [10, p. 21], R is called a *K-ring with a degree function*.

If R is a K-ring with a degree function, then there is a unique standard form for $\mathrm{GE}_2(R)$ [10, Theorem (7.1)].

Examples 5.7. (a) Let R be a K-ring with a degree-function such that $R \neq K$. From Remark 5.6 and Theorem 5.5, all connected components of the projective line $\mathbb{P}(R)$ have infinite diameter.

The associated generalized chain geometry $\Sigma(K, R)$ has a lot of strange properties. For example, *any two* distant points are joined by a unique K-chain. However, we do not enter a detailed discussion here.

(b) Let $K[X]$ be the polynomial ring over a field K in a central indeterminate X. From (a) and Example 5.2 (b), the projective line $\mathbb{P}(K[X])$ is connected and its diameter is infinite. On the other hand, if K is commutative then $K[X]$ has stable rank 3 [20, 2.9]; see also [3, Chapter V, (3.5)]. So there does not seem to be an immediate connection between stable rank and diameter.

(c) Let $R := K[X_1, X_2, \dots, X_m]$ be the polynomial ring over a field K in $m > 1$ independent central indeterminates. Then, by an easy induction and by [10, Proposition (7.3)],

$$A_n := \begin{pmatrix} 1 + X_1X_2 & X_1^2 \\ -X_2^2 & 1 - X_1X_2 \end{pmatrix}^n = \begin{pmatrix} 1 + nX_1X_2 & nX_1^2 \\ -nX_2^2 & 1 - nX_1X_2 \end{pmatrix} \tag{21}$$

is in $\mathrm{GL}_2(R) \backslash \mathrm{GE}_2(R)$ for all $n \in \mathbb{Z}$ that are not divisible by the characteristic of K. Also, the inner automorphism of $\mathrm{GL}_2(R)$ arising from the matrix A_1 takes $B_{12}(1) \in \mathrm{E}_2(R)$ to a matrix that is not even in $\mathrm{GE}_2(R)$; see [18, p. 121–122]. So neither $\mathrm{E}_2(R)$ nor $\mathrm{GE}_2(R)$ is a normal subgroup of $\mathrm{GL}_2(R)$.

We infer that the projective line over R is not connected. Further, it follows from (21) that the number of right cosets of $\mathrm{GE}_2(R)$ in $\mathrm{GL}_2(R)$ is infinite, if the characteristic of K is zero, and $\geqslant \operatorname{char} K$ otherwise. From Theorem 3.2, this number of right cosets is at the same time the number of connected components in $\mathbb{P}(R)$. Even in case of $\operatorname{char} K = 2$ there are at least three connected components, since the index of $\mathrm{GE}_2(R)$ in $\mathrm{GL}_2(R)$ cannot be two. From (a), all connected components of $\mathbb{P}(R)$ have infinite diameter.

So, for each commutative field K, we obtain a disconnected chain geometry $\Sigma(K, R)$, whereas for each skew field K a disconnected generalized chain geometry arises.

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