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THE REGULARITY OF WEAK SOLUTIONS TO NONLINEAR SCALAR FIELD ELLIPTIC EQUATIONS CONTAINING $p\&q$-LAPLACIANS

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Abstract. In this paper, we consider the regularity of weak solutions $u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N)$ of the elliptic partial differential equation

$$-\Delta_p u - \Delta_q u = f(x),\ x \in \mathbf{R}^N,$$

where $1 < q < p < N$. We prove that these solutions are locally in $C^{1,\alpha}$ and decay exponentially at infinity. Furthermore, we prove the regularity for the solutions $u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N)$ of the following equations

$$-\Delta_p u - \Delta_q u = f(x,u),\ x \in \mathbf{R}^N,$$

where $N \geq 3$, $1 < q < p < N$, and $f(x,u)$ is of critical or subcritical growth about u. As an application, we can show that the solution we got in [8] has the same regularity.

1. Introduction

In this paper, we study the regularity of weak solutions to the following nonlinear elliptic equations with $p\&q$-Laplacians:

$$(1.1)\qquad \begin{cases} -\Delta_p u + m|u|^{p-2}u - \Delta_q u + n|u|^{q-2}u = g(x,u), & x \in \mathbf{R}^N,\\ u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N), \end{cases}$$

where $m, n > 0$, $N \geq 3$, $1 < q < p < N$, $\Delta_t u = \operatorname{div}(|\nabla u|^{t-2}\nabla u)$ is the t-Laplacian of u for $t > 1$.

The $p\&q$-Laplacian problem (1.1) comes, for example, from a general reaction diffusion system

$$(1.2)\qquad u_t = \operatorname{div}[D(u)\nabla u] + c(x,u),$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, the

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first term on the right-hand side of (1.2) corresponds to the diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration u.

Recently, the eigenvalue problem for a $p\&q$-Laplacian type equation with $p=2$ was studied by Bence [1] and the stationary solution of (1.2) was studied by Cherfils and Il'yasov in [4] on a bounded domain $\Omega \subset \mathbf{R}^N$ with $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$ and $c(x,u) = -p(x)|u|^{p-2}u - q(x)|u|^{q-2}u + \lambda g(x)|u|^{\gamma-2}u$ for $1 < p < \gamma < q$ and $\gamma < p^*$, where $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* = +\infty$, if $p \geq n$.

In [8], using the concentration compactness principle and Mountain Pass Theorem, we proved the existence of a nontrivial solution to (1.1) under suitable assumptions on $g(x,u)$(see (C_1)–(C_5) in [8]). It is natural to study the regularity of weak solutions of (1.1). To this end, we consider the following equation

$$-\Delta_p u - \Delta_q u = f(x), \tag{1.3}$$

where $f \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$. By a weak solution u to (1.3), we mean a function $u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N)$ (or $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N)$) such that

$$\int_{\mathbf{R}^N} \left[|\nabla u|^{p-2}\nabla u \nabla\varphi + |\nabla u|^{q-2}\nabla u\nabla\varphi - f(x)\varphi\right] dx = 0 \text{ for any } \varphi \in C_0^\infty(\mathbf{R}^N).$$

It is obvious that (1.1) is a special case of (1.3) if we take $f(x) = g(x,u(x)) - m|u(x)|^{p-2}u(x) - n|u(x)|^{q-2}u(x)$.

For degenerate elliptic equations

$$-\Delta_p u = f(x,u) \tag{1.4}$$

and systems with some special structure, the $C^{1,\alpha}$ regularity of weak solutions was proved in [7] when $p=2$, and in [11, 17, 18] and [6] when $p \geq 2$. The existence and integrability of second-order derivatives of weak solutions to (1.4) were studied in [13, 15, 19] for all $1 < p < +\infty$, from which the $C^{1,\alpha}$ regularity of weak solutions to (1.4) is obtained.

With an extra assumption that $u \in L^\infty(\Omega)$, [5] and [16] proved the local $C^{1,\alpha}$ regularity of the solutions u to a general class of quasilinear elliptic equations

$$\int_\Omega \sum_{j=1}^N \left\{a_j(x,u,\nabla u)\cdot\varphi_{x_j}\right\} - h(x,u,\nabla u)\varphi\, dx = 0,\ \varphi \in C_0^\infty(\Omega), \tag{1.5}$$

where a_j belongs to $C^0(\Omega\times\mathbf{R}\times\mathbf{R}^N)\cap C^1(\Omega\times\mathbf{R}\times\mathbf{R}^N - \{0\})$ and h is a Caratheodory function, i.e., for each $(t,p) \in \mathbf{R}^{N+1}$, $h(x,t,p)$ is measurable in x and continuous in t and p for a.e. $x \in \mathbf{R}^N$. It was shown that their results can be applied to (1.4) for all $1 < p < \infty$.

The decay of the solution u of p-Laplacian type equations were considered by many authors. When $p=2$, [2] showed that under some conditions on f, if u is a radially symmetric solution of

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), & u \neq 0, \end{cases} \tag{1.6}$$

then $u \in C^2(\mathbf{R}^N)$ and

$$|D^\alpha u(x)| \le Ce^{-\delta|x|},\ x \in \mathbf{R}^N, \tag{1.7}$$

for some $C,\ \delta > 0$ and for $|\alpha| \le 2$. By introducing exponential weighted spaces, [3] showed that positive solutions of

$$\begin{cases} -\Delta u + f(x,u) = 0 & \text{in } \mathbf{R}^N, \\ u \to 0 & \text{at infinity}, \end{cases} \tag{1.8}$$

decay exponentially at infinity.

Under suitable assumptions on $V(x)$ and f, the existence and $C^{1,\alpha}$ regularity of weak solutions of the p-Laplacian type Schrödinger equations

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \\ u \in W^{1,p}(\mathbf{R}^N),\ 1 < p < +\infty, \end{cases} \tag{1.9}$$

were proved in [11]. Furthermore, it was shown in [11] that the solutions decay exponentially in x when $|x| \ge R$ for some $R > 0$. We extend this result to $p\&q$-Laplacian type equations, too.

Our main results are as follows:

Theorem 1. *Suppose that $f \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ is a weak solution of (1.3) where $p > 1$. Then*

(i) $|\nabla u| \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ *and for every compact $K \subset \mathbf{R}^N$, there exists a constant C depending only on N, p, q, $\operatorname*{ess\,sup}_K |u|$ and $\operatorname*{ess\,sup}_K |f|$ such that*

$$\|\nabla u\|_{L^\infty(K)} \le C; \tag{1.10}$$

(ii) *$x \to \nabla u(x)$ is locally Hölder continuous in $\mathbf{R}^N$, i.e., there exists an $\alpha \in (0,1)$ and a constant C depending only upon N, p, q, $\operatorname*{ess\,sup}_K |u|$ and $\operatorname*{ess\,sup}_K |f|$ for every compact $K \subset \mathbf{R}^N$, such that*

$$|\nabla u(x) - \nabla u(y)| \le C|x-y|^\alpha, \quad x, y \in K. \tag{1.11}$$

Theorem 2. *Suppose that $f(x,t)$ satisfy:*

(A1) *$f(x,t)\colon \mathbf{R}^N \times \mathbf{R}^1 \to \mathbf{R}^1$ satisfies the Caratheodory conditions, i.e., for a.e. $x \in \mathbf{R}^N$, $f(x,t)$ is continuous in $t \in \mathbf{R}^1$ and for each $t \in \mathbf{R}^1$, $f(x,t)$ is Lebesgue measurable with respect to $x \in \mathbf{R}^N$.*

(A2) *$f(x,t)$ is of critical or subcritical growth about u at infinity, i.e., for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that $|f(x,t)| \le \varepsilon|t|^{q-1} + C_\varepsilon|t|^{p^*-1}$ for all $(x,t) \in \mathbf{R}^N \times R^1$, where $p^* = \frac{NP}{N-p}$ if $N > p$, $0 < p^* < +\infty$ if $N \le p$.*

If $u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N)$, $1 < q < p < N$, *is a weak solution of*

$$-\Delta_p u - \Delta_q u = f(x,u), \tag{1.12}$$

then there is an $\alpha > 0$ *and a constant* C *depending only on* N, p, q, $\operatorname{ess}\sup_{B_R(x_0)} |u|$ *for any* $R > 0$, *such that*

$$|\nabla u(x)| \leq C, \tag{1.13}$$

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^\alpha \tag{1.14}$$

for all $x, y \in B_R(x_0)$ *and any* $x_0 \in \mathbf{R}^N$.

In [8] the existence of a weak solution of (1.1) was obtained under the following assumptions:

(C_1) $g\colon \mathbf{R}^N \times \mathbf{R}^1 \to \mathbf{R}^1$ satisfies the Caratheodory conditions; $g(x,t) \geq 0$, for $t \geq 0$ and $g(x,t) \equiv 0$, for $t < 0$ and all $x \in \mathbf{R}^N$,

(C_2) $\lim_{t\to 0^+} \frac{g(x,t)}{t^{p-1}} = 0$ uniformly in $x \in \mathbf{R}^N$; $\lim_{s\to +\infty} \frac{g(x,t)}{t^{p-1}} = \ell$ uniformly in $x \in \mathbf{R}^N$ for some $\ell \in (0,+\infty)$,

and some extra technical conditions.

By Theorem 1 and 2, it is easy to see that weak solutions of (1.1) are locally in $C^{1,\alpha}$. We also get the exponential decay of weak solutions at infinity under the hypotheses (C_1) and (C_2).

In fact, we have the following result:

Theorem 3. *Suppose* $g(x,t)$ *satisfies (A1), (A2) of Theorem 2 and* u *is a weak solution of (1.1). Then*

(i) u *is bounded on* $\mathbf{R}^N$, *i.e.,* $\|u\|_{L^\infty(\mathbf{R}^N)} < +\infty$ *and* $\lim_{R\to+\infty} \|u\|_{L^\infty(|x|>R)} = 0$;

(ii) $u(x)$ *decays exponentially as* $|x| \to +\infty$, *i.e.,* $\exists C > 0$, $\varepsilon > 0$, $R > 0$ *such that*

$$|u(x)| \leq Ce^{-\varepsilon|x|} \quad \text{when } |x| \geq R. \tag{1.15}$$

One cannot obtain Theorem 1 by the results in [5, 16] or [11], since the p&q-Laplace equations do not satisfy the assumptions in [5, 16] and [11]. Our results are new to our knowledge; they are the generalization of the results of [5, 16] and [11]. Theorem 2 is an application of Theorem 1, which may be applied to more cases.

To prove Theorem 1, we mainly use the frame works of [5, 16, 11], respectively, to different steps. Since the main purpose of [5, 16] and [11] is to consider the regularity of weak solutions for p-Laplacian type equations, the ellipticity and growth conditions imposed on a_j are homogeneous about ∇u. For example, in [16], it is required that

$$\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}(x,\mu,\eta)\cdot \xi_i\xi_j \geq \gamma\cdot(\kappa+|\eta|)^{p-2}|\xi|^2, \tag{1.16}$$
$$\sum_{i,j=1}^{N} \left|\frac{\partial a_j}{\partial \eta_i}(x,\mu,\eta)\right| \leq \Gamma\cdot(\kappa+|\eta|)^{p-2}$$

for some $\gamma,\ \Gamma > 0$ and $\kappa \in [0,1]$. It is obvious that $p\&q$-Laplace equations do not satisfy the above conditions. Since $p\&q$-Laplace equations can not be included in the frame works of [5, 16] or [11], much more careful analysis is needed in the proof.

We use the method of Proposition 1 in [16] to get a useful identity (see (2.5) in §2 below). Although in [16] only a similar inequality is required to show the boundedness of the gradiant ∇u of any weak solution u to (1.3), we expect that this identity can be used somewhere. After the local boundedness of $|\nabla u|$ is proved, we follow the usual way (see [7, 9]) to obtain the $C^{1,\alpha}$ regularity of the weak solution.

To prove Theorem 2, we use Theorem 1. To apply Theorem 1, we need only to prove the local boundedness of the weak solutions u, i.e., $\|u\|_{L^\infty(B_R(x_0))} \leq C(x_0)$ for any given $x_0 \in \mathbf{R}^N$ and then apply Theorem 1 with $f(x) = f(x,u(x))$. Usually, one uses the test function $\varphi = \eta^p u^+(u_L^+)^{p(\beta-1)}$ with

$$u_L^+ = \begin{cases} u^+, & u < L,\\ L, & u \geq L,\end{cases}$$

to prove the local boundedness of u^+ (see, e.g., [10, 12]). As one may see, this test function does not work in our case. We follow [14] to define $\bar u = u^+ + k$, and

$$\bar u_L = \begin{cases} \bar u, & u^+ < L,\\ L+k, & u \geq L,\end{cases}$$

and $\varphi(x) = \eta^p(\bar u \bar u_L^{p(\beta-1)} - k^{p(\beta-1)+1})$ for some $k > 0$ as a test function. It turns out that this test function does work.

To prove Theorem 3, we mainly use the method of [11]. The key step is to get a decay estimate of the weak solution as in [10](see (5.25) below). However, as both p and q-Laplacian are involved, the test functions used in [10, 11, 14] do not work. We overcome this difficulty by using two test functions separately, to get a couple of inequalities and then combine them to get (5.25). As soon as (5.25) is obtained, the exponential decay of the solutions will be obtained as in [11].

The paper is organized as follows: In §2, we prove Theorem 1(i); in §3, we prove Theorem 1(ii); in §4, we prove the boundedness of weak solutions and then apply Theorem 1 to prove Theorem 2. In §5, we give the proof of Theorem 3.

Our symbols are standard. For example, $B_r(x_0)$ for $x_0 \in \mathbf{R}^N,\ r > 0$ is the open ball $\{x \in \mathbf{R}^N \big|\ |x-x_0| < r\}$; $L^p(\Omega)$ is the usual L^p-space over the domain $\Omega \subset \mathbf{R}^N$ with norm $\|\cdot\|_{L^p(\Omega)}$; $\operatorname{meas} E$ means the N-dimensional Lebesgue measure of the set $E \subset \mathbf{R}^N$, and so on.

2. The proof of Theorem 1(i)

In this section, we give the proof of Theorem 1(i). To this end, we consider the following equation

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x), & x \in \mathbf{R}^N, \\ u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N), & 1 < q < p. \end{cases} \tag{2.1}$$

Notice that we have by the assumptions that

$$f \in L^\infty_{\mathrm{loc}}(R^N), \quad u \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N). \tag{2.2}$$

We will show that

$$\|\nabla u\|_{L^\infty(B_R(x_0))} \le C, \tag{2.3}$$

where C is a constant depending only on N, p, q, and $\|u\|_{L^\infty(B_R(x_0))}$. For simplicity, we give the proof on $B \equiv B_1(x_0)$, the unit ball in $\mathbf{R}^N$ with centre x_0 for any given $x_0 \in \mathbf{R}^N$. Firstly, we prove an identity inspired by [16].

Proposition 2.1. *If ψ is a nonnegative C^2-function with compact support and $G\colon \mathbf{R}^1 \to \mathbf{R}^1$ is a piecewise C^1-function with only finitely many breaks and*

$$0 \le G' \le c_0 \tag{2.4}$$

for some constant c_0, then any weak solution u of (2.1) satisfies

$$\begin{aligned} &\int_B \sum_{i,j=1}^N \{(|\nabla u|^{p-2} + |\nabla u|^{q-2})\delta_{ij} + [(p-2)|\nabla u|^{p-4} + (q-2)|\nabla u|^{q-4}]u_{x_i}u_{x_j}\} \\ &\quad \cdot u_{x_s,x_i}u_{x_s,x_j}G'(u_{x_s})\psi\, dx \\ &= \int_B \sum_{j=1}^N (|\nabla u|^{p-2} + |\nabla u|^{q-2})u_{x_j} \cdot \frac{d}{dx_s}\{G(u_{x_s})\psi_{x_j}\}dx \\ &\quad - \int_B f\frac{d}{dx_s}\{G(u_{x_s})\psi\}dx, \end{aligned} \tag{2.5}$$

where δ_{ij} are the Kronecker symbols.

Proof. The proof follows by multiplying equation (2.1) by $\frac{d}{dx_s}(G(u_{x_s})\psi)$ and integrating by parts. □

Next we show the L^∞-estimate of the gradient of solutions u of (2.1). Before that we give the following result.

Lemma 2.2. ([16], Corollary 1) *For any $v \in W^{1,p}(B_R)$, where $B_R = B_R(x_0)$ for any fixed $x_0 \in \mathbf{R}^N$, suppose that*

$$\int_{B_R} |v|\, dx \le M \cdot R^N \tag{2.6}$$

and

$$\int_{A_{k,r}} |\nabla v|^p \, dx \le M^p \cdot (r' - r)^{-p} \cdot R^{N\alpha} \cdot (\operatorname{meas} A_{k,r'})^{1-\alpha} \tag{2.7}$$

for some constant M, some $\alpha \in (0, p/N)$, all $k \ge 0$ and all r and r' satisfying

$$R/2 < r < r' \le R,$$

where $A_{k,r} = \{x \in B_r(x_0) \big| v(x) > k\}$. Then there is a constant C depending only on N, p, and α such that

$$v \le C \cdot M \quad \text{in } B_{R/2}(x_0). \tag{2.8}$$

For the proof of Theorem 1(i), it is enough to prove the following result.

Proposition 2.3. *Suppose that (2.2) holds for the weak solution u of (2.1). Then for any $x_0 \in \mathbf{R}^N$, there exists a constant C depending only on N, p, q, $\operatorname{ess\,sup}_B |u|$ and $\operatorname{ess\,sup}_B |f|$ such that*

$$|\nabla u| \le C \quad \text{in } B_{1/2}(x_0), \tag{2.9}$$

where $B = B_1(x_0)$.

Proof. Choose a nonnegative C^∞-function ρ having the properties

$$\rho(t) \begin{cases} = 0, & \text{for } t \ge 1, \\ \in (0,1), & \text{for } t \in (0,1), \\ = 1, & \text{for } t \le 0. \end{cases} \tag{2.10}$$

For $R \in (0, 1/8)$ and $i \in \mathbf{Z}^+ \cup \{0\}$, we set

$$\begin{aligned} R_i &= 2R + 2^{-i-1} R, \\ B_i &= B_{R_i}(x_0), \\ \varphi_i(x) &= \rho(2^{i+1} R^{-1}(|x - x_0| - R_i)). \end{aligned} \tag{2.11}$$

In the following, C stands for a generic constant depending only on N, p, q, $\operatorname{ess\,sup}_B |u|$ and $\operatorname{ess\,sup}_B |f|$ and may differ in different spaces, where $B = B_1(x_0)$. In contrast to C, the generic constant $C(R)$ may also depend on R, and $C(\varepsilon)$ may depend on ε.

To prove (2.9), we will first show that there is an $R_0 > 0$ depending only on N, p, q, $\operatorname{ess\,sup}_B |u|$ and $\operatorname{ess\,sup}_B |f|$ such that

$$\int_{B_i} |\nabla u|^{p+2i} dx \le C(R) \tag{2.12}$$

for $i = 0, 1, \ldots, [Np]$ provided that

$$R \le R_0, \tag{2.13}$$

where $[Np]$ is the integer part of Np. It can be seen that (2.12) is true for $i = 0$. Hence we may suppose that (2.12) holds for some $i \in \{1, \dots, [Np] - 1\}$ and then we prove that it is true for $i + 1$.

We pick an $M > 0$ and define for $t \in \mathbf{R}^1$ that

$$g(t) = \begin{cases} t - 1, & \text{if } t \geq 1, \\ 0, & \text{if } t \in [-1, 1], \\ t + 1, & \text{if } t \leq -1, \end{cases}$$

$$g_M(t) = \begin{cases} M, & \text{if } g(t) \geq M, \\ g(t), & \text{if } g(t) \in [-M, M], \\ -M, & \text{if } g(t) \leq -M, \end{cases}$$

and

$$G(t) = g(t)|g_M(t)|^{2i}.$$

It is obvious that $G(t)$ satisfies the assumption of Proposition 2.1. Then for any $s \in \{1, 2, \dots, N\}$, we define

$$u_s = g(u_{x_s}) = \begin{cases} u_{x_s} - 1, & \text{if } u_{x_s} \geq 1, \\ 0, & \text{if } u_{x_s} \in [-1, 1], \\ u_{x_s} + 1, & \text{if } u_{x_s} \leq -1, \end{cases}$$

$$u_{s,M} = g_M(u_{x_s}) = \begin{cases} M, & \text{if } u_s \geq M, \\ u_s, & \text{if } u_s \in [-M, M], \\ -M, & \text{if } u_s \leq -M. \end{cases}$$

Inserting

$$G(u_{x_s}) = u_s|u_{s,M}|^{2i}, \quad \psi = \varphi_{i+1}^2$$

into the left hand of (2.5) and noting that $G'(u_{x_s}) \geq u_{s,M}^{2i} \geq 0$, we have

$$\begin{aligned} &\int_B \sum_{i,j=1}^N \big\{(|\nabla u|^{p-2} + |\nabla u|^{q-2})\delta_{ij} + [(p-2)|\nabla u|^{p-4} + (q-2)|\nabla u|^{q-4}]u_{x_i}u_{x_j}\big\} \\ &\quad \cdot u_{x_s,x_i}u_{x_s,x_j}G'(u_{x_s})\psi\,dx \\ &= \int_B \big\{(|\nabla u|^{p-2} + |\nabla u|^{q-2})|\nabla u_{x_s}|^2 + \big[(p-2)|\nabla u|^{p-4} + (q-2)|\nabla u|^{q-4}\big] \\ &\quad \cdot |\nabla u \cdot \nabla u_{x_s}|^2\big\}G'(u_{x_s})\psi\,dx \\ &\geq \int_B \big\{ \big[|\nabla u|^{p-2}|\nabla u_{x_s}|^2 + (p-2)|\nabla u|^{p-4}|\nabla u \cdot \nabla u_{x_s}|^2\big] \\ &\quad + \big[|\nabla u|^{q-2}|\nabla u_{x_s}|^2 + (q-2)|\nabla u|^{q-4}|\nabla u \cdot \nabla u_{x_s}|^2\big] \big\}u_{s,M}^{2i}\varphi_{i+1}^2\,dx \\ &\geq \min\{1, p-1\} \int_B |\nabla u|^{p-2}|\nabla u_{x_s}|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx. \end{aligned} \tag{2.14}$$

On the other hand, by the definition of $u_{s,M}$, we have that $|\nabla u| \geq 1$ on the support of $u_{s,M}$. Hence

$$
\begin{aligned}
&\int_B \sum_{j=1}^N (|\nabla u|^{p-2} + |\nabla u|^{q-2}) u_{x_j} \cdot \frac{d}{dx_s}\{G(u_{x_s})\psi_{x_j}\}\, dx \\
&= \int_B \sum_{j=1}^N (|\nabla u|^{p-2} + |\nabla u|^{q-2}) u_{x_j} \cdot G'(u_{x_s}) u_{x_s x_s} \psi_{x_j} dx \\
&\quad + \int_B \sum_{j=1}^N (|\nabla u|^{p-2} + |\nabla u|^{q-2}) u_{x_j} \cdot G(u_{x_s}) \psi_{x_s x_j}\, dx \\
&\leq C \int_B (|\nabla u|^{p-2} + |\nabla u|^{q-2}) |\nabla u| u_{s,M}^{2i} |\nabla u_s| \varphi_{i+1} |\nabla \varphi_{i+1}|\, dx \\
&\quad + C(R) \int_B \sum_{j=1}^N (|\nabla u|^{p-1} + |\nabla u|^{q-1}) u_{s,M}^{2i} |\nabla u|\, dx \\
&\leq C \int_B |\nabla u|^{p-1} u_{s,M}^{2i} |\nabla u_s| \cdot \varphi_{j+1} |\nabla \varphi_{j+1}|\, dx + C(R) \int_B |\nabla u|^{p+2i}\, dx \\
&\leq \varepsilon \int_B |\nabla u|^{p-2} |\nabla u_s|^2 u_{s,M}^{2i} \varphi_{i+1}^2\, dx + C(\varepsilon) \int_B |\nabla u|^p u_{s,M}^{2i} |\nabla \varphi_{i+1}|^2\, dx + C(R),
\end{aligned}
\tag{2.15}
$$

and by (2.2) and the fact that $|\nabla u| \geq 1$ on the support of $u_{s,M}$, we have that

$$
\begin{aligned}
&\int_B (-f) \frac{d}{dx_s}\{G(u_{x_s})\psi\}\, dx \\
&\leq C \int_B |f| u_{s,M}^{2i} |\nabla u_{x_s}| \varphi_{i+1}^2\, dx + C \int_B |f| |u_s| u_{s,M}^{2i} \varphi_{i+1} |\nabla \varphi_{i+1}|\, dx \\
&\leq C \int_B |\nabla u|^{p-1} u_{s,M}^{2i} |\nabla u_s| \varphi_{i+1}^2\, dx + C \int_B |\nabla u|^p u_{s,M}^{2i} \varphi_{i+1} |\nabla \varphi_{i+1}|\, dx \\
&\leq \varepsilon \int_B |\nabla u|^{p-2} |\nabla u_s|^2 u_{s,M}^{2i} \varphi_{i+1}^2\, dx + C(\varepsilon) \int_B |\nabla u|^p u_{s,M}^{2i} \varphi_{i+1}^2\, dx \\
&\quad + C(R) \int_B |\nabla u|^{p+2i}\, dx.
\end{aligned}
\tag{2.16}
$$

Thus by (2.5), (2.14), (2.15) and (2.16), we have that

$$
\begin{aligned}
&\min\{1, p-1\} \int_B |\nabla u|^{p-2} |\nabla u_{x_s}|^2 u_{s,M}^{2i} \varphi_{i+1}^2\, dx \\
&\leq \int_B \sum_{j=1}^N (|\nabla u|^{p-2} + |\nabla u|^{q-2}) u_{x_j} \cdot \frac{d}{dx_s}\{G(u_{x_s})\psi_{x_j}\}\, dx - \int_B f \frac{d}{dx_s}\{G(u_{x_s})\psi\}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\varepsilon \int_B |\nabla u|^{p-2}|\nabla u_s|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(\varepsilon)\int_B |\nabla u|^p u_{s,M}^{2i}|\nabla\varphi_{i+1}|^2\,dx\\
&\quad + C(\varepsilon)\int_B |\nabla u|^p u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(R)\\
&\le 2\varepsilon \int_B |\nabla u|^{p-2}|\nabla u_s|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(\varepsilon,R)\int_B |\nabla u|^{p+2i}\,dx + C(R)\\
&\le 2\varepsilon \int_B |\nabla u|^{p-2}|\nabla u_s|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(\varepsilon,R).
\end{aligned}
$$

Then ε can be chosen such that

$$
\int_B |\nabla u|^{p-2}|\nabla u_{x_s}|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx \le C(R). \tag{2.17}
$$

Now, we prove (2.12) for $i+1$. Notice that

$$
\sum_{s=1}^N |\nabla u|^{p+2} u_{s,M}^{2i} = \sum_{s=1}^N \Big(\sum_{j=1}^N u_{x_j}^2\Big)^{(p+2)/2} u_{s,M}^{2i} \le \sum_{s=1}^N\sum_{j=1}^N |u_{x_j}|^{p+2}u_{s,M}^{2i} \tag{2.18}
$$

and the fact that

$$
|u_{x_j}|^{p+2}u_{s,M}^{2i} \le |u_{x_s}|^{p+2}u_{s,M}^{2i} \le \sum_{s=1}^N |u_{x_s}|^{p+2}u_{s,M}^{2i},\ \text{ if } |u_{x_j}|\le|u_{x_s}|,
$$

as well as

$$
|u_{x_j}|^{p+2}u_{s,M}^{2i} \le |u_{x_j}|^{p+2}u_{j,M}^{2i} \le \sum_{s=1}^N |u_{x_s}|^{p+2}u_{s,M}^{2i},\ \text{ if } |u_{x_j}|\ge|u_{x_s}|.
$$

Thus we have

$$
\sum_{s=1}^N\sum_{j=1}^N |u_{x_j}|^{p+2}u_{s,M}^{2i} \le N^2\sum_{s=1}^N |u_{x_s}|^{p+2}u_{s,M}^{2i}. \tag{2.19}
$$

Hence with the help of (2.18) and (2.19), we have that

$$
\begin{aligned}
&\sum_{s=1}^N\int_B |\nabla u|^{p+2}u_{s,M}^{2i}\varphi_{i+1}^2\,dx\\
&\le C\sum_{s=1}^N\int_B |u_{x_s}^{p+2}u_{s,M}^{2i}\varphi_{i+1}^2\,dx \qquad\qquad \text{by (2.18), (2.19)}\\
&\le C\sum_{s=1}^N\int_B |u_s|^p u_s u_{s,M}^{2i}\varphi_{i+1}^2\cdot u_{x_s}\,dx + C\sum_{s=1}^N\int_B u_{s,M}^{2i}\varphi_{i+1}^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{s=1}^{N}\int_B |u_s|^p u_{s,x_s} u_{s,M}^{2i}\varphi_{i+1}^2 u\,dx + C\sum_{s=1}^{N}\int_B |u_s|^{p-2}u_s u_{s,x_s} u_{s,M}^{2i}\varphi_{i+1}^2 u\,dx\\
&\quad + C\sum_{s=1}^{N}\int_B |u_s|^p u_s u_{s,M}^{2i-2} u_{s,M} u_{s,M,x_s}\varphi_{i+1}^2 u\,dx\\
&\quad + C\sum_{s=1}^{N}\int_B |u_s|^p u_s u_{s,M}^{2i}\varphi_{i+1}|\nabla\varphi_{i+1}| u\,dx + C(R)\int_{B_i}|\nabla u|^{2i}\,dx\\
&\le C\sum_{s=1}^{N}\int_B |\nabla u|^p|\nabla u_s| u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C\sum_{s=1}^{N}\int_B |\nabla u|^p u_{s,M}^{2i}\varphi_{i+1}^2\,dx\\
&\quad + C\sum_{s=1}^{N}\int_B |\nabla u|^{p+1} u_{s,M}^{2i}\varphi_{i+1}|\nabla\varphi_{i+1}|\,dx + C(R)\\
&\le 2\varepsilon\sum_{s=1}^{N}\int_B |\nabla u|^{p+2} u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(\varepsilon,R)\sum_{s=1}^{N}\int_B |\nabla u|^p u_{s,M}^{2i}\,dx\\
&\quad + C(\varepsilon)\sum_{s=1}^{N}\int_B |\nabla u|^{p-2}|\nabla u_s|^2 u_{s,M}^{2i}\varphi_{i+1}^2\,dx + C(R).
\end{aligned}
\tag{2.20}
$$

Here, integration by parts and Young's inequality are used. Then, by virtue of (2.12) for i and (2.17), (2.20) implies that

$$
\sum_{s=1}^{N}\int_B |\nabla u|^{p+2} u_{s,M}^{2i}\varphi_{i+1}^2\,dx \le C(R). \tag{2.21}
$$

Set $i = 0$ in (2.21). We get

$$
\sum_{s=1}^{N}\int_B |\nabla u|^{p+2}\varphi_1^2\,dx \le C(R), \tag{2.22}
$$

and letting $M \to +\infty$ in (2.21), we get

$$
\sum_{s=1}^{N}\int_B |\nabla u|^{p+2} u_s^{2i}\varphi_{i+1}^2\,dx \le C(R). \tag{2.23}
$$

So by (2.22) and (2.23) we get

$$
\begin{aligned}
\int_{B_{i+1}} |\nabla u|^{p+2(i+1)}\,dx &\le \int_B |\nabla u|^{p+2}|\nabla u|^{2i}\varphi_{i+1}^2\,dx\\
&\le C\sum_{s=1}^{N}\int_B |\nabla u|^{p+2}|u_{x_s}|^{2i}\varphi_{i+1}^2\,dx
\end{aligned}
$$

$$\le C\sum_{s=1}^{N}\int_B |\nabla u|^{p+2}|u_s|^{2i}\varphi_{i+1}^2\,dx + C\sum_{s=1}^{N}\int_B |\nabla u|^{p+2}\varphi_{i+1}^2\,dx \le C(R).$$

Thus (2.12) is proved.

Now, we use (2.12) to prove (2.9). From now on, we fix R by taking

$$R = R_0 \tag{2.24}$$

for some given $R_0 \in \mathbf{R}^1$. As the dependence on R of the generic constant C does not matter any more, we do not indicate it in the following. For $k \ge 0$ and

$$R \le r \le r' \le 2R,$$

we set

$$\begin{aligned}\varphi(x) &= \rho((r'-r)^{-1}\cdot(|x-x_0|-r)),\\ A_{k,r} &= \{x\in B_r(x_0)\,|\,u_s(x)>k\}.\end{aligned}$$

For $t \in \mathbf{R}^1$, we define

$$g(t) = \begin{cases} t-1, & \text{if } t\ge 1,\\ 0, & \text{if } t\in[-1,1],\\ t+1, & \text{if } t\le -1,\end{cases}$$

and

$$G(t) = \max\{g(t)-k, 0\}.$$

It is obvious that $G(t)$ satisfies the assumption of Proposition 2.1. Then we define $u_s = g(u_{x_s})$ and insert

$$G(u_{x_s}) = \max\{u_s - k, 0\}, \quad \psi = \varphi^2$$

into (2.5), and following in the same way which leads to (2.17), we get

$$\int_{A_{k,r'}} |\nabla u|^{p-2}|\nabla u_s|^2\varphi^2\,dx \le C\cdot(r'-r)^{-2}\int_{A_{k,r}}|\nabla u|^p\,dx. \tag{2.25}$$

Noticing that (2.12) gives that

$$\int_{B_{Np}} |\nabla u|^{Np}\,dx \le C, \tag{2.26}$$

and the fact that $r' < R_i$ implies that

$$B_{r'}(x_0) \subset B_i(x_0) \tag{2.27}$$

for any $i \in \{0,1,\dots,[Np]\}$, we have by (2.26) and (2.27) that

$$\left(\int_{A_{k,r'}} |\nabla u|^{Np}\,dx\right)^{1/N} \le \left(\int_{B_{Np}} |\nabla u|^{Np}\,dx\right)^{1/N} \le C. \tag{2.28}$$

Then, (2.28) and Hölder's inequality show that

$$\begin{aligned}\int_{A_{k,r'}} |\nabla u|^p\,dx &\le (\int_{A_{k,r'}} |\nabla u|^{Np}\,dx)^{1/N}\cdot(\operatorname{meas} A_{k,r'})^{\frac{N-1}{N}}\\ &\le C\cdot(\operatorname{meas} A_{k,r'})^{\frac{N-1}{N}}.\end{aligned}\tag{2.29}$$

Thus, by (2.25), (2.29), Young's and Hölder's inequalities, we get that

$$\int_{A_{k,r'}} |\nabla u|^{p-2}|\nabla u_s|^2\varphi^2\,dx \le C\cdot(r'-r)^{-2}(\operatorname{meas} A_{k,r'})^{1-\frac{1}{N}},\tag{2.30}$$

and then

$$\int_{A_{k,r}} |\nabla u|^{p-2}|\nabla u_s|^2\,dx \le C\cdot(r'-r)^{-2}(\operatorname{meas} A_{k,r'})^{1-\frac{1}{N}}.\tag{2.31}$$

If $p\ge 2$, (2.31) implies that

$$\int_{A_{k,r}} |\nabla u_s|^2\,dx \le \int_{A_{k,r}} |\nabla u|^{p-2}|\nabla u_s|^2\,dx \le C\cdot(r'-r)^{-2}(\operatorname{meas} A_{k,r'})^{1-\frac{1}{N}}.\tag{2.32}$$

If $p\le 2$, we additionally use (2.29), Hölder's and Young's inequalities to obtain that

$$\begin{aligned}&\int_{A_{k,r}} |\nabla u_s|^p\,dx\\ &\le \left(\int_{A_{k,r}} (r'-r)^{2-p}|\nabla u|^{p-2}|\nabla u_s|^2\,dx\right)^{p/2}\cdot\left(\int_{A_{k,r}} (r'-r)^{-p}|\nabla u|^p\,dx\right)^{(2-p)/2}\\ &\le \frac{p}{2}(r'-r)^{2-p}\cdot\int_{A_{k,r}} |\nabla u|^{p-2}|\nabla u_s|^2\,dx + \frac{2-p}{2}(r'-r)^{-p}\cdot\int_{A_{k,r}} |\nabla u|^p\,dx\\ &\le C(r'-r)^{-p}(\operatorname{meas} A_{k,r'})^{1-\frac{1}{N}}.\end{aligned}\tag{2.33}$$

If we choose $R_0\in(1/2,1)$ in (2.24) at first, we have

$$\begin{aligned}\int_{B_{2R}} |u_{x_s}|\,dx &\le \left(\int_{B_{2R}} |\nabla u|^p\,dx\right)^{1/p}\cdot(\operatorname{meas} B_{2R})^{\frac{(p-1)}{p}}\\ &\le C\cdot\left[\kappa_N\cdot(2R)^N\right]^{\frac{(p-1)}{p}}\\ &\le CR^N,\end{aligned}\tag{2.34}$$

where κ_N denotes the volume of the unit ball in $\mathbf{R}^N$.

So (2.32), (2.33), (2.34) and Lemma 2.2 show that

$$u_s\le C \quad \text{in } B_R(x_0).$$

As $-u$ satisfies all the same estimates above as u does, we have shown that Proposition 2.3 is true. Hence Theorem 1(i) is proved. □

3. The proof of Theorem 1(ii)

We will prove Theorem 1(ii) in this section. To this end, it is enough to prove the following result:

Proposition 3.1. *Suppose that u is a weak solution of (2.1) and u, $f(x)$ and $|\nabla u|$ are locally bounded. Then there is an $\alpha > 0$ and a constant C depending only on N, p, q, $\operatorname{ess\,sup}_B |u|$ and $\operatorname{ess\,sup}_B |f|$ such that*

$$|\nabla u(x) - \nabla u(x_0)| \leq C \cdot |x - x_0|^\alpha, \quad \forall x \in B_{1/2}(x_0), \tag{3.1}$$

where $B = B_1(x_0)$ for any given $x_0 \in \mathbf{R}^N$.

In the following, ρ is defined as in (2.10). By C, we denote a positive generic constant depending only on N, p, q, $\operatorname{ess\,sup}_{B_1(x_0)} |u|$ and $\operatorname{ess\,sup}_{B_1(x_0)} |f|$. We pick an $R \in (0, 1/2)$ and set

$$M = \max_s \operatorname{ess\,sup}_{B_R(x_0)} |u_{x_s}|. \tag{3.2}$$

Before we prove Proposition 3.1, we give the following results:

Lemma 3.2. ([[9], Lemma 3.9) *There is a C depending only on N, such that*

$$(l-k) \cdot (\operatorname{meas} A_{l,\rho})^{1-\frac{1}{N}} \leq \beta \rho^N \operatorname{meas}^{-1}\{B_\rho(x_0) \backslash A_{k,\rho}\} \cdot \int_{A_{l,k,\rho}} |\nabla v|\, dx$$

for all $l > k$ and $v \in W^{1,1}(B_\rho(x_0))$, where $A_{k,\rho} = \{x \in B_\rho(x_0) \,|\, v(x) > k\}$ and $A_{l,k,\rho} = \{x \in B_\rho(x_0) \,|\, k < v(x) \leq l\}$.

Lemma 3.3. ([9], Lemma 4.7) *If a nonnegative sequence $\{y_h\}$, $h = 0, 1, 2, \ldots$, satisfies*

$$y_{h+1} \leq c b^h y_h^{1+\varepsilon}, \quad h = 0, 1, \ldots,$$

where c, ε and $b > 1$ are positive constants, then

$$y_h \leq c^{\frac{(1+\varepsilon)^h - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^h - 1}{\varepsilon^2} - \frac{h}{\varepsilon}} y_0^{(1+\varepsilon)h}.$$

Especially, if $y_0 \leq \theta = c^{-1/\varepsilon} b^{-1/\varepsilon^2}$, then

$$y_h \leq \theta b^{-1/\varepsilon}$$

and

$$y_h \to 0, \quad \text{as } h \to \infty.$$

Lemma 3.4. ([9], Lemma 4.8) *Suppose $u(x)$ is measurable and bounded on $B_{\rho_0}(x_0)$. Considering $B_\rho(x_0)$ and $B_{b\rho}(x_0)$, where $b > 1$ is a constant, if for all $\rho \leq b^{-1}\rho_0$, $u(x)$ satisfies one of the following inequalities*

$$\operatorname{osc}\{u; B_\rho(x_0)\} \leq \bar{c}\rho^\varepsilon,$$
$$\operatorname{osc}\{u; B_\rho(x_0)\} \leq \theta \operatorname{osc}\{u; B_{b\rho}(x_0)\},$$

where $\bar{c}$, $\varepsilon \le 1$ and $\theta < 1$ are positive constants, then

$$\operatorname{osc}\{u; B_\rho(x_0)\} \le c\rho_0^{-\alpha}\rho^\alpha$$

whenever $\rho \le \rho_0$, where

$$\alpha = \min\{\varepsilon, -\log_b \theta\}, \quad c = b^\alpha \max\{\bar{c}\rho_0^\varepsilon, \operatorname{osc}\{u;\ B_{\rho_0}(x_0)\}\}.$$

Lemma 3.5. ([5], Proposition 4.1) *Suppose that u is a weak solution of (2.1) and u, $f(x)$ and $|\nabla u|$ are locally bounded. Then for any given $x_0 \in \mathbf{R}^N$, there is a $\mu > 0$ depending only on N, p, q, M, $\operatorname{ess}\sup\limits_{B_1(x_0)} |u|$ and $\operatorname{ess}\sup\limits_{B_1(x_0)} |f|$, such that if for some $1 \le s \le N$*

$$\operatorname{meas}\{x \in B_R(x_0) \mid u_{x_s}(x) \le M/2\} \le \mu R^N, \tag{3.3}$$

then

$$u_{x_s}(x) \ge M/8, \quad \forall x \in B_{R/2}(x_0),$$

where M is defined in (3.2). Analogously, if

$$\operatorname{meas}\{x \in B_R(x_0) \mid u_{x_s}(x) \ge -M/2\} \le \mu R^N, \tag{3.4}$$

then

$$u_{x_s}(x) \le -M/8, \quad \forall x \in B_{R/2}(x_0).$$

Now, we begin to prove Proposition 3.1.

We have shown in §2 that the gradient of a weak solution u of (2.1) is locally bounded under the condition of Proposition 3.1. Therefore, by Lemma 3.5 there are two cases: Case I: Either (3.3) or (3.4) is satisfied; Case II: Neither (3.3) nor (3.4) is satisfied. We follow [5] to consider these two cases to prove Proposition 3.1.

Case I: Either (3.3) or (3.4) is satisfied. Notice that if either (3.3) or (3.4) holds, we have by Lemma 3.5 that

$$|u_{x_s}(x)| \ge M/8, \quad \forall x \in B_{R/2}(x_0).$$

Moreover, by the definition of M (see (3.2)) we have

$$M/8 \le |\nabla u| \le M \quad \text{in } B_{R/2}(x_0). \tag{3.5}$$

For $l > k \ge 0$ and r, $r' \in \mathrm{R}$ satisfying $0 < r < r' \le R$, we set for a solution u of (2.1) that

$$\varphi(x) = \rho((r'-r)^{-1} \cdot (|x - x_0| - r)),$$
$$A_{k,r} = \{x \in B_r(x_0) \mid u_{x_s}(x) > k\}$$

and

$$A_{l,k,r} = \{x \in B_r(x_0) \mid k < u_{x_s}(x) \le l\}.$$

For $t \in \mathbf{R}^1$, we define

$$g(t) = \begin{cases} t-1, & \text{if } t \ge 1, \\ 0, & \text{if } t \in [-1,1], \\ t+1, & \text{if } t \le -1, \end{cases}$$

and

$$G(t) = \max\{g(t) - k, 0\}.$$

It is obvious that $G(t)$ satisfies the assumption of Proposition 2.1. Then we define $u_s = g(u_{x_s})$ and insert

$$G(u_{x_s}) = \max\{u_s - k, 0\}, \quad \psi = \varphi^2$$

into (2.5). Integrating the first term on the right of (2.5) by parts, then following in the same way which leads to (2.17), we get

$$\begin{aligned}\int_{A_{k,r'}} |\nabla u_{x_s}|^2 \varphi^2 \, dx &\le C \int_{A_{k,r'}} (u_{x_s} - k)^2 |\nabla \varphi|^2 \, dx + C \int_{A_{k,r'}} \varphi^2 \, dx \\ &\le C \cdot (r' - r)^{-2} \int_{A_{k,r'}} (u_{x_s} - k)^2 \, dx + C \cdot \operatorname{meas} A_{k,r'}.\end{aligned} \tag{3.6}$$

Notice that if u_{x_s} satisfies (3.6), so does $-u_{x_s}$. On the other hand, for $W(x) = \pm u_{x_s}(x)$, at least one of the following inequalities

$$\operatorname{meas}\big\{x \in B_{R/2}(x_0) \,|\, u_{x_s}(x) > \max_{B_R(x_0)} u_{x_s} - \tfrac{1}{2} \operatorname{osc}\{u_{x_s}; B_R(x_0)\}\big\} \le \tfrac{1}{2} \operatorname{meas} B_{R/2}(x_0),$$

$$\operatorname{meas}\big\{x \in B_{R/2}(x_0) \,|\, u_{x_s}(x) < \min_{B_R(x_0)} u_{x_s} + \tfrac{1}{2} \operatorname{osc}\{u_{x_s}; B_R(x_0)\}\big\} \le \tfrac{1}{2} \operatorname{meas} B_{R/2}(x_0)$$

must be true. That is, either $W(x) = u_{x_s}(x)$ or $W(x) = -u_{x_s}(x)$ satisfies

$$\begin{aligned}&\operatorname{meas}\big\{x \in B_{R/2}(x_0) \,|\, W(x) > \max_{B_R}(x_0) W - \tfrac{1}{2} \operatorname{osc}\{u_{x_s}; B_R(x_0)\}\big\} \\ &\le \tfrac{1}{2} \operatorname{meas} B_{R/2}(x_0).\end{aligned} \tag{3.7}$$

If we set

$$\omega = \tfrac{1}{2} \operatorname{osc}\{u_{x_s}; B_R(x_0)\}, \quad k' = \max_{B_R(x_0)} W - \omega \quad \text{and} \quad k'' = \max_{B_R(x_0)} W, \tag{3.8}$$

then (3.7) implies that

$$\operatorname{meas} A_{k',R/2} \le \tfrac{1}{2} \operatorname{meas} B_{R/2}(x_0). \tag{3.9}$$

In the following, we first assume that

$$\omega \ge 2^{t_0} R, \tag{3.10}$$

where t_0 is determined below.

Lemma 3.6. *For any $\theta \in (0,1)$, there is a $t_0 > 0$, such that if W satisfies (3.6), (3.10) (i.e., W satisfies all the estimates that u_{x_s} does in (3.6) and (3.10)), then for*

$$k'' = \max_{B_R(x_0)} W \ge \max_{B_R} W - 2^{-t_0} \omega, \tag{3.11}$$

$$k^0 = \max_{B_R} W - 2^{-t_0+1} \omega, \tag{3.12}$$

we have

$$\operatorname{meas} A_{k^0,R/2} \le \theta R^N, \tag{3.13}$$

where $A_{k^0,R/2}$ is defined for W as for u_{x_s}.

In fact, from (3.7) we know that we can assume $W = u_{x_s}$ in Lemma 3.6 without loss of generality.

Proof. Set $k_t = \max\limits_{B_R(x_0)} W - 2^{-t}\omega$, $D_t = A_{k_t,R/2} \backslash A_{k_{t+1},R/2}$, $t = 0, 1, \dots, t_0 - 1$. Putting $r = R/2$, $r' = R$, $k = k_t$, $l = k_{t+1}$, $t = 0, 1, \dots, t_0 - 2$, into (3.6), we have

$$\int_{A_{k_t,R/2}} |\nabla W|^2\, dx \le C[1 + (R/2)^{-2}(2^{-t}\omega)^2] \cdot \operatorname{meas} A_{k_t,R}. \tag{3.14}$$

By (3.10) and (3.14), we have

$$\int_{A_{k_t,R/2}} |\nabla W|^2\, dx \le C\kappa_N (2^{-t}\omega)^2 R^{N-2}, \tag{3.15}$$

where κ_N is the volume of the unit ball in $\mathbf{R}^N$.

Now we use Lemma 3.2 to estimate the left hand side of (3.15). Putting $k = k_t$, $l = k_{t+1}$, $\rho = R/2$ into Lemma 3.2 and with the help of (3.9), we have

$$\begin{aligned}
&\operatorname{meas}^{1-\frac{1}{N}} A_{k_{t_0-1},R/2} \le \operatorname{meas}^{1-\frac{1}{N}} A_{k_{t+1},R/2} \\
&\le \frac{\beta(R/2)^N}{(k_{t+1} - k_t)\operatorname{meas}(B_{R/2}(x_0)\backslash A_{k_t,R/2})} \int_{A_{k_{t+1},k_t,R/2}} |\nabla W|\, dx \\
&\le \frac{\beta \cdot (R/2)^N}{2^{-(t+1)}\omega \operatorname{meas}(B_{R/2}(x_0)\backslash A_{k_t,R/2})} \int_{D_t} |\nabla W|\, dx \\
&\le \frac{2^{t+2}\beta}{\kappa_N \cdot \omega} \int_{D_t} |\nabla W|\, dx,
\end{aligned} \tag{3.16}$$

where $D_t = A_{k_{t+1},k_t,R/2}$. Then (3.15) and (3.16) give

$$\operatorname{meas}^{\frac{2(N-1)}{N}} A_{k_{t_0-1},R/2} \le C\beta^2 \kappa_N^{-1} \cdot R^{N-2} \operatorname{meas} D_t. \tag{3.17}$$

Summing up t from 0 to $t_0 - 2$ and noticing that $\sum\limits_t \operatorname{meas} D_t \le \operatorname{meas} B_{R/2}(x_0) = \kappa_N(\frac{R}{2})^N$, we have

$$\operatorname{meas}^{\frac{2(N-1)}{N}} A_{k_{t_0-1},R/2} \le \frac{C\beta^2}{t_0 - 1} \cdot R^{2(N-1)}. \tag{3.18}$$

So, if we take $t_0 = 2 + [C\beta^2\theta^{-\frac{2(N-1)}{N}}]$ and $k^0 = k_{t_0-1}$ in (3.18), we get (3.13), and Lemma 3.6 is proved. □

Following Lemma 3.6, we show another result.

Lemma 3.7. *For $R/4 \le r < r' \le R/2$, $k \in \left[k^0, k^0 + \frac{H}{2}\right]$ and $H = \max\limits_{B_R(x_0)} W - k^0$, if W satisfies (3.6), (3.13) (where $A_{k,l}$ are defined for W), we have either*

$$\max_{B_{R/4}(x_0)} W(x) \le k^0 + \frac{H}{2} \tag{3.19}$$

or

$$H \le R. \tag{3.20}$$

Proof. Considering $B_{\rho_h}(x_0)$, where $\rho_h = \frac{R}{4} + \frac{R}{2^{h+2}}$, $h = 0, 1, \ldots$, and a sequence of levels

$$k_h = k^0 + \frac{H}{2} - \frac{H}{2^{h+1}}, \quad h = 0, 1, \ldots,$$

and denoting $y_h = R^{-N} \operatorname{meas} A_{k_h,\rho_h}$ and $D_{h+1} = A_{k_h,\rho_{h+1}} \backslash A_{k_{h+1},\rho_{h+1}}$, it is obvious that $k^0 \le k_h \le k^0 + \frac{H}{2}$ is true for all $h = 0, 1, \ldots$. By (3.6) with $k = k_h$, $l = k_{h+1}$, $r' = \rho_h$, $r = \rho_{h+1}$, we have

$$\begin{aligned} \int_{D_{h+1}} |\nabla W|^2\, dx &\le C\big[1 + (R/2^{(h+3)})^{-2} \cdot (\max_{B_R} u_{x_s} - k_h)^2\big] \cdot R^N y_h \\ &\le C\left[1 + 2^{2(h+3)} R^{-2} H^2\right] R^N y_h. \end{aligned} \tag{3.21}$$

If (3.20) were not true, that is

$$1 < R^{-2} H^2, \tag{3.22}$$

then (3.21), (3.22) would imply that

$$\begin{aligned} \int_{D_{h+1}} |\nabla W|^2\, dx &\le C\left[1 + 2^{2(h+3)}\right] H^2 R^{N-2} y_h \\ &\le C 2^{2(h+4)} H^2 R^{N-2} y_h. \end{aligned} \tag{3.23}$$

Noticing that

$$\operatorname{meas} D_{h+1} \le \operatorname{meas} A_{k_h,\rho_{h+1}} \le \operatorname{meas} A_{k_h,\rho_h} = R^N y_h,$$

we have by Hölder's inequality and (3.23) that

$$\begin{aligned} \int_{D_{h+1}} |\nabla W|\, dx &\le \left(\int_{D_{h+1}} |\nabla W|^2\, dx\right)^{1/2} \cdot (\operatorname{meas} D_{h+1})^{1/2} \\ &\le C 2^{h+4} H R^{(N-2)/2} y_h^{1/2} \cdot (R^N y_h)^{1/2} \\ &\le C 2^{h+4} H R^{N-1} y_h. \end{aligned} \tag{3.24}$$

On the other hand, for

$$\theta \le 2^{-2N-1} \kappa_N, \tag{3.25}$$

if we take $k = k_h$, $l = k_{h+1}$, $\rho = \rho_{h+1}$ in Lemma 3.2 and by (3.13), (3.25) and Lemma 3.2, then we have that

$$\begin{aligned} \int_{D_{h+1}} |\nabla W|\, dx &\ge \beta^{-1}(k_{h+1} - k_h) R^{N-1} y_{h+1}^{1-1/N} \rho_{h+1}^{-N} \cdot \operatorname{meas}(B_{\rho_{h+1}}(x_0) \backslash A_{k_h,\rho_{h+1}}) \\ &\ge \beta^{-1} 2^{-(h+2)} H R^{N-1} \left(\tfrac{R}{2}\right)^{-N} \cdot \operatorname{meas}(B_{R/4}(x_0) \backslash A_{k^0,R/2}) \\ &\ge \beta^{-1} 2^{-(h+N+3)} \kappa_N H R^{N-1} y_{h+1}^{1-1/N}. \end{aligned} \tag{3.26}$$

So, (3.24) and (3.26) show that

$$
\begin{aligned}
y_{h+1} &\leq (C\beta 2^{N+7}{\kappa_N}^{-1}) \cdot (4^{\frac{N}{N+1}})^h \cdot y_h^{\frac{N}{N-1}} \\
&\triangleq c_0 b_0^h y_h^{1+\varepsilon_0}, \quad h = 0, 1, \ldots,
\end{aligned} \tag{3.27}
$$

where $\varepsilon_0 = \frac{1}{N-1} > 0$, $b_0 = 4^{\frac{N}{N-1}}$, $c_0 = (C\beta 2^{N+7}{\kappa_N}^{-1})^{\frac{N}{N-1}}$.

Then, if

$$
y_0 \leq c_0^{-1/\varepsilon_0} b_0^{-1/{\varepsilon_0}^2},
$$

that is, (3.13) is satisfied with $\theta \leq c_0^{-1/\varepsilon_0} b_0^{-1/{\varepsilon_0}^2}$, (3.25) and Lemma 3.3 show that

$$
y_h \to 0, \quad \text{as } h \to +\infty
$$

and

$$
\max_{B_{R/4}(x_0)} W(x) = \lim_{h\to\infty} k_h = k^0 + \frac{H}{2}.
$$

So, Lemma 3.7 is proved. □

Thus by Lemma 3.6 and Lemma 3.8 under the assumption (3.10), we finally get that

$$
\begin{aligned}
\max_{B_{R/4}(x_0)} W(x) = \lim_{h\to\infty} k_h = k^0 + \frac{H}{2} &= k^0 + \tfrac{1}{2}\big[\max_{B_R(x_0)} W - k^0\big] \\
= \tfrac{1}{2}\big[\max_{B_R(x_0)} W + k^0\big] &= \max_{B_R(x_0)} W - 2^{-t_0}\omega,
\end{aligned}
$$

that is,

$$
\omega \leq 2^{t_0}\{\max_{B_R(x_0)} W - \max_{B_{R/4}(x_0)} W\}. \tag{3.28}
$$

Thus we have

$$
\begin{aligned}
\omega &\leq \max\{2^{t_0}(\max_{B_R(x_0)} W - \max_{B_{R/4}(x_0)} W); 2^{t_0} R\} \\
&\leq 2^{t_0} \max\{\max_{B_R(x_0)} W - \max_{B_{R/4}(x_0)} W; R\}
\end{aligned} \tag{3.29}
$$

even if (3.10) does not hold.

Remember that by definition

$$
\omega = \tfrac{1}{2}\operatorname{osc}\{u_{x_s}; B_R(x_0)\} \quad \text{and} \quad W = u_{x_s} \text{ or } W = -u_{x_s}.
$$

Inequality (3.29) shows that either

$$
\operatorname{osc}\{u_{x_s}; B_R(x_0)\} \leq 2^{t_0} R \tag{3.30}
$$

or

$$
\begin{aligned}
\operatorname{osc}\{u_{x_s}; B_R(x_0)\} &\leq 2^{t_0+1}\big[\max_{B_R(x_0)} W - \max_{B_{R/4}(x_0)} W\big] \\
&\leq 2^{t_0+1}\big[\operatorname{osc}\{u_{x_s}; B_R(x_0)\} - \operatorname{osc}\{u_{x_s}; B_{R/4}(x_0)\}\big],
\end{aligned}
$$

that is,

$$\operatorname{osc}\{u_{x_s}; B_{R/4}(x_0)\} \le \left(1 - \frac{1}{2^{t_0+1}}\right) \operatorname{osc}\{u_{x_s}; B_R(x_0)\}. \tag{3.31}$$

Then (3.30), (3.31) and Lemma 3.4 imply that $u \in C^{1,\alpha}(B_{1/8}(x_0))$ for some $\alpha \in (0,1)$, and Proposition 3.1 is proved in Case I. □

Case II: Neither (3.3) nor (3.4) is satisfied. In this section, we will prove Proposition 3.1 under the assumption that neither (3.3) nor (3.4) is true, i.e.,

$$\operatorname{meas}\{x \in B_R(x_0) | u_{x_s} > \tfrac{M}{2}\} \le (\kappa_N - \mu) R^N$$

and

$$\operatorname{meas}\{x \in B_R(x_0) | u_{x_s} < -\tfrac{M}{2}\} \le (\kappa_N - \mu) R^N,$$

where κ_N denotes the volume of the unit ball in $\mathbf{R}^N$. Obviously, the above two inequalities show that

$$\operatorname{meas}\{x \in B_R(x_0) \,|\, u_{x_s} > (1 - 1/2^t)\bar{M}\} \le (\kappa_N - \mu) R^N, \tag{3.32}$$

$$\operatorname{meas}\{x \in B_R(x_0) \,|\, u_{x_s} < -(1 - 1/2^t)\bar{M}\} \le (\kappa_N - \mu) R^N, \tag{3.33}$$

where $\bar{M} = \max_s \operatorname{ess} \sup_{B_{2R}(x_0)} |u_{x_s}|$ and $t \ge 1$.

For the proof of Proposition 3.1 in Case II, we first assume that

$$\bar{M} > 2^{t_1} R, \tag{3.34}$$

where t_1 will be determined in the following lemma.

Lemma 3.8. *For any $\theta \in (0,1)$, there exists $t_1 \ge 2$ such that*

$$\operatorname{meas}\{x \in B_R(x_0) \,|\, u_{x_s} > (1 - 1/2^{t_1})\bar{M}\} \le \theta R^N, \tag{3.35}$$

$$\operatorname{meas}\{x \in B_R(x_0) \,|\, u_{x_s} < -(1 - 1/2^{t_1})\bar{M}\} \le \theta R^N, \tag{3.36}$$

where $\bar{M} = \max_s \operatorname{ess} \sup_{B_{2R}(x_0)} |u_{x_s}|$.

Proof. We set $\varphi(x) = \rho(r' - r)^{-1}(|x - x_0| - r)$,

$$A^+_{k,r} = \{x \in B_r x_0 \,|\, u_{x_s} > k\} \text{ for (3.35), where } k \ge (1 - 1/2^{t_1})\bar{M} > 0$$

and

$$A^-_{k,r} = \{x \in B_r x_0 \,|\, u_{x_s} < k\} \text{ for (3.36), where } k \le -(1 - 1/2^{t_1})\bar{M} < 0.$$

We will prove (3.35) only; (3.36) can be proved similarly. Notice that we have

$$\frac{\bar{M}}{2} \le |u_{x_s}| \le \bar{M} \quad \text{on } A^+_{k,r}.$$

For $t \in \mathbf{R}^1$, we define

$$g(t) = \begin{cases} t - 1, & \text{if } t \ge 1, \\ 0, & \text{if } t \in [-1, 1], \\ t + 1, & \text{if } t \le -1, \end{cases}$$

and

$$G(t) = \max\{g(t) - k, 0\}.$$

It is obvious that $G(t)$ satisfies the assumption of Proposition 2.1. Then we define $u_s = g(u_{x_s})$ and insert

$$G(u_{x_s}) = \max\{u_s - k, 0\}, \quad \psi = \varphi^2$$

into (2.5), and following the steps to get (3.6) again, we have

$$\int_{A^+_{k,r}} |\nabla u_{x_s}|^2\,dx \le C\cdot(r'-r)^{-2}\int_{A^+_{k,r'}} [u_{x_s} - k]^2\,dx + C\,\mathrm{meas}\,A^+_{k,r'}. \tag{3.37}$$

Taking $r = R$ and $r' = 2R$ in (3.37), we have

$$\int_{A^+_{k,R}} |\nabla u_{x_s}|^2\,dx \le CR^{-2}\int_{A^+_{k,2R}} [u_{x_s} - k]^2\,dx + C\,\mathrm{meas}\,A^+_{k,2R}. \tag{3.38}$$

Noticing that (3.32) implies that

$$\mathrm{meas}(B_R\backslash A_{(1-2^{-t})\bar{M},R}) \ge \mu R^N, \tag{3.39}$$

we get by (3.34), (3.38), (3.39) and Lemma 3.2 with $v = u_{x_s}$, $l = (1-2^{-(t+1)})\bar{M}$, $k = (1-2^{-t})\bar{M}$, where $2 \le t \le t_0$, $\rho = R$ (and, for convenience, we will still use k, l in the following calculations) that

$$\begin{aligned}
&2^{-(t+1)}\bar{M}\big(\mathrm{meas}\,A^+_{(1-1/2^{s+1})\bar{M},R}\big)^{1-1/N} \\
&\le CR^N\cdot\frac{1}{\mu R^N}\int_{A^+_{l,k,R}} |\nabla u_{x_s}|\,dx \\
&\le C\mu^{-1}\left(\int_{A^+_{l,k,R}} |\nabla u_{x_s}|^2\,dx\right)^{1/2}\cdot(\mathrm{meas}\,A^+_{l,k,R})^{1/2} \\
&\le C\mu^{-1}\big[CR^{-2}\cdot\int_{A^+_{k,2R}}(\bar{M}-k)^2\,dx + C\,\mathrm{meas}\,A^+_{k,2R}\big]^{1/2}\big(\mathrm{meas}\,A^+_{l,k,R}\big)^{1/2} \\
&= C\mu^{-1}\big[R^{-2}\cdot 2^{-2t}\bar{M}^2 + 1\big]^{1/2}(\mathrm{meas}\,A^+_{k,2R})^{1/2}(\mathrm{meas}\,A^+_{l,k,R})^{1/2} \\
&\le C\mu^{-1}R^{-1}2^{-t}\bar{M}\left[\kappa_N(2R)^N\right]^{1/2}\left[\mathrm{meas}\,A^+_{l,k,R}\right]^{1/2}.
\end{aligned} \tag{3.40}$$

Squaring both sides of (3.40) and dividing both sides by $2^{-2(t+1)}$, we get

$$(\mathrm{meas}\,A^+_{(1-1/2^{t+1})\bar{M},R})^{2(N-1)/N} \le C\mu^{-1}\kappa_N R^{N-2}\big[\mathrm{meas}\,A^+_{l,k,R}\big]^{1/2}.$$

We sum up $t = 2, 3, \dots, t_1 - 1$ and notice that $\sum \mathrm{meas}\,A^+_{l,k,R} \le \kappa_N R^N$ to obtain

$$(t_1 - 2)(\mathrm{meas}\,A^+_{(1-1/2^{t_1})\bar{M},R})^{2(N-1)/N} \le C\mu^{-1}\kappa_N^2 R^{2(N-1)}. \tag{3.41}$$

So to prove Lemma 3.8, it is enough to take

$$t_1 = 3 + C\mu^{-1}\kappa_N^2\theta^{-2(N-1)/N} \tag{3.42}$$

in (3.41). □

Lemma 3.9. *If u_{x_s} satisfies (3.37), then there exists a $\theta \in (0,1)$ such that if for some t_1*

$$\operatorname{meas}\{x \in B_R(x_0) | u_{x_s} > (1 - 1/2^{t_1})\bar{M}\} \le \theta R^N, \tag{3.43}$$

then

$$\operatorname*{ess\,sup}_{B_{R/2}(x_0)} u_{x_s} \le (1 - 1/2^{t_1+1})\bar{M}. \tag{3.44}$$

Proof. From (3.37),

$$\int_{A^+_{l,k,r}} |\nabla u_{x_s}|^2\, dx \le C(r'-r)^{-2} \int_{A^+_{k,r'}} \big[u_{x_s} - k\big]^2\, dx + C \operatorname{meas} A^+_{k,r'}. \tag{3.45}$$

We set

$$\rho_h = \frac{R}{2} + \frac{R}{2^{h+1}}, \quad H = \sup_{B_{2R}(x_0)} \big[u_{x_s} - (1 - 1/2^{t_1})\bar{M}\big],$$
$$k_h = \big[1 - 1/2^{t_1}\big] \cdot \bar{M} + (1 - 1/2^h) \cdot H/2,\ h = 0, 1, \ldots,$$

and denote

$$y_h = R^{-N} \operatorname{meas} A^+_{k_h,\rho_h}, \quad D_{h+1} = A^+_{k_h,\rho_{h+1}} \backslash A^+_{k_{h+1},\rho_{h+1}}.$$

It is obvious that

$$k_0 \le k_h \le k_0 + H/2,\ h = 0, 1, \ldots.$$

So by (3.45) with

$$k = k_h,\ l = k_{h+1}\ r' = \rho_h,\ r = \rho_{h+1},\ h = 0, 1, \ldots,$$

we get

$$\begin{aligned} \int_{D_{h+1}} |\nabla u_{x_s}|^2\, dx &\le C(\rho_h - \rho_{h+1})^{-2} \int_{A^+_{k_h,\rho_h}} [u_{x_s} - k_h]^2\, dx + C \operatorname{meas} A^+_{k_h,\rho_h} \\ &\le C[2^{2(h+2)} H^2 R^{-2} + 1] \operatorname{meas} A^+_{k_h,\rho_h}. \end{aligned} \tag{3.46}$$

If $2^{2(h+2)} H^2 R^{-2} \le 1$, then by virtue of (3.34) we have

$$H \le 2^{-(h+1)} R \le R/2 \le 1/2^{t_1+1} \bar{M}.$$

Then by the definition of H, we have

$$\begin{aligned} \sup_{B_{2R}(x_0)} u_{x_s} &= H + (1 - 1/2^{t_1})\bar{M} \\ &\le 1/2^{t_1+1}\bar{M} + (1 - 1/2^{t_1})\bar{M} \\ &\le (1 - 1/2^{t_1+1})\bar{M}. \end{aligned} \tag{3.47}$$

If $2^{2(h+2)} H^2 R^{-2} \ge 1$, then (3.46) shows that

$$\begin{aligned} \int_{D_{h+1}} |\nabla u_{x_s}|^2\, dx &\le C 2^{-2(h+2)} H^2 R^{-2} \operatorname{meas} A^+_{k_h,\rho_h} \\ &\le C 2^{2(h+2)} H^2 R^{N-2} y_h. \end{aligned} \tag{3.48}$$

By (3.48), Hölder's inequality and the fact that

$$\operatorname{meas} D_{h+1} \le R^N y_h,$$

we have

$$\begin{aligned}\int_{D_{h+1}} |\nabla u_{x_s}|\, dx &\le \left(\int_{D_{h+1}} |\nabla u_{x_s}|^2\, dx\right)^{1/2} \cdot (\operatorname{meas} D_{h+1})^{1/2} \\ &\le C2^{h+3} H R^{N-1} y_h.\end{aligned} \tag{3.49}$$

Taking $k = k_h$, $l = k_{h+1}$, $\rho = \rho_{h+1}$ in Lemma 3.2 and noticing that in (3.43) we can assume that $\theta \le 2^{-(N+1)}\kappa_N$, we have

$$\begin{aligned}\int_{D_{h+1}} |\nabla u_{x_s}|\, dx &\ge \beta^{-1}(k_{h+1} - k_h) R^{N-1} y_{h+1}^{1-1/N} \rho_{h+1}^{-N} \operatorname{meas}\big(B_{\rho_{h+1}}(x_0)\backslash A^+_{k_h,\rho_{h+1}}\big) \\ &\ge \beta^{-1} 2^{-(h+2)} H R^{N-1} y_{h+1}^{1-1/N} R^{-N} \operatorname{meas}\big(B_{R/2}(x_0)\backslash A^+_{k_0,R}\big) \\ &\ge \beta^{-1} 2^{-(h+2)} H R^{N-1} y_{h+1}^{1-1/N} R^{-N} \cdot 2^{-(N+1)} \kappa_N R^N \\ &= \beta^{-1} 2^{-(h+N+3)} H \kappa_N R^{N-1} y_{h+1}^{1-1/N}.\end{aligned} \tag{3.50}$$

So (3.49), (3.50) imply that

$$y_{h+1}^{1-1/N} \le C4^{h+3} y_h,$$

that is,

$$y_{h+1} \le C^{\frac{N}{N-1}} (4^{\frac{N}{N-1}})^h y_h^{\frac{N}{N-1}} \triangleq c_1 b_1^h y_h^{1+\varepsilon_1}, \tag{3.51}$$

where $\varepsilon_1 = \frac{1}{N-1} > 0$, $b_1 = 4^{\frac{N}{N-1}}$, $c_1 = C^{\frac{N}{N-1}}$. If

$$y_0 \le c_1^{-1/\varepsilon_1} b_1^{-1/\varepsilon_1{}^2},$$

that is, (3.43) is satisfied with $\theta \le c_1^{-1/\varepsilon_0} b_1^{-1/\varepsilon_0{}^2}$, then (3.51) and Lemma 3.3 show that

$$y_h \to 0, \quad \text{as } h \to +\infty,$$

and

$$\begin{aligned}\sup_{B_{R/2}(x_0)} u_{x_s}(x) &\le \lim_{h\to\infty} k_h = k^0 + \frac{H}{2} \\ &\le (1 - 2^{-t_1})\bar{M} + \tfrac{1}{2}[\bar{M} - (1 - 2^{-t_1})\bar{M}] \\ &= (1 - 1/2^{t_1+1})\bar{M}.\end{aligned} \tag{3.52}$$

Inequalities (3.47) and (3.52) show that Lemma 3.9 is true. □

Conclusion of the proof of Case II. If (3.34) is not satisfied, then

$$\max_s \operatorname{ess} \sup_{B_{2R}(x_0)} |u_{x_s}| = \bar{M} \le 2^{t_1} R. \tag{3.53}$$

Otherwise, if (3.34) is satisfied, we take $\theta = \min\{2^{-N-1}\kappa_N;\ c_1^{-1/\varepsilon_0} b_1^{-1/\varepsilon_0{}^2}\}$ by Lemma 3.9, then take t_1 by Lemma 3.8 to obtain (3.44), that is,

$$\operatorname*{ess\,sup}_{B_{R/2}(x_0)} u_{x_s}(x) \le (1 - 1/2^{t_1+1}) \max_s \operatorname*{ess\,sup}_{B_{2R}(x_0)} |u_{x_s}|$$

and

$$\operatorname*{ess\,inf}_{B_{R/2}(x_0)} u_{x_s}(x) \ge -(1 - 1/2^{t_1+1}) \max_s \operatorname*{ess\,sup}_{B_{2R}(x_0)} |u_{x_s}|.$$

Thus we get

$$\max_s \operatorname*{ess\,sup}_{B_{R/2}(x_0)} |u_{x_s}| \le \delta_0 \cdot \max_s \operatorname*{ess\,sup}_{B_{2R}(x_0)} |u_{x_s}|, \tag{3.54}$$

where $\delta_0 = 1 - 1/2^{t_1+1}$.

Similarly to Case I, (3.53), (3.54) and Lemma 3.4 with some modifications show that

$$\max_s \operatorname*{ess\,sup}_{B_\rho(x_0)} |u_{x_s}| \le C \cdot \rho^\alpha \text{ for any } \rho \in (0, 2R), \tag{3.55}$$

which obviously implies Proposition 3.1 in Case II. □

For completeness, we give the proof of (3.55) in the following. If we set $R = R_0$, $\rho_0 = 2R$, $\rho_k = 4^{-k}\rho_0$, $k = 1, 2, \ldots$ and $w_k = \max_s \sup_{B_{\rho_k}(x_0)} |u_{x_s}|$, then (3.53) and (3.54) show that

$$w_k = \max\{2^{s_0}\rho_k, \delta_0 w_{k-1}\}$$

and

$$w_0 \le 2^{s_0}\rho_0 \equiv \widetilde{C} \cdot 4^{-\alpha},$$

where $\alpha = \min\{1, -\log_4 \delta_0\}$. Then for $y_k = 4^{k\alpha} w_k$, $k = 1, 2, \ldots$, we have

$$\begin{aligned} y_k &\le \max\{2^{s_0} \cdot 4^{k\alpha}\rho_k, \delta_0 \cdot 4^{k\alpha} w_{k-1}\} \\ &= \max\{2^{s_0} \cdot 4^{k(\alpha-1)}\rho_0, 4^\alpha \delta_0 y_{k-1}\} \\ &\le \max\{2^{s_0}\rho_0, y_{k-1}\} \\ &= \max\{\widetilde{C} \cdot 4^{-\alpha}, y_{k-1}\} \end{aligned} \tag{3.56}$$

and

$$y_0 = w_0 \le \widetilde{C} \cdot 4^{-\alpha}. \tag{3.57}$$

So (3.56), (3.57) show that for all $k = 0, 1, 2, \ldots$

$$y_k \le \widetilde{C} \cdot 4^{-\alpha},$$

that is,

$$w_k \le \widetilde{C} \cdot 4^{-\alpha} \cdot 4^{-k\alpha} = \widetilde{C} \cdot 4^{-\alpha} \left(\frac{\rho_k}{\rho_0}\right)^\alpha. \tag{3.58}$$

Now for any given $\rho \in (0, \rho_0]$, there exists a $k \geq 1$ such that $\rho_k \leq \rho \leq \rho_{k-1}$. Thus

$$\begin{aligned}\max_s \sup_{B_\rho(x_0)} |u_{x_s}| &\leq \max_s \sup_{B_{\rho_{k-1}}(x_0)} |u_{x_s}| \\ &= w_k \leq \widetilde{C} \cdot 4^{-\alpha}\rho_0^{-\alpha} \cdot \rho_{k-1}^{\alpha} \\ &= \widetilde{C}(4\rho_0)^{-\alpha} \cdot \rho^{\alpha}.\end{aligned} \tag{3.59}$$

Thus, Theorem 1(ii) is proved. □

4. The proof of Theorem 2

In this section, we will give the proof of Theorem 2. Firstly, we prove the local boundedness of weak solutions to (1.12). We consider any weak solution u to the equation

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x, u), & x \in \mathbf{R}^N, \\ u \in W^{1,p}(\mathbf{R}^N) \cap W^{1,q}(\mathbf{R}^N), \end{cases} \tag{4.1}$$

where $1 < q < p < N$ and $N \geq 3$. We will prove that if $f(x,t)$ satisfies the following

$$|f(x,t)| \leq \varepsilon|t|^{q-1} + C(\varepsilon)|t|^{p^*-1}, \tag{4.2}$$

where $p^* = \frac{Np}{N-p}$ if $N > p$ and $0 < p^* < \infty$ if $N \leq p$, then any weak solution u to (4.1) is locally bounded. We only consider the usual case $N > p$; the other case is even simpler. To prove this, we set $B_R = B_R(x_0)$ for some given $x_0 \in \mathbf{R}^N$ for simplicity and choose a nonnegative C^∞-function η with the properties

$$|\nabla\eta| \leq \frac{2}{r} \quad \text{for } r \in (0, R)$$

and

$$\eta = \begin{cases} 1, & \text{if } x \in B_R, \\ (0,1), & \text{others}, \\ 0, & \text{if } x \notin B_{R+r}. \end{cases}$$

Without loss of generality, we assume $u \geq 0$ and denote $\bar{u} = u + k$ for some $k > 0$. Then

$$\bar{u}_L = \begin{cases} \bar{u}, & \text{if } u < L, \\ L + k, & \text{if } u \geq L. \end{cases}$$

Otherwise, we will consider u^+, u^- and $\bar{u} = u^+ + k$, $\bar{u} = u^- + k$ separately. For all cases, we have $D\bar{u}_L = 0$ in $\{x \in \mathbf{R}^N \mid u(x) = 0 \text{ or } u(x) \geq L\}$.

Set the test function $\varphi(x) = \eta^p(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1})$, where $\beta > 1$ will be determined later. From now on, we denote by C a generic positive constant which may depend only on p, q, N. Inserting φ into (4.1) and integrating on $\mathbf{R}^N$, we get

$$\begin{aligned}(4.3)\qquad &\int_{\mathbf{R}^N} p\eta^{p-1}\big(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1}\big)|\nabla u|^{p-2}\nabla u\nabla\eta + \eta^p\bar{u}_L^{p(\beta-1)}|\nabla u|^{p-2}\nabla u\nabla\bar{u} \\ &\quad + p(\beta-1)\eta^p\bar{u}\bar{u}_L^{p(\beta-1)-1}|\nabla u|^{p-2}\nabla u\nabla\bar{u} \\ &\quad + p\eta^{p-1}\big(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1}\big)|\nabla u|^{q-2}\nabla u\nabla\eta + \eta^p\bar{u}_L^{p(\beta-1)}|\nabla u|^{q-2}\nabla u\nabla\bar{u} \\ &\quad + p(\beta-1)\eta^p\bar{u}\bar{u}_L^{p(\beta-1)-1}|\nabla u|^{q-2}\nabla u\nabla\bar{u} \\ &= \int_{\mathbf{R}^N} f(x,u)\varphi\,dx \le C\int_{\mathbf{R}^N}\big[\bar{u}^{p^*-1}+1\big]\eta^p\bar{u}\bar{u}_L^{p(\beta-1)}\,dx.\end{aligned}$$

Now

$$\begin{aligned}(4.4)\qquad &\left|\int_{\mathbf{R}^N} p\eta^{p-1}\big(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1}\big)|\nabla u|^{p-2}\nabla u\nabla\eta\,dx\right| \\ &\le p\int_{\mathbf{R}^N}\eta^{p-1}\bar{u}\bar{u}_L^{p(\beta-1)}|\nabla\bar{u}|^{p-2}|\nabla\bar{u}||\nabla\eta|\,dx \\ &\le \varepsilon\int_{\mathbf{R}^N}\big(\eta\bar{u}_L^{\beta-1}|\nabla\bar{u}|\big)^p\,dx + C(\varepsilon)\int_{\mathbf{R}^N}\big(\bar{u}\bar{u}_L^{(\beta-1)}|\nabla\eta|\big)^p\,dx,\end{aligned}$$

and, similarly,

$$\begin{aligned}(4.5)\qquad &\left|\int_{\mathbf{R}^N} p\eta^{p-1}\big(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1}\big)|\nabla u|^{q-2}\nabla u\nabla\eta\,dx\right| \\ &\le p\int_{\mathbf{R}^N}\eta^{p-1}\bar{u}\bar{u}_L^{p(\beta-1)}|\nabla\bar{u}|^{q-2}|\nabla\bar{u}||\nabla\eta|\,dx \\ &= p\int_{\mathbf{R}^N}\bar{u}_L^{p(\beta-1)}\big[\eta^{\frac{p(q-1)}{q}}|\nabla\bar{u}|^{q-2}|\nabla\bar{u}|\cdot\eta^{\frac{p}{q}-1}\bar{u}|\nabla\eta|\big]\,dx \\ &\le \int_{\mathbf{R}^N}\bar{u}_L^{p(\beta-1)}\big[\varepsilon\eta^p|\nabla\bar{u}|^q + C(\varepsilon)\eta^{p-q}\bar{u}^q|\nabla\eta|^p\big]\,dx \\ &= \varepsilon\int_{\mathbf{R}^N}\eta^p\bar{u}_L^{p(\beta-1)}|\nabla\bar{u}|^q\,dx + C(\varepsilon)\int_{\mathbf{R}^N}\eta^{p-q}\bar{u}^q\bar{u}_L^{p(\beta-1)}|\nabla\eta|^q\,dx.\end{aligned}$$

Thus ε can be chosen such that by (4.4), (4.5) and (4.3) we have

$$\begin{aligned}&C\int_{\mathbf{R}^N}\big[\bar{u}^{p^*-1}+1\big]\eta^p\bar{u}\bar{u}_L^{p(\beta-1)} \\ &\ge \int_{\mathbf{R}^N} p(\beta-1)\eta^p\bar{u}\bar{u}_L^{p(\beta-1)-1}|\nabla\bar{u}_L|^p + \frac{1}{2}\eta^p\bar{u}_L^{p(\beta-1)}|\nabla\bar{u}|^p - C\cdot(\bar{u}\bar{u}_L^{\beta-1}|\nabla\eta|)^p \\ &\quad + p(\beta-1)\eta^p\bar{u}\bar{u}_L^{p(\beta-1)-1}|\nabla\bar{u}_L|^q + \frac{1}{2}\eta^p\bar{u}_L^{p(\beta-1)}|\nabla\bar{u}|^q - C\eta^{p-q}\bar{u}^q\bar{u}_L^{p(\beta-1)}|\nabla\eta|^q\,dx.\end{aligned}$$

Taking $k = 1$ and noting that $\bar{u} \geq k$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} p(\beta-1)\eta^p \bar{u}_L^{p(\beta-1)}\big(|\nabla \bar{u}_L|^p + |\nabla \bar{u}_L|^q\big) + \frac{1}{2}\eta^p \bar{u}_L^{p(\beta-1)}\big(|\nabla \bar{u}|^p + |\nabla \bar{u}|^q\big)\, dx \\
&\leq C \int_{\mathbf{R}^N} \Big[\eta^p \bar{u}^{p^*} \bar{u}_L^{p(\beta-1)} + \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p + \eta^{p-q}\bar{u}^q \bar{u}_L^{p(\beta-1)}|\nabla \eta|^q\Big]\, dx.
\end{aligned} \tag{4.6}$$

Set $W_L = \eta \bar{u}\bar{u}_L^{\beta-1}$ for $\beta > 1$. Observing that $\eta \bar{u}_L^{\beta-1} \leq \eta \bar{u}\bar{u}_L^{\beta-1}$ and

$$\begin{aligned}
\int_{\mathbf{R}^N} \eta^{p-q}\bar{u}^q \bar{u}_L^{p(\beta-1)}|\nabla \eta|^q\, dx &= \int_{\mathbf{R}^N} \bar{u}^q \bar{u}_L^{q(\beta-1)}|\nabla \eta|^q \cdot \eta^{p-q}\bar{u}_L^{(p-q)(\beta-1)}\, dx \\
&\leq \int_{\mathbf{R}^N} \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p\, dx + \int_{\mathbf{R}^N} \big(\eta \bar{u}_L^{\beta-1}\big)^p\, dx \\
&\leq \int_{\mathbf{R}^N} \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p\, dx + \int_{\mathbf{R}^N} \big(\eta \bar{u}\bar{u}_L^{\beta-1}\big)^p\, dx,
\end{aligned}$$

(4.6) implies that

$$\begin{aligned}
&\left(\int_{\mathbf{R}^N} \big(\eta \bar{u}\bar{u}_L^{\beta-1}\big)^{p^*}\, dx\right)^{\frac{p}{p^*}} = \left(\int_{\mathbf{R}^N} W_L^{p^*}\, dx\right)^{\frac{p}{p^*}} \leq C \int_{\mathbf{R}^N} |\nabla W_L|^p\, dx \\
&\leq C \int_{\mathbf{R}^N} \bar{u}^p \bar{u}_L^{p(\beta-1)}|\nabla \eta|^p + C\beta^p \int_{\mathbf{R}^N} \Big[\eta^p \bar{u}_L^{p(\beta-1)}|\nabla \bar{u}|^p + \eta^p \bar{u}_L^{p(\beta-1)}|\nabla \bar{u}_L|^p\Big]\, dx \\
&\leq C \int_{\mathbf{R}^N} \bar{u}^p \bar{u}_L^{p(\beta-1)}|\nabla \eta|^p + C\beta^p \\
&\quad \cdot \int_{\mathbf{R}^N} \Big[\eta^p \bar{u}^{p^*} \bar{u}_L^{p(\beta-1)} + \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p + \eta^{p-q}\bar{u}^q \bar{u}_L^{p(\beta-1)}|\nabla \eta|^q\Big]\, dx \\
&\leq C\beta^p \int_{\mathbf{R}^N} \Big[\eta^p \bar{u}^{p^*} \bar{u}_L^{p(\beta-1)} + \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p + \big(\eta \bar{u}\bar{u}_L^{\beta-1}\big)^p\Big]\, dx \\
&\leq C\beta^p \left[\int_{\mathbf{R}^N} \big(\bar{u}\bar{u}_L^{\beta-1}|\nabla \eta|\big)^p\, dx + \int_{\mathbf{R}^N} \eta^p \bar{u}^{p^*} \bar{u}_L^{p(\beta-1)}\, dx\right].
\end{aligned} \tag{4.7}$$

We claim that there exists an $R_0 > 0$ such that

$$\bar{u} \in L^{(p^*)^2/p}(B_{R_0}). \tag{4.8}$$

In fact, since

$$\int_{\mathbf{R}^N} \eta^p \bar{u}^{p^*} \bar{u}_L^{p(\beta-1)}\, dx \leq \left[\int_{\mathbf{R}^N} (\eta \bar{u}\bar{u}_L^{\beta-1})^{p^*}\, dx\right]^{p/p^*} \cdot \left[\int_{B_{R+r}} \bar{u}^{p^*}\, dx\right]^{(p^*-p)/p^*},$$

taking $\beta = p^*/p$ in (4.7) and $R = R_0$ small enough such that

$$\left[\int_{B_{2R}} \bar{u}^{p^*}\, dx\right]^{(p^*-p)/p^*} \leq \frac{1}{2C},$$

we get that

$$\left(\int_{\mathbf{R}^N}\left(\eta\bar{u}\bar{u}_L^{\beta-1}\right)^{p^*}dx\right)^{\frac{p}{p^*}}\le C\int_{\mathbf{R}^N}\left(\bar{u}\bar{u}_L^{\beta-1}|\nabla\eta|\right)^p dx\le C\int_{\mathbf{R}^N}\bar{u}^{p\beta}|\nabla\eta|^p\,dx. \tag{4.9}$$

Letting $L\to+\infty$ in (4.9), we get

$$\left(\int_{B_{R_0}}\bar{u}^{(p^*)^2/p}\,dx\right)^{\frac{p}{p^*}}\le C\int_{B_{\mathbf{R}^N}}|\nabla\eta|^p\bar{u}^{p^*}\,dx<+\infty.$$

Then we will show that $\bar{u}\in L^\infty(B_R)$, $0<R<R_0/2$.

Set $t=(p^*)^2/(p^*-p)p>1$. Suppose $\bar{u}\in L^{p\beta t/(t-1)}(B_{R+r})$, $0<r<R$. By (4.8) and Sobolev's inequality, we have

$$\begin{aligned}\int_{\mathbf{R}^N}\eta^p\bar{u}^{p^*}\bar{u}_L^{p(\beta-1)}\,dx&\le\left[\int_{B_{R+r}}(\eta^p\bar{u}^{p\beta})^{t/(t-1)}\right]^{1-1/t}\cdot\int_{B_{R+r}}(\bar{u}^{(p^*-p)t}\,dx)^{1/t}\\&\le\left[\int_{B_{R+r}}(\eta^p\bar{u}^{p\beta})^{t/(t-1)}\right]^{t-1/t}\cdot\int_{B_{R+r}}(\bar{u}^{(p^*)^2/p}\,dx)^{1/t}\\&\le C\left[\int_{B_{R+r}}(\eta^p\bar{u}^{p\beta})^{t/(t-1)}\right]^{1-1/t}\end{aligned}\tag{4.10}$$

and

$$\int_{\mathbf{R}^N}|\nabla\eta|^p\bar{u}^p\bar{u}_L^{p(\beta-1)}\,dx\le Cr^{-p}\left[\int_{B_{R+r}}(\bar{u}^{p\beta})^{t/(t-1)}\,dx\right]^{1-1/t}. \tag{4.11}$$

So by (4.10), (4.11) and (4.7), we get

$$\left[\int_{\mathbf{R}^N}(\eta\bar{u}\bar{u}_L^{\beta-1})^{p^*}\,dx\right]^{p/p^*}\le C\beta^p r^{-p}\left[\int_{B_{R+r}}(\bar{u}^{p\beta})^{t/(t-1)}\,dx\right]^{1-1/t},$$

i.e.,

$$\left[\int_{B_R}\bar{u}^{p^*\beta}\,dx\right]^{1/\beta}\le C^{1/\beta}\beta^{p^*/\beta}r^{-p^*/\beta}\left[\int_{B_{R+r}}\bar{u}^{p\beta t/(t-1)}\,dx\right]^{\frac{(t-1)p^*}{tp\beta}}, \tag{4.12}$$

where C is independent of r, β.

Set $\chi=p^*(t-1)/pt$ ($\chi>1$), $\beta=\chi^i$, $B_i=B_{R+2^{-i}r}$, $i=0,1,\ldots$, in (4.12) and

$$I_i=\left(\int_{B_i}\left(|\bar{u}|^{(pt)/(t-1)}\right)^{\chi^i}dx\right)^{1/\chi^i}. \tag{4.13}$$

Then (4.12) implies that

$$
\begin{aligned}
I_{i+1} &= \|\bar{u}^{pt/(t-1)}\|_{\chi^{i+1}(B_{i+1})} = \|\bar{u}\|^{p^*/\chi}_{p^*\chi^i(B_{i+1})} \\
&\leq C^{\frac{1}{\chi^{i+1}}} \left(\frac{r}{2^{i+1}}\right)^{-\frac{p^*}{\chi^{i+1}}} \|\bar{u}^{(pt)/(t-1)} \chi^{p^* i \chi^{-(i+1)}}\|_{\chi^i(B_i)} \\
&= C^{\frac{1}{\chi^{i+1}}} \cdot \left[2^{-(i+1)} r\right]^{\frac{-p^*}{\chi^{i+1}}} \chi^{p^* i \chi^{-(i+1)}} I_i \\
&\leq C^{\Sigma_{j=0}^{i+1}\chi^{-j}} (2^{p^*})^{\Sigma_{j=0}^{i+1} j\chi^{-j}} (r^{-p^*})^{\Sigma_{j=0}^{i+1}\chi^{-j}} \chi^{p^*\Sigma_{j=0}^{i+1} j\chi^{-(j+1)}} I_0.
\end{aligned} \tag{4.14}
$$

Note that $I_0 \leq C(\int_{B_{2R}} |\bar{u}|^{p^*}\, dx)^{\frac{1}{p^*}} < +\infty$; so let $i \to +\infty$ in (4.13). We get

$$\bar{u} \in L^\infty(B_R(x_0)), \tag{4.15}$$

and since $x_0 \in \mathbf{R}^N$ is arbitrary, we have

$$u \in L^\infty_{\text{loc}}(\mathbf{R}^N) \tag{4.16}$$

by the definition of $\bar{u}$. Thus, with the help of (4.16), Theorem 1 implies Theorem 2. □

For equation (1.1), one can set

$$f(x,u) = g(x,u) - m|u|^{p-2}u - n|u|^{q-2}u.$$

It is obvious that $g(x,u)$ and $f(x,u)$ satisfy (4.2) if $g(x,u)$ satisfies (C_1)–(C_2) in [8]. So one can see that the solutions of (1.1) are locally bounded. Then Theorem 1 implies that these solutions are locally in $C^{1,\alpha}$.

5. The proof of Theorem 3

In this section, we will give the proof of Theorem 3 by virtue of (4.16). To show (i) of Theorem 3, we mainly follow the steps of [10]. The difference is that, as one can see, neither the test function $v = \eta^p u^+ (u_L^+)^{p(\beta-1)}$ used in [10] nor the test function $\varphi = \eta^p(\bar{u}\bar{u}_L^{p(\beta-1)} - k^{p(\beta-1)+1})$ used in §4 works in our case.

To overcome this difficulty, our main idea is to use two test functions separately to get a couple of inequalities and then combine them to get the decay estimate of the weak solutions. As soon as this is done, we can follow the way of [11] to prove Theorem 3(ii) with the help of Theorem 3(i). In the following, C stands for a generic constant depending only on N, p, q, and m, n.

We choose a nonnegative C^∞-function ξ having the following properties:

$$
|\nabla\xi| \leq \frac{2}{r} \quad \text{for some } r \in (0, R/2),
$$

$$
\xi = \begin{cases} 1, & \text{if } x \in {B_R}^c, \\ (0,1), & \text{others}, \\ 0, & \text{if } x \in B_{R-r}, \end{cases}
$$

where $B_\rho = B_\rho(0)$ and ${B_\rho}^c \equiv \mathbf{R}^N \backslash B_\rho$ for $\rho > 0$. Without loss of generality, we assume $u \geq 0$ and define the test function $\varphi(x) = \xi^p u u_L^{p(\beta-1)}$ and $W_L = \xi u u_L^{\beta-1}$, where u_L is defined as before and $\beta > 1$ is to be determined later.

Inserting φ into (1.1) and integrating on $\mathbf{R}^N$ as in §4, we get the estimate

$$\begin{aligned}
&\int_{\mathbf{R}^N} p(\beta-1)\eta^p u_L^{p(\beta-1)}(|\nabla u_L|^p + |\nabla u_L|^q) + \frac{1}{2}\eta^p u_L^{p(\beta-1)}(|\nabla u|^p + |\nabla u|^q)\,dx \\
&\leq C\int_{\mathbf{R}^N}\left[f\varphi + (u u_L^{\beta-1}|\nabla\eta|)^p + \eta^{p-q}u^q u_L^{p(\beta-1)}|\nabla\eta|^q\right]dx,
\end{aligned} \tag{5.1}$$

where $f(x,t) = g(x,t) - m|t|^{p-2}t - n|t|^{q-2}t$. Note that $|g(x,t)| \leq \varepsilon|t|^{p-1} + C(\varepsilon)|t|^{p^*-1}$ for any $\varepsilon > 0$ and $t \geq 0$. We have

$$\int_{\mathbf{R}^N} f\varphi\,dx \leq (\varepsilon - n)\xi^p u^q u_L^{p(\beta-1)} - m\xi^p u^p u_L^{p(\beta-1)} + C(\varepsilon)u^{p^*}u_L^{p\beta-1}. \tag{5.2}$$

By (5.1), (5.2) with $\varepsilon = n/2$ and the fact that $u^q u_L^{p(\beta-1)} \leq u^q u_L^{q(\beta-1)} + u^p u_L^{p(\beta-1)}$, we have

$$\begin{aligned}
\int_{\mathbf{R}^N}|\nabla W_L|^p\,dx \leq C\beta^p &\int_{\mathbf{R}^N} u^p u_L^{p(\beta-1)}(|\nabla\xi|^p + \xi^{p-q}|\nabla\xi|^q)\,dx \\
&+ C\beta^p\int_{\mathbf{R}^N} u^q u_L^{q(\beta-1)}\xi^{p-q}|\nabla\xi|^q\,dx \\
&+ C\beta^p\int_{\mathbf{R}^N}\xi^p u^{p^*}u_L^{p(\beta-1)}\,dx.
\end{aligned} \tag{5.3}$$

Define $\psi(x) = \xi^p u u_L^{q(\beta-1)}$ and $V_L = \xi^{p/q}u u_L^{\beta-1}$, insert ψ into (1.1) and estimate as before. We get

$$\begin{aligned}
\int_{\mathbf{R}^N}|\nabla V_L|^q\,dx \leq C\beta^q &\int_{\mathbf{R}^N} u^q u_L^{q(\beta-1)}(|\nabla\xi|^p + \xi^{p-q}|\nabla\xi|^q)\,dx \\
&+ C\beta^p\int_{\mathbf{R}^N} u^p u_L^{p(\beta-1)}|\nabla\xi|^p\,dx + C\beta^p\int_{\mathbf{R}^N}\xi^p u^{p^*}u_L^{q(\beta-1)}\,dx,
\end{aligned} \tag{5.4}$$

where we have used the fact that $u^p u_L^{q(\beta-1)} \leq u^p u_L^{p(\beta-1)} + u^q u_L^{q(\beta-1)}$. Taking r small enough, (5.3), (5.4) and Sobolev's inequalities imply that

$$\begin{aligned}
&\left(\int_{\mathbf{R}^N} W_L^{p^*}\,dx\right)^{p/p^*} + \left(\int_{\mathbf{R}^N} V_L^{q^*}\,dx\right)^{q/q^*} \\
&\leq C\left(\int_{\mathbf{R}^N}|\nabla W_L|^p\,dx + \int_{\mathbf{R}^N}|\nabla V_L|^q\,dx\right) \\
&\leq C\beta^p\int_{\mathbf{R}^N}\left(u^p u_L^{p(\beta-1)} + u^q u_L^{q(\beta-1)}\right)(|\nabla\xi|^p + \xi^{p-q}|\nabla\xi|^q)\,dx \\
&\quad + C\beta^p\int_{\mathbf{R}^N}\xi^p u^{p^*}u_L^{p(\beta-1)}\,dx + C\beta^p\int_{\mathbf{R}^N}\xi^p u^{p^*}u_L^{q(\beta-1)}\,dx
\end{aligned} \tag{5.5}$$

$$\le C\beta^p \int_{\mathbf{R}^N} \big(u^p u_L^{p(\beta-1)} + u^q u_L^{q(\beta-1)}\big)|\nabla\xi|^p\,dx + C\beta^p \int_{\mathbf{R}^N} \xi^p u^{p^*} u_L^{p(\beta-1)}\,dx + C\beta^p \int_{\mathbf{R}^N} \xi^p u^{q^*} u_L^{q(\beta-1)}\,dx,$$

where we have used the fact that $u^{p^*}u_L^{q(\beta-1)} \le u^{p^*}u_L^{p(\beta-1)} + u^{q^*}u_L^{q(\beta-1)}$.

We claim that

$$u \in L^{(p^*)^2/p} \cap L^{(q^*)^2/q}(|x| \ge R). \tag{5.6}$$

In fact, since

$$\int_{\mathbf{R}^N} \eta^p u^{p^*} \bar{u}_L^{p(\beta-1)}\,dx \le \left[\int_{\mathbf{R}^N} \big(\eta u u_L^{\beta-1}\big)^{p^*}\,dx\right]^{p/p^*} \cdot \left[\int_{|x|\ge R-r} u^{p^*}\,dx\right]^{(p^*-p)/p^*}, \tag{5.7}$$

$$\int_{\mathbf{R}^N} \eta^p u^{q^*} \bar{u}_L^{q(\beta-1)}\,dx \le \left[\int_{\mathbf{R}^N} \big(\eta^{p/q} u u_L^{\beta-1}\big)^{q^*}\,dx\right]^{q/q^*} \cdot \left[\int_{|x|\ge R-r} u^{q^*}\,dx\right]^{(q^*-q)/q^*} \tag{5.8}$$

and $u \in L^{p^*} \cap L^{q^*}(\mathbf{R}^N)$, letting $\beta = p^*/p$, we have, for R large enough, that

$$\left[\int_{|x|\ge R-r} u^{p^*}\,dx\right]^{(p^*-p)/p^*} \le \frac{1}{2C\beta^p}, \quad \left[\int_{|x|\ge R-r} u^{q^*}\,dx\right]^{(q^*-q)/q^*} \le \frac{1}{2C\beta^p}. \tag{5.9}$$

So, (5.5), (5.7), (5.8) and (5.9) imply that

$$\begin{aligned}
&\left(\int_{|x|\ge R} \big(u u_L^{p^*/p-1}\big)^{p^*}\,dx\right)^{p/p^*} \le Cr^{-p} \int_{\mathbf{R}^N} \big(u^{p^*} + u^{qp^*/p}\big)\,dx \\
&\le Cr^{-p} \int_{\mathbf{R}^N} u^{p^*}\,dx + Cr^{-p} \int_{\mathbf{R}^N} u^{\frac{Np}{N-p}-q} u^q\,dx \\
&\le Cr^{-p} \int_{\mathbf{R}^N} u^{p^*}\,dx + Cr^{-p} \left(\int_{\mathbf{R}^N} u^{p^*}\,dx\right)^{\frac{q}{N}} \left(\int_{\mathbf{R}^N} u^{q^*}\,dx\right)^{\frac{N-q}{N}} \\
&< +\infty.
\end{aligned} \tag{5.10}$$

Similarly, letting $\beta = q^*/q$ and noticing that $q^* < pq^*/q < p^*$ implies $u^{pq^*/q} \le u^{q^*} + u^{p^*}$, we get

$$\begin{aligned}
&\left(\int_{|x|\ge R} \big(u u_L^{q^*/q-1}\big)^{q^*}\,dx\right)^{q/q^*} \le Cr^{-p} \int_{\mathbf{R}^N} \big(u^{q^*} + u^{pq^*/q}\big)\,dx \\
&\le Cr^{-p} \int_{\mathbf{R}^N} u^{q^*}\,dx + Cr^{-p} \int_{\mathbf{R}^N} \big(u^{p^*} + u^{q^*}\big)\,dx < +\infty.
\end{aligned} \tag{5.11}$$

If we let $L \to \infty$ in (5.10) and (5.11), (5.6) follows.

Now we give the proof of $u \in L^\infty(|x| \ge R)$. Notice that (5.5) implies that either

$$\left(\int_{\mathbf{R}^N} W_L^{p^*}\,dx\right)^{p/p^*} \le C\beta^p \int_{\mathbf{R}^N} \big(u^p u_L^{p(\beta-1)}|\nabla\xi|^p + \xi^p u^{p^*} u_L^{p(\beta-1)}\big)\,dx \tag{5.12}$$

or

$$\left(\int_{\mathbf{R}^N} V_L^{q^*}\,dx\right)^{q/q^*} \le C\beta^p \int_{\mathbf{R}^N} \left(u^q u_L^{q(\beta-1)}|\nabla\xi|^p + \xi^p u^{q^*} u_L^{q(\beta-1)}\right) dx \tag{5.13}$$

is true. Let $t_1 = (p^*)^2/(p^*-p)p$; then $t_1 > 1$. Suppose that $u \in L^{\beta p t_1/(t_1-1)}(|x| \ge R-r)$ for some $\beta \ge 1$. Then

$$\begin{aligned}\int_{\mathbf{R}^N} \eta^p u^{p^*} \bar{u}_L^{p(\beta-1)}\,dx &\le \left[\int_{B_{R^N}} (\eta^p u^{p\beta})^{t_1/(t_1-1)}\,dx\right]^{1-1/t_1} \cdot \int_{|x|\ge R-r} \left(u^{(p^*-p)t_1}\,dx\right)^{1/t_1}\\ &\le \left[\int_{|x|\ge R-r} (u^{p\beta})^{t_1/(t_1-1)}\,dx\right]^{1-1/t_1} \cdot \int_{|x|\ge R-r} \left(u^{(p^*)^2/p}\,dx\right)^{1/t_1}\\ &\le C\left[\int_{|x|\ge R-r} (u^{p\beta})^{t_1/(t_1-1)}\,dx\right]^{1-1/t_1}\end{aligned}$$

and

$$\int_{\mathbf{R}^N} |\nabla\eta|^p u^p u_L^{p(\beta-1)}\,dx \le Cr^{-p}\left[R^N-(R-r)^N\right]^{1/t_1}\left[\int_{|x|\ge R-r} (u^{p\beta})^{t_1/(t_1-1)}\,dx\right]^{1-1/t_1}.$$

So by (5.12) we get

$$\left[\int_{\mathbf{R}^N} (\eta u u_L^{\beta-1})^{p^*}\,dx\right]^{p/p^*} \le C\beta^p\left(1+r^{-p}R^{N/t_1}\right)\left[\int_{|x|\ge R-r} (u^{p\beta})^{t_1/(t_1-1)}\,dx\right]^{1-1/t_1},$$

that is,

$$\|u\|_{p^*\beta} \le C^{\beta^{-1}}\beta^{\beta^{-1}}\left(1+r^{-p}R^{N/t_1}\right)^{(p\beta)^{-1}}\|u\|_{\beta s_1}(|x|\ge R-r), \tag{5.14}$$

where $s_1 = pt_1/(t_1-1)$ and C is independent of r, β. Similarly, if we set $t_2 = (q^*)^2/(q^*-q)q$ and $s_2 = qt_2/(t_2-1)$, (5.13) implies that

$$\|u\|_{q^*\beta} \le C^{\beta^{-1}}\beta^{p/q\beta^{-1}}\left(1+r^{-p}R^{N/t_2}\right)^{(q\beta)^{-1}}\|u\|_{\beta s_2}(|x|\ge R-r), \tag{5.15}$$

that is, for any given ξ defined as before, we have that (5.14) or (5.15) is true.

We set $R>0$, $0<r<R/2$, $R_i = R-2^{-i}r$, $B_i = B_{R_i}(0)$ for $i=0,1,\dots$ and use (5.14) and (5.15) to iterate as follows: For $i=0$, we set $I_0 = \|u\|_{p^*(B_0^c)}$; For $i=1$, if (5.14) holds, we set $\beta_1 = p^*(t_1-1)/(pt_1) = p^*/s_1$ and $\nu_1 = p^*\beta_1$. Then by (5.14) with $\beta=\beta_1$ we have

$$I_1 \equiv \|u\|_{\nu_1(B_1^c)} = \|u\|_{p^*\beta_1(B_1^c)} \le C^{\beta_1^{-1}}\beta_1^{\beta_1^{-1}}\left(1+(2^1/r)^pR^{N/t_1}\right)^{(p\beta_1)^{-1}} I_0. \tag{5.16}$$

If (5.15) holds, we set $\beta_1 = p^*/s_2$ and $\nu_1 = q^*\beta_1$, then by (5.15) with $\beta=\beta_1$ to get

$$I_1 \equiv \|u\|_{\nu_1(B_1^c)} = \|u\|_{q^*\beta_1(B_1^c)} \le C^{\beta_1^{-1}}\beta_1^{(p/q)\beta_1^{-1}}\left(1+(2^1/r)^pR^{N/t_2}\right)^{(q\beta_1)^{-1}} I_0. \tag{5.17}$$

For $i=2$, if (5.16) and (5.14) hold, we set β_2 with $\beta_2 s_1 = p^*\beta_1 = \nu_1$ (i.e., $\beta_2 = \nu_1/s_1$), $\nu_2 = p^*\beta_2$, then by (5.14) and (5.16) with $\beta=\beta_2$ to get

$$I_2 \equiv \|u\|_{\nu_2(B_2^c)} \le C^{\beta_2^{-1}}\beta_2^{\beta_2^{-1}}\left(1+(2^2/r)^pR^{N/t_1}\right)^{(p\beta_2)^{-1}} I_1. \tag{5.18}$$

If (5.16) and (5.15) hold, we set β_2 with $\beta_2 s_2 = p^*\beta_1 = \nu_1$ (i.e., $\beta_2 = \nu_1/s_2$), $\nu_2 = q^*\beta_2$, then by (5.15) and (5.16) with $\beta = \beta_2$ to get

$$I_2 \equiv \|u\|_{\nu_2(B_2^c)} \le C^{\beta_2^{-1}} \beta_2^{(p/q)\beta_2^{-1}} \big(1 + (2^2/r)^p R^{N/t_2}\big)^{(q\beta_2)^{-1}} I_1. \tag{5.19}$$

If (5.17) and (5.14) hold, we set β_2 with $\beta_2 s_1 = q^*\beta_1 = \nu_1$ (i.e., $\beta_2 = \nu_1/s_1$), $\nu_2 = p^*\beta_2$, then by (5.14) and (5.17) with $\beta = \beta_2$ to get

$$I_2 \equiv \|u\|_{\nu_2(B_2^c)} \le C^{\beta_2^{-1}} \beta_2^{\beta_2^{-1}} \big(1 + (2^2/r)^p R^{N/t_1}\big)^{(p\beta_2)^{-1}} I_1. \tag{5.20}$$

If (5.17) and (5.15) hold, we set β_2 with $\beta_2 s_2 = q^*\beta_1 = \nu_1$ (i.e., $\beta_2 = \nu_1/s_2$), $\nu_2 = q^*\beta_2$, then by (5.15) and (5.17) with $\beta = \beta_2$ to get

$$I_2 \equiv \|u\|_{\nu_2(B_2^c)} \le C^{\beta_2^{-1}} \beta_2^{(p/q)\beta_2^{-1}} \big(1 + (2^2/r)^p R^{N/t_2}\big)^{(q\beta_2)^{-1}} I_1. \tag{5.21}$$

Note that all the ν_i and $\beta_i,\ i = 1, 2$ above have the forms

$$\nu_i = p^*(p^*/s_1)^k (q^*/s_2)^{i-k}, \quad i = 1, 2,\ k = 0, 1, \dots, i,$$
$$\beta_i = \nu_i/p^* \quad \text{or} \quad \beta_i = \nu_i/q^*, \quad i = 1, 2.$$

Now $1 < (q^*/s_2)^i \le \beta_i \le p^*/q(p^*/s_1)^i$ for all $i \ge 1$, and there are only two cases:

$$\begin{aligned} I_{i+1} \equiv \|u\|_{\nu_{i+1}} = \|u\|_{p^*\beta_i}(B_{i+1}^c) &\le C^{\beta_{i+1}^{-1}} \beta_{i+1}^{\beta_{i+1}^{-1}} \big(1 + (2^{i+1}/r)^p R^{N/t_1}\big)^{(p\beta_{i+1})^{-1}} I_i \\ &\le \big[Cp^*/q(1 + r^{-p} R^{N/t_1})\big]^{\Sigma_{j=1}^{i+1}(q^*/s_2)^{-1}} (2p^*/s_1)^{\Sigma_{j=1}^{i+1} j(q^*/s_2)^{-j}} I_0 \end{aligned} \tag{5.22}$$

or

$$\begin{aligned} I_{i+1} = \|u\|_{q^*\beta_i}(B_{i+1}^c) &\le C^{\beta_{i+1}^{-1}} \beta_{i+1}^{(p/q)\beta_{i+1}^{-1}} \big(1 + (2^{i+1}/r)^p R^{N/t_2}\big)^{(q\beta_{i+1})^{-1}} I_i \\ &\le \big[C(p^*/q)^{p/q}(1 + r^{-p} R^{N/t_2})\big]^{\Sigma_{j=1}^{i+1}(q^*/s_2)^{-j}} (2p^*/s_1)^{p/q\Sigma_{j=1}^{i+1} j(q^*/s_2)^{-j}} I_0. \end{aligned} \tag{5.23}$$

If we let $i \to \infty$, then (5.22) and (5.23) imply that

$$I_\infty \equiv \|u\|_{\infty(B_R^c)} \le (C(p, q, r, R))^{\Sigma_{j=1}^{\infty}(q^*/s_2)^{-j}} (2p^*/s_1)^{p/q\Sigma_{j=1}^{\infty} j(q^*/s_2)^{-j}} I_0. \tag{5.24}$$

Since $q^* > s_2$, (5.24) implies that

$$\|u\|_{L^\infty(|x|\ge R)} \le C\|\bar{u}\|_{p^*(|x|\ge R-r)} \le C\|\bar{u}\|_{p^*(|x|\ge R/2)}. \tag{5.25}$$

Inequality (5.25) and the local boundedness of u imply (i) of Theorem 3. With the help of (5.25), one can follow the steps of ([11] Theorem 3.1) to prove the exponential decay of u. We just sketch the proof of this fact here. In fact, (i) shows that there is a constant $\widetilde{C}$, such that $\|u\|_\infty \le \widetilde{C}$. We define a smooth function $U(x) = \widetilde{C} e^{\varepsilon R} e^{-\varepsilon|x|}$ and the test function $\phi = (u - U)^+$. It is obvious that $\phi \in W_0^{1,p}(\mathbf{R}^N \backslash B_R)$. Then we have, if $|x| > R$ is large enough and $\varepsilon > 0$ is small enough, that

$$\begin{aligned} &- \Delta_p U - \Delta_q U + \frac{m}{2}|U|^{p-2}U + \frac{n}{2}|U|^{q-2}U \\ &= U^{p-1}\left[\frac{m}{2} - \frac{(N-1)}{|x|}\varepsilon^{p-1} - (p-1)\varepsilon^p\right] + U^{q-1}\left[\frac{n}{2} - \frac{(N-1)}{|x|}\varepsilon^{q-1} - (q-1)\varepsilon^q\right] \\ &> 0. \end{aligned}$$

That is why,

$$\int_{|x|\geq R}\Big(-\Delta_p U - \Delta_q U + \frac{m}{2}|U|^{p-2}U + \frac{n}{2}|U|^{q-2}U\Big)\,\phi\,dx \geq 0. \tag{5.26}$$

On the other hand, by (C$_2$) we have

$$f(x,u) \leq -\frac{m}{2}|u|^{p-2}u - \frac{n}{2}|u|^{q-2}u \quad \text{as } u \to 0^+. \tag{5.27}$$

Thus, (1.1) and (5.27) imply that

$$\int_{|x|\geq R}(-\Delta_p u - \Delta_q u + m/2|u|^{p-2}u + n/2|u|^{q-2}u)\phi dx \leq 0. \tag{5.28}$$

So, (5.26), (5.28) and the definition of ϕ show that

$$\begin{aligned}
0 &\geq \int_{|x|\geq R}\sum_{i=1}^{N}\big(|\nabla u|^{p-2}u_{x_i} - |\nabla U|^{p-2}U_{x_i}\big)\phi_{x_i}\,dx \\
&\quad + \frac{m}{2}\int_{|x|\geq R}(u^{p-1} - U^{p-1})\phi\,dx \\
&\quad + \int_{|x|\geq R}\sum_{i=1}^{N}\big(|\nabla u|^{q-2}u_{x_i} - |\nabla U|^{q-2}U_{x_i}\big)\phi_{x_i}\,dx \\
&\quad + \frac{n}{2}\int_{|x|\geq R}(u^{q-1} - U^{q-1})\phi\,dx \\
&= \int_{\{|x|\geq R\}\cap\{u>U\}}\sum_{i=1}^{N}\big(|\nabla u|^{p-2}u_{x_i} - |\nabla U|^{p-2}U_{x_i}\big)\phi_{x_i} \\
&\quad + \frac{m}{2}(u^{p-1} - U^{p-1})\phi\,dx \\
&\quad + \int_{\{|x|\geq R\}\cap\{u>U\}}\sum_{i=1}^{N}\big(|\nabla u|^{q-2}u_{x_i} - |\nabla U|^{q-2}U_{x_i}\big)\phi_{x_i} \\
&\quad + \frac{n}{2}(u^{q-1} - U^{q-1})\phi\,dx.
\end{aligned} \tag{5.29}$$

Since $(|\xi|^{t-2}\xi_i - |\eta|^{t-2}\eta)(\xi_i - \eta_i) > 0$ when $t > 1$, $\xi \neq \eta$, (5.29) implies that

$$u \leq U \quad \text{a.e. in } \{x \in \mathbf{R}^N : \ |x| > R\}. \tag{5.30}$$

Notice that $U \in C^\infty(\mathbf{R}^N)$ and Theorem 2 implies that $u \in C^1(\mathbf{R}^N)$. Therefore

$$u \leq \widetilde{C}e^{\varepsilon R}e^{-\varepsilon|x|} = Ce^{-\varepsilon|x|}$$

when $|x| \geq R$. This completes the proof of Theorem 3. □

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