Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 191–211

FIXED POINTS OF MEROMORPHIC SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we discuss the problems on the fixed points of meromorphic solutions of higher order linear differential equations with meromorphic coefficients and their derivatives. Because of the restriction of differential equations, we obtain that the properties of fixed points of meromorphic solutions of higher order linear differential equations with meromorphic coefficients and their derivatives are more interesting than those of general transcendental meromorphic functions. Some estimates of the exponent of convergence of fixed points of solutions and their derivatives are obtained.

1. Introduction and main results

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [14]). However, there are few studies on the fixed points of solutions of differential equations. In [3], Chen Zong-Xuan at first studied the problems on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients. In [11], Wang and Yi studied the problems on the fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [9], I. Laine and J. Rieppo had given an extension and improvement of the results in [11]; they studied the problems on the fixed points and iterated order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [10], Wang and Lü studied the problems on the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives. The main purpose of this paper is to extend some results in [10] to the case of higher order linear differential equations with meromorphic coefficients.

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of R. Nevanlinna's theory of meromorphic functions (see [4], [13], [12], [8]). In addition, let $\sigma(f)$ denote the order of growth

2000 Mathematics Subject Classification: Primary 30D35, 34M05.

This work is supported by the National Natural Science Foundation of China (No. 10471048).

of the meromorphic function $f(z)$, $\bar{\lambda}(f)$ denote the exponent of convergence of the sequence of distinct zeros of f, $v_f(r)$ denote the central index of $f(z)$. In order to give some estimates of fixed points, we recall the following definitions (see [3], [7], [9]).

Definition 1.1. Let $z_1, z_2, \ldots, |z_j| = r_j$, $0 \leq r_1 \leq r_2 \leq \cdots$, be the sequence of distinct fixed points of a transcendental meromorphic function f. Then $\overline{\tau}(f)$, the exponent of convergence of the sequence of distinct fixed points of f, is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0 \,\Big|\, \sum_{j=1}^{\infty} |z_j|^{-\tau} < +\infty\right\}.$$

It is evident that

$$\overline{\tau}(f) = \overline{\lim_{r\to\infty}} \frac{\log \overline{N}\left(r, \dfrac{1}{f-z}\right)}{\log r}.$$

Definition 1.2. Suppose that $f(z)$ be a meromorphic function of infinite order. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

Definition 1.3. Let f be a meromorphic function. Then $\overline{\lambda_2}(f)$, the hyper-exponent of convergence of the sequence of distinct zeros of f, is defined by

$$\overline{\lambda_2}(f) = \overline{\lim_{r\to\infty}} \frac{\log\log \overline{N}\left(r, \dfrac{1}{f}\right)}{\log r},$$

and $\overline{\tau}_2(f)$, the hyper-exponent of convergence of the sequence of distinct fixed points of f, is defined by

$$\overline{\tau}_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log \overline{N}\left(r, \dfrac{1}{f-z}\right)}{\log r}.$$

Definition 1.4 ([2], [6]). Let $P(z)$ be rational and of the form $P_1(z)/P_2(z)$, where $P_1(z)$ and $P_2(z)$ are two polynomials. We denote by di(P), the "degree at infinity" of P, defined by

$$\mathrm{di}(P) = \deg P_1(z) - \deg P_2(z). \tag{1.1}$$

In [10], Wang and Lü obtained the following results.

Theorem A. *Suppose that $P(z) = P_1(z)/P_2(z) \not\equiv 0$ be a rational function with $n = \mathrm{di}(P)$. Then every transcendental meromorphic solution f of the equation*

$$f'' + P(z)f = 0 \tag{1.2}$$

satisfies that f and f', f'' all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \sigma(f) = \max\{\tfrac{1}{2}(n+2), 0\}.$$

Theorem B. *Suppose that $A(z)$ be a transcendental meromorphic function satisfying $\delta(\infty, A) > 0$, $\sigma(A) = \sigma < +\infty$. Then every meromorphic solution $f \not\equiv 0$ of the equation*

$$f'' + A(z)f = 0 \tag{1.3}$$

satisfies that f and f', f'' all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \sigma(f) = +\infty, \quad \overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = \sigma_2(f) = \sigma.$$

Remark 1.1. In [10], Wang and Lü did not give the detailed proof of Theorem A, they only proved that $\sigma(f) = \max\{\frac{1}{2}(n+2), 0\}$, and omit the proof of $\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \sigma(f)$. In fact, when $n = -2$, we cannot prove that $\overline{\tau}(f') = \overline{\tau}(f'') = \sigma(f)$ by using the similar method as the proof of Theorem B in [10]; please look at Remark 3.1 in Section 3.

In this paper, we shall prove the following two theorems.

Theorem 1.1. *Suppose that $P(z) = P_1(z)/P_2(z) \not\equiv 0$ be a rational function with $n = \mathrm{di}(P)$, and k be an integer with $k \geq 2$. Then:*

(1) *Every transcendental meromorphic solution f of the equation*

$$f^{(k)} + P(z)f = 0 \tag{1.4}$$

satisfies $\sigma(f) = \max\{(n+k)/k, 0\}$.

(2) *If $n \neq -k$, then every meromorphic solution $f \not\equiv 0$ of the equation* (1.4) *satisfies that f and $f', f'', \ldots, f^{(k)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \cdots = \overline{\tau}(f^{(k)}) = \max\left\{\frac{n+k}{k}, 0\right\}.$$

(3) *If $n = -k$, then every transcendental meromorphic solution f of the equation* (1.4) *satisfies that f and $f', f'', \ldots, f^{(k-2)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \cdots = \overline{\tau}(f^{(k-2)}) = 0.$$

Remark 1.2. Setting $k = 2$ in Theorem 1.1, we get Theorem 1 of [10] or Theorem A, which contains the related results in [1]. From Theorem 1.1, we know that every transcendental meromorphic solution f of equation (1.4) has infinitely many fixed points. In addition, if $n = -k$, there exists some equations of the form (1.4) that may have solutions with finite fixed points. For example, the equation

$$f''' - \frac{6}{z(z-2)(z-5)} f = 0$$

has a family of polynomial solutions $\{f_c = cz(z-2)(z-5); c \text{ is a constant}\}$, which have at most three fixed points.

Theorem 1.2. *Suppose that $k \geq 2$ and $A(z)$ be a transcendental meromorphic function satisfying $\delta(\infty, A) = \delta > 0$, $\sigma(A) = \sigma < +\infty$. Then every meromorphic solution $f \not\equiv 0$ of the equation*

$$f^{(k)} + A(z) f = 0 \tag{1.5}$$

satisfies that f and $f', f'', \ldots, f^{(k)}$ all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f^{(j)}) = \sigma(f) = +\infty, \quad \overline{\tau}_2(f) = \overline{\tau}_2(f^{(j)}) = \sigma_2(f) = \sigma, \tag{1.6}$$

for $j = 1, \ldots, k$.

Remark 1.3. Setting $k = 2$ in Theorem 1.2, we get Theorem 2 of [10] or Theorem B.

2. Some lemmas

Lemma 2.1 ([11]). *Suppose that $f(z) = g(z)/d(z)$ is a meromorphic function with $\sigma(f) = \varrho$, where $g(z)$ is an entire function and $d(z)$ is a polynomial. Then there exists a sequence $\{r_j\}$, $r_j \to \infty$, such that for all z satisfying $|z| = r_j$, $|g(z)| = M(r_j, g)$, when j sufficiently large, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{v_g(r_j)}{z}\right)^n \bigl(1 + o(1)\bigr), \quad n \geq 1,$$
$$\sigma(f) = \lim_{j\to\infty} \frac{\log v_g(r_j)}{\log r_j}.$$

Lemma 2.2. *Suppose that $k \geq 2$ and $A(z)$ is a transcendental meromorphic function with $\delta(\infty, A) = \delta > 0$ and $\sigma(A) = \sigma < +\infty$. Then every meromorphic solution $f \not\equiv 0$ of equation (1.5) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma$.*

Proof. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.5). Rewrite (1.5) as

$$A = -\frac{f^{(k)}}{f}. \tag{2.1}$$

By the lemma of logarithmic derivatives, there exists a set E with finite linear measure such that $m(r,A) \le c\log\bigl(rT(r,f)\bigr)$ for $r \notin E$, where c is a positive constant.

It follows from the definition of deficiency that for sufficiently large r, we have $m(r,A) \ge \frac{1}{2}\delta T(r,A)$. So when $r \notin E$ sufficiently large, we have

$$T(r,A) \le \frac{2c}{\delta}\log\bigl(rT(r,f)\bigr). \tag{2.2}$$

Hence by the definition of hyper-order, we obtain that $\sigma(f) = +\infty$, and

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}}\frac{\log\log T(r,f)}{\log r} \ge \sigma(A) = \sigma. \tag{2.3}$$

On the other hand, we know that the poles of f can only occur at the poles of A, so $\lambda(1/f) \le \sigma(A)$. According to the Hadamard factorization theorem, we can write f as $f(z) = g(z)/d(z)$, where $d(z)$ and $g(z)$ are entire functions satisfying $\lambda(d) = \sigma(d) = \lambda(1/f) \le \sigma \le \sigma_2(g) = \sigma_2(f)$. Substituting $f(z) = g(z)/d(z)$ into (1.5), by Theorem 2.1 in [7], we get

$$\sigma_2(f) = \sigma_2(g) \le \sigma. \tag{2.4}$$

Hence $\sigma_2(f) = \sigma$. This completes the proof.

Lemma 2.3. *Suppose that k be a positive integer and $f(z)$ be a nonzero solution of equation* (1.5)*. Let $w_0 = f - z$, $w_1 = f' - z$, $w_2 = f'' - z$. Then w_0, w_1 and w_2 satisfy the following equations*:

$$-w_0^{(k)} - Aw_0 = zA, \ \text{if } k \ge 2; \tag{2.5}$$

$$-Aw_1^{(k)} + A'w_1^{(k-1)} - A^2w_1 = zA^2, \ \text{if } k \ge 3; \tag{2.6}$$

$$\begin{aligned} &-A^2w_2^{(k)} + 2AA'w_2^{(k-1)} + (AA'' - 2(A')^2)w_2^{(k-2)} - A^3w_2 \\ &\quad = zA^3, \ \text{if } k \ge 4; \end{aligned} \tag{2.7}$$

$$-Aw_1'' + A'w_1' - A^2w_1 = zA^2 - A', \ \text{if } k = 2; \tag{2.8}$$

$$\begin{aligned} &-A^2w_2'' + 2AA'w_2' + (AA'' - 2(A')^2 - A^3)w_2 \\ &\quad = zA^3 - 2AA' + 2z(A')^2 - zAA'', \ \text{if } k = 2; \end{aligned} \tag{2.9}$$

$$-A^2w_2''' + 2AA'w_2'' + (AA'' - 2A'^2)w_2' - A^3w_2 \tag{2.10}$$

$$= zA^3 - AA'' + 2A'^2, \text{ if } k = 3;$$

$$\begin{aligned} (2.11)\qquad & -A^3 w_3''' + 3A^2 A' w_3'' + 3A(AA'' - 2(A')^2) w_3' \\ &\qquad + (A^2 A''' - 6AA'A'' + 6A'^3 - A^4) w_3 \\ &= zA^4 - 3A(AA'' - 2A'^2) \\ &\qquad - z(A^2 A''' - 6AA'A'' + 6A'^3), \text{ if } k = 3. \end{aligned}$$

Proof. From $f = w_0 + z$, we have that $f^{(k)} = w_0^{(k)}$ for $k \geq 2$. Substituting it into (1.5), we get $f = -w_0^{(k)}/A$, that is, $w_0 + z = -w_0^{(k)}/A$. From this, we obtain (2.5).

According to the equality $f' = w_1 + z$, we obtain that $f^{(k)} = w_1^{(k-1)}$ for $k \geq 3$. Substituting it into (1.5), we get $f = -w_1^{(k-1)}/A$, so that $f' = (-w_1^{(k)} A + w_1^{(k-1)} A')/A^2$, that is, $w_1 + z = (-w_1^{(k)} A + w_1^{(k-1)} A')/A^2$. From this, we obtain (2.6).

By the equality $f'' = w_2 + z$, we get that $f^{(k)} = w_2^{(k-2)}$ for $k \geq 4$. Substituting it into (1.5), we get $f = -w_2^{(k-2)}/A$, so that

$$\begin{aligned} f'' = (f')' &= \left(\frac{-w_2^{(k-1)} A + w_2^{(k-2)} A'}{A^2} \right)' \\ &= \frac{-A^2 w_2^{(k)} + 2AA' w_2^{(k-1)} + \big(AA'' - 2(A')^2\big) w_2^{(k-2)}}{A^3}, \end{aligned}$$

that is,

$$w_2 + z = \frac{-A^2 w_2^{(k)} + 2AA' w_2^{(k-1)} + \big(AA'' - 2(A')^2\big) w_2^{(k-2)}}{A^3}.$$

From this, we obtain (2.7).

We may prove equations (2.9)–(2.11) similarly. Hence the proof of Lemma 2.3 is complete.

Let i, j are two non-negative integers. Now we define the notation $H_{ij}(A)$ as follows.

(1) For any non-negative integer i, we define $H_{i0}(A) = -A^i$;

(2) For any non-negative integer i, $H_{ii}(A)$ and $H_{i(i-1)}(A)$ are defined by the following recurrence formula:

$$\begin{aligned} (2.12)\qquad & H_{(i+1)(i+1)}(A) = A \cdot [H_{ii}(A)]' - (i+1)A' \cdot H_{ii}(A), \\ & H_{(i+2)(i+1)}(A) = A \cdot [H_{(i+1)i}(A)]' - (i+2)A' \cdot H_{(i+1)i}(A) \\ &\qquad\qquad + A \cdot H_{(i+1)(i+1)}(A), \\ & H_{00}(A) = -1, \qquad H_{10}(A) = -A; \end{aligned}$$

(3) For $i \geq 3$ and $1 \leq j \leq i-2$, $H_{ij}(A)$ are defined as a sum of a finite number of terms of the type

$$B = bA^{l_0}(A')^{l_1} \cdots (A^{(i+1)})^{l_{i+1}}, \tag{2.13}$$

where b is a constant, $l_0, l_1, \ldots, l_{i+1}$ are non-negative integers such that $l_0 + l_1 + \cdots + l_{i+1} = i$ and $l_1 + 2l_2 + \cdots + (i+1)l_{i+1} = j$. It is obvious that $H_{ij}(A)$ is a differential polynomial of A. Moreover, this notation $H_{ij}(A)$, $1 \leq j \leq i-2$, may represent a different differential polynomial of A in different occurrences, even within one single formula.

By the definition of $H_{ij}(A)$, simple computation yields

$$\begin{aligned}
H_{00}(A) &= -1,\\
H_{10}(A) &= -A,\\
H_{11}(A) &= A';\\
H_{20}(A) &= -A^2,\\
H_{21}(A) &= 2AA',\\
H_{22}(A) &= AA'' - 2A'^2,\\
H_{32}(A) &= 3A^2A'' - 6AA'^2,\\
H_{33}(A) &= A^2A''' - 6AA'A'' + 6A'^3;\\
A \cdot \big(H_{ij}(A)\big)' &= H_{(i+1)(j+1)}(A) \text{ for } 0 < j \leq i-2;\\
A' \cdot H_{ij}(A) &= H_{(i+1)(j+1)}(A) \text{ for } 0 < j \leq i-2;\\
A \cdot H_{ij}(A) &= H_{(i+1)j}(A) \text{ for } 0 \leq j \leq i-1.
\end{aligned} \tag{2.14}$$

Lemma 2.4. *Suppose that $f(z)$ is a nonzero solution of equation* (1.5) *and $k \geq 2$. Let $w_i = f^{(i)} - z$, $i = 0, 1, \ldots, k-2$. Then w_i satisfy the following equations*:

$$\sum_{j=0}^{i} H_{ij}(A) w_i^{(k-j)} - A^{i+1} w_i = zA^{i+1}, \quad i = 0, 1, \ldots, k-2, \tag{2.15}$$

where $H_{ij}(A)$ are defined by (2.12)–(2.14).

Proof. When $k = 2$ or 3, 4, by Lemma 2.3 and (2.12)–(2.14), it is evident that (2.15) hold. In the following, we suppose that $k \geq 5$. We shall use an inductive method to prove it.

At first, by Lemma 2.3 and (2.14), we get that (2.15) hold for $i = 0, 1, 2$.

Next, suppose that w_i, $2 \leq i \leq k-3$, satisfy (2.15). Now we verify that w_{i+1} also satisfies (2.15).

From $w_i + z = f^{(i)}$ and $w_{i+1} + z = f^{(i+1)}$, we know that

$$w_i^{(k-j)} = w_{i+1}^{(k-j-1)}, \quad j = 0, 1, \ldots, i, \ i \le k-3.$$

Since w_i satisfies (2.15), we have

$$f^{(i)} = w_i + z = \frac{\sum_{j=0}^{i} H_{ij}(A) w_i^{(k-j)}}{A^{i+1}},$$

so by (2.14) and (2.12), we obtain

$$\begin{aligned}
f^{(i+1)} = (f^{(i)})' &= \left(\frac{\sum_{j=0}^{i} H_{ij}(A) w_i^{(k-j)}}{A^{i+1}} \right)' \\
&= \left(\frac{\sum_{j=0}^{i} H_{ij}(A) w_{i+1}^{(k-j-1)}}{A^{i+1}} \right)' \\
&= \frac{1}{A^{2(i+1)}} \Bigg\{ \Bigg[\sum_{j=0}^{i} \big(H_{ij}(A)\big)' w_{i+1}^{(k-j-1)} + \sum_{j=0}^{i} H_{ij}(A) w_{i+1}^{(k-j)} \Bigg] A^{i+1} \\
&\qquad - (i+1) A^i A' \sum_{j=0}^{i} H_{ij}(A) w_{i+1}^{(k-j-1)} \Bigg\} \\
&= \frac{1}{A^{i+2}} \Bigg\{ \sum_{j=0}^{i-2} H_{(i+1)(j+1)}(A) w_{i+1}^{(k-j-1)} + A\big(H_{i(i-1)}(A)\big)' w_{i+1}^{(k-i)} \\
&\qquad + A\big(H_{ii}(A)\big)' w_{i+1}^{(k-i-1)} + \sum_{j=0}^{i} A H_{ij}(A) w_{i+1}^{(k-j)} \\
&\qquad - (i+1) A' \sum_{j=0}^{i} H_{ij}(A) w_{i+1}^{(k-j-1)} \Bigg\} \\
&= \frac{1}{A^{i+1+1}} \Bigg\{ \sum_{j=1}^{i-1} H_{(i+1)j}(A) w_{i+1}^{(k-j)} + H_{(i+1)i}(A) w_{i+1}^{(k-i)} \\
&\qquad + H_{(i+1)(i+1)}(A) w_{i+1}^{(k-i-1)} \\
&\qquad + \sum_{j=1}^{i-1} H_{(i+1)j}(A) w_{i+1}^{(k-j)} + H_{i0}(A) \cdot A w_{i+1}^{(k)} \\
&\qquad - (i+1) \sum_{j=1}^{i-1} H_{(i+1)j}(A) w_{i+1}^{(k-j)} \Bigg\}
\end{aligned}$$

$$= \frac{1}{A^{i+1+1}}\left\{-A^{i+1}w_{i+1}^{(k)} + \sum_{j=1}^{i+1} H_{(i+1)j}(A)w_{i+1}^{(k-j)}\right\}$$

$$= \frac{1}{A^{i+1+1}}\sum_{j=0}^{i+1} H_{(i+1)j}(A)w_{i+1}^{(k-j)} = w_{i+1} + z.$$

From the above equality, we obtain that w_{i+1} also satisfies (2.15). This completes the proof.

Lemma 2.5. *Suppose that* $f(z) \not\equiv 0$ *is a solution of equation* (1.5) *and* $k \geq 2$. *Let* $w_{k-1} = f^{(k-1)} - z$. *Then* w_{k-1} *satisfies*

$$\sum_{j=0}^{k-1} H_{(k-1)j}(A)w_{k-1}^{(k-j)} - A^k w_{k-1} = zA^k - H_{(k-1)(k-1)}(A), \tag{2.16}$$

where H_{ij} *are defined by* (2.12)–(2.14).

Proof. When $k = 2$ or 3, by (2.8), (2.10) and (2.14), it is evident that (2.16) holds. When $k \geq 4$, from $w_{k-2} = f^{(k-2)} - z$ and $w_{k-1} = f^{(k-1)} - z$, we obtain

$$w_{k-2}^{(k-j)} = w_{k-1}^{(k-j-1)}, \quad j = 0, 1, 2, \ldots, k-3, \qquad w''_{k-2} = w'_{k-1} + 1.$$

Since w_{k-2} satisfies (2.15), rewrite it as

$$f^{(k-2)} = w_{k-2} + z = \frac{\sum_{j=0}^{k-2} H_{(k-2)j}(A)w_{k-2}^{(k-j)}}{A^{k-1}},$$

so by (2.14) and (2.12), we have

$$\begin{aligned}
f^{(k-1)} &= (f^{(k-2)})' \\
&= \left(\frac{\sum_{j=0}^{k-2} H_{(k-2)j}(A)w_{k-2}^{(k-j)}}{A^{k-1}}\right)' \\
&= \left(\frac{\sum_{j=0}^{k-3} H_{(k-2)j}(A)w_{k-1}^{(k-j-1)} + H_{(k-2)(k-2)}(A)(w'_{k-1}+1)}{A^{k-1}}\right)' \\
&= \frac{1}{A^{2(k-1)}}\Bigg\{\Bigg[\sum_{j=0}^{k-3}(H_{(k-2)j}(A))'w_{k-1}^{(k-j-1)} + \sum_{j=0}^{k-3} H_{(k-2)j}(A)w_{k-1}^{(k-j)} \\
&\quad + \big(H_{(k-2)(k-2)}(A)\big)'(w'_{k-1}+1) + H_{(k-2)(k-2)}(A)w''_{k-1}\Bigg]A^{k-1} \\
&\quad - \Bigg[\sum_{j=0}^{k-3} H_{(k-2)j}(A)w_{k-1}^{(k-j-1)} \\
&\qquad + H_{(k-2)(k-2)}(A)(w'_{k-1}+1)\Bigg](k-1)A^{k-2}A'\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{A^k}\Big\{\sum_{j=0}^{k-4} A\big(H_{(k-2)j}(A)\big)' w_{k-1}^{(k-j-1)} + \sum_{j=0}^{k-3} AH_{(k-2)j}(A) w_{k-1}^{(k-j)} \\
&\quad + A\big(H_{(k-2)(k-3)}(A)\big)' w''_{k-1} - (k-1)A'H_{(k-2)(k-3)}(A)w''_{k-1} \\
&\qquad + AH_{(k-2)(k-2)}(A)w''_{k-1} + A\big(H_{(k-2)(k-2)}(A)\big)'(w'_{k-1}+1) \\
&\qquad - (k-1)A'H_{(k-2)(k-2)}(A)(w'_{k-1}+1) \\
&\qquad - \sum_{j=0}^{k-4}(k-1)A'H_{(k-2)j}(A)w_{k-1}^{(k-j-1)}\Big\} \\
&= \frac{1}{A^k}\Big\{\sum_{j=0}^{k-4} H_{(k-1)(j+1)}(A) w_{k-1}^{(k-j-1)} + \sum_{j=1}^{k-3} H_{(k-1)j}(A) w_{k-1}^{(k-j)} \\
&\qquad + AH_{(k-2)0}(A)w_{k-1}^{(k)} + H_{(k-1)(k-2)}(A)w''_{k-1} \\
&\qquad + H_{(k-1)(k-1)}(A)(w'_{k-1}+1) \\
&\qquad - \sum_{j=0}^{k-4}(k-1)H_{(k-1)(j+1)}(A)w_{k-1}^{(k-j-1)}\Big\} \\
&= \frac{1}{A^k}\Big\{-A^{k-1}w_{k-1}^{(k)} + \sum_{j=1}^{k-3} H_{(k-1)j}(A)w_{k-1}^{(k-j)} \\
&\qquad + H_{(k-1)(k-2)}(A)w''_{k-1} + H_{(k-1)(k-1)}(A)(w'_{k-1}+1)\Big\} \\
&= \frac{1}{A^k}\Big\{\sum_{j=0}^{k-1} H_{(k-1)j}(A)w_{k-1}^{(k-j)} + H_{(k-1)(k-1)}(A)\Big\} = w_{k-1} + z.
\end{aligned}
$$

From the above equality we get (2.16). This completes the proof.

Lemma 2.6. *Suppose that $f(z) \not\equiv 0$ is a solution of equation* (1.5) *and* $k \geq 2$. *Let $w_k = f^{(k)} - z$. Then w_k satisfies*

$$
\sum_{j=0}^{k-1} H_{kj}(A)w_k^{(k-j)} + \big(H_{kk}(A) - A^{k+1}\big)w_k = zA^{k+1} - H_{k(k-1)}(A) - zH_{kk}(A), \tag{2.17}
$$

where H_{ij} are defined by (2.12)–(2.14).

Proof. When $k=2$, by (2.9) and (2.14), it is evident that (2.17) holds. When $k \geq 3$, we obtain from $w_{k-1} = f^{(k-1)} - z$ and $w_k = f^{(k)} - z$,

$$
\begin{aligned}
w_{k-1}^{(k-j)} &= w_k^{(k-j-1)}, \quad j = 0, 1, \ldots, k-3, \\
w''_{k-1} &= w'_k + 1, \\
w'_{k-1} &= w_k + z - 1.
\end{aligned}
$$

Since w_{k-1} satisfies (2.16), rewrite it as

$$f^{(k-1)} = w_{k-1} + z = \frac{\sum_{j=0}^{k-1} H_{(k-1)j}(A) w_{k-1}^{(k-j)} + H_{(k-1)(k-1)}(A)}{A^k},$$

so by (2.14) and (2.12), we have

$$\begin{aligned}
f^{(k)} &= (f^{(k-1)})' \\
&= \left(\frac{\sum_{j=0}^{k-1} H_{(k-1)j}(A) w_{k-1}^{(k-j)} + H_{(k-1)(k-1)}(A)}{A^k}\right)' \\
&= \left(\frac{\sum_{j=0}^{k-3} H_{(k-1)j}(A) w_k^{(k-j-1)} + H_{(k-1)(k-2)}(A)(w_k'+1)}{A^k}\right. \\
&\qquad \left. + \frac{H_{(k-1)(k-1)}(A)(w_k+z-1) + H_{(k-1)(k-1)}(A)}{A^k}\right)' \\
&= \left(\frac{\sum_{j=0}^{k-3} H_{(k-1)j}(A) w_k^{(k-j-1)} + H_{(k-1)(k-2)}(A)(w_k'+1)}{A^k}\right. \\
&\qquad \left. + \frac{H_{(k-1)(k-1)}(A)(w_k+z)}{A^k}\right)'
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{A^{2k}}\Bigg\{\Bigg[\sum_{j=0}^{k-3} (H_{(k-1)j}(A))' w_k^{(k-j-1)} + \sum_{j=0}^{k-3} H_{(k-1)j}(A) w_k^{(k-j)} \\
&\quad + \big(H_{(k-1)(k-2)}(A)\big)'(w_k'+1) + H_{(k-1)(k-2)}(A) w_k'' \\
&\qquad + (H_{(k-1)(k-1)}(A))'(w_k+z) + H_{(k-1)(k-1)}(A)(w_k'+1)\Bigg] A^k \\
&\quad - \Bigg[\sum_{j=0}^{k-3} H_{(k-1)j}(A) w_k^{(k-j-1)} + H_{(k-1)(k-2)}(A)(w_k'+1) \\
&\quad + H_{(k-1)(k-1)}(A)(w_k+z)\Bigg] kA^{k-1}A'\Bigg\} \\
&= \frac{1}{A^{k+1}}\Bigg\{\sum_{j=0}^{k-3} H_{k(j+1)}(A) w_k^{(k-j-1)} \\
&\quad + AH_{(k-1)0}(A) w_k^{(k)} + \sum_{j=1}^{k-3} H_{kj}(A) w_k^{(k-j)} \\
&\quad + A\big(H_{(k-1)(k-2)}(A)\big)'(w_k'+1) + AH_{(k-1)(k-2)}(A) w_k'' \\
&\quad + A\big(H_{(k-1)(k-1)}(A)\big)'(w_k+z)
\end{aligned}$$

$$+ AH_{(k-1)(k-1)}(A)(w_k' + 1) - kA'H_{(k-1)(k-2)}(A)(w_k' + 1)$$
$$- kA'H_{(k-1)(k-1)}(A)(w_k + z) - \sum_{j=0}^{k-3} kH_{k(j+1)}(A)w_k^{(k-j-1)}\Bigg\}$$
$$= \frac{1}{A^{k+1}}\Bigg\{\sum_{j=1}^{k-2} H_{kj}(A)w_k^{(k-j)} - A^k w_k^{(k)} + H_{k(k-1)}(A)(w_k' + 1)$$
$$+ H_{k(k-2)}(A)w_k'' + H_{kk}(A)(w_k + z)\Bigg\}$$
$$= \frac{1}{A^{k+1}}\Bigg\{\sum_{j=0}^{k} H_{kj}(A)w_k^{(k-j)} + H_{k(k-1)}(A) + zH_{kk}(A)\Bigg\} = w_k + z.$$

From the above equality we obtain (2.17). This completes the proof.

Lemma 2.7. *Suppose that $k \geq 2$ and $P(z)$ is a rational function with $n = \mathrm{di}(P) < -k \leq -2$. Then there exist two nonzero constants c_1 and c_2 such that*

$$H_{(k-1)(k-1)}(P) = \frac{c_1 + o(1)}{z^{-nk-1+k+n}} \quad \text{as } z \to \infty, \tag{2.18}$$

and

$$H_{k(k-1)}(P) + zH_{kk}(P) = \frac{c_2 + o(1)}{z^{-n(k+1)-1+k+n}} \quad \text{as } z \to \infty. \tag{2.19}$$

Proof. At first, we prove (2.18) by means of induction.

Since $P(z)$ is a rational function with $n = \mathrm{di}(P) < -k \leq -2$, we have

$$P(z) = \frac{c_0 + o(1)}{z^{-n}} \quad \text{as } z \to \infty, \tag{2.20}$$

where $c_0 \neq 0$ is a constant. From this and (2.14), simple computation yields

$$H_{11}(P) = P' = \frac{nc_0 + o(1)}{z^{-n+1}} \quad \text{as } z \to \infty,$$

where $nc_0 \neq 0$. Hence (2.18) holds, when $k = 2$.

Furthermore, if (2.18) is assumed to be valid when $k = m$ (> 2), that is, there exists a constant $c_1'' \neq 0$ such that

$$H_{(m-1)(m-1)}(P) = \frac{c_1'' + o(1)}{z^{-nm-1+m+n}} \quad \text{as } z \to \infty,$$

it must also hold when $k = m+1$, since, by (2.12),

$$\begin{aligned}
H_{mm}(P) &= P \cdot [H_{(m-1)(m-1)}(P)]' - mP' \cdot H_{(m-1)(m-1)}(P) \\
&= \frac{c_0 + o(1)}{z^{-n}} \cdot \left[\frac{c_1'' + o(1)}{z^{-nm-1+m+n}} \right]' - m \cdot \frac{nc_0 + o(1)}{z^{-n+1}} \cdot \frac{c_1'' + o(1)}{z^{-nm-1+m+n}} \\
&= \frac{(1-m-n)c_0 c_1'' + o(1)}{z^{-n(m+1)-1+(m+1)+n}} \\
&= \frac{c_1 + o(1)}{z^{-n(m+1)-1+(m+1)+n}} \quad \text{as } z \to \infty,
\end{aligned}$$

where $c_1 = (1-m-n)c_0 c_1'' \neq 0$. Hence (2.18) holds.

Next, we prove (2.19) by means of induction. By (2.20) and (2.14), simple computation yields

$$\begin{aligned}
H_{21}(P) + zH_{22}(P) = 2PP' + z(PP'' - 2P'^2) &= \frac{n(1-n)c_0^2 + o(1)}{z^{-2n+1}} \\
&= \frac{c_2' + o(1)}{z^{-4n+3}} \quad \text{as } z \to \infty,
\end{aligned}$$

where $c_2' = n(1-n)c_0^2 \neq 0$. Hence (2.19) holds, when $k = 2$.

Furthermore, if (2.19) is assumed to be valid when $k = m$ (> 2), that is, there exists a constant $c_2'' \neq 0$ such that

$$H_{m(m-1)}(P) + zH_{mm}(P) = \frac{c_2'' + o(1)}{z^{-n(m+1)-1+m+n}} \quad \text{as } z \to \infty,$$

it must also hold when $k = m+1$, since, by (2.12),

$$\begin{aligned}
H_{(m+1)m}(P) + zH_{(m+1)(m+1)}(P) &= P[H_{m(m-1)}(P)]' - (m+1)P'H_{m(m-1)}(P) \\
&\quad + PH_{mm}(P) + zP \cdot [H_{mm}(P)]' \\
&\quad - (m+1)zP' \cdot H_{mm}(P) \\
&= P[H_{m(m-1)}(P) + zH_{mm}(P)]' \\
&\quad - (m+1)[H_{m(m-1)}(P) + zH_{mm}(P)] \\
&= \frac{c_0 + o(1)}{z^{-n}} \cdot \left[\frac{c_2'' + o(1)}{z^{-n(m+1)-1+m+n}} \right]' \\
&\quad - (m+1) \cdot \frac{nc_0 + o(1)}{z^{-n+1}} \cdot \frac{c_2'' + o(1)}{z^{-n(m+1)-1+m+n}} \\
&= \frac{(1-m-n)c_0 c_2'' + o(1)}{z^{-n(m+2)-1+(m+1)+n}} \\
&= \frac{c_2 + o(1)}{z^{-n(m+2)-1+(m+1)+n}} \quad \text{as } z \to \infty,
\end{aligned}$$

where $c_2 = (1-m-n)c_0 c_2'' \neq 0$. Hence (2.19) holds, and the proof is complete.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.5). Then, by Lemma 2.2, we obtain that

$$\sigma(f) = \infty, \quad \sigma_2(f) = \sigma.$$

Set $w_i = f^{(i)} - z$, $i = 0, 1, \ldots, k$. Then for every i, a point z_0 is a fixed points of $f^{(i)}$ if and only if z_0 is a zero of w_i, and

$$\sigma(w_i) = \sigma(f^{(i)}) = \sigma(f) = +\infty, \quad \overline{\tau}(f^{(i)}) = \bar{\lambda}(w_i), \tag{3.1}$$

and

$$\sigma_2(w_i) = \sigma_2(f^{(i)}) = \sigma_2(f) = \sigma, \quad \overline{\tau}_2(f^{(i)}) = \overline{\lambda_2}(w_i). \tag{3.2}$$

By Lemmas 2.4–2.6, we know that w_i, $i = 0, 1, \ldots, k$, satisfy the following equations:

$$\sum_{j=0}^{i} H_{ij}(A) w_i^{(k-j)} - A^{i+1} w_i = z A^{i+1}, \quad i = 0, 1, \ldots, k-2; \tag{3.3}$$

$$\sum_{j=0}^{k-1} H_{(k-1)j}(A) w_{k-1}^{(k-j)} - A^k w_{k-1} = z A^k - H_{(k-1)(k-1)}(A); \tag{3.4}$$

$$\begin{aligned} \sum_{j=0}^{k-1} H_{kj}(A) w_k^{(k-j)} &+ (H_{kk}(A) - A^{k+1}) w_k \\ &= z A^{k+1} - H_{k(k-1)}(A) - z H_{kk}(A). \end{aligned} \tag{3.5}$$

Since A is a transcendental meromorphic function, $A^{i+1} z \not\equiv 0$, $i = 0, 1, \ldots, k-2$. We claim that

$$z A^k - H_{(k-1)(k-1)}(A) \not\equiv 0 \quad \text{and} \quad z A^{k+1} - H_{k(k-1)}(A) - z H_{kk}(A) \not\equiv 0.$$

In fact, if $z A^k - H_{(k-1)(k-1)}(A) \equiv 0$, rewrite it as

$$A = \frac{H_{(k-1)(k-1)}(A)}{z A^{k-1}}.$$

From this, we obtain that $m(r, A) = S(r, A)$, which contradicts $\delta(\infty, A) > 0$. If $z A^{k+1} - H_{k(k-1)}(A) - z H_{kk}(A) \equiv 0$, rewrite it as

$$A = \frac{H_{k(k-1)}(A)}{z A^k} + \frac{H_{kk}(A)}{A^k}.$$

Similarly, we may obtain that $m(r,A)=S(r,A)$, which contradicts $\delta(\infty,A)>0$; hence the claims hold.

Rewrite (3.3), (3.4) and (3.5) as

$$\frac{1}{zA^{i+1}}\left(\sum_{j=0}^{i} H_{ij}(A)\frac{w_i^{(k-j)}}{w_i}-A^{i+1}\right)=\frac{1}{w_i},\quad i=0,1,\ldots,k-2, \tag{3.6}$$

$$\frac{1}{zA^k-H_{(k-1)(k-1)}(A)}\left(\sum_{j=0}^{k-1} H_{(k-1)j}(A)\frac{w_{k-1}^{(k-j)}}{w_{k-1}}-A^k\right)=\frac{1}{w_{k-1}}, \tag{3.7}$$

$$\begin{aligned}&\frac{1}{zA^{k+1}-H_{k(k-1)}(A)-zH_{kk}(A)}\\ &\qquad\times\left(\sum_{j=0}^{k-1} H_{kj}(A)\frac{w_k^{(k-j)}}{w_k}+H_{kk}(A)-A^{k+1}\right)=\frac{1}{w_k}.\end{aligned} \tag{3.8}$$

Hence there exists a set E with finite linear measure such that for $r\notin E$, we have

$$m\left(r,\frac{1}{w_i}\right)\le O\left(m\left(r,\frac{1}{A}\right)\right)+C\big(\log rT(r,w_i)\big),\quad i=0,1,\ldots,k. \tag{3.9}$$

On the other hand, since $H_{ij}(A)$ are differential polynomials of A, according to their definition, it is easy to get that if z_0 is a pole of order l (≥ 1) of A, then z_0 is a pole of order $li+j$ of $H_{ij}(A)$. In the following, we split it into three cases to discuss the poles of w_i:

Case (1): $i=0,1,\ldots,k-2$. Since the coefficients of $w_i^{(k-j)}$ are $H_{ij}(A)$, $j=0,1,\ldots,i$, and the right-hand sides of the equations (3.3) are $A^{i+1}z$, by (3.3), all zeros (except for $z=0$) of w_i, whose order is larger than k, are zeros of $A(z)$ (each is not the pole of A, if not, it will lead to a contradiction as follows). Hence

$$N\left(r,\frac{1}{w_i}\right)\le k\overline{N}\left(r,\frac{1}{w_i}\right)+N\left(r,\frac{1}{zA^{i+1}}\right). \tag{3.10}$$

Case (2): $i=k-1$. Suppose that w_{k-1} has a zero of order k_1 $(\ge k^2-k+2)$ at point z_1, and z_1 is also the pole of order l of $A(z)$. We claim that $0\le l\le k-1$. In fact, if $l\ge k$, then z_1 is a pole of order kl of $zA^k-H_{(k-1)(k-1)}(A)$, and z_1 is also a pole of order l_1, $l_1\le kl-k_1<kl$, of the left-hand side in equality (3.4); this is a contradiction; hence $0\le l\le k-1$. At this time, z_1 is a pole of order $l(k-1)+j$ of $H_{(k-1)j}(A))$ for $j=0,1,\ldots,k-1$. Therefore, by (3.4), direct computation yields that z_1 is a zero of order $k_1-k(l+1)+l$ of the left-hand side of

equality (3.4), so z_1 is also a zero of order $k_1-k(l+1)+l$ of $zA^k-H_{(k-1)(k-1)}(A)$. Especially, $k_1-k(l+1)+l=k_1-(k^2-k+1)$ for $l=k-1$. Hence

$$N\left(r,\frac{1}{w_{k-1}}\right)\le (k^2-k+2)\overline{N}\left(r,\frac{1}{w_{k-1}}\right)+N\left(r,\frac{1}{zA^k-H_{(k-1)(k-1)}(A)}\right). \tag{3.11}$$

Case (3): $i=k$. Suppose that w_k has a zero of order k_2 ($\ge k^2+k+1$) at point z_2, and z_2 is also the pole of order t of $A(z)$. We claim that $0\le t\le k$. In fact, if $t\ge k+1$, then z_2 is a pole of order $(k+1)t$ of $zA^{k+1}-H_{k(k-1)}(A)-zH_{kk}(A)$, and z_2 is also a pole of order $(k+1)t-k_2$ of the left-hand side in equality (3.5); this is a contradiction. Hence $0\le t\le k$. At this time, z_2 is a pole of order $tk+j$ of $H_{kj}(A)$ for $j=0,1,\dots,k$. Therefore, by (3.5), direct computation yields that z_2 is a zero of order $k_2-k(t+1)$ of the left-hand side of equality (3.5), so z_2 is also a zero of order $k_2-k(t+1)$ of $zA^{k+1}-H_{k(k-1)}(A)-zH_{kk}(A)$. Especially, $k_1-k(t+1)=k_1-(k^2+k)$ for $t=k$. Hence

$$N\left(r,\frac{1}{w_k}\right)\le (k^2+k+1)\overline{N}\left(r,\frac{1}{w_k}\right)+N\left(r,\frac{1}{zA^{k+1}-H_{k(k-1)}(A)-zH_{kk}(A)}\right). \tag{3.12}$$

Note that for sufficiently large r, $C\log\bigl(rT(r,w_i)\bigr)\le \frac{1}{2}T(r,w_i)$, $i=0,1,\dots,k$. Because w_i, $i=0,1,\dots,k$, are transcendental meromorphic functions and $\sigma(A)=\sigma<+\infty$, so for any given $\varepsilon>0$, it follows from (3.10)–(3.12) and (3.9) that

$$T(r,w_i)\le 2(k^2+k+1)\overline{N}\left(r,\frac{1}{w_i}\right)+O(r^{\sigma+\varepsilon}),\quad i=0,1,\dots,k, \tag{3.13}$$

for $r\notin E_2$ sufficiently large, where E_2 has finite linear measure.

By Definition 1.2, (3.1), (3.2) and (3.13), we obtain that for $i=0,1,\dots,k$, we have

$$\overline{\tau}(f^{(i)})=\bar{\lambda}(w_i)=\sigma(w_i)=+\infty,\quad \overline{\tau}_2(f^{(i)})=\overline{\lambda_2}(w_i)=\sigma_2(w_i)=\sigma. \tag{3.14}$$

That is, each meromorphic solution $f\not\equiv 0$ of (1.5) and its $f',\dots,f^{(k)}$ all have infinitely many fixed points and satisfy (1.6). This completes the proof.

Proof of Theorem 1.1. (1) Suppose that $f(z)$ is a transcendental meromorphic solution of (1.4). Then we know that the poles of f can only occur at the poles of $P(z)$, so f has only finite poles. According to the Hadamard factorization theorem, we can write f as $f(z)=g(z)/d(z)$, where $g(z)$ is an entire function satisfying $\sigma(g)=\sigma(f)$ and $d(z)$ is a polynomial. By Lemma 2.1, for any $\varepsilon>0$, there exists a sequence $\{r_j\}$, $r_j\to\infty$, such that for all z satisfying $|z|=r_j$, $|g(z)|=M(r_j,g)$, when j is large enough, we have

$$\frac{f^{(k)}(z)}{f(z)}=\left(\frac{v_g(r_j)}{z}\right)^k\bigl(1+o(1)\bigr), \tag{3.15}$$

$$\sigma(f)=\lim_{j\to\infty}\frac{\log v_g(r_j)}{\log r_j}. \tag{3.16}$$

Substituting (3.15) into (1.4), we have

$$\left(\frac{v_g(r_j)}{r_j}\right)^k \bigl(1+o(1)\bigr) = r_j^n\bigl(a+o(1)\bigr), \tag{3.17}$$

where a is a nonzero constant. From (3.16) and (3.17), we obtain that $\sigma(f) = \max\{(n+k)/k, 0\}$.

(2): $\mathrm{di}(P) = n \neq -k$. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.4). Then f must be a transcendental meromorphic function. In fact, if not, then f is a rational function. We conclude that $f^{(k)}(z)/f(z) = \bigl(b+o(1)\bigr)/z^k$ as $z \to \infty$ (refer to p. 62 in [4]). On the other hand, since $\mathrm{di}(P) = n \neq -k$ and $P(z) \not\equiv 0$, we have that $-P(z) = (b_0 + o(1))/z^{-n}$ as $z \to \infty$, where $b_0 \neq 0$; this contradicts $f^{(k)}(z)/f(z) = -P(z)$ and $-n \neq k$. Hence $f(z)$ is a transcendental meromorphic function. From part (1), we have

$$\sigma(f) = \max\left\{\frac{n+k}{k}, D\right\}. \tag{3.18}$$

Set $w_i = f^{(i)} - z$, $i = 0, 1, \dots, k$. Then for every i, a point z_0 is a fixed point of $f^{(i)}$ if and only if z_0 is a zero of w_i, and

$$\sigma(w_i) = \sigma(f^{(i)}) = \sigma(f) = \max\left\{\frac{n+k}{k}, D\right\}, \quad \overline{\tau}(f^{(i)}) = \bar{\lambda}(w_i). \tag{3.19}$$

According to Lemmas 2.4–2.6, we know that w_i, $i = 0, 1, \dots, k$, satisfy the following equations:

$$\sum_{j=0}^{i} H_{ij}(P) w_i^{(k-j)} - P^{i+1} w_i = zP^{i+1}, \quad i = 0, 1, \dots, k-2; \tag{3.20}$$

$$\sum_{j=0}^{k-1} H_{(k-1)j}(P) w_{k-1}^{(k-j)} - P^k w_{k-1} = zP^k - H_{(k-1)(k-1)}(P), \quad \text{if } k \geq 3; \tag{3.21}$$

$$\sum_{j=0}^{k-1} H_{kj}(P) w_k^{(k-j)} + \bigl(H_{kk}(P) - P^{k+1}\bigr) w_k = zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P). \tag{3.22}$$

Since $P \not\equiv 0$, we conclude that $P^{i+1} z \not\equiv 0$ for $i = 0, 1, \dots, k-2$. We claim that

$$zP^k - H_{(k-1)(k-1)}(P) \not\equiv 0 \quad \text{and} \quad zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P) \not\equiv 0.$$

We discuss three subcases below.

Subcase I: $\mathrm{di}(P) = n \geq 0$. If $zP^k - H_{(k-1)(k-1)}(P) \equiv 0$, rewrite it as $P = H_{(k-1)(k-1)}(P)/P^{k-1}z$. Notice that $P(z)$ is a rational function. We have that $P^{(j)}(z)/P(z) \to 0$ as $z \to \infty$. From this we obtain that $H_{(k-1)(k-1)}(P)/zP^{k-1} \to 0$ as $z \to \infty$, which contradicts that $P(z) \to \infty$ or a nonzero constant. If $zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P) \equiv 0$, rewrite it as

$$P = \frac{H_{k(k-1)}(P)}{zP^k} + \frac{H_{kk}(P)}{P^k}.$$

Similarly, we may obtain that

$$\frac{H_{k(k-1)}(P)}{zP^k} + \frac{H_{kk}(P)}{P^k} \to 0 \quad \text{as } z \to \infty,$$

which contradicts that $P(z) \to \infty$ or a nonzero constant; hence the claims hold.

Subcase II: When $-k < \mathrm{di}(P) = n \leq -1$, we have $P(z) = \big(c_0 + o(1)\big)/z^{-n}$ as $z \to \infty$, where c_0 is a nonzero constant. From this, we have

$$zP^k = \frac{c_0^k + o(1)}{z^{-nk-1}}, \quad zP^{k+1} = \frac{c_0^{k+1} + o(1)}{z^{-n(k+1)-1}} \quad \text{as } z \to \infty. \tag{3.23}$$

On the other hand, by (2.12)–(2.14), using a similar argument as in the proof of Lemma 2.7, we have

$$H_{(k-1)(k-1)}(P) = \frac{d_1 + o(1)}{z^{-nk-1+k+n}} \quad \text{as } z \to \infty, \tag{3.24}$$

and

$$H_{k(k-1)}(P) + zH_{kk}(P) = \frac{d_2 + o(1)}{z^{-n(k+1)-1+k+n}} \quad \text{as } z \to \infty, \tag{3.25}$$

where d_1, d_2 are two constants (note that d_1 or d_2 may be zero). Hence it follows from (3.23)–(3.25) and $k + n > 0$ that

$$zP^k - H_{(k-1)(k-1)}(P) \not\equiv 0 \quad \text{and} \quad zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P) \not\equiv 0,$$

that is, the claims hold.

Subcase III: $n = \mathrm{di}(P) < -k$. By Lemma 2.7, there exist two nonzero constants c_1, c_2 such that

$$H_{(k-1)(k-1)}(P) = \frac{c_1 + o(1)}{z^{-nk-1+k+n}} \quad \text{as } z \to \infty, \tag{3.26}$$

and

$$H_{k(k-1)}(P) + zH_{kk}(P) = \frac{c_2 + o(1)}{z^{-n(k+1)-1+k+n}} \quad \text{as } z \to \infty. \tag{3.27}$$

Hence it follows from (3.23), (3.26), (3.27) and $k + n < 0$ that

$$zP^k - H_{(k-1)(k-1)}(P) \not\equiv 0 \quad \text{and} \quad zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P) \not\equiv 0,$$

that is, the claims hold.

Rewrite (3.20), (3.21) and (3.22) as

$$\frac{1}{zP^{i+1}} \left(\sum_{j=0}^{i} H_{ij}(P) \frac{w_i^{(k-j)}}{w_i} - P^{i+1} \right) = \frac{1}{w_i}, \quad i = 0, 1, \ldots, k-2, \tag{3.28}$$

$$\frac{1}{zP^k - H_{(k-1)(k-1)}(P)} \left(\sum_{j=0}^{k-1} H_{(k-1)j}(P) \frac{w_{k-1}^{(k-j)}}{w_{k-1}} - P^k \right) = \frac{1}{w_{k-1}}, \tag{3.29}$$

$$\frac{1}{zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P)} \left(\sum_{j=0}^{k-1} H_{kj}(P) \frac{w_k^{(k-j)}}{w_k} + H_{kk}(P) - P^{k+1} \right) = \frac{1}{w_k}. \tag{3.30}$$

Hence there exists a set E with finite linear measure such that for $r \notin E$, and we have

$$m\left(r, \frac{1}{w_i}\right) \le O\left(m\left(r, \frac{1}{P}\right)\right) + C\big(\log rT(r, w_i)\big), \quad i = 0, 1, \ldots, k. \tag{3.31}$$

Applying similar arguments as in the proof of Theorem 1.2, we obtain that

$$N\left(r, \frac{1}{w_i}\right) \le k\overline{N}\left(r, \frac{1}{w_i}\right) + N\left(r, \frac{1}{zP^{i+1}}\right), \quad i = 0, 1, \ldots, k-2; \tag{3.32}$$

$$N\left(r, \frac{1}{w_{k-1}}\right) \le (k^2 - k + 2)\overline{N}\left(r, \frac{1}{w_{k-1}}\right) + N\left(r, \frac{1}{zP^k - H_{(k-1)(k-1)}(P)}\right); \tag{3.33}$$

$$N\left(r, \frac{1}{w_k}\right) \le (k^2 + k + 1)\overline{N}\left(r, \frac{1}{w_k}\right) + N\left(r, \frac{1}{zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P)}\right). \tag{3.34}$$

Note that for sufficiently large r, $C\log\big(rT(r,w_i)\big) \le \frac{1}{2}T(r,w_i)$, $i=0,1,\dots,k$. Because w_i, $i=0,1,\dots,k$, are transcendental meromorphic functions and P is a rational function, so for any given $\varepsilon>0$, from (3.31)–(3.34), we have

$$T(r,w_i) \le 2(k^2+k+1)\overline{N}\left(r,\frac{1}{w_i}\right) + O(\log r), \quad i=0,1,\dots,k,$$

for $r \notin E_3$ sufficiently large, where E_3 has finite linear measure.

By (3.19), we obtain that for $i=0,1,\dots,k$, $f^{(i)}(z)$ all have infinitely many fixed points and

$$\overline{\tau}(f^{(i)}) = \bar{\lambda}(w_i) = \sigma(w_i) = \max\left\{\frac{n+k}{k},0\right\}. \tag{3.35}$$

That is, each meromorphic solution $f \not\equiv 0$ of (1.4) and its $f',\dots,f^{(k)}$ all have infinitely many fixed points and satisfy (3.35).

Case (3): $\mathrm{di}(P)=n=-k$. Suppose that $f(z)$ is a transcendental meromorphic solution of (1.4). Set $w_i(z)=f^{(i)}(z)-z$, $i=0,1,\dots,k-2$, since $P^{i+1}z\not\equiv 0$ for $i=0,1,\dots,k-2$. Using similar arguments as in part (2), we may prove that for any given $\varepsilon>0$,

$$T(r,w_i) \le 2k\overline{N}\left(r,\frac{1}{w_i}\right) + O(\log r), \quad i=0,1,\dots,k-2, \tag{3.36}$$

for $r\notin E_4$ sufficiently large, where E_4 has finite linear measure.

Note that w_i, $i=0,1,\dots,k-2$, are transcendental meromorphic functions. It follows from (3.36) that f and $f',f'',\dots,f^{(k-2)}$ all have infinitely many fixed points and

$$\overline{\tau}(f)=\overline{\tau}(f')=\cdots=\overline{\tau}(f^{(k-2)})=\sigma(w_i)=\max\left\{\frac{n+k}{k},0\right\}=0.$$

Hence the proof of Theorem 1.1 is complete.

Remark 3.1. When $n=-k$ in Theorem 1.1, we cannot prove that $\overline{\tau}(f^{k-1}) = \overline{\tau}(f^{(k)}) = \sigma(f)$ by using similar arguments as in part (2) of the proof of Theorem 1.1. Since we cannot prove that $zP^k - H_{(k-1)(k-1)}(P) \not\equiv 0$ and $zP^{k+1} - H_{k(k-1)}(P) - zH_{kk}(P)\not\equiv 0$ in (3.21) and (3.22), respectively. In fact, if $n=-k$, there exist some equations of the form (1.4) such that $zP^k - H_{(k-1)(k-1)}(P)\equiv 0$ or $zP^{k+1}-H_{k(k-1)}(P)-zH_{kk}(P)\equiv 0$. For example, when $k=2$, by (2.12)–(2.14), simple computation yields

$$zP^2 - H_{11}(P) = zP^2 - P' \equiv 0 \quad \text{for } P(z)=\frac{-2}{z^2},$$

and

$$zP^3 - H_{21}(P) - zH_{22}(P) = zP^3 - 2PP' + 2z(P')^2 - zPP'' \equiv 0 \quad \text{for } P(z) = \frac{-6}{z^2}.$$

When $k = 3$, simple computation yields $zP^3 - H_{22}(P) = zP^3 - PP'' + 2P'^2 \equiv 0$ for $P(z) = -6/z^3$, and for $P(z) = -24/z^3$, we have

$$\begin{aligned} zP^4 - H_{32}(P) - zH_{33}(P) &= zP^4 - 3P(PP'' - 2P'^2) \\ &\quad + 6zPP'P'' - zP^2P''' - 6zP'^3 \equiv 0. \end{aligned}$$

Acknowledgements. The authors thank the referees for their helpful comments and suggestions to improve our manuscript.

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Received 30 December 2004