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MODULUS OF CONTINUITY FOR QUASIREGULAR MAPPINGS WITH FINITE DISTORTION EXTENSION

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Abstract. We establish a sharp modulus of continuity for those planar quasiregular mappings defined in a domain with a cone condition that admit an extension to a mapping of locally exponentially integrable distortion.

1. Introduction

In this paper, we consider planar mappings $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $f \in W^{1,1}_{\rm loc}(\mathbf{R}^2;\mathbf{R}^2)$ with $|Df(x)|^2 \le K(x)J_f(x)$ a.e., where $K(x) \ge 1$, $J_f(x)$ is locally integrable and $\exp(\lambda K)$ is locally integrable for some $\lambda > 0$. We call such an f a mapping of locally exponentially integrable distortion. These mappings are known to be continuous and some modulus of continuity results were established in [2], [7], [10], [5] and [8]. Our results deal with the mappings that are additionally assumed to be quasiregular in some domain Ω. Recall that a mapping $f\colon \Omega \to f(\Omega) \subset \mathbf{R}^2$ is quasiregular if $f \in W^{1,1}_{\rm loc}(\Omega;\mathbf{R}^2)$, $J_f(x)$ is locally integrable and in the distortion inequality above the function $K(x)$ is bounded, that is $1 \le K(x) \le \mathbf{K}$ for some $\mathbf{K}$, almost everywhere in Ω. If in addition we assume f to be a homeomorphism, we say that f is $\mathbf{K}$-quasiconformal. The main result of the paper can be stated as follows (see the next section for the definitions).

Theorem 1. *Let Ω be a simply connected bounded domain, satisfying a δ-cone condition, and suppose $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a mapping of finite distortion such that $\exp(\lambda K(x))$ is locally integrable for some $\lambda > 0$. If the restriction of f to Ω is quasiregular, then there exist positive constants $\hat{C}$ and $\tilde{C}$ such that*

$$|f(x) - f(y)| \le \frac{\hat{C}}{\log^{\frac{\lambda\pi}{2(\pi - \arcsin\delta)}} \frac{\tilde{C}}{|x-y|}}, \tag{1}$$

whenever $x, y \in \overline{\Omega}$. On the other hand, for a given $s > 0$ there exists a bounded domain Ω_0, satisfying a δ_0-cone condition, and a mapping f_0, quasiconformal in Ω_0 and having locally exponentionally integrable distortion for all $\mu < \lambda_0 = \frac{2(\pi - \arcsin\delta_0)}{s\pi}$, such that the modulus continuity estimate (1) *fails for f_0 with the logarithm to the power $\frac{1}{s} + \varepsilon$ for any given $\varepsilon > 0$.*

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For the unit disk $B = B(0,1)$ we have the following consequence.

Corollary 1. *Suppose $f\colon B \to \mathbf{R}^2$ is a quasiregular mapping of the unit disk B. If f has an extension to a mapping of finite locally exponentially integrable distortion for some $\lambda > 0$, then there exist positive constants $\hat{C}$ and $\tilde{C}$ such that*

$$|f(x) - f(y)| \leq \frac{\hat{C}}{\log^{\lambda} \frac{\tilde{C}}{|x-y|}}, \tag{2}$$

whenever $x, y \in \overline{B}$.

This result improves an estimate in [8]. It is a counterpart for the result in [1], stating that a conformal mapping f in the unit disk with a $\mathbf{K}$–quasiconformal extension is Hölder continuous in the unit disk with the sharp exponent $1 - k$, where $k = (\mathbf{K} - 1)/(\mathbf{K} + 1)$, which is better than $1/\mathbf{K}$ given by a well-known result for quasiconformal mappings. In our case, for a general mapping of exponentially integrable distortion, the exponent of the logarithm in the estimate (2) would be $\lambda/2$ ([10]).

In the last section of this paper we make some comments on the case when the domain Ω in question is a quasidisk.

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2. Preliminaries

Let $\Omega \subset \mathbf{R}^2$ be a domain, i.e. a connected and open subset of $\mathbf{R}^2$. We say that a mapping $f\colon \Omega \to f(\Omega) \subset \mathbf{R}^2$ has *finite distortion* if the following conditions are satisfied:

1. $f \in W^{1,1}_{\text{loc}}(\Omega; \mathbf{R}^2)$.
2. The Jacobian determinant $J_f(x)$ of f is locally integrable.
3. $|Df(x)|^2 \leq K(x) J_f(x)$ a.e. $x \in \Omega$

for some measurable function $K(x) \geq 1$ which is finite almost everywhere. The function $K(x)$ is referred to as a distortion (function) of f and the phrase *exponentially integrable distortion* means that $\exp(\lambda K(x)) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > 0$.

Above, $Df(x)$ denotes the differential matrix of f at the point x (which for $f \in W^{1,1}_{\text{loc}}$ exists a.e.) and $J_f(x) := \det Df(x)$ is the Jacobian. The norm of $Df(x)$ is defined as

$$|Df(x)| := \max\{|Df(x)e| : e \in \mathbf{R}^2,\ |e| = 1\}.$$

We say that a domain Ω satisfies a δ-cone condition, if there exists such a constant $b > 0$ that for any $x \in \partial\Omega$ we can take a line segment $]x, y] \subset \Omega$ of the length $l([x, y]) \geq b$ such that for any $z \in]x, y]$ we have $\operatorname{dist}(z, \partial\Omega) \geq \delta l([x, z])$.

We call a curve in the extended plane a quasicircle if it is the image of a circle under a quasiconformal mapping of the plane. If the mapping can be taken $\mathbf{K}$-quasiconformal, the curve is called a $\mathbf{K}$-quasicircle. A quasidisk is a domain, bounded by a quasicircle.

Let us define the modulus of a path family (see [11]). If Γ is a path family in Ω, then we set

$$\begin{aligned}\operatorname{mod}(\Gamma,\Omega) = \inf\Big\{ \int_\Omega \rho^2(x)\,dx : \rho\colon \mathbf{R}^2 \to [0,\infty[\text{ is a Borel function} \\ \text{s.t. } \int_\gamma \rho\,ds \geq 1 \text{ for every } \gamma \in \Gamma \Big\}.\end{aligned} \tag{3}$$

Finally, we will need the following integral-type isoperimetric inequality.

Lemma 1. *Let $f\colon \Omega \to \mathbf{R}^2$ be a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbf{R}^2)$. Then for each $B(x_0,R) \subset\subset \Omega$ the inequality*

$$⨍_{B(x_0,r)} J_f(x)dx \leq \left(⨍_{\partial B(x_0,r)} |Df(x)|ds \right)^2 \tag{4}$$

holds for almost every $0 < r < R$.

Proof. First, as f is homeomorphism, we have the following inequality (see [6], Theorem 6.3.2)

$$\int_{B(x_0,r)} J_f(x)dx \leq |f(B(x_0,r))|. \tag{5}$$

Next, we use the usual isoperimetric inequality (see [3], 3.2.43 and 3.2.44) for such r that f is absolutely continuous on $\partial B(x_0,r)$ (this is true for a.e. $0 < r < R$):

$$\begin{aligned}|f(B(x_0,r))| &\leq \frac{(H^1(\partial f B(x_0,r)))^2}{4\pi} = \frac{(H^1(f(\partial B(x_0,r))))^2}{4\pi} \\ &\leq \frac{1}{4\pi}\left(\int_{\partial B(x_0,r)} |Df(x)|\,ds \right)^2.\end{aligned} \tag{6}$$

Finally, the combination of (5) and (6) gives us the required inequality. □

3. Homeomorphic case

We first establish the first part of Theorem 1 and Corollary 1 for the homeomorphic case. In the next section, it will be shown how to handle the non-homeomorphic case. First, we record the following auxiliary result (see [7], Lemma 4.2 and its proof).

Lemma 2. *Let $f\colon G \to \mathbf{R}^2$, where G is some domain, be a mapping with finite distortion whose distortion function satisfies*

$$I = \int_G \exp(\lambda K(x))\,dx < \infty. \tag{7}$$

If $B = B(x_0, r_2) \subset G$, then

$$|f(x) - f(y)|^2 \int\limits_{r_1}^{r_2/2} \frac{\lambda\, dt}{t \log(I/\pi t^2)} \leq C_{\lambda,I} \int_B J_f(x)\, dx, \tag{8}$$

whenever $x, y \in B(x_0, r_1) \subset B(x_0, r_2)$.

Let us take a large enough ball $B = B(x_0, R_0)$, containing our fixed domain Ω as its subset and such that $\operatorname{dist}(\Omega, \partial B) \geq R$ for some fixed R. Denote

$$I = \int_B \exp(\lambda K(x))\, dx.$$

In order to prove the theorem for f homeomorphic it suffices to establish the following two lemmas.

Lemma 3. *Under the hypotheses of Theorem 1 we have*

$$|f(x) - f(y)| \leq \frac{C_1(I, \lambda, \delta, \mathbf{K}, R, f)(\int_B J_f(x)\, dx)^{1/2}}{\log^{\frac{\pi\lambda}{2(\pi - \arcsin\delta)}} \frac{C_2(I,\lambda,\delta,\mathbf{K},R)}{|x-y|}}, \tag{9}$$

for all $x, y \in \partial\Omega$, provided f is a homeomorphism.

The proof of Lemma 3 actually shows that the estimate (9) holds also when Ω is unbounded for those $x, y \in \partial\Omega \cap B$ for which

$$\min\{\operatorname{dist}(x, \partial B), \operatorname{dist}(y, \partial B)\} \geq R.$$

In addition, we do not have to require the distortion function to be locally exponentially integrable in the entire plane; it is enough to consider only the set $\{x \in B\colon \operatorname{dist}(x, \partial\Omega) < R + \varepsilon\}$ for some $\varepsilon > 0$.

Lemma 4. *Let Ω be a simply connected bounded domain and suppose that $f \in C(\overline{\Omega})$ is quasiconformal in Ω. If for some positive constants C_1, C_2 and γ the estimate*

$$|f(x) - f(y)| \leq \frac{C_1}{\log^\gamma \frac{C_2}{|x-y|}}, \tag{10}$$

holds for all $x, y \in \partial\Omega$, then there exist such constants $\hat{C}$ and $\tilde{C}$ that

$$|f(x) - f(y)| \leq \frac{\hat{C}}{\log^\gamma \frac{\tilde{C}}{|x-y|}} \tag{11}$$

holds for all $x, y \in \overline{\Omega}$.

Proof of Lemma 3. Let us take such $x, y \in \partial\Omega$ that $|x-y| < \frac{R^2}{8}(\frac{\pi}{I})^{1/2} \leq \frac{1}{16}(\frac{I}{\pi})^{1/2}$ and apply Lemma 2 for $x_0 = x$, $r_1 = 2|x-y|$ and $r_2 = 2(I/\pi)^{\frac{1}{4}} r_1^{\frac{1}{2}}$. The choice of x and y guarantees that $2r_1 < r_2 \leq R$. We have

$$|f(x) - f(y)|^2 \leq C_{\lambda,I} \int_{B(x, 2\sqrt{2}(\frac{I}{\pi})^{\frac{1}{4}}|x-y|^{\frac{1}{2}})} J_f(x)\, dx. \tag{12}$$

Denote $B_r = B(x,r)$. Using Lemma 1 together with the Hölder inequality and the distortion inequality, we obtain

$$\fint_{B_r} J_f(x)\,dx \le \fint_{\partial B_r} K(x)\,ds \fint_{\partial B_r} J_f(x)\,ds. \tag{13}$$

This yields the following differential-type inequality:

$$\frac{d}{dr}\Big(\log\Big(\int_{B_r} J_f(x)\,dx\Big)\Big) \ge \frac{2}{r\fint_{\partial B_r} K(x)\,ds}. \tag{14}$$

Let us choose integers i_R and i_r so that $\log R - 1 < i_R \le \log R$ and $\log r_2 \le i_{r_2} < \log r_2 + 1$. We have

$$\int_{r_2}^{R} \frac{dr}{r\fint_{\partial B_r} K(x)\,ds} \ge \sum_{i=i_{r_2}}^{i_R-1} \int_{e^i}^{e^{i+1}} \frac{dr}{r\fint_{\partial B_r} K(x)\,ds}. \tag{15}$$

Each of the terms on the right-hand side can be estimated in the following way. Fix $i \in \{i_{r_2}, i_{r_2}+1, \dots, i_R - 1\}$. The change of variables $r = e^t$ leads to

$$\int_{e^i}^{e^{i+1}} \frac{dr}{r\fint_{\partial B_r} K(x)\,ds} = \int_{i}^{i+1} \frac{dt}{\fint_{\partial B_{e^t}} K(x)\,ds}. \tag{16}$$

Next, the Jensen inequality yields

$$\int_{i}^{i+1} \frac{dt}{\fint_{\partial B_{e^t}} K(x)\,ds} \ge \Big[\int_{i}^{i+1}\fint_{\partial B_{e^t}} K(x)\,ds\,dt\Big]^{-1}. \tag{17}$$

Using the fact, that f is quasiconformal in Ω, we obtain

$$\begin{aligned}\int_{i}^{i+1}\fint_{\partial B_{e^t}} K(x)\,ds\,dt &= \int_{i}^{i+1} \frac{1}{2\pi e^t}\Big(\int_{\partial B_{e^t}\cap\Omega} K(x)\,ds + \int_{\partial B_{e^t}\cap(\mathbf{R}^2\setminus\overline{\Omega})} K(x)\,ds\Big)dt \\ &\le \mathbf{K} + \int_{i}^{i+1} \frac{1}{2\pi e^t}\int_{l_t} K(x)\,ds\,dt \\ &\le \mathbf{K} + \int_{i}^{i+1} \frac{d(l_t)}{2\pi e^t}\fint_{l_t} K(x)\,ds\,dt,\end{aligned} \tag{18}$$

where l_t is some arc of the circle ∂B_{e^t}, containing the arc $\partial B_{e^t}\cap(\mathbf{R}^2\setminus\overline{\Omega})$ and having the length at least πe^t, and $d(l)$ denotes the length of an arc l. The cone condition for Ω makes it possible to take l_t so that $\pi e^t \le d(l_t) = \max\{d(\partial B_{e^t}\cap(\mathbf{R}^2\setminus\overline{\Omega})), \pi e^t\} \le 2(\pi - \arcsin\delta)e^t$. As the function $\tau \to \exp\lambda\tau$ is convex, we may use the Jensen inequality in order to estimate the remaining term. Applying it twice and making a change of variables, we obtain

$$
\begin{aligned}
\int_i^{i+1} \frac{d(l_t)}{2\pi e^t} ⨍_{l_t} K(x)\,ds\,dt &\le \frac{\pi - \arcsin\delta}{\pi} \int_i^{i+1} ⨍_{l_t} K(x)\,ds\,dt \\
&\le \frac{\pi - \arcsin\delta}{\pi\lambda} \log \int_i^{i+1} ⨍_{l_t} \exp(\lambda K(x))\,ds\,dt \\
&= \frac{\pi - \arcsin\delta}{\pi\lambda} \log \int_{e^i}^{e^{i+1}} \frac{1}{r d(l_{\log r})} \int_{l_{\log r}} \exp(\lambda K(x))\,ds\,dr \qquad (19)\\
&\le \frac{\pi - \arcsin\delta}{\pi\lambda} \log \frac{1}{\pi e^{2i}} \int_{e^i}^{e^{i+1}} \int_{\partial B_r} \exp(\lambda K(x))\,ds\,dr \\
&\le \frac{\pi - \arcsin\delta}{\pi\lambda} \log \frac{I}{\pi e^{2i}}.
\end{aligned}
$$

Finally, combining (15), (16), (17), (18) and (19), we arrive at

$$
\begin{aligned}
\int_{r_2}^{R} \frac{dr}{r ⨍_{\partial B_r} K(x)\,ds} &\ge \sum_{i=i_{r_2}}^{i_R-1} \Big[\frac{\pi - \arcsin\delta}{\pi\lambda} \log \frac{C_{I,\lambda,\delta,\mathbf{K}}}{e^{2i}}\Big]^{-1} \\
&\ge \int_{i_{r_2}-1}^{i_R-2} \Big[\frac{\pi - \arcsin\delta}{\pi\lambda} \log \frac{C_{I,\lambda,\delta,\mathbf{K}}}{e^{2r}}\Big]^{-1} dr \qquad (20)\\
&\ge \frac{\pi\lambda}{\pi - \arcsin\delta} \int_{r_2}^{R/e^3} \frac{dt}{t \log \frac{C_{I,\lambda,\delta,\mathbf{K}}}{t^2}} \\
&= \log \left(\frac{\log \frac{C_{I,\lambda,\delta,\mathbf{K}}}{r_2^2}}{\log \frac{e^6 C_{I,\lambda,\delta,\mathbf{K}}}{R^2}} \right)^{\frac{\pi\lambda}{2(\pi - \arcsin\delta)}}.
\end{aligned}
$$

Together with (14) this gives the estimate

$$
\int_{B_{r_2}} J_f(x)\,dx \le \left(\frac{\log \frac{e^6 C_{I,\lambda,\delta,\mathbf{K}}}{R^2}}{\log \frac{C_{I,\lambda,\delta,\mathbf{K}}}{r_2^2}} \right)^{\frac{\pi\lambda}{\pi - \arcsin\delta}} \int_{B_R} J_f(x)\,dx. \tag{21}
$$

Combining it with (12) we obtain the desired estimate for such $x, y \in \overline{\Omega}$ that $|x-y| < \frac{R^2}{8}(\frac{\pi}{I})^{1/2} \le \frac{1}{16}(\frac{I}{\pi})^{1/2}$. Finally, as Ω is bounded and f is continuous in $\overline{\Omega}$, the estimate (9) actually holds for all $x, y \in \overline{\Omega}$. $\square$

Proof of Lemma 4. Given a point $x \in \Omega$, let us put

$$B_x = B(x, \frac{1}{2}\operatorname{dist}(x, \partial\Omega))$$

and $G_x = 5B_x \cap \partial\Omega$. From the basic modulus estimates and Lemma 2 from [4] it follows that

$$\frac{|f(x) - f(y)|}{\operatorname{diam} f(B_x)} \le C\Big(\frac{|x-y|}{\operatorname{diam} B_x}\Big)^{1/\mathbf{K}}, \tag{22}$$

whenever $y \in B_x$ (here $\mathbf{K}$ is the quasiconformality coefficient of f in Ω), and

$$\operatorname{diam} f(B_x) \asymp \operatorname{dist}(f(B_x), \partial f(\Omega)) = \operatorname{dist}(f(B_x), f(\partial\Omega)). \tag{23}$$

Let us denote the path family connecting B_x and G_x in Ω by Γ. As $\operatorname{diam} B_x \asymp \operatorname{diam} G_x$, Ω is simply connected and $2\operatorname{dist}(B_x, G_x) = \operatorname{diam} B_x$, the modulus $\operatorname{mod}(\Gamma, \Omega)$ has a positive lower bound. Thus, the modulus $\operatorname{mod}(f(\Gamma), f(\Omega))$ has also a positive lower bound. This and (23) imply

$$\operatorname{dist}(f(B_x), f(G_x)) \le C \operatorname{diam} f(B_x) \tag{24}$$

and

$$\operatorname{diam} f(B_x) \le C \operatorname{dist}(f(B_x), f(G_x)) \le C \operatorname{diam} f(G_x), \tag{25}$$

for some constant $C > 0$; otherwise $\operatorname{mod}(f(\Gamma), f(\Omega))$ would be arbitrarily small.

Let us first consider such points $x, y \in \Omega$ that either $x \in B_y$ or $y \in B_x$ holds. Because of the symmetry, we may assume that $y \in B_x$. Combining (22) and (25) and using the estimate on the boundary, we obtain

$$\begin{aligned} |f(x) - f(y)| &\le \hat{C}_1\Big(\frac{|x-y|}{\operatorname{diam} B_x}\Big)^{1/\mathbf{K}} \operatorname{diam} f(G_x) \\ &\le \hat{C}_2\Big(\frac{|x-y|}{\operatorname{diam} B_x}\Big)^{1/\mathbf{K}} \log^{-\gamma}\frac{C_3}{\operatorname{diam} B_x} \le \hat{C}_2 \log^{-\gamma}\frac{C_3}{|x-y|}. \end{aligned} \tag{26}$$

The last step follows from the monotonicity of the function $t \log^{-\gamma\mathbf{K}} \frac{C_3}{|x-y|} t$ for $t \in [\frac{|x-y|}{\operatorname{diam} B_x}, 1]$, provided the constant C_2 in the a priori estimate (10) is big enough (we may always assume it to be as big as we want by changing C_1 in a suitable way).

Let us then consider such points $x, y \in \Omega$, that

$$|x-y| \ge \max\Big\{\frac{1}{2}\operatorname{dist}(x, \partial\Omega), \frac{1}{2}\operatorname{dist}(y, \partial\Omega)\Big\}. \tag{27}$$

Fix some points $x' \in G_x$ and $y' \in G_y$. Notice that

$$|x' - y'| \le |x - x'| + |x - y| + |y - y'| \le 11|x-y|. \tag{28}$$

Next we use the estimate on the boundary for the points $x', y' \in \partial\Omega$, obtaining

$$|f(x') - f(y')| \le C_1 \log^{-\gamma}\frac{C_2}{|x'-y'|} \le C_1 \log^{-\gamma}\frac{C_3}{|x-y|}, \tag{29}$$

again by assuming C_2 to be sufficiently large. Next, using (24) and (25), we obtain

$$\begin{aligned}|f(x)-f(x')| &\le \operatorname{dist}(f(x),f(G_x))+\operatorname{diam} f(G_x)\\ &\le \operatorname{dist}(f(B_x),f(G_x))+\operatorname{diam} f(B_x)+\operatorname{diam} f(G_x)\\ &\le C\operatorname{diam} f(G_x),\end{aligned} \tag{30}$$

for some constant $C>0$. Thus, using the estimate on the boundary and the fact, that $\operatorname{diam} G_x \le 5\operatorname{diam} B_x \le 10|x-y|$, we conclude that

$$|f(x)-f(x')| \le \hat{C}_2 \log^{-\gamma}\frac{\tilde{C}_3}{|x-y|}. \tag{31}$$

Finally, this together with the same kind of estimate for $|f(y)-f(y')|$ and (29) gives us the desired estimate for $|f(x)-f(y)|$ with the help of triangle inequality. The statement of the lemma for the remaining cases, for example when $x\in\partial\Omega$ and $y\in\Omega$, can be obtained in the same way. □

Finally, let us show that Corollary 1 holds for homeomorphic f. Given the unit disk $B=B(0,1)$, let us map it conformally onto the upper half-plane H with the help of a Möbius transformation ψ having the point $(0,1)$ as its pole. The mapping $f\circ\psi^{-1}$ is quasiconformal in H and its distortion is locally exponentially integrable in some half-plane $P=\{(x_1,x_2)\in\mathbf{R}^2\colon x_2>-h\}$, where $h>0$. Indeed, take $h<-y_2$, where $(y_1,y_2)\in\mathbf{R}^2\setminus\overline{H}$ is the pole of the Möbius transformation ψ^{-1}. For each $x\in P$ we have

$$\begin{aligned}|D(f\circ\psi^{-1})(x)|^2 &= |Df(\psi^{-1}(x))D\psi^{-1}(x)|^2 \le |Df(\psi^{-1}(x))|^2|D\psi^{-1}(x)|^2\\ &\le K(\psi^{-1}(x))J_f(\psi^{-1}(x))J_{\psi^{-1}}(x) = K(\psi^{-1})J_{f\circ\psi^{-1}}(x).\end{aligned} \tag{32}$$

So, the composition $f\circ\psi^{-1}$ has the finite distortion function

$$K_{f\circ\psi^{-1}}(x)\le K(\psi^{-1}(x))$$

for $x\in P$. Let us show that it is locally exponentially integrable with the same λ. Choose a compact set $E\subset P$. Using a change of variables, we obtain

$$\begin{aligned}\int_E \exp[\lambda K(\psi^{-1}(x))]\,dx &= \int_E \exp[\lambda K(\psi^{-1}(x))]J_{\psi^{-1}}(x)J^{-1}_{\psi^{-1}}(x)\,dx\\ &= \int_{\psi^{-1}(E)} \exp(\lambda K(y))J^{-1}_{\psi^{-1}}(\psi(y))\,dy\\ &= \int_{\psi^{-1}(E)} \exp(\lambda K(y))J_{\psi}(y)\,dy\\ &\le \sup_{\psi^{-1}(E)} J_\psi \int_{\psi^{-1}(E)} \exp(\lambda K(y))\,dy<\infty.\end{aligned} \tag{33}$$

As H satisfies the cone condition for $\delta=1$, we may apply the local version of Lemma 3 for the mapping $f\circ\psi^{-1}$. In order to do it, we take a ball $B_0=(x_0,R_0)\subset\mathbf{R}^2$ so big that for all $x\in\partial B\cap\{(x_1,x_2)\in\mathbf{R}^2\colon x_2<2/3\}$ we had $\psi(x)\in B_0$ and

$\operatorname{dist}(x, \partial B_0) > R$ for a fixed $R < h$. So, for $x, y \in \partial B \cap \{(x_1, x_2) \in \mathbf{R}^2 \colon x_2 < 2/3\}$ we obtain

$$\begin{aligned} |f(x) - f(y)| &= |(f \circ \psi^{-1})(\psi(x)) - (f \circ \psi^{-1})(\psi(y))| \\ &\leq \frac{\hat{C}}{\log^\lambda \frac{\tilde{C}}{|\psi(x)-\psi(y)|}} \leq \frac{\hat{C}}{\log^\lambda \frac{C'}{|x-y|}}. \end{aligned} \tag{34}$$

Here we used the fact, that $|\psi(x) - \psi(y)| \leq M|x - y|$ for some constant $M > 0$, whenever $x, y \in \mathbf{R}^2 \setminus B((0, 1), \frac{1}{3})$.

Repeating the reasoning for the upper part of the ball B (and taking the point $(0, -1)$ as a pole), we obtain an estimate of the same kind for $x, y \in \partial B \cap \{(x_1, x_2) \in \mathbf{R}^2 \colon x_2 > -2/3\}$ and thus for all x and y on the boundary ∂B. Finally, the claim follows by invoking Lemma 4.

4. Proof of Theorem 1

We will pass from the homeomorphic case to the non-homeomorphic, using the so-called Stoilow factorization (see, for example, [6], Chapter 11). Let us first note that the given mapping f defined in the plane and having finite locally exponentially integrable distortion belongs to the Orlicz–Sobolev class $W^{1,Q}_{\mathrm{loc}}(\mathbf{C})$, where $Q(t) = \frac{t^2}{\log(e+t)}$ (see, for example, [6], §11.5). The mapping f satisfies almost everywhere the equation

$$\overline{\partial} f(z) = \mu_f(z) \partial f(z), \tag{35}$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $|\mu_f(z)| \leq \frac{K(z)-1}{K(z)+1}$. Equation (35) is called the Beltrami equation. Let us take a ball B, containing the domain Ω, where the given mapping f is quasiconformal. Consider the Beltrami equation with the Beltrami coefficient $\mu = \mu_f \chi_B$. By Theorem 11.8.3 in [6], this equation has a homeomorphic solution h in the class $z + W^{1,Q}_{\mathrm{loc}}(\mathbf{C})$ (i.e. $|h_{\overline{z}}| + |h_z - 1| \in L^Q(\mathbf{C})$). Next, the mapping $f\big|_B$ is a solution of this equation in B, so by Theorem 11.5.1 in [6] it can be represented as $f\big|_B = \varphi \circ h$, where $\varphi \colon h(B) \to \mathbf{C}$ is holomorphic. As a solution of the same Beltrami equation, h satisfies

$$|Df(z)|^2 \leq K(z) J_h(z)$$

almost everywhere in B. Using the fact that φ is Lipschitz in $h(\overline{\Omega}) \subset\subset h(B)$ and the obtained continuity estimate for the mapping h, we easily get the required inequality for f. The corollary is dealt in the same way.

We will base the construction of our example, showing the sharpness of the obtained result, on a mapping constructed in [8] (f_2 from the proof of Theorem 1). Based on what is done in [8], we can state the following lemma.

Lemma 5. *For a given $s > 0$ there exists a homeomorphic mapping f of finite distortion which is quasiconformal in the right half-plane $H = \{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 >$*

$0\}$ *such that its distortion function* K *in the left half-plane satisfies*

$$K(x) \le 2s\log(2/|x|) + C, \tag{36}$$

where $C > 0$ *is some constant, for all* $x \in B(0,r) \cap (\mathbf{R}^2 \setminus H)$ *for some* $r > 0$ *and is bounded in* $\mathbf{R}^2 \setminus B(0,r)$*, and for all positive* C_1*,* C_2 *and* ε *there exists such* $x_0 \in \partial H$ *that*

$$|f(x) - f(0)| = |f(x)| > C_1 \log^{-\frac{1}{s}-\varepsilon} \frac{C_2}{|x|} \tag{37}$$

holds for all $x \in \partial H$*, such that* $|x| < |x_0|$.

As we can notice

$$\exp(\lambda K(x)) \le \frac{C}{|x|^{2s\lambda}}, \tag{38}$$

that is, the distortion of f is locally exponentially integrable for all $\lambda < 1/s$.

Let us then consider the domain $\Omega = \{(R\cos\theta, R\sin\theta) \in \mathbf{R}^2 \colon R \in \mathbf{R},\ -\alpha < \theta < 0\}$, where $0 < \alpha < \pi$ is some fixed angle. It satisfies the cone condition for $\delta = \sin\frac{\alpha}{2}$. This domain can always be cut in such a way that the remaining domain $\Omega_0 \subset \Omega$ is bounded, satisfies the cone condition for the same δ and its boundary near the origin coincides with the boundary of the domain Ω. For example, if $0 < \alpha < \frac{\pi}{2}$, then Ω_0 can be taken in the form $\Omega_0 = \Omega \cap B(0, R_0)$ for some $R_0 > 0$.

Denote $\beta = \frac{\pi}{2\pi - \alpha}$ and take the mapping $g \colon \mathbf{R}^2 \setminus \overline{\Omega} \to \mathbf{R}^2$ defined by $g(R\cos\theta, R\sin\theta) = (R^\beta \sin\beta\theta, -R^\beta \cos\beta\theta)$. This mapping maps the set $\mathbf{R}^2 \setminus \overline{\Omega} = \{(R\cos\theta, R\sin\theta) \in \mathbf{R}^2 \colon R \in \mathbf{R},\ 0 < \theta < 2\pi - \alpha\}$ conformally onto the right half-plane $H = \{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 > 0\}$ and is extendable to a quasiconformal mapping of the whole plane.

Next, consider the superposition $f \circ g$, where f is the mapping from Lemma 5. It is quasiconformal in Ω and, hence, in Ω_0; indeed, in the same way as before for $x \in \Omega$ we calculate

$$|D(f \circ g)(x)|^2 = |Df(g(x))Dg(x)|^2 \le \mathbf{K}_f \mathbf{K}_g J_{f\circ g}(x), \tag{39}$$

where $\mathbf{K}_f$ and $\mathbf{K}_g$ denote the quasiconformality coefficients of f and g respectively. Similarly, we can estimate the distortion outside $\overline{\Omega}$:

$$|D(f \circ g)(x)|^2 \le K(g(x)) J_{f\circ g}(x). \tag{40}$$

Thus, for the distortion function of $f \circ g$, denoted by $K_{f\circ g}$, we have the estimate

$$K_{f\circ g}(x) \le K(g(x)) \le 2s\log\frac{2}{|g(x)|} + C = 2s\log\frac{2}{|x|^\beta} + C \tag{41}$$

and

$$\exp(\mu K_{f\circ g}(x)) \le \frac{C}{|x|^{2s\mu\beta}}, \tag{42}$$

for $x \in \mathbf{R}^2 \setminus \overline{\Omega}_0$ close to the origin, so it is exponentially integrable for all $\mu < 1/s\beta = \frac{2(\pi - \arcsin\delta)}{s\pi} = \frac{2\pi-\alpha}{s\pi}$. Thus, Lemma 3 gives us the estimate (10) for the boundary points with $\gamma = 1/s - \varepsilon$ for any given positive ε.

Finally, using Lemma 5, we calculate

$$
\begin{aligned}
|(f\circ g)(x) - (f\circ g)(0)| = |f(g(x))| &> C_1 \log^{-\frac{1}{s}-\varepsilon} \frac{C_2}{|g(x)|} \\
&= C_1\beta^{-\frac{1}{s}-\varepsilon} \log^{-\frac{1}{s}-\varepsilon} \frac{C_2^{\frac{1}{\beta}}}{|x|}
\end{aligned}
\tag{43}
$$

for x, close enough to the origin. This completes the proof of the theorem.

5. Result for quasidisks

Recall that each quasidisk can be mapped onto the exterior of the unit disk under a conformal mapping, which is extendable to a quasiconformal mapping of the entire plane (see, for example, [9], Chapter I, §6). Thus, let us state the following theorem.

Theorem 2. *Let Ω be a bounded quasidisk such that some conformal mapping $\varphi\colon \mathbf{R}^2 \setminus \overline{\Omega} \to \mathbf{R}^2$, mapping the exterior of $\overline{\Omega}$ onto the exterior of the closed unit disk $\overline{B}$, has the property $J_\varphi \in L^p(\hat{B} \setminus \overline{\Omega})$, where $\hat{B}$ is some ball, containing $\overline{\Omega}$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a mapping of finite distortion such that* $\exp(\lambda K(x))$ *is locally integrable for some $\lambda > 0$. If the restriction of f to the quasidisk Ω is quasiregular, then there exist positive constants C_1 and C_2 such that*

$$
|f(x) - f(y)| \le \frac{C_1}{\log^{\frac{p-1}{p}\lambda} \frac{C_2}{|x-y|}}, \tag{44}
$$

whenever $x, y \in \overline{\Omega}$.

Proof. As it was shown before, it is enough to consider the homeomorphic case. Denote by $\tilde{\varphi}$ a quasiconformal extension of φ to the entire plane. Let us first note that the superposition $f\circ\tilde{\varphi}^{-1}$ satisfies the conditions of the Corollary 1. Indeed, for $x \in B$ we have

$$
|D(f\circ\tilde{\varphi}^{-1})(x)|^2 = |Df(\tilde{\varphi}^{-1}(x))D\tilde{\varphi}^{-1}(x)|^2 \le \mathbf{K}_f \mathbf{K}_{\tilde{\varphi}^{-1}} J_{f\circ\tilde{\varphi}^{-1}}(x), \tag{45}
$$

that is, the mapping $f\circ\tilde{\varphi}^{-1}$ is quasiconformal in B. Let us now consider the exterior of B. For $x \in \mathbf{R}^2 \setminus \overline{B}$ we have that

$$
|D(f\circ\tilde{\varphi}^{-1})(x)|^2 \le K(\varphi^{-1}(x)) J_{f\circ\varphi^{-1}}(x). \tag{46}
$$

So, the composition $f\circ\tilde{\varphi}^{-1}$ has the finite distortion function

$$
K_{f\circ\tilde{\varphi}^{-1}}(x) \le K(\varphi^{-1}(x))
$$

for $x \in \mathbf{R}^2 \setminus \overline{B}$. Let us show that it is exponentially integrable in $\varphi(\hat{B})$ with some λ_1. Indeed, using a change of variables and the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\varphi(\hat{B})} \exp(\lambda_1 K_{f\circ\tilde{\varphi}^{-1}}(x))\,dx \\
&\le \int_B \exp(\lambda_1 \mathbf{K}_f \mathbf{K}_{\tilde{\varphi}^{-1}})\,dx + \int_{\varphi(\hat{B})\setminus\overline{B}} \exp[\lambda_1 K(\varphi^{-1}(x))]\,dx \\
&= \int_{\hat{B}\setminus\overline{\Omega}} \exp[\lambda_1 K(y)] J_\varphi(y)\,dy + C \\
&\le \Big(\int_{\hat{B}\setminus\overline{\Omega}} \exp\Big[\lambda_1 \frac{p}{p-1} K(y)\Big]\,dy\Big)^{(p-1)/p} \Big(\int_{\hat{B}\setminus\overline{\Omega}} J_\varphi^p(y)\,dy\Big)^{1/p} + C < \infty,
\end{aligned} \tag{47}$$

when $\lambda_1 = \frac{p-1}{p}\lambda$. After applying Corollary 1 we arrive at

$$\begin{aligned}
|f(x) - f(y)| &= |(f\circ\tilde{\varphi}^{-1})(\tilde{\varphi}(x)) - (f\circ\tilde{\varphi}^{-1})(\tilde{\varphi}(y))| \\
&\le \frac{\hat{C}}{\log^{\frac{p-1}{p}\lambda} \frac{\tilde{C}}{|\tilde{\varphi}(x)-\tilde{\varphi}(y)|}} \le \frac{C_1}{\log^{\frac{p-1}{p}\lambda} \frac{C_2}{|x-y|}},
\end{aligned} \tag{48}$$

whenever $x, y \in \overline{\Omega}$ (here we used the local Hölder continuity of the quasiconformal mapping $\tilde{\varphi}$ and the boundedness of Ω). □

Let us return to the domain Ω from Section 4. This domain is a quasidisk. Let us map it conformally onto the upper half-plane by means of the mapping h_2 having the form $h_2(z) = z^\beta$ in terms of the complex plane. Let us now map the upper half-plane onto the exterior of the unit disk using the Möbius transformation $h_1(z) = \frac{z+i}{z-i}$ in terms of the complex plane. The pole of this map is the point $a = (0,1)$. Its preimage in Ω is $b = h_2^{-1}(a) = (\cos(\pi - \frac{\alpha}{2}), \sin(\pi - \frac{\alpha}{2})) = (\cos\frac{\pi}{2\beta}, \sin\frac{\pi}{2\beta})$. Let us take the Möbius transformation h_3 of the complex plane, mapping infinity to this point, for example, $h_3(z) = \frac{(\cos\frac{\pi}{2\beta} + i\sin\frac{\pi}{2\beta})z}{z+1}$. The superposition $h = h_1\circ h_2\circ h_3$ has the form $h(z) = \frac{z^\beta + (1+z)^\beta}{z^\beta - (1+z)^\beta}$. This mapping preserves infinity and maps conformally the exterior of the bounded domain $h_3^{-1}(\Omega)$ onto the the exterior of the unit disk B. The Jacobian determinant of g is p-integrable when $p < \frac{2\pi-\alpha}{\pi-\alpha}$. Thus, Theorem 2 gives for the mapping $f\circ g$ near the origin the continuity estimate (10) with the exponent $1/s - \varepsilon$ for any given positive ε, which is sharp by Theorem 1.

Remark. The conclusion of Theorem 2 is interesting only when $\frac{p-1}{p} > \frac{1}{2}$, i.e., when $p > 2$. It appears to be unknown if this is always the case; by Brennan's conjecture any $p < 2$ would do even when Ω is not a quasidisk. One could also modify the proof of Theorem 1 to cover the case of a "twisted" cone condition, satisfied by quasidisks. This would give an exponent strictly better than $\frac{\lambda}{2}$ but the dependence from K would be complicated.

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