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CHARACTERIZATION OF REARRANGEMENT INVARIANT SPACES WITH FIXED POINTS FOR THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

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Abstract. We characterize the rearrangement invariant spaces for which there exists a non-constant fixed point, for the Hardy–Littlewood maximal operator (the case for the spaces $L^p(\mathbf{R}^n)$ was first considered in [7]). The main result that we prove is that the space $L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ is minimal among those having this property.

1. Introduction

The centered Hardy–Littlewood maximal operator $\mathscr{M}$ is defined on the Lebesgue space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ by

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|\,dy,$$

where $|B_r|$ denotes the measure of the Euclidean ball B_r centered at the origin of $\mathbf{R}^n$.

In this paper we study the existence of non-constant fixed points of the maximal operator $\mathscr{M}$ (i.e., $\mathscr{M} f = f$) in the framework of the rearrangement invariant (r.i.) functions spaces (see Section 2 below). We will use some of the estimates proved in [7], where the case $L^p(\mathbf{R}^n)$ was studied, and show that they can be sharpened to obtain all the rearrangement invariant norms with this property (in particular we extend Korry's result to the end point case $p=n/(n-2)$, where the weak-type spaces have to considered). The main argument behind this problem is the existence of a minimal space $L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ contained in all the r.i. spaces with the fixed point property.

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2. Background on rearrangement invariant spaces

Since we work in the context of rearrangement invariant spaces it will be convenient to start by reviewing some basic definitions about these spaces.

A rearrangement invariant space $X = X(\mathbf{R}^n)$ (r.i. space) is a Banach function space on $\mathbf{R}^n$ endowed with a norm $\|\cdot\|_{X(\mathbf{R}^n)}$ such that

$$\|f\|_{X(\mathbf{R}^n)} = \|g\|_{X(\mathbf{R}^n)}$$

whenever $f^* = g^*$. Here f^* stands for the non-increasing rearrangement of f, i.e., the non-increasing, right-continuous function on $[0,\infty)$ equimeasurable with f.

An r.i. space $X(\mathbf{R}^n)$ has a representation as a function space on $\overline{X}\,(0,\infty)$ such that

$$\|f\|_{X(\mathbf{R}^n)} = \|f^*\|_{\overline{X}\,(0,\infty)}.$$

Any r.i. space is characterized by its *fundamental function*

$$\phi_X(s) = \|\chi_E\|_{X(\mathbf{R}^n)}$$

(here E is any subset of $\mathbf{R}^n$ with $|E| = s$) and the *fundamental indices*

$$\bar{\beta}_X = \inf_{s>1} \frac{\log M_X(s)}{\log s} \quad \text{and} \quad \underline{\beta}_X = \sup_{s<1} \frac{\log M_X(s)}{\log s},$$

where

$$M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \quad s > 0.$$

It is well known that

$$0 \le \underline{\beta}_X \le \bar{\beta}_X \le 1.$$

(We refer the reader to [2] for further information about r.i. spaces.)

3. Main result

Before formulating our main result, it will be convenient to start with the following remarks (see [7]):

Remark 3.1. By Lebesgue's differentiation theorem one easily obtains that

$$|f(x)| \le \mathscr{M} f(x) \quad \text{a.e. } x \in \mathbf{R}^n;$$

thus f is a fixed point of $\mathscr{M}$, if and only if f is positive and

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy \le f(x) \quad \text{a.e. } x \in \mathbf{R}^n,$$

or equivalently f is a positive *super-harmonic* function (i.e. $\triangle f \le 0$, where $\triangle$ is the Laplacian operator).

Remark 3.2. If f is a non-constant fixed point of $\mathscr{M}$, and $\varphi \geq 0$ belongs to the Schwartz class $\mathscr{S}(\mathbf{R}^n)$, with $\int_{\mathbf{R}^n} \varphi(x)\,dx = 1$, then the function $f_t(x) = (f * \varphi_t)(x)$, with $\varphi_t(x) = t^{-n}\varphi(x/t)$ is also a non-constant fixed point of $\mathscr{M}$ which belongs to $\mathscr{C}^\infty(\mathbf{R}^n)$ (notice that using the Lebesgue differentiation theorem, there exists some $t > 0$ such that f_t is non-constant, since f is non-constant). In particular if $X(\mathbf{R}^n)$ is an r.i. space and $f \in X(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$, since $\mathscr{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ we get that $f_t \in X(\mathbf{R}^n) \cap \mathscr{C}^\infty(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$.

Remark 3.3. Using the theory of weighted inequalities for $\mathscr{M}$ (see [5]), if $\mathscr{M}f = f$, in particular $f \in A_1$ (the Muckenhoupt weight class), and hence $f(x)\,dx$ defines a doubling measure. Hence, $f \notin L^1(\mathbf{R}^n)$. Also, using the previous remark we see that if $f \in L^p(\mathbf{R}^n)$ is a fixed point, then $f \in L^q(\mathbf{R}^n)$, for all $p \leq q \leq \infty$.

Definition 3.4. Given an r.i. space $X(\mathbf{R}^n)$, we define

$$D_{I_2}\big(X(\mathbf{R}^n)\big) = \big\{f \in L^0(\mathbf{R}^n) : \|I_2 f\|_{X(\mathbf{R}^n)} < \infty\big\},$$

where I_2 is the Riesz potential,

$$(I_2 f)(x) = \int_{\mathbf{R}^n} |x-y|^{2-n} f(y)\,dy.$$

It is not hard to see that the space $D_{I_2}\big(X(\mathbf{R}^n)\big)$ is either trivial or is the largest r.i. space which is mapped by I_2 into $X(\mathbf{R}^n)$, and is also related with the theory of the optimal Sobolev embeddings (see [4] and the references quoted therein).

Theorem 3.5. *Let $X(\mathbf{R}^n)$ be an r.i. space. The following statements are equivalent*:

(1) *There is a non-constant fixed point $f \in X(\mathbf{R}^n)$ of $\mathscr{M}$.*
(2) $n \geq 3$ *and* $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$.
(3) $n \geq 3$ *and* $\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t) \in \overline{X}\,(0,\infty)$.
(4) $n \geq 3$ *and* $\big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big) \subset X(\mathbf{R}^n)$.
(5) $n \geq 3$ *and* $D_{I_2}\big(X(\mathbf{R}^n)\big) \neq \{0\}$.

Proof. (1) $\Rightarrow$ (2) Since if $n = 1$ or $n = 2$, the only positive super-harmonic functions are the constant functions (see [8, Remark 1, p. 210]), necessarily $n \geq 3$. Moreover, it is proved in [7] that, if $f \in \mathscr{C}^\infty(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$, then

$$f(x) \geq c|x|^{2-n}\chi_{\{x:|x|>1\}}(x).$$

Since $f \in X(\mathbf{R}^n)$, then $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$.

(2) $\Rightarrow$ (3) Since if $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$, then

$$F(x) = \chi_{\{x:|x|\leq 1\}}(x) + |x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n).$$

An easy computation shows that

$$F^*(t) \simeq \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t).$$

(3) $\Rightarrow$ (4) Since $f \in \big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big)$ if and only if

$$\sup_{t>0} f^*(t)W(t) < \infty,$$

where $W(t) = \max(1, t^{1-2/n})$, we have that

$$f^*(t) \le \|f\|_{L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)} W^{-1}(t)$$

and since $W^{-1}(t) = \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)} \in \overline{X}\,(0,\infty)$ we have that

$$\|f\|_{X(\mathbf{R}^n)} = \|f^*\|_{\overline{X}\,(0,\infty)} \le c\|f\|_{L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)}$$

with $c = \|W^{-1}\|_{\overline{X}\,(0,\infty)}$.

(4) $\Rightarrow$ (5) Since (see [9] and [1])

$$(I_2 f)^*(t) \le c_1\left(t^{2/n-1}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)s^{2/n-1}\,ds\right) \le c_2(I_2 f^0)^*(t)$$

where $f^0(x) = f^*(c_n|x|^n)$, $c_n =$ measure of the unit ball in $\mathbf{R}^n$. (Observe that $(f^0)^* = f^*$). Rewriting the middle term in the above inequalities, using Fubini's theorem, we get

$$(I_2 f)^*(t) \le d_1\left(\frac{n}{n-2}\int_t^\infty f^{**}(s)s^{2/n-1}\,ds\right) \le d_2(I_2 f^0)^*(t),$$

where $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$. Thus, $f \in D_{I_2}\big(X(\mathbf{R}^n)\big)$ if and only if

$$\left\|\int_t^\infty f^{**}(s)s^{2/n-1}\,ds\right\|_{\overline{X}\,(0,\infty)} < \infty. \tag{1}$$

Since

$$F(t) = \int_t^\infty \chi^{**}_{[0,1]}(s)s^{2/n-1}\,ds = c\big(\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t)\big)$$

is a decreasing function, and

$$\begin{aligned} F^0(x) &= F(c_n|x|^n) \\ &\simeq \big(\chi_{\{x:|x|\le 1\}}(x) + |x|^{2-n}\chi_{\{x:|x|>1\}}(x)\big) \in L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \end{aligned}$$

we get that $\chi^\circ_{[0,1]} \in D_{I_2}\big(X(\mathbf{R}^n)\big)$.

Another argument to prove this part is the following:
Since, if $n \geq 3$ (see [2, Theorem 4.18, p. 228])

$$I_2\colon L^1(\mathbf{R}^n) \to L^{n/(n-2),\infty}(\mathbf{R}^n) \quad \text{and} \quad I_2\colon L^{n/2,1}(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$$

is bounded, we have that

$$I_2\colon \big(L^1(\mathbf{R}^n) \cap L^{n/2,1}(\mathbf{R}^n)\big) \to \big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big) \subset X(\mathbf{R}^n)$$

is bounded, and hence $L^1(\mathbf{R}^n) \cap L^{n/2,1}(\mathbf{R}^n) \subset D_{I_2}\big(X(\mathbf{R}^n)\big)$.

(5) $\Rightarrow$ (1) Since $n \geq 3$, we can use the classical formula of potential theory (see [10, p. 126])

$$-h = \triangle(I_2 h)$$

to conclude that there is a positive function $f = I_2 \chi^\circ_{[0,1]} \in X(\mathbf{R}^n)$. Then $0 \leq f_t = I_2(\chi^\circ_{[0,1]} * \varphi_t) \in X(\mathbf{R}^n) \cap \mathscr{C}^\infty(\mathbf{R}^n)$ and $\triangle f_t \leq 0$. ▫

We now consider particular examples, like the Lorentz spaces:

Corollary 3.6. *Let $1 \leq p < \infty$, and assume $\Lambda^p(\mathbf{R}^n, w)$ is a Banach space (i.e., $w \in B_p$ if $1 < p < \infty$ or $p \in B_{1,\infty}$ if $p = 1$, see [3]). Then, there exists a non-constant function $f \in \Lambda^p(\mathbf{R}^n, w)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and*

$$\int_1^\infty \frac{w(t)}{t^{p(1-2/n)}}\, dt < \infty.$$

In particular, this condition always holds, for $p > 1$ and n large enough.

Proof. The integrability condition follows by using the previous theorem. Now, if $w \in B_p$, then there exists an $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$, and hence it suffices to take $n > 2/\varepsilon$. Observe that if $w = 1$ and $p = 1$, then $\Lambda^1(\mathbf{R}^n, w) = L^1(\mathbf{R}^n)$, which does not have the fixed point property for any dimension n. ▫

Corollary 3.7. *Let $1 \leq p, q \leq \infty$ (if $p = 1$ we only consider $q = 1$). Then, there exists a non-constant function $f \in L^{p,q}(\mathbf{R}^n)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and*

$$\begin{cases} n/(n-2) < p \leq \infty \text{ or} \\ p = n/(n-2) \text{ and } q = \infty. \end{cases}$$

Corollary 3.8. (See [7]) *Let $1 \leq p \leq \infty$. There exists a non-constant function $f \in L^p(\mathbf{R}^n)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and $n/(n-2) < p \leq \infty$.*

It is interesting to know when given an r.i. space $X(\mathbf{R}^n)$, the space $D_{I_2}\big(X(\mathbf{R}^n)\big)$ is not trivial, or equivalently

$$(2) \qquad \overline{D_{I_2}\big(X(\mathbf{R}^n)\big)} := \left\{ f \in L^0([0,\infty)) : \left\| \int_t^\infty f^{**}(s) s^{2/n-1}\, ds \right\|_{\overline{X}\,(0,\infty)} < \infty \right\}$$

is not trivial. This will be done in terms of the fundamental indices of X. We start by computing the fundamental function of $D_{I_2}\big(X(\mathbf{R}^n)\big)$.

Lemma 3.9. *Let X be an r.i. space on $\mathbf{R}^n$, $n\geq 3$. Let Y be given by* (2)*. Then*

$$\phi_Y(s)\simeq s^{n/2}\|P_{1-2/n}\chi_{[0,s]}\|_X$$

where $P_{1-2/n}f(t)=t^{2/n-1}\int_0^t f(s)s^{-2/n}\,ds$.

Proof.

$$\begin{aligned}s^{n/2}P_{1-2/n}\chi_{[0,s]}(t)&\simeq s^{n/2}\left(\chi_{[0,s]}(t)+\left(\frac{s}{t}\right)^{1-2/n}\chi_{[s,\infty)}(t)\right)\\&\simeq\int_t^\infty \chi^{**}_{[0,s]}(r)r^{2/n-1}\,dr.\ \square\end{aligned}$$

Theorem 3.10. *Let X be an r.i. space on $\mathbf{R}^n$, $n\geq 3$. Let Y be given by* (2)*. Then*

(1) *If $\bar{\beta}_X<1-2/n$, then $Y\neq\{0\}$.*
(2) *If $Y\neq\{0\}$ then $\underline{\beta}_X\leq 1-2/n$.*

Proof. (1) Let $\chi_r=\chi_{[0,r]}$. Then

$$P_{1-2/n}\chi_r(t)=\int_0^1\chi_r(\xi t)\frac{d\xi}{\xi^{n/2}}\leq c\sum_{k=0}^\infty 2^{-k(1-n/2)}\chi_{2^k r}(t).$$

Thus

$$\|P_{1-2/n}\chi_r\|_X\leq c\sum_{k=0}^\infty 2^{-k(1-n/2)}\phi_X(2^k r)\leq c\phi_X(r)\sum_{k=0}^\infty 2^{-k(1-n/2)}M_X(2^k).$$

Let $\varepsilon>0$ be such that $\bar{\beta}_X+\varepsilon<1-2/n$. Then by the definition of $\bar{\beta}_X$ it follows readily that there is a constant $c>0$ such that

$$M_X(2^k)\leq c2^{k(\bar{\beta}_X+\varepsilon)},$$

and hence

$$\sum_{k=0}^\infty 2^{-k(1-n/2)}M_X(2^k)\leq\sum_{k=0}^\infty 2^{-k(1-n/2-\bar{\beta}_X-\varepsilon)}<\infty,$$

which implies that $\chi_r\in Y$.

(2) Since $Y\neq\{0\}$ if and only if $\|P_{1-2/n}\chi_{[0,1]}\|_X<\infty$ and

$$\sup_{t>0}(P_{1-2/n}\chi_{[0,1]})^{**}(t)\phi_X(t)\leq\|P_{1-2/n}\chi_{[0,1]}\|_X<\infty, \tag{3}$$

and easy computations show that (3) implies that

$$1 \le \sup_{t\ge 1} \frac{\phi_X(t)}{t^{1-2/n}} = c < \infty, \tag{4}$$

then, by (4)

$$\begin{aligned} M_X(a) &= \max\Big(\sup_{t\ge 1/a} \frac{\phi_X(ta)}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \Big) \\ &= \max\Big(\sup_{t\ge 1/a} \frac{\phi_X(ta)}{(at)^{1-2/n}} \frac{(at)^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \Big) \\ &\simeq \max\Big(a^{1-2/n} \sup_{t\ge 1/a} \frac{t^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \Big). \end{aligned}$$

Thus, if $a<1$, using again (4) we get

$$M_X(a) \ge a^{1-2/n} \sup_{t\ge 1/a} \frac{t^{1-2/n}}{\phi_X(t)} \ge a^{1-2/n}$$

which implies that

$$\underline{\beta}_X \le 1 - 2/n. \ \square$$

Let us see that the converse in the previous theorem is not true.

Proposition 3.11. *There are rearrangement invariant spaces X such that*

(1) $Y \ne \{0\}$ *and* $\bar\beta_X \ge 1 - 2/n$.
(2) $Y = \{0\}$ *and* $\underline{\beta}_X < 1 - 2/n$.

Proof. Let $\varphi(t) = t^a \chi_{[0,1]}(t) + t^b \chi_{[1,\infty)}(t)$, with $0 \le a, b \le 1$. Let

$$X = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t)\varphi(t) < \infty \Big\}.$$

Since φ is a quasi-concave function, we have that

$$\varphi(t) = \phi_X(t)$$

and

$$\underline{\beta}_X = \min(a,b), \quad \bar\beta_X = \max(a,b).$$

On the other hand, the space Y defined by (2) is not trivial if and only if

$$b \le 1 - 2/n.$$

Now, to prove (1) take $b \le 1-2/n$ and $a \ge 1-2/n$. And to see (2) take $b > 1-2/n$ and $a \le 1-2/n$. $\square$

Remark 3.12. If we consider

$$X_0 = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t) t^{1-2/n}(1+\log^+ t) < \infty \Big\}$$

and

$$X_1 = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t) \frac{t^{1-2/n}}{(1+\log^+ t)} < \infty \Big\}$$

then $\underline{\beta}_{X_i} = \bar{\beta}_{X_i} = 1 - 2/n$, $Y_0 = \{0\}$ and $Y_1 \neq \{0\}$.

Remark 3.13. It was proved in [7] that if we consider the strong maximal function (i.e., the maximal operator associated to centered intervals in $\mathbf{R}^n$), then there were no fixed points in any $L^p(\mathbf{R}^n)$ space, regardless of the dimension. The same argument works to show that $L^p(\mathbf{R}^n)$ cannot be replaced by any different r.i. space. Also, if we study this question for other kind of sets, like, e.g., Buseman–Feller differentiation bases (see [6]), then the only possible fixed points are the constant functions. This observation applies to any non-centered maximal operator (with respect to balls, cubes, etc.).

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