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EQUALITY CASES IN THE SYMMETRIZATION INEQUALITIES FOR BROWNIAN TRANSITION FUNCTIONS AND DIRICHLET HEAT KERNELS

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Abstract. We prove equality statements for the symmetrization inequalities for Brownian transition functions and Dirichlet heat kernels. The proofs involve the equality statements for the related polarization inequalities which we also prove. These results lead to symmetrization inequalities for Green functions, condenser capacities, and exit times of Brownian motion.

1. Introduction

Let D be a domain in $\mathbf{R}^n, n \geq 2$ and $\{\mathtt{X}_t\}_{t\geq 0}$ be Brownian motion in D. We denote by $P_t^D(x,B)$ the corresponding transition function; that is,

$$P_t^D(x,B) = \mathbf{P}^x(\mathtt{X}_t \in B; t < T^D),$$

where B is a Borel subset of $\mathbf{R}^n$, x is a point in D, $\mathbf{P}^x$ is the probability measure corresponding to Brownian motion starting at x, and T^D is the exit time from D. For fixed D and B, the function $u(t,x) = P_t^D(x,B)$, $x \in D$, $t > 0$, satisfies the heat equation, the initial condition $u(0,x) = \chi_B(x)$, and the boundary condition

$$\lim_{x\to\zeta} u(t,x) = 0, \quad t > 0,$$

for all points $\zeta \in \partial D$ which are regular for the Dirichlet problem in D.

The probability measure $P_t^D(x,\cdot)$ is absolutely continuous with respect to the n-dimensional Lebesgue measure (denoted in the sequel by m_n). The corresponding density (Radon–Nikodym derivative) will be denoted by $p_t^D(x,y)$. This density can be chosen to be a function continuous in t,x,y; it is the heat kernel for D. For more details on transition functions and heat kernels, we refer to [10], [11], [15].

In the present article, we study the behavior of transition functions and heat kernels under symmetrization. For the sake of concreteness we will state and prove symmetrization results only for 1-dimensional Steiner symmetrization. We give here the definition of 1-dimensional Steiner symmetrization. Let H be an $(n-1)$-dimensional hyperplane in $\mathbf{R}^n$. We define the symmetrization $S_H A$ of an open or closed set $A \subset \mathbf{R}^n$ by determining its intersections with every line perpendicular to

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H. Let $\Sigma(x)$ be the line which is perpendicular to H and passes through the point $x \in H$. Let r_x be the 1-dimensional Lebesgue measure of the set $\Sigma(x) \cap A$.

- If $0 < r_x < \infty$, let $(-r_x, r_x)$ be the open linear segment on $\Sigma(x)$ centered at x with length $2r_x$. Let $[-r_x, r_x]$ be the corresponding closed segment. Then
$$S_H A \cap \Sigma(x) := \begin{cases} (-r_x, r_x), & \text{if } A \text{ is open,} \\ [-r_x, r_x], & \text{if } A \text{ is closed.} \end{cases}$$
- If $r_x = 0$, then
$$S_H A \cap \Sigma(x) := \begin{cases} \varnothing, & \text{if } A \cap \Sigma(x) \text{ is empty,} \\ \{x\}, & \text{if } A \cap \Sigma(x) \text{ is nonempty.} \end{cases}$$
- If $r_x = \infty$, then
$$S_H A \cap \Sigma(x) = \Sigma(x).$$

We refer to [3], [9], [14], [16], [18] and references therein for more information about symmetrization.

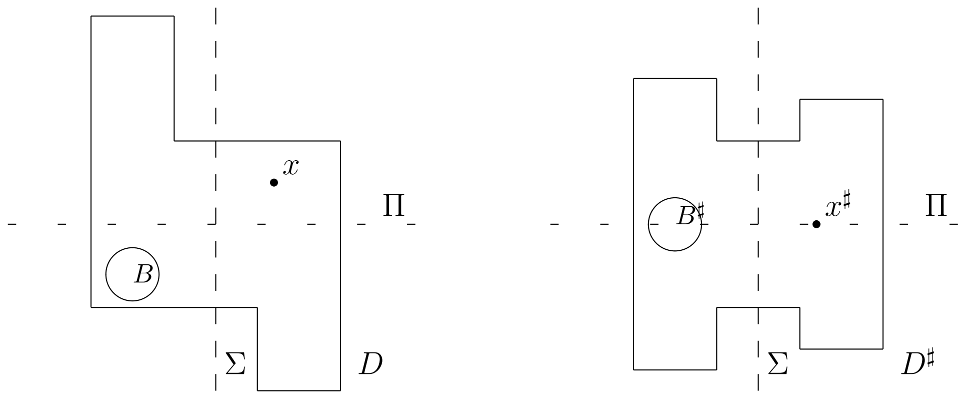

Figure 1. An open set D, a subset B of D and their symmetrizations $D^\sharp$ and $B^\sharp$.

Let $\Pi = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n : x_n = 0\}$. Every $(n-1)$-dimensional hyperplane in $\mathbf{R}^n$ will be simply called *plane.* Every plane parallel to Π will be called *horizontal.* A line will be called *vertical* if it is perpendicular to Π. If $H = \Pi$, we write $S_H A = A^\sharp$. If $x = (x_1, \dots, x_{n-1}, x_n) \in \mathbf{R}^n$, we denote by $x^\sharp$ the orthogonal projection of x on Π, $x^\sharp = (x_1, \dots, x_{n-1}, 0)$.

The behavior of solutions of parabolic equations under symmetrization has been studied by various authors; see [1], [3], [6], [7], [9], [18] and references therein. Let D be a domain in $\mathbf{R}^n$. Let Σ be a vertical line intersecting D. Let $\Phi\colon \mathbf{R} \to \mathbf{R}$ be a nonconstant, convex, increasing function with $\Phi(0) = 0$. Let B be an open or closed subset of D. The following inequalities are known, see [1], [3], [9]:

$$(1.1)\qquad \int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) \le \int_\Sigma \Phi(P_t^{D^\sharp}(x,B^\sharp))\, m_1(dx),$$

$$(1.2)\qquad P_t^D(x,B) \le P_t^{D^\sharp}(x^\sharp,B^\sharp),\ x\in D,$$

$$(1.3)\qquad \int_\Sigma \Phi(p_t^D(x,y))\, m_1(dx) \le \int_\Sigma \Phi(p_t^{D^\sharp}(x,y^\sharp))\, m_1(dx),\ y\in D,$$

$$(1.4)\qquad p_t^D(x,y) \le p_t^{D^\sharp}(x^\sharp,y^\sharp),\ x\in D,\ y\in D.$$

The next two theorems deal with the equality cases for the inequalities (1.1)–(1.4). Before stating them, we need to introduce some terminology and notation. For two Borel sets A,B in $\mathbf{R}^n$, the notation $A\cong B$ means that $C_2((A\setminus B)\cup(B\setminus A))=0$ and the notation $A\sim B$ means $m_n((A\setminus B)\cup(B\setminus A))=0$. Here and below C_2 is the logarithmic capacity for $n=2$ or the Newtonian capacity for $n\ge 3$. We say that a set $D\subset\mathbf{R}^n$ is a *striplike set* if for every vertical line Σ that intersects D, we have $\Sigma\cap D=\Sigma$. We say that a set D is an *essentially striplike set* if there exists a striplike set G such that $G\cong D$. In the sequel we always assume that the left-hand sides of (1.1) and (1.3) are finite.

Theorem 1. *Let D be a domain in $\mathbf{R}^n$. Let Σ be a vertical line intersecting D. Let $\Phi\colon \mathbf{R}\to\mathbf{R}$ be a nonconstant, convex, increasing function with $\Phi(0)=0$. Let B be an open or closed subset of D. Assume that $m_n(B)>0$.*

(a) *Suppose that D is an essentially striplike set, B is bounded, and Φ is linear function (that is, of the form $\Phi(x)=ax$). Then equality holds in (1.1) for all $t>0$.*

(b) *Suppose that D is an essentially striplike set and Φ is not linear in any interval. Then equality holds in (1.1) for some $t>0$ if and only if there exists a horizontal plane H such that $S_HB\sim B$.*

(c) *Suppose that D is not an essentially striplike set. Then equality holds in (1.1) for some $t>0$ if and only if there exists a horizontal plane H such that $S_HD\cong D$ and $S_HB\sim B$.*

(d) *Equality holds in (1.2) for some $x\in D$ and some $t>0$ if and only if there exists a horizontal plane H such that $x\in H$, $S_HD\cong D$ and $S_HB\sim B$.*

Theorem 2. *Let D be a domain in $\mathbf{R}^n$. Let Σ be a vertical line intersecting D. Let $\Phi\colon \mathbf{R}\to\mathbf{R}$ be a nonconstant, convex, increasing function with $\Phi(0)=0$.*

(a) *Suppose that D is an essentially striplike set. Then equality holds in (1.3) for all $t>0$ and all $y\in D$.*

(b) *Suppose that D is not an essentially striplike set. Then equality holds in (1.3) for some $t>0$ and some $y\in D$ if and only if there exists a horizontal plane H such that $S_HD\cong D$ and $y\in H$.*

(c) *Equality holds in (1.4) for some $x\in D$, some $y\in D$, and some $t>0$ if and only if there exists a horizontal plane H such that $x\in H$, $y\in H$, and $S_HD\cong D$.*

The proofs of the above symmetrization results is based on the approach to symmetrization via polarization; for a description of this method and various applications in potential theory and partial differential equations, we refer to [3], [7], [8], [9], [12], [17]. In Section 2 we describe polarization and we state the equality statements for known inequalities describing the behavior of transition functions and heat kernels under polarization. In Sections 3 and 4 we prove the polarization results. In Section 5 we prove Theorem 1; the proof of Theorem 2 is similar and omitted. In the rest of the present section we review some consequences of Theorems 1 and 2.

To avoid some trivial cases, in the rest of this section we assume that D is not an essentially striplike set.

1.1. Green functions. Suppose that D a Greenian domain in $\mathbf{R}^n$. Let $G^D(x,y)$ denote the Green function for D. The classical symmetrization inequalities for Green functions have been proved by Baernstein and Taylor (see [2], [5], [4]). These inequalities (with notation as in Theorems 1 and 2) are:

$$\int_\Sigma \Phi(G^D(x,y))\, m_1(dx) \le \int_\Sigma \Phi(G^{D^\sharp}(x,y^\sharp))\, m_1(dx),\ y\in D, \tag{1.5}$$

$$G^D(x,y) \le G^{D^\sharp}(x^\sharp,y^\sharp),\ x\in D,\ y\in D. \tag{1.6}$$

They also follow easily from the inequalities (1.3), (1.4) and the following formula relating heat kernels and Green functions [15, p. 111]:

$$G^D(x,y) = \int_0^\infty p_t^D(x,y)\, dt,\ x,y\in D. \tag{1.7}$$

It follows from Theorem 2 that equality holds in (1.5) for some $y\in D$ if and only if there exists a horizontal plane H such that $S_H D\cong D$ and $y\in H$. Also, equality holds in (1.6) for some $x\in D$ and some $y\in D$ if and only if there exists a horizontal plane H such that $x\in H$, $y\in H$, and $S_H D\cong D$. Such equality statements have been proved by Solynin [17] with the additional assumption that D is regular for the Dirichlet problem.

1.2. Condenser capacities. We continue to assume that D is a Greenian domain. Let K be a compact subset of D with $C_2(K)>0$. The Green capacity of K with respect to D (see [13, p. 174]) is

$$C^D(K) = \left[\min \int_K\int_K G^D(x,y)\ \mu(dx)\mu(dy)\right]^{-1},$$

where the minimum is taken over all probability Borel measures μ on K. This quantity is equal (modulo a multiplicative constant) to the capacity of the condenser with plates K and $(\mathbf{R}^n\cup\{\infty\})\setminus D$; see [13, p. 97]. The condenser capacity is usually defined via the Dirichlet integral. For condenser capacities, we have the following symmetrization inequality (see [12], [16]):

$$C^D(K) \ge C^{D^\sharp}(K^\sharp). \tag{1.8}$$

If we use the inequalities (1.5), (1.6) and the corresponding equality statements, we find that equality holds in (1.8) if and only if there exists a horizontal plane H such that $S_H K \cong K$ and $S_H D \cong D$.

1.3. Sojourn times and lifetimes. Let B be an open subset of the Greenian domain D. The quantity

$$G^D(x,B) := \int_B G^D(x,y)\, m_n(dy) = \int_0^\infty P_t^D(x,B)\, dt,\ x \in D$$

represents the expected length of time that a Brownian motion starting from x spends in B before exiting D. In particular for $B = D$, we obtain $G^D(x,D) = \mathbf{E}^x\, T^D$, the expected lifetime of Brownian motion in D.

Using the symmetrization results for the transition function, we find that

$$G^D(x,B) \le G^{D^\sharp}(x^\sharp, B^\sharp),$$

with equality if and only if there exists a horizontal plane H such that $x \in H$, $S_H B \sim B$, and $S_H D \cong D$.

1.4. Exit times. The inequality

$$\int_D p_t^D(x,y)\, m_n(dy) \le \int_{D^\sharp} p_t^{D^\sharp}(x^\sharp, y)\, m_n(dy),\ x \in D,\ t > 0$$

comes easily from (1.3). It is equivalent to the inequality

$$\mathbf{P}^x(T^D > t) \le \mathbf{P}^{x^\sharp}(T^{D^\sharp} > t).$$

Equality holds for some $t > 0$ and some $x \in D$ if and only if there exists a horizontal plane H such that $x \in H$ and $S_H D \cong D$.

2. Polarization inequalities

For $E \subset \mathbf{R}^n$, we denote by $\widehat{E}$ the reflection of E in the $(n-1)$-dimensional plane Π. Thus we have

$$\widehat{E} = \{(x_1, \ldots, x_{n-1}, x_n) : (x_1, \ldots, x_{n-1}, -x_n) \in E\}.$$

We will also use the following notation: if $x = (x_1, \ldots, x_{n-1}, x_n)$, then $\hat{x} := (x_1, \ldots, x_{n-1}, -x_n)$ and $x^* := (x_1, \ldots, x_{n-1}, |x_n|)$; $E_+ := \{(x_1, \ldots, x_{n-1}, x_n) \in E : x_n > 0\}$; $E_o := E \cap \Pi$; $E_- = \{(x_1, \ldots, x_{n-1}, x_n) \in E : x_n < 0\}$.

Let E be any set in $\mathbf{R}^n$. We divide E into three disjoint subsets S, U, V as follows: The set S is the *symmetric part* of E: $S = S_E = \{x \in E : \hat{x} \in E\} = E \cap \widehat{E}$. The set U is the *upper non-symmetric part* of E: $U = U_E = \{x \in E : x \in E_+,\ \hat{x} \notin E\} = E_+ \setminus S_E$. The set V is the *lower non-symmetric part* of E: $V = V_E = \{x \in E : x \in E_-,\ \hat{x} \notin E\} = E_- \setminus S_E$. Then $E = S \cup U \cup V$. The *polarization* E^* of E is the set

$$E^* := S \cup U \cup \widehat{V}.$$

Equivalently, $E^* = (E \cup \widehat{E})_+ \cup (E \cap \widehat{E})_-$. It is clear that the polarization of an open set is open. The polarization of a domain D need not be a domain. The open set

D^* has a unique connected component intersecting $\mathbf{R}^n{}_+$. In the sequel $P_t^{D^*}(x, B^*)$ denotes the transition function for Brownian motion in this component.

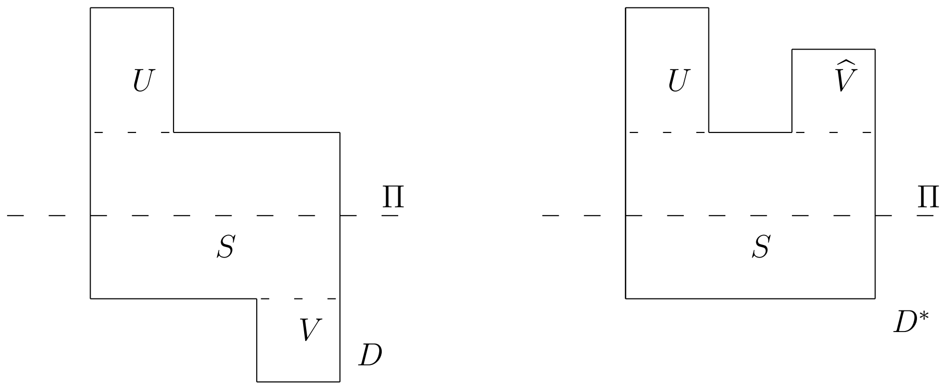

Figure 2. A set D and its polarization D^* with respect to the plane Π.

The polarization as defined above may be called polarization *with respect to* Π. In a similar way, one can define polarization with respect to any other oriented $(n-1)$-dimensional plane in $\mathbf{R}^n$. Let H be such a plane. We denote by $P_H E$ the polarization of E with respect to H. We also denote by $R_H E$ the reflection of E in H.

The following polarization inequalities for transition functions and heat kernels come from [7] and [9]. Let D be a domain with symmetric part S and let B be a Borel subset of D. Then for $t > 0$,

$$P_t^D(x, B) \le P_t^{D^*}(x^*, B^*), \ x \in D, \tag{2.1}$$

$$P_t^D(x, B) + P_t^D(\hat{x}, B) \le P_t^{D^*}(x, B^*) + P_t^{D^*}(\hat{x}, B^*), \ x \in S, \tag{2.2}$$

$$p_t^D(x, y) \le p_t^{D^*}(x^*, y^*), \ x, y \in D, \tag{2.3}$$

$$p_t^D(x, y) + p_t^D(\hat{x}, y) \le p_t^{D^*}(x, y^*) + p_t^{D^*}(\hat{x}, y^*), \ x \in S, \ y \in D. \tag{2.4}$$

In the following theorems we determine the equality cases in the above inequalities.

Theorem 3. *Let D be a domain and let B be a Borel subset of D with $m_n(B) > 0$.*

(a) *Equality holds in* (2.1) *for some $x \in D$ and some $t > 0$ if and only if either ($x = x^*$, $B \sim B^*$, $D \cong D^*$) or ($x = \widehat{x^*}$, $B \sim \widehat{B^*}$, $D \cong \widehat{D^*}$).*

(b) *Equality holds in* (2.2) *for some $x \in D \cap \widehat{D}$ and some $t > 0$ if and only if either ($B \sim B^*$, $D \cong D^*$) or ($B \sim \widehat{B^*}$, $D \cong \widehat{D^*}$).*

Theorem 4. (a) *Equality holds in* (2.3) *for some $x \in D$, some $y \in D$, and some $t > 0$ if and only if either ($x = x^*$, $y = y^*$, $D \cong D^*$) or ($x = \widehat{x^*}$, $y = \widehat{y^*}$, $D \cong \widehat{D^*}$).*

(b) *Equality holds in* (2.4) *for some $x \in S$, some $y \in D$, and some $t > 0$ if and only if either ($y \sim y^*$, $D \cong D^*$) or ($y = \widehat{y^*}$, $D \cong \widehat{D^*}$).*

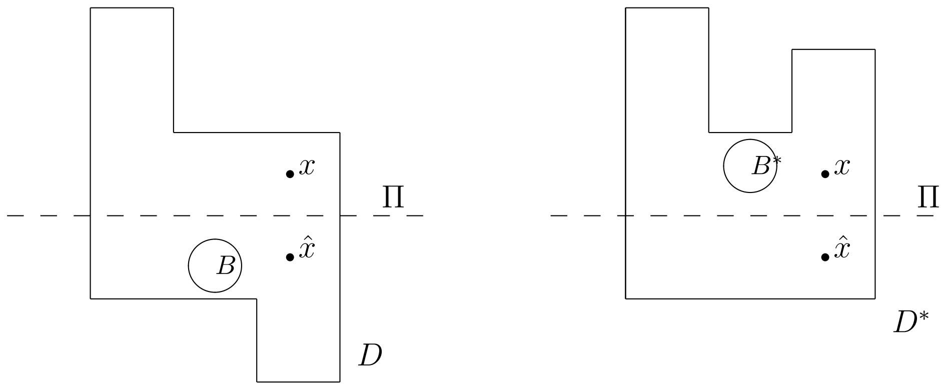

Figure 3. $P_t^D(x,B) + P_t^D(\hat{x},B) \le P_t^{D^*}(x,B^*) + P_t^{D^*}(\hat{x},B^*)$.

The inequalities (2.1)–(2.4) lead to convex integral mean inequalities which we now describe: Let D be a domain in $\mathbf{R}^n$ and let B be a Borel subset of D with $m_n(B) > 0$. Let $\Phi\colon \mathbf{R} \to \mathbf{R}$ be a nonconstant, convex, increasing function with $\Phi(0) = 0$. Let Σ be a vertical line that intersects D. Then for all $t > 0$,

$$\int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) \le \int_\Sigma \Phi(P_t^{D^*}(x,B^*))\, m_1(dx), \tag{2.5}$$

$$\int_\Sigma \Phi(p_t^D(x,y))\, m_1(dx) \le \int_\Sigma \Phi(p_t^{D^*}(x,y^*))\, m_1(dx), \ y \in D. \tag{2.6}$$

Theorem 5. *Let D, B, Φ, Σ be as above.*

(a) *Suppose that $D \cong \widehat{D}$ and that Φ is a linear function. Then equality holds in* (2.5) *for all $t > 0$.*

(b) *Suppose that $D \cong \widehat{D}$ and that Φ is not linear in any interval. Then equality holds in* (2.5) *for some $t > 0$ if and only if $B \sim B^*$ or $B \sim \widehat{B^*}$.*

(c) *Suppose that $D \not\cong \widehat{D}$. Then equality holds in* (2.5) *for some $t > 0$ if and only if ($D \cong D^*$, $B \sim B^*$) or ($D \cong \widehat{D^*}$, $B \sim \widehat{B^*}$).*

Theorem 6. *Let D, Φ, Σ be as above.*

(a) *Suppose that $D \cong \widehat{D}$. Then equality holds in* (2.6) *for all $t > 0$ and all $y \in D$.*

(b) *Suppose that $D \not\cong \widehat{D}$. Then equality holds in* (2.6) *for some $t > 0$ and some $y \in D$ if and only if ($D \cong D^*$, $y = y^*$) or ($D \cong \widehat{D^*}$, $y = \widehat{y^*}$).*

In Sections 3 and 4 we prove Theorems 3 and 5. The proof of Theorems 4 and 6 is similar.

3. Proof of Theorem 3

We denote by S, U, V the symmetric, upper non-symmetric, and lower non-symmetric part of D, respectively. Hence $D = S \cup U \cup V$ and $D^* = S \cup U \cup \widehat{V}$.

It is easy to prove that if either ($x = x^*$, $B \sim B^*$, $D \cong D^*$) or ($x = \widehat{x^*}$, $B \sim \widehat{B^*}$, $D \cong \widehat{D^*}$), then (2.1) and (2.2) hold with equality. So we prove only the converse statements.

Since $P_t^D(x, \cdot)$ is a measure absolutely continuous with respect to m_n, it suffices to prove part (a) of Theorem 3 by considering the following nine cases: (a1) $x \in D_+$, $B \subset \mathbf{R}^n{}_+$; (a2) $x \in D_-$, $B \subset \mathbf{R}^n{}_-$; (a3) $x \in D_-$, $B \subset \mathbf{R}^n{}_+$; (a4) $x \in D_+$, $B \subset \mathbf{R}^n{}_-$; (a5) $x \in D_+$, B symmetric with respect to Π; (a6) $x \in D_-$, B symmetric with respect to Π; (a7) $x \in D_o$, B symmetric with respect to Π; (a8) $x \in D_o$, $B \subset \mathbf{R}^n{}_+$; (a9) $x \in D_o$, $B \subset \mathbf{R}^n{}_-$. Similarly we prove part (b) of Theorem 3 by considering the following three cases: (b1) $B \subset \mathbf{R}^n{}_+$; (b2) $B \subset \mathbf{R}^n{}_-$; (b3) B symmetric with respect to Π.

Case (a3). In this case we assume that $B \subset \mathbf{R}^n{}_+$. We have to prove the strict inequality

$$P_t^D(x, B) < P_t^{D^*}(\hat{x}, B),\ t > 0,\ x \in D_-.$$

Suppose that there exist $t_1 > 0$ and $x_1 \in D_-$ such that $P_{t_1}^D(x_1, B) = P_{t_1}^{D^*}(\widehat{x_1}, B)$. By applying the parabolic minimum principle (see e.g. [11, Chapter XV]) to the function $P_t^{D^*}(\hat{x}, B) - P_t^D(x, B)$, $t > 0$, $x \in D_-$, we conclude that

$$P_t^D(x, B) = P_t^{D^*}(\hat{x}, B),\ 0 < t < t_1,\ x \in D_-.$$

Taking limits as $t \to 0$ for $\hat{x} \in B$, we arrive to the contradiction $0 = 1$.

Case (a4). The proof in this case is the same as the proof for Case (a3).

Case (a1). In this case we assume that $x \in D_+$ and $B \subset \mathbf{R}^n{}_+$. We have to prove that if $P_t^D(x, B) = P_t^{D^*}(x, B)$ for $t = T$, then $D \cong D^*$. By the parabolic minimum principle applied to the function $P_t^{D^*}(x, B) - P_t^D(x, B)$, $t > 0$, $x \in D_+$, we infer that

$$P_t^D(x, B) = P_t^{D^*}(x, B),\ 0 < t < T,\ x \in D_+. \tag{3.1}$$

Suppose that there exists a point $v \in \partial V \cap \partial S$ which is regular for the Dirichlet problem in S. We take limits in (3.1) as $x \to \hat{v}$ and conclude that $P_t^{D^*}(\hat{v}, B) = 0$. This is absurd because $P_t^{D^*}(x, B) > 0$ for all $t > 0$ and all $x \in D^*$. Therefore all points of the set $\partial V \cap \partial S$ are irregular for the Dirichlet problem in S. By Kellogg's theorem (see e.g. [13, Chapter V]), $C_2(\partial S \cap \partial V) = 0$. A standard application of the strong Markov property implies that every compact subset K of V has harmonic measure zero (every Brownian path from $x \in S$ to K should pass through the set $\partial S \cap \partial V$). Hence $C_2(K) = 0$ and therefore $C_2(V) = 0$; this means $D \cong D^*$.

Cases (a2), (a5), (a6). The proofs in these cases are the same as the proof in case (a1).

Case (b1). We assume that $B \subset \mathbf{R}^n{}_+$ and have to prove that if

$$P_t^D(x, B) + P_t^D(\widehat{x}, B) = P_t^{D^*}(x, B^*) + P_t^{D^*}(\widehat{x}, B^*),$$

for some $x \in S$ and some $t = T > 0$, then $D \cong D^*$. By the parabolic minimum principle,

$$P_t^D(x,B) + P_t^D(\hat{x},B) = P_t^{D^*}(x,B^*) + P_t^{D^*}(\hat{x},B^*), \tag{3.2}$$

for all $x \in S$ and all t with $0 < t \le T$.

Suppose that the set $\partial S \cap \partial V$ contains a point v which regular for the Dirichlet problem in S. We take limits in (3.2) as $x \to v$ and conclude that $P_t^D(v,B) = P_t^{D^*}(\hat{v},B)$. This contradicts the strict inequality of Case (a3). Therefore all points of the set $\partial S \cap \partial V$ are irregular for the Dirichlet problem in S. Hence $C_2(V) = 0$ which means $D \cong D^*$.

Case (b2). The proof in this case is the same as the proof in case (b1).

Case (b3). We assume that B is symmetric with respect to Π and have to prove that if

$$P_t^D(x,B) + P_t^D(\widehat{x},B) = P_t^{D^*}(x,B^*) + P_t^{D^*}(\widehat{x},B^*),$$

for some $x \in S$ and some $t = T > 0$, then either $D \cong D^*$ or $D \cong \widehat{D^*}$.

By the parabolic minimum principle,

$$P_t^D(x,B) + P_t^D(\hat{x},B) = P_t^{D^*}(x,B^*) + P_t^{D^*}(\hat{x},B^*), \tag{3.3}$$

for all $x \in S$ and all t with $0 < t \le T$.

Suppose that all points the set $\partial S \cap \partial U$ are irregular for the Dirichlet problem in S. This implies that $C_2(U) = 0$ which means $D \cong \widehat{D^*}$. Similarly if all points of the set $\partial S \cap \partial V$ are irregular for the Dirichlet problem in S, then $C_2(V) = 0$ and therefore $D \cong D^*$.

If the set $\partial S \cap \partial U$ has a regular point u, then we take limits in (3.3) as $x \to u$ and conclude that

$$P_t^D(u,B) = P_t^{D^*}(u,B).$$

By Case (a5), this implies $D \cong D^*$.

If the set $\partial S \cap \partial V$ has a regular point v, we take limits as $x \to \hat{v}$ and use Case (a6) to conclude that $D \cong \widehat{D^*}$.

Cases (a7), (a8), (a9). These cases are covered by cases (b1), (b2), (b3).

4. Proof of Theorem 5

The proof uses the following elementary lemma; (see [17]).

Lemma 1. *Let $a_1, b_1, a_2, b_2 \in \mathbf{R}$ be such that*

$$a_2 + b_2 \le a_1 + b_1 \quad \text{and} \quad 0 \le a_1 \le a_2 \le b_2 < b_1.$$

Let $\Phi\colon \mathbf{R} \to \mathbf{R}$ be a nonconstant, convex, increasing function. Then

$$\Phi(a_2) + \Phi(b_2) \le \Phi(a_1) + \Phi(b_1). \tag{4.1}$$

Equality holds in (4.1) *if and only if Φ is affine* (*that is, of the form $\Phi(x) = ax + b$*) *on $[a_1, b_1]$ and $a_1 + b_1 = a_2 + b_2$.*

We proceed with the proof of the theorem.

(a) Suppose that $D \cong \widehat{D}$ and that Φ is a linear function. It is easy to see (using the strong Markov property) that for $s \in S$ and $t > 0$,

$$P_t^D(s,B) = P_t^S(s,B) \quad \text{and} \quad P_t^{D^*}(s,B^*) = P_t^S(s,B^*).$$

Also, because of symmetry, for $s \in S_+$ and $t > 0$,

$$P_t^S(s,B) + P_t^S(\hat{s},B) = P_t^S(s,B^*) + P_t^S(\hat{s},B^*).$$

Since Φ is linear, it follows that equality holds in (2.5).

(b) Suppose that $D \cong \widehat{D}$ and Φ is not linear. If $B \sim B^*$ or $B \sim \widehat{B^*}$, then it is easy to prove that (2.5) holds with equality for all $t > 0$. Conversely, assume that (2.5) holds with equality for some $t > 0$. Then by the inequalities (2.1), (2.2) and Lemma 1, for $s \in \Sigma_+$,

$$\Phi(P_t^S(s,B)) + \Phi(P_t^S(\hat{s},B)) = \Phi(P_t^S(s,B^*)) + \Phi(P_t^S(\hat{s},B^*)).$$

Since Φ is not linear in any interval, it follows from Lemma 1 that for every $s \in \Sigma_+$,

$$P_t^S(s,B^*) = P_t^S(s,B) \quad \text{or} \quad P_t^S(s,B^*) = P_t^S(\hat{s},B).$$

By Theorem 3, $B \sim B^*$ or $B \sim \widehat{B^*}$.

(c) Suppose that $D \ncong \widehat{D}$. If $D \cong D^*$, $B \sim B^*$ or if $D \cong \widehat{D^*}$, $B \sim \widehat{B^*}$, then it is easy to show that equality holds in (2.5) for all $t > 0$. Conversely, assume that equality holds in (2.5) for some $t > 0$. Then by the inequalities (2.1), (2.2) and Lemma 1, for all $s \in \Sigma_+$,

$$\Phi(P_t^D(s,B)) + \Phi(P_t^D(\hat{s},B)) = \Phi(P_t^{D^*}(s,B^*)) + \Phi(P_t^{D^*}(\hat{s},B^*)).$$

By Lemma 1, for each $s \in \Sigma_+$, at least one of the following three equalities must be satisfied:

$$P_t^D(s,B) + P_t^D(\hat{s},B) = P_t^{D^*}(s,B^*) + P_t^{D^*}(\hat{s},B^*), \tag{4.2}$$

$$P_t^{D^*}(s,B^*) = P_t^D(s,B), \tag{4.3}$$

$$P_t^{D^*}(s,B^*) = P_t^D(\hat{s},B). \tag{4.4}$$

By Theorem 3, we conclude that either ($D \cong D^*$, $B \sim B^*$) or ($D \cong \widehat{D^*}$, $B \sim \widehat{B^*}$). □

5. Proof of Theorem 1

For the proof of Theorem 1 we need some definitions and lemmas. For a point $p = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, we set $\mathrm{h}(p) = x_n$.

Definition 1. (i) Let Ω be a Borel set in $\mathbf{R}^n$. We say that $\Omega \in \mathscr{A}_1$ if there exists a horizontal plane H such that for every vertical line Σ that intersects Ω, the set $\Sigma \cap (\mathbf{R}^n \setminus \Omega)$ is either empty or a nonempty, bounded, vertical segment, symmetric with respect to H. We say that $\Omega \in \mathscr{A}_2$ if for every vertical line Σ that intersects Ω, the set $\Sigma \cap \Omega$ is either the whole line Σ or an upward half-line. We say that $\Omega \in \mathscr{A}_3$

if for every vertical line Σ that intersects Ω, the set $\Sigma \cap \Omega$ is either the whole line Σ or a downward half-line.

(ii) Let D be an open set such that $D \ncong C$ for any striplike set C. We say that $D \in \mathscr{G}_j$ if $D \cong \Omega$, for some $\Omega \in \mathscr{A}_j$, $j = 1, 2, 3$.

(iii) Let B be an open or closed set such that $B \nsim C$ for any striplike set C. We say that $B \in \mathscr{F}_j$ if $B \sim \Omega$, for some $\Omega \in \mathscr{A}_j$, $j = 1, 2, 3$.

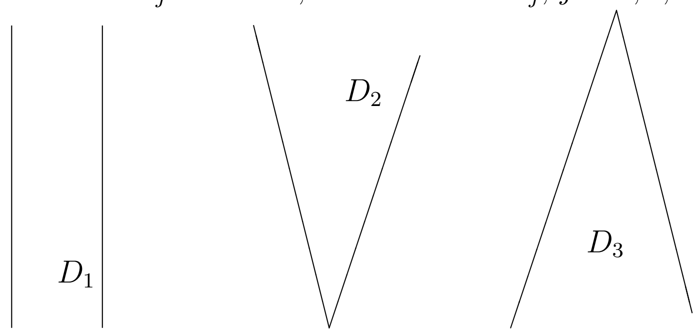

Figure 4. The set D_j belongs to the class $\mathscr{G}_j$, $j = 1, 2, 3$.

Lemma 2. *Let D be an open set in $\mathbf{R}^n$. Assume that $D \notin \mathscr{G}_1 \cup \mathscr{G}_2 \cup \mathscr{G}_3$. There exists a horizontal plane H_o such that $S_{H_o} D \cong D$ if and only if for every horizontal plane H, either $D \cong P_H D$ or $R_H D \cong P_H D$.*

Proof. We call *Condition* A the statement: *for every horizontal plane H, either $D \cong P_H D$ or $R_H D \cong P_H D$.* It is easy to see that if there exists a horizontal plane H_o such that $S_{H_o} D \cong D$, then Condition A holds.

Conversely, assume that D satisfies Condition A. Let A be the set of all points $x \in D^c$ for which there exists a ball $B(x)$ centered at x such that $C_2(B(x) \setminus D) = 0$. Then A is a subset of ∂D.

Claim 1: $C_2(A) = 0$.

Proof. We cover A by countably many balls B_j centered at points of A and having the property $C_2(B_j \setminus D) = 0$. Since

$$A \subset \bigcup_j (B_j \setminus D),$$

Claim 1 follows from the subadditivity of capacity.

We set

$$\Omega = D \cup A.$$

It is clear that Ω is an open set of $\mathbf{R}^n$ and $\Omega \cong D$. Moreover, since $D \notin \mathscr{G}_1 \cup \mathscr{G}_2 \cup \mathscr{G}_3$, Ω is not essentially striplike. Also, Ω satisfies Condition A in a stronger form: *For every horizontal plane H, either $\Omega = P_H \Omega$ or $R_H \Omega = P_H \Omega$.*

Claim 2: Ω is vertically convex.

Proof. Let $\Sigma = \{(x_1, \ldots, x_{n-1}, y) : y \in \mathbf{R}\}$ be a vertical line that intersects Ω. Let $p_1, p_2 \in \Sigma \cap \Omega$ and suppose that $\mathrm{h}(p_1) > \mathrm{h}(p_2)$. Let Σ_j be the component of

$\Sigma \cap \Omega$ that contains p_j, $j = 1, 2$. We need to show that $\Sigma_1 = \Sigma_2$. Suppose that $\Sigma_1 \neq \Sigma_2$. Set

$$y_1 = \inf\{\mathrm{h}(p) : p \in \Sigma_1\}, \quad y_2 = \sup\{\mathrm{h}(p) : p \in \Sigma_2\}$$

and

$$p_1' = (x_1, \ldots, x_{n-1}, y_1), \quad p_2' = (x_1, \ldots, x_{n-1}, y_2).$$

Note that it may happen that $y_1 = y_2$. Consider the plane

$$H_o = \left\{ (x_1, \ldots, x_{n-1}, x_n) : x_n = \frac{y_1 + y_2}{2} \right\}.$$

By successive applications of Condition A, we see that

$$\Sigma \cap \Omega = \Sigma \setminus [p_1', p_2'].$$

Here $[p_1', p_2']$ is the vertical segment with endpoints p_1', p_2' (or a singleton if $p_1' = p_2'$).

By working in the same way for every vertical line that intersects Ω, we see that $\Omega \in \mathscr{G}_1$. Since $C_2(\Omega \setminus D) = C_2(A) = 0$, we also have $D \in \mathscr{G}_1$. This contradicts the assumption of the lemma. Hence Claim 2 is proved.

Claim 3: There exists a plane H_o parallel to Π such that Ω is symmetric with respect to H_o.

Proof. Suppose that Σ is a vertical line such that $\Sigma \cap \Omega$ is a half-line. By using the Condition A, we see that Ω (and hence D) belongs to $\mathscr{G}_2 \cup \mathscr{G}_3$; contradiction. Recalling also that Ω is vertically convex, we conclude that for every vertical line intersecting D, either $\Sigma \cap \Omega = \Sigma$ or $\Sigma \cap \Omega$ is a vertical segment. Since Ω is not essentially striplike, there exist vertical lines Σ for which $\Sigma \cap \Omega$ is a vertical segment.

Fix a vertical line Σ with $\Sigma \cap \Omega = (p_1, p_2)$. Define

$$H_o = \left\{ (x_1, \ldots, x_{n-1}, x_n) : x_n = \frac{\mathrm{h}(p_1) + \mathrm{h}(p_2)}{2} \right\}.$$

Now using Condition A, we see that Ω is symmetric with respect to H_o and Claim 3 is proved.

By Claims 2 and 3, $S_{H_o}\Omega = \Omega$. Therefore Claim 1 implies that $S_{H_o}D \cong D$. $\square$

Lemma 3. *Let B be an open or closed set in $\mathbf{R}^n$. Assume that $B \notin \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. There exists a horizontal plane H_o such that $S_{H_o}B \sim B$ if and only if for every horizontal plane H, either $B \sim P_H B$ or $R_H B \sim P_H B$.*

Proof. If $S_{H_o}B \sim B$ for some horizontal plane H_o, then it is easy to see that for every horizontal plane H, either $B \sim P_H B$ or $R_H B \sim P_H B$.

Conversely, assume that the latter condition holds. Let B_1 be the set of all points $p \in B$ for which there exists a ball $B(p)$ centered at p such that $m_n(B \cap B(p)) = 0$. Let $F := B \setminus B_1$. It is easy to see that F is a closed set. Note also that if $p \in B_1$, then p is a density point of B^c; by Lebesgue's density theorem, $m_n(B_1) = 0$. Hence

$$F \sim B. \tag{5.1}$$

By standard arguments, (5.1) implies

$$S_H F \sim S_H B, \tag{5.2}$$

for all planes H parallel to Π.

By arguments similar to those in the proof of Lemma 2, we find that F is vertically convex and symmetric with respect to some horizontal plane H_o. Therefore

$$S_{H_o} F \sim F. \tag{5.3}$$

Now (5.1), (5.2), (5.3) imply $S_{H_o} B \sim B$. □

Proof of Theorem 1. (a) By [9, Lemma 7.2], there exist horizontal, oriented planes H_j with corresponding polarizations P_j, $j \in \mathbf{N}$, such that for the sequence of sets $F_k := P_k \dots P_2 P_1(B)$, we have (convergence in the Hausdorff metric)

$$\lim_{k\to\infty} d_{\mathrm{Haus}}(F_k, B^\sharp) = 0. \tag{5.4}$$

For every $k \in \mathbf{N}$, the sets $B, F_k, B^\sharp$ have common orthogonal projection on the plane Π. Since $B^\sharp$ is bounded and convex in the vertical direction, (5.4) implies that

$$\lim_{k\to\infty} m_n(B^\sharp \setminus F_k) = 0. \tag{5.5}$$

By Theorem 3(a), inequality (2.5), and the assumption that Φ is affine,

$$\begin{aligned}
\int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) &= \int_\Sigma \Phi(P_t^D(x,F_k))\, m_1(dx) \\
&\le \int_\Sigma \Phi(P_t^D(x,B^\sharp))\, m_1(dx) \\
&= \int_\Sigma \Phi(P_t^D(x,F_k) + P_t(B^\sharp \setminus F_k))\, m_1(dx) \\
&= \int_\Sigma \Phi(P_t^D(x,F_k))\, m_1(dx) \\
&\quad + \int_\Sigma \Phi(P_t^D(x,B^\sharp \setminus F_k))\, m_1(dx) \\
&= \int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) \\
&\quad + \int_\Sigma \Phi(P_t^D(x,B^\sharp \setminus F_k))\, m_1(dx).
\end{aligned}$$

We take limits as $k \to \infty$ and using (5.5) we obtain part (a) of Theorem 1.

(b) If $S_H B \sim B$, then it is clear that equality holds in (1.1) for all $t > 0$. Conversely, assume that (1.1) holds with equality for some $t > 0$. Seeking for a contradiction, assume also that $S_H B \not\sim B$ for any horizontal plane H. Then Lemma 3 implies that either $B \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, or there exists a horizontal plane H such that $B \not\sim P_H B$ and $R_H B \not\sim P_H B$.

If $B \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, then $B^\sharp$ is an essentially striplike set with $B \subsetneqq B^\sharp$ and $m_n(B^\sharp) > m_n(B)$. So (1.1) cannot hold with equality. Hence there exists a horizontal plane H such that $B \not\sim P_H B$ and $R_H B \not\sim P_H B$. Then, by Theorem 5(b) and inequality (1.1), for every $t > 0$,

$$\begin{aligned}\int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) &< \int_\Sigma \Phi(P_t^{P_H D}(x, P_H B))\, m_1(dx)\\ &\le \int_\Sigma \Phi(P_t^{D^\sharp}(x, B^\sharp))\, m_1(dx).\end{aligned} \tag{5.6}$$

Contradiction.

(c) If $S_H D \cong D$ and $S_H B \sim B$ for some horizontal plane H, then we trivially have equality in (1.1) for all $t > 0$, that is:

$$\int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) = \int_\Sigma \Phi(P_t^{D^\sharp}(x, B^\sharp))\, m_1(dx). \tag{5.7}$$

Conversely, assume that (5.7) holds for some $t > 0$. Seeking for a contradiction, suppose that $S_H D \not\cong D$ for any horizontal plane H. Then either $D \in \mathscr{G}_1 \cup \mathscr{G}_2 \cup \mathscr{G}_3$ (this leads easily to a contradiction), or (by Lemma 2) there exists a horizontal plane H such that $D \not\cong P_H D$ and $R_H D \not\cong P_H D$. By Theorem 5(c) and inequality (1.1), we obtain

$$\begin{aligned}\int_\Sigma \Phi(P_t^D(x,B))\, m_1(dx) &< \int_\Sigma \Phi(P_t^{P_H D}(x, P_H B))\, m_1(dx)\\ &\le \int_\Sigma \Phi(P_t^{D^\sharp}(x, B^\sharp))\, m_1(dx).\end{aligned} \tag{5.8}$$

This contradicts (5.7). Therefore there exists a horizontal plane H such that $S_H D \cong D$. We may assume that $H = \Pi$; so we have $D \cong D^\sharp$ and it remains to prove that $B^\sharp \sim B$.

Suppose that $B^\sharp \not\sim B$. Then either $B \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ (this leads easily to a contradiction), or (by Lemma 3) there exists a horizontal plane H such that $B \not\sim P_H B$ and $R_H B \not\sim P_H B$. We continue as above and arrive at a strict inequality that contradicts (5.7).

(d) The proof is similar to the proof of (c). □

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