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A SHARPENING OF A THEOREM OF BOULIGAND. WITH AN APPLICATION TO HARMONIC MAPS

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In memory of Professor Gert Kjærgård Pedersen

Abstract. The result by G. Bouligand (1926) about the boundary behaviour of the solution to the generalized Dirichlet problem is sharpened, the requirement that the prescribed boundary function f be bounded being now replaced by $|f|$ having a superharmonic majorant. By way of application, a recent result on the boundary behaviour of the solution to the variational Dirichlet problem for harmonic maps is sharpened by leaving out the previous requirement that the prescribed boundary map be bounded.

Introduction

In the course of his investigation from 1926 of the classical Dirichlet problem for harmonic functions in a bounded open set $\Omega \subset \mathbf{R}^m$, G. Bouligand [Bou] essentially found that, for any *bounded* resolutive (e.g. Borel measurable) function $f\colon \partial\Omega \to \mathbf{R}$ and any regular point $x_0 \in \partial\Omega$ at which f is continuous, the generalized solution $H_f^\Omega\colon \Omega \to \mathbf{R}$ in the sense of Perron and Wiener satisfies

$$(*) \qquad H_f^\Omega(x) \to f(x_0) \qquad \text{for } x \to x_0,\ x \in \Omega.$$

It is known that the boundedness hypothesis on f cannot be omitted, see [Br2]. The Perron–Wiener method was extended and further developed by Brelot in his axiomatic theory of harmonic functions from the late 1950's, comprising also Bouligand's result [Br2, p. 115].

In the present article it is shown that the Bouligand–Brelot result remains valid when boundedness of f is weakened to the existence of a superharmonic function $s > 0$ in a neighbourhood of $\overline{\Omega}$ such that $|f| \leqslant s$ on $\partial\Omega$ (Theorem 1). The author was led to this complement to the Bouligand–Brelot result while studying the Dirichlet problem for *harmonic maps*, in the sense of Eells and Sampson [ES] from 1964, between Riemannian manifolds X and Y, rather than harmonic

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functions. Owing to the nonlinearity of the concept of harmonic map, the Perron–Wiener–Brelot procedure is no longer available. Instead one applies the direct method of variational calculus, whereby the *energy* (a generalized Dirichlet integral) is to be minimized among all finite energy maps $X \to Y$ which agree, in the complement of $\Omega \Subset X$, with a prescribed map $X \to Y$ of finite energy.

We first consider the variational Dirichlet problem for harmonic *functions* in an open set Ω of compact closure in a *hypoelliptic* symmetric regular real *Dirichlet space* of local type, in the sense of Feyel and de La Pradelle [FP2], who proved that such a space X is also a harmonic space in the sense of Brelot, with all the general properties known from classical potential theory. The variational solution is shown to be identical with the solution in the sense of Perron–Wiener–Brelot, and Theorem 1 therefore applies (Theorem 3). The identity of the two solutions is known for continuous and hence *bounded* functions, even in the setting of nonlinear potential theory, [HKM, Corollary 9.29]. Our proof of Theorem 3 uses H. Cartan's theorem [Ca1] from 1945 on completeness of the space $\mathscr{E}^+$ of all positive measures of finite energy on $\mathbf{R}^m$, $m \geqslant 3$, extended in Theorem 2 below to the present hypoelliptic Dirichlet space X. We further use an approximation result [F4, Theorem 6.5] from 1972, underlying the theory of harmonic functions relative to the Cartan fine topology on X; this result draws on Choquet [Ch3] and Brelot [Br3], [Br4] and [Br5].

Thus prepared, we pass to the variational Dirichlet problem for harmonic maps. The source space X is specialized to be a Riemannian manifold (possibly with boundary), or more generally: an admissible *Riemannian polyhedron* with simplexwise smooth Riemannian metric, cf. the recent monograph [EF] by J. Eells and the present author. The target space Y is a simply connected complete *geodesic space* of nonpositive curvature in the sense of A. D. Alexandrov [Al1], [Al2]. In this setting we have a good concept of *energy* of maps $X \to Y$, introduced and investigated by Korevaar and Schoen [KS] in 1993 (for manifold domains), and in [EF, Chapter 9] for polyhedral domains. Existence, uniqueness, and interior Hölder continuity of the solution to the variational Dirichlet problem for harmonic maps $\Omega \to Y$ ($\Omega \Subset X$, Ω open) with a prescribed map $\psi\colon X \to Y$ of finite energy was established in [F9, Theorem 1], and continuity of the solution up to the boundary at any regular boundary point for Ω at which ψ is continuous was proved there for *bounded* ψ. In Theorem 4 of the present paper this boundedness requirement is removed, by application of Theorem 3, and hence of Theorem 1.

1. A sharpening of a theorem of Bouligand

Theorem 1. *In a Brelot harmonic space X without compact components, let Ω be a relatively compact open set and x_0 a regular boundary point for Ω. If a numerical function f on $\partial\Omega$ is majorized by a superharmonic function $s > 0$ on X then*

$$\limsup_{\Omega \ni x \to x_0} \overline{H}{}^{\Omega}_f(x) \leqslant \limsup_{\partial\Omega \ni y \to x_0} f(y). \tag{1}$$

In classical potential theory the hypothesis $f \leqslant s$ can be omitted if Ω is Lipschitz near x_0, see [Ar], but not for general Ω, as shown by Brelot by a variety of examples in [Br2]. See also [CC, Exercise 3.2.14] and (with Ω disconnected): [AG, p. 187]. It suffices, in Theorem 1, that s be defined in some open subset of X containing $\overline{\Omega}$. If there exists at all a superharmonic function $s > 0$ on X then s is bounded from below on $\partial\Omega$, and Theorem 1 is therefore a sharper version of a well-known result by Brelot [Br2, Proposition 19, p. 115] in which $f \leqslant s$ is replaced by f being bounded from above. Brelot's result, in turn, is an axiomatic version of a slight extension of a result due to Bouligand [Bou, p. 89] in classical potential theory. We recall Brelot's short proof, his result being used in the proof of Theorem 1: Denote by c the right-hand member of (1). For given $c' > c$ there is a neighbourhood ω of x_0 in $\partial\Omega$ such that $f \leqslant c'$ in ω. If $c' < \sup f$, there exists a continuous function $g\colon \partial\Omega \to [c', \sup f]$ such that $g(x_0) = c'$ and $g = \sup f$ in $(\partial\Omega) \setminus \omega$. If instead $c' \geqslant \sup f$, define $g = c'$ on $\partial\Omega$. In either case we have $f \leqslant g$ on $\partial\Omega$, and hence, by definition of regularity of $x_0 \in \partial\Omega$, [Br2, p. 114]:

$$\limsup_{x\to x_0} \overline{H}{}_f^{\Omega}(x) \leqslant \lim_{x\to x_0} H_g^{\Omega}(x) = g(x_0) = c'.$$

Proof of Theorem 1. Choose in X a relatively compact open set $X_0 \supset \overline{\Omega}$. The function $h := R_s^{\complement X_0} > 0$ on X (defined as the pointwise infimum of the family of all superharmonic functions on X which majorize s on $\complement X_0$) is harmonic in X_0. Replacing X_0 by X we may thus assume from the beginning that there exists a harmonic function $h > 0$ on X.

The right-hand member of (1) has the form $c\,h(x_0)$ for some $c \in [-\infty, +\infty]$. We may assume that $c < +\infty$. Suppose first that $c \geqslant 0$. For given $c' > c$ choose an open neighbourhood V of x_0 in X so that $f \leqslant c'h$ on $\overline{V} \cap \partial\Omega$. The superharmonic function $u := \widehat{R}_s^{\complement V}$ (the greatest lower semicontinuous minorant of $R_s^{\complement V}$ in X) is harmonic and hence continuous in V; and $\overline{H}{}_s^{V} = u$ in V, [Br2, Proposition 5, p. 85]. Furthermore,

$$f \leqslant u + c'h \qquad \text{on } \partial\Omega \tag{2}$$

because $f \leqslant c'h$ in $\overline{V} \cap \partial\Omega$ and $f \leqslant s = u$ in $\complement \overline{V}$.

Since x_0 is regular, $\complement\Omega$ is not thin at x_0, [Br2, Theorem 32, p. 142], and $v := \widehat{R}_u^{\complement\Omega}$ $(\leqslant u)$ therefore has the value $v(x_0) = u(x_0)$, [Br2, Theorem 28]. It follows that

$$\begin{aligned} u(x_0) = v(x_0) &\leqslant \liminf_{x\to x_0} v(x) \leqslant \limsup_{x\to x_0} v(x) \\ &\leqslant \lim_{x\to x_0} u(x) = u(x_0) < +\infty, \end{aligned} \tag{3}$$

and hence u and v are bounded in some neighbourhood $W \subset V$ of x_0. Define an auxiliary function g on $\partial\Omega$ by

$$g = \begin{cases} f - u & \text{on } W \cap \partial\Omega, \\ c'h & \text{elsewhere on } \partial\Omega, \end{cases}$$

and note that $f - u \leqslant g \leqslant c'h$ on $\partial\Omega$ by (2); thus g is bounded from above on $\partial\Omega$, like h. Applying the Bouligand–Brelot result to g we therefore obtain by (3)

$$\limsup_{\Omega\ni x\to x_0} \overline{H}{}_g^{\Omega}(x) \leqslant \limsup_{\partial\Omega\ni y\to x_0} g(y) \leqslant c\,h(x_0) - u(x_0). \tag{4}$$

Since $f \leqslant u + g$ on $\partial\Omega$ and $\overline{H}{}_u^{\Omega} = \widehat{R}{}_u^{\complement\Omega} = v$ in Ω, we conclude from (3) and (4) that

$$\limsup_{\Omega\ni x\to x_0} \overline{H}{}_f^{\Omega}(x) \leqslant \limsup_{\Omega\ni x\to x_0} v(x) + \limsup_{\Omega\ni x\to x_0} \overline{H}{}_g^{\Omega}(x) \leqslant c\,h(x_0),$$

the right-hand member of (1).

In the case where $-\infty < c < 0$ we have $H_{ch}^{\Omega}(x) = c\,h(x) \to c\,h(x_0)$ as $x \to x_0$, $x \in \Omega$; and hence from the above case $c \geqslant 0$, applied to $f - ch$ in place of f:

$$\begin{aligned}\limsup_{\Omega\ni x\to x_0} \overline{H}{}_f^{\Omega}(x) &\leqslant \limsup_{\Omega\ni x\to x_0} \overline{H}{}_{f-ch}^{\Omega}(x) + c\,h(x_0)\\ &\leqslant \limsup_{\partial\Omega\ni y\to x_0} \big(f(y) - c\,h(y)\big) + c\,h(x_0) = \limsup_{\partial\Omega\ni y\to x_0} f(y).\end{aligned}$$

In the remaining case where $c = -\infty$ we obtain from the case $-\infty < c < 0$, applied for $-\infty < a < 0$ to $f_a := \max\{f, ah\}$ in place of f:

$$\limsup_{\Omega\ni x\to x_0} \overline{H}{}_f^{\Omega}(x) \leqslant \limsup_{\Omega\ni x\to x_0} \overline{H}{}_{f_a}^{\Omega}(x) \leqslant \limsup_{\partial\Omega\ni y\to x_0} f_a(y),$$

the last $\limsup$ being $a\,h(x_0) > -\infty$. For $a \to -\infty$ the resulting inequality reads $\limsup_{x\to x_0} \overline{H}{}_f^{\Omega}(x) = -\infty$, which implies (1). □

2. Measures of finite energy

In this section, $X = (X, E, \tau)$ denotes a symmetric regular real Dirichlet space in the sense of [De2, pp. 137, 158], or see [FOT]. Here τ is a positive measure with $\operatorname{supp}\tau = X$, and E denotes the given closed symmetric Dirichlet form with domain $\mathscr{D}$ dense in $L^2(X,\tau)$ and topologically included in $L^1_{\mathrm{loc}}(X,\tau)$. Write $E(f,f) = E(f)$.

We assume that the underlying space X is connected and locally compact with countable base; furthermore that (X, E, τ) is *hypoelliptic* in the sense of Feyel–de La Pradelle [FP2], see also [La]; and in particular of *local type* in the sense that, for every $f \in \mathscr{D}$, we have $E(|f|) = E(f)$, or equivalently $E(f^+, f^-) = 0$. Hypoellipticity means that (i) every local E-solution (see Section 3 below) has a continuous version, and (ii) for any $x \in X$ there exists in an open neighbourhood of x an E-solution whose continuous version is strictly positive. Recall from [FP2, p. 121] that X is noncompact and locally connected, and that there exists an associated Brelot harmonic sheaf $\mathscr{H}$ on X with a symmetric lower semicontinuous

Green kernel G, and satisfying the axiom of domination, see also [La]. The E-potentials on (X, E, τ) are represented uniquely by the Green potentials $y \mapsto G\mu(y) = \int G(\,\cdot\,, y)\, d\mu$ on X of measures $\mu \in \mathscr{E}^+$, i.e., the positive measures of finite G-*energy* $\int G\mu\, d\mu = \int G\, d(\mu \otimes \mu)$.

The (outer) *capacity* $\operatorname{cap} A$ of a set $A \subset X$ is defined in [De2, pp. 162–164]. We have $\operatorname{cap} A = 0$ if and only if A is *polar* in the sense of Brelot, i.e., there exists a superharmonic function $s \geqslant 0$ on X such that $s = +\infty$ in A; if so, one may take $s = G\mu$, $\mu \in \mathscr{E}^+$, [FP2, 32°, 21°]. A polar set has μ-measure 0 for any $\mu \in \mathscr{E}^+$, [FP2, 4°]. A property involving points of X is said to hold *quasi-everywhere* (q.e.) in a set A if it holds in A except perhaps in some polar set.

A map $\varphi\colon X \to Y$ of X into a topological space Y is said to be *quasicontinuous* if φ is continuous relative to the complements of open subsets of X of arbitrarily small capacity. Every element of $\mathscr{D}$ (in particular every potential $G\mu$, $\mu \in \mathscr{E}^+$) has quasicontinuous versions [De2, p. 170], or see [FOT, Theorem 2.1.3]. Henceforth, *we tacitly use quasicontinuous versions of elements of* $\mathscr{D}$.

For any function $f \in \mathscr{D}$ and any measure $\mu \in \mathscr{E}^+$,

$$\int f\, d\mu = E(f, G\mu), \tag{5}$$

[FP2, 37°]. It follows that, for any $\mu, \nu \in \mathscr{E}^+$,

$$\int G\mu\, d\nu = E(G\mu, G\nu) = \int G\nu\, d\mu, \quad \int G\mu\, d\mu = E(G\mu), \tag{6}$$

$$\Big(\int G\mu\, d\nu\Big)^2 \leqslant \int G\mu\, d\mu \int G\nu\, d\nu. \tag{7}$$

For signed measures μ, ν write $\mu \in \mathscr{E}$ if and only if $\mu^+, \mu^- \in \mathscr{E}^+$; in that case write $G\mu = G\mu^+ - G\mu^-$, which is well defined q.e.; then $\int G\mu\, d\nu$ is well defined in an obvious way on account of (7) applied to the couples (μ^+, ν^+), (μ^+, ν^-), (μ^-, ν^+), and (μ^-, ν^-). Furthermore, (5), (6), (7) remain valid for $\mu \in \mathscr{E}$, the latter because $\int G\mu\, d\mu \geqslant 0$ for $\mu \in \mathscr{E}$.—For an alternative proof of (7), see [FP1, Corollary 14].

Following Cartan [Ca1], [Ca2], we thus have the prehilbert space $\mathscr{E}$ with the inner product $\langle \mu, \nu \rangle = E(G\mu, G\nu)$, which is (strictly) positive definite. The corresponding *energy norm* on $\mathscr{E}$ is given by $\|\mu\|^2 = \int G\mu\, d\mu$. Extending a key result in [Ca1], [Ca2, Théorème 2] for the Newtonian kernel on $\mathbf{R}^m$, $m \geqslant 3$, and for the Green kernel on an open ball in $\mathbf{R}^m$, $m \geqslant 2$, we have in the present more general setting:

Theorem 2. *The cone* $\mathscr{E}^+$ *of all positive measures of finite energy on* X *is complete in the energy norm.*

Proof. The proof of this in [Ca1, p. 91] reduces to verifying that every measure $\mu \in \mathscr{E}^+$ can be approximated in energy norm by *signed* measures $\mu' \in \mathscr{E}$ with $G\mu' \in \mathscr{K}^+(X)$, where $\mathscr{K}(X)$ denotes the space of all continuous functions $X \to \mathbf{R}$ of compact support. It is known that the class of all measures $\mu \in \mathscr{E}^+$ with $G\mu$ finite and continuous is dense in $\mathscr{E}^+$ in energy norm, cf. [Ca1, p. 90], [Oht], [Ch2], [F1, Theorem 3.4.1]; the proof uses Lusin's theorem together with the continuity principle of Evans and Vasilescu, valid in the present setting according to [JN]. Thus, *it remains to approximate any $\mu \in \mathscr{E}^+$ having a finite continuous potential by measures $\mu' \in \mathscr{E}$ for which $G\mu' \in \mathscr{K}^+(X)$.*

For any relatively compact open set $\Omega \subset X$ take $\mu' = \mu - \mu^{\complement\Omega}$, where $\mu^{\complement\Omega}$ is obtained by balayage of μ on $\complement\Omega$, [He, Théorème 10.1], [CC, Corollary 7.1.2]. Then $G\mu' = G\mu - \widehat{R}^{\complement\Omega}_{G\mu} \geqslant 0$, [He, Proposition 31.3, 2)] (the proof of which does not require that $\operatorname{supp}\mu$ be compact) is finite and continuous ($\widehat{R}^{\complement\Omega}_{G\mu}$ being lower semicontinuous) and equals 0 off the compact closure of Ω. Furthermore, $G\mu - G\mu' = \widehat{R}^{\complement\Omega}_{G\mu}$ is harmonic in Ω and decreases for increasing Ω to a function $h \geqslant 0$ which is harmonic in X, by Brelot's convergence property. And $h = 0$ because $h \leqslant G\mu$, a potential. Finally,

$$\|\mu - \mu'\|^2 = \|\mu^{\complement\Omega}\|^2 = \int G\mu^{\complement\Omega}\, d\mu^{\complement\Omega} = \int \widehat{R}^{\complement\Omega}_{G\mu}\, d\mu^{\complement\Omega} = \int \widehat{R}^{\complement\Omega}_{G\mu}\, d\mu \searrow 0$$

as $\Omega \nearrow X$ because $\int G\mu\, d\mu < +\infty$ and $\int \widehat{R}^{\complement\Omega}_{G\mu}\, d\mu^{\complement\Omega} = \int \widehat{R}^{\complement\Omega}_{\widehat{R}^{\complement\Omega}_{G\mu}}\, d\mu = \int \widehat{R}^{\complement\Omega}_{G\mu}\, d\mu$, by idempotency of the operator of balayage of superharmonic functions $\geqslant 0$ on a given set, see e.g. [CC, Theorem 9.1.1 and Corollary 9.2.3]. ▫

Remark 1. Cartan's proof of completeness of $\mathscr{E}^+$ extends immediately to that of the Green kernel on a regular domain X on a Riemann surface. The regularity assumption was removed by Edwards [Ed], who considered the Green kernel on a hyperbolic Riemann surface X, a particular instance of the present setting.

3. The variational Dirichlet problem

With $X = (X, E, \tau)$ as in the preceding section we proceed to study the variational Dirichlet problem for harmonic functions in an open set $\Omega \subset X$. Recall that we tacitly employ quasicontinuous versions of elements of $\mathscr{D}$. Write

$$\mathscr{D}(\Omega) = \{u \in \mathscr{D} : u = 0 \text{ q.e. in } \complement\Omega\},$$

and $\mathscr{D}^+(\Omega) = \{u \in \mathscr{D}(\Omega) : u \geqslant 0 \text{ q.e.}\}$. A (quasicontinuous) function $u \in \mathscr{D}$ is said to be an *E-solution* in Ω if

$$E(u, v) = 0 \text{ for every } v \in \mathscr{D}(\Omega),$$

whereas u is termed an E*-subsolution* if $E(u,v) \leqslant 0$ for every $v \in \mathscr{D}^+(\Omega)$. A *continuous* E-solution in Ω is said to be *harmonic* in Ω.

Given a (quasicontinuous) function $f \in \mathscr{D}$ write

$$\mathscr{D}_f(\Omega) = \{u \in \mathscr{D} : u = f \text{ q.e. in } \complement\Omega\} = \{u \in \mathscr{D} : u - f \in \mathscr{D}(\Omega)\}.$$

The E-minimizer u in $\mathscr{D}_f(\Omega)$ considered in the following proposition is called the *variational solution* to the Dirichlet problem in Ω with prescribed function $f \in \mathscr{D}$.

Proposition 1. *For any* $f \in \mathscr{D}$, $\mathscr{D}_f(\Omega)$ *has a unique element* u *of least energy* $E(u)$; *and* u *is the only element of class* $\mathscr{D}_f(\Omega)$ *which is an* E*-solution in* Ω (*and therefore, by hypoellipticity, has a version which is harmonic in* Ω).

Proof. Being a convex closed nonvoid subset of the Hilbert space $\mathscr{D}$, $\mathscr{D}_f(\Omega)$ has a unique element u of minimal norm $E(u)^{1/2}$. For any $v \in \mathscr{D}(\Omega)$ and any $t \in \mathbf{R}$ we have $u + tv \in \mathscr{D}_f(\Omega)$, and hence

$$E(u) \leqslant E(u+tv) = E(u) + 2t\, E(u,v) + t^2 E(v),$$

from which it follows that $E(u,v) = 0$. Consequently, u is an E-solution in Ω and so has a (quasicontinuous) version u on X which is harmonic in Ω. For any other element $u' \in \mathscr{D}_f(\Omega)$ which is an E-solution in Ω, $u - u' \in \mathscr{D}(\Omega)$ is an E-solution in Ω, and hence orthogonal to itself, and so $u = u'$. □

The following proposition is the key to our application of Theorem 1 to harmonic maps (Section 4). Only property (8) will be used in this paper; it was stated in [F5, Lemme 3] (for the typical case $X = \mathbf{R}^m$) with a slight indication of proof, and used there for characterizing the E-(sub)solutions in Ω as those functions in $\mathscr{D}(\Omega)$ which are *finely* (*sub*)*harmonic* off some polar set [F5, Théorème 11].

For any function $f\colon X \to [0,+\infty]$ write

$$R_f = \inf\{u : u \text{ superharmonic, } u \geqslant f\}$$

(pointwise infimum, understood as identically $+\infty$ if no such u exists). Denote by $\widehat{R}_f$ the greatest lower semicontinuous minorant of R_f, or equivalently, by [Br2, Proposition 23, p. 135]:

$$\widehat{R}_f = \min\{u : u \text{ superharmonic, } u \geqslant f \text{ q.e.}\}.$$

Proposition 2. *For any* $f \in \mathscr{D}^+$ *there exists a unique measure* $\lambda \in \mathscr{E}^+$ *such that*

$$G\lambda \geqslant f \qquad \text{q.e.,} \tag{8}$$

$$G\lambda = f \qquad \lambda\text{-a.e.,} \tag{9}$$

and hence $\int f\,d\lambda = \int G\lambda\,d\lambda \leqslant E(f)$. *This measure* λ *is the unique measure in* $\mathscr{E}^+$ *having one of the following equivalent properties* (*with pointwise minimum in* (10))*:*

$$G\lambda = \min\{G\mu : \mu \in \mathscr{E}^+,\ G\mu \geqslant f \text{ q.e.}\}, \tag{10}$$

$$\int G\lambda\,d\lambda = \min\Big\{\int G\mu\,d\mu : \mu \in \mathscr{E}^+,\ G\mu \geqslant f \text{ q.e.}\Big\}. \tag{11}$$

Remark 2. It can be shown that λ is also uniquely determined by any one of the following two properties:

$$\int f\,d\lambda = \max\Big\{\int f\,d\mu : \mu \in \mathscr{E}^+,\ G\mu \leqslant f\ \mu\text{-a.e.}\Big\},$$

$$\int (2f - G\lambda)\,d\lambda = \max\Big\{\int (2f - G\mu)\,d\mu : \mu \in \mathscr{E}^+\Big\}.$$

As already noted, the left-hand members of these two equations and of (11) all equal $\int G\lambda\,d\lambda = \int f\,d\lambda$; their square-root $\|\lambda\|$ is termed the *energy capacity* of f in [F3, Section 6.7] [1] (where G is the Newtonian kernel on $\mathbf{R}^3$). As a seminorm on $\mathscr{K}(X)$, $\|\lambda\|$ is studied in [FP1].

Proof. Suppose first that $f \in \mathscr{D}^+ \cap \mathscr{K}(X)$; then $\widehat{R}_f$ is the bounded potential $G\lambda$ of a unique measure $\lambda \in \mathscr{E}^+$, [Br2, Proposition 10, p. 94]; this implies (8). Furthermore, $G\lambda \in \mathscr{D}^+$ is harmonic in the open set $\Omega = \{G\lambda > f\}$, as shown by Poisson modification, see e.g. [CC, Proposition 2.2.3]; and so $\operatorname{supp}\lambda \subset \complement\Omega$, [He, Proposition 30.1, 2)], [CC, Proposition 11.4.12, c)]. Thus (9) holds because $G\lambda \geqslant f$ λ-a.e. in view of (8). Furthermore, $G\lambda \leqslant f$ in $\complement\Omega$, and hence $G\lambda = f$ q.e. in $\complement\Omega$, by (8). Thus $G\lambda \in \mathscr{D}_f(\Omega)$; and according to Proposition 1, $G\lambda$ is the unique solution to the variational Dirichlet problem in Ω with prescribed function f. It follows in view of (6) that

$$\int G\lambda\,d\lambda = E(G\lambda) = \min\{E(u) : u \in \mathscr{D},\ u = f \text{ q.e. in } \complement\Omega\}.$$

In particular, by (9), $\int f\,d\lambda = \int G\lambda\,d\lambda \leqslant E(f)$ (since $u = f$ competes).

For arbitrary $f \in \mathscr{D}^+$, choose a sequence $(f_n) \subset \mathscr{D}^+ \cap \mathscr{K}(X)$ so that $E(f_n - f) \to 0$, [De2, Lemme 1, p. 159]; and hence $f_n \to f$ in $L^1_{\mathrm{loc}}(X,\tau)$. We may assume that $f_n \to f$ pointwise τ-a.e. As shown above, we may write $\widehat{R}_{f_n} = G\lambda_n$ with $\lambda_n \in \mathscr{E}^+$. There is a constant $c > 0$ such that, in view of (6) and the inequality following (9) (now with f replaced by f_n):

$$E(G\lambda_n) = \int G\lambda_n\,d\lambda_n \leqslant E(f_n) \leqslant c^2. \tag{12}$$

[1] The comprehensive exposition announced at the end of that subsection in [F3] exists only in manuscript (1971).

Since G is lower semicontinuous it follows that (λ_n) is vaguely bounded, and we may therefore assume that (λ_n) converges vaguely to a measure λ, necessarily of class $\mathscr{E}^+$; and consequently $G\lambda \leqslant \liminf_n G\lambda_n$ pointwise in X. We show that, actually,

$$G\lambda = \liminf_n G\lambda_n \qquad \text{q.e. in } X \tag{13}$$

(cf. [Ca1, Théorème 6] for the newtonien potentiel). Choquet's theorem on capacitability e.g. of Borel sets [Ch1] applies in the present setting, [FOT, Theorem 2.1.1]; and (13) will therefore follow from the integrated form of the remaining inequality "$\geqslant$":

$$\langle\lambda, \mu\rangle \geqslant \liminf_n \langle\lambda_n, \mu\rangle \qquad \text{for every } \mu \in \mathscr{E}^+, \tag{14}$$

to be established now. As shown in the proof of Theorem 2, μ can be approximated in energy norm by signed measures $\mu' \in \mathscr{E}$ with $G\mu' \in \mathscr{D}^+ \cap \mathscr{K}(X)$, and hence

$$\langle\lambda, \mu'\rangle = \int G\mu'\, d\lambda = \lim_n \int G\mu'\, d\lambda_n = \lim_n \langle\lambda_n, \mu'\rangle,$$

and so

$$\langle\lambda, \mu\rangle \geqslant \langle\lambda, \mu'\rangle - \|\lambda\|\, \|\mu - \mu'\| \geqslant \liminf_n \langle\lambda_n, \mu\rangle - 2c\|\mu - \mu'\|$$

by (12), with $\|\mu - \mu'\|$ as small as we please. This establishes (14) and hence (13), which in turn implies that $G\lambda = \liminf_n G\lambda_n \geqslant \liminf_n f_n = f$ τ-a.e., and so indeed $G\lambda \geqslant f$ q.e., by quasicontinuity of $G\lambda$ and f, [De1], [De2, Théorème, p. 170], or [FOT, Theorem 2.1.3]. This establishes (8) for arbitrary $f \in \mathscr{D}^+$.

Since λ does not charge the polar sets, (8) implies $G\lambda \geqslant f$ λ-a.e. Thus (9) will follow from its integrated form, or just from the inequality

$$\int G\lambda\, d\lambda \leqslant \int f\, d\lambda. \tag{15}$$

To obtain (15) we show that $G\lambda_n \to G\lambda$ weakly in the Hilbert space $\mathscr{D}$, i.e., in view of (5),

$$\int u\, d\lambda_n = E(G\lambda_n, u) \to E(G\lambda, u) = \int u\, d\lambda \qquad \text{for } u \in \mathscr{D}. \tag{16}$$

Because $\lambda_n \to \lambda$ vaguely, (16) holds for $u \in \mathscr{D} \cap \mathscr{K}(X)$ (dense in $\mathscr{D}$), and hence for any $u \in \mathscr{D}$, in view of (12). Since moreover $f_n \to f$ in $\mathscr{D}$, it is well known that (16) extends to

$$\int f_n\, d\lambda_n = E(G\lambda_n, f_n) \to E(G\lambda, f) = \int f\, d\lambda.$$

As shown in the beginning of the proof, (9) holds with λ replaced by λ_n and f by f_n; and hence $\int G\lambda_n\, d\lambda_n = \int f_n\, d\lambda_n$. This leads to (15), and therefore to (9), again for any $f \in \mathscr{D}^+$:

$$\int G\lambda\, d\lambda \leqslant \liminf_n \int G\lambda_n\, d\lambda_n = \lim_n \int f_n\, d\lambda_n = \int f\, d\lambda.$$

Any $\lambda \in \mathscr{E}^+$ satisfying (8) and (9) also satifies (10) and (11). To see this, let $\mu \in \mathscr{E}^+$ and $G\mu \geqslant f$ q.e., hence λ-a.e.; then $G\mu \geqslant G\lambda$ λ-a.e., by (9). By Lemma 1 below it follows that $G\mu \geqslant G\lambda$ everywhere, and so (8), (9) imply (10), which in turn implies (11) because $\int G\mu\, d\mu \geqslant \int G\lambda\, d\mu = \int G\mu\, d\lambda \geqslant \int G\lambda\, d\lambda$. Note that (10) implies uniqueness of λ, a measure $\lambda \in \mathscr{E}^+$ being determined by its potential $G\lambda$. Likewise, (11) implies uniqueness of λ, the convex subset $\{\mu \in \mathscr{E}^+ : G\mu \geqslant f \text{ q.e.}\}$ of the prehilbert space $\mathscr{E}$ having at most one element of minimal norm. □

Lemma 1 (Cartan's maximum principle, cf. [Ca2, Proposition 2]). *Let $\lambda \in \mathscr{E}^+$, and let $s \geqslant 0$ be superharmonic on X. If $G\lambda \leqslant s$ λ-a.e., then $G\lambda \leqslant s$ everywhere.*

Proof. The pointwise minimum $\min\{G\lambda, s\}$ is a potential $G\nu \leqslant G\lambda$, and hence $\int G\nu\, d\nu \leqslant \int G\lambda\, d\nu = \int G\nu\, d\lambda \leqslant \int G\lambda\, d\lambda$, and so $\nu \in \mathscr{E}^+$. Furthermore, $G\nu = G\lambda$ λ-a.e.; and consequently, as in [Ca1], $\int (G\lambda - G\nu)(d\lambda - d\nu) \leqslant 0$, i.e., $\|\lambda - \nu\|^2 \leqslant 0$, $\lambda = \nu$, and so indeed $G\lambda = G\nu \leqslant s$ everywhere. □

Theorem 3. *For any relatively compact open set $\Omega \subset X$ and any function $f \in \mathscr{D}$, the harmonic restriction to Ω of the variational solution u from Proposition 1 is also the solution H_f^Ω in the sense of Perron–Wiener–Brelot with prescribed boundary function $f_{|\partial\Omega}$. Consequently, by Theorem 1,*

$$u(x) \to f(x_0) \qquad \text{for } x \to x_0,\ x \in \Omega$$

for every regular boundary point x_0 for Ω at which f is continuous.

Proof. Since $f^+, f^- \in \mathscr{D}$, we may assume that $f \geqslant 0$. By Proposition 2 there exists $\lambda \in \mathscr{E}^+$ such that $p := G\lambda \geqslant f$ q.e., and even everywhere (after adding a suitable potential). In particular, p is *semibounded* in the sense of Brelot [Br5, p. 41], or see [F4, Section 2]. It follows that

$$\overline{H}{}_f^\Omega \leqslant \overline{H}{}_p^\Omega = R_p^{\complement\Omega} \qquad \text{on } \Omega,$$

the equality by [Br2, Proposition 5, p. 85], or see [CC, Proposition 5.3.3]. Denoting by μ_x^Ω the harmonic measure for Ω at x, we thus have

$$0 \leqslant \int_* f\, d\mu_x^\Omega \leqslant \int^* f\, d\mu_x^\Omega = \overline{H}{}_f^\Omega(x) \leqslant R_p^{\complement\Omega}(x) < \infty, \tag{17}$$

the equality by [Br2, pp. 111–112], and finiteness by harmonicity of $R_p^{\complement\Omega}$ in Ω. Furthermore, f is μ_x^Ω-measurable for each $x \in \Omega$. Indeed, f is continuous relative to the complements of open sets ω_n with $\operatorname{cap}\omega_n \to 0$, and hence $e := \bigcap_n \omega_n$ is polar. Write $f_n = f$ in $\complement\omega_n$ and $f_n = 0$ in ω_n. Then f_n is Borel measurable, and so is therefore $f' = \sup_n f_n$ (pointwise). Moreover, $f' = f$ in $\complement e$, and $\mu_x^\Omega(e) = 0$ by [CC, Corollary 6.2.4], so f is μ_x^Ω-measurable; and μ_x^Ω-integrable by (17). Consequently, $\overline{H}{}_f^\Omega(x) = \underline{H}{}_f^\Omega(x) = \int f\,d\mu_x^\Omega$. Thus the PWB-solution H_f^Ω exists and is harmonic in Ω.

To prove that H_f^Ω is also the variational solution, it remains according to Proposition 1 to verify that the extension u of H_f^Ω by f in $\complement\Omega$ is quasicontinuous, or equivalently: *finely continuous q.e.* in X.[2] Thus the quasicontinuous function $f\colon X \to \mathbf{R}$ is finely continuous off some polar (hence finely closed) set e_1. The irregular points of $\partial\Omega$ form another polar set e_2, [Br2, Theorem 32]; and $e_3 := \{p = +\infty\}$ is likewise polar. We prove below that, for any $x_0 \in \partial\Omega$ not in the polar set $e := e_1 \cup e_2 \cup e_3$,

$$H_f^\Omega(x) \to f(x_0) \qquad \text{for } x \to x_0 \textit{ finely},\ x \in \Omega. \tag{18}$$

Because H_f^Ω is harmonic and therefore continuous (in particular finely continuous) in Ω, it will follow from (18) that the said extension u of H_f^Ω is indeed finely continuous at any point $x \in X \setminus e$ (whether $x \in \Omega$, $x \in \partial\Omega$, or $x \in \complement\overline{\Omega}$).

For the proof of (18) we first approximate f from below by upper semicontinuous (u.s.c.) functions h of compact support such that $0 \leqslant h \leqslant f$. Because $f \leqslant p$, the function $q := \widehat{R}_{f-h}$ is a potential satisfying $f - h \leqslant q \leqslant \widehat{R}_{p-h} \leqslant p$ q.e., and hence, for $x \in \Omega$,

$$\overline{H}{}_{f-h}^\Omega(x) \leqslant \overline{H}{}_q^\Omega(x) = \widehat{R}_q^{\complement\Omega}(x) \to \widehat{R}_q^{\complement\Omega}(x_0) = q(x_0)$$

as $x \to x_0$ finely (by definition of the fine topology), the latter equality by [Br2, Corollary (ii), p. 115], $x_0 \notin e_2$ being regular. Because $f = h + (f - h)$ it follows, for any function $\varphi \in \mathscr{K}(X)$ with $\varphi \geqslant h$, that $\overline{H}{}_f^\Omega \leqslant H_\varphi^\Omega + \overline{H}{}_{f-h}^\Omega$, and hence

$$\begin{aligned}\operatorname*{fine\,lim\,sup}_{\Omega\ni x\to x_0} H_f^\Omega(x) &\leqslant \lim_{\Omega\ni x\to x_0} H_\varphi^\Omega(x) + q(x_0)\\ &= \varphi(x_0) + \widehat{R}_{f-h}(x_0) \leqslant f(x_0) + 2\varepsilon.\end{aligned}$$

Indeed, for given $\varepsilon > 0$, φ can be chosen so that $\varphi(x_0) \leqslant h(x_0) + \varepsilon \leqslant f(x_0) + \varepsilon$. Moreover, f is finely u.s.c. q.e., and at $x_0 \in \complement e_1$ in particular. The u.s.c. minorant

[2] Fine continuity means continuity relative to the Cartan fine topology, the weakest topology on X in which every local superharmonic function is continuous. The stated equivalence, essentially going back to Choquet [Ch3], has been established in more generality in [Br6, Theorems IV.7 and IV.8] and in [F2, Theorem 5.5].

h can therefore be chosen so that ($\widehat{R}_{f-h}(x_0) \leqslant$) $R_{f-h}(x_0) < \varepsilon$ according to the former assertion of [F4, Theorem 6.5(c)], established by application of [Br3].

Next, we approximate f from above by lower semicontinuous (l.s.c.) functions g such that $f \leqslant g \leqslant p$. Then $r := \widehat{R}_{g-f}$ satisfies $g - f \leqslant r$ q.e., and hence, for $x \in \Omega$,

$$\overline{H}\,^{\Omega}_{g-f}(x) \leqslant \overline{H}\,^{\Omega}_{r}(x) = \widehat{R}^{\complement\Omega}_{r}(x) \to \widehat{R}^{\complement\Omega}_{r}(x_0) = r(x_0)$$

as $x \to x_0$ finely, $x_0 \notin e_2$ being regular. Because $f = g - (g - f)$ in $\complement e_3$ it follows, for any $\varphi \in \mathscr{K}^+(X)$ with $\varphi \leqslant g$, that $\underline{H}^{\Omega}_{f} \geqslant H^{\Omega}_{\varphi} - \overline{H}\,^{\Omega}_{g-f}$, and hence

$$\begin{aligned}\operatorname*{fine\,lim\,inf}_{\Omega \ni x \to x_0} H^{\Omega}_{f}(x) &\geqslant \lim_{\Omega \ni x \to x_0} H^{\Omega}_{\varphi}(x) - r(x_0) \\ &= \varphi(x_0) - \widehat{R}_{g-f}(x_0) \geqslant f(x_0) - 2\varepsilon.\end{aligned}$$

Indeed, since $g(x_0) \leqslant p(x_0) < +\infty$, φ can be chosen so that $\varphi(x_0) \geqslant g(x_0) - \varepsilon$. Furthermore, f is finely l.s.c. q.e., and at x_0 in particular. The l.s.c. majorant g can therefore be chosen so that ($\widehat{R}_{g-f}(x_0) \leqslant$) $R_{g-f}(x_0) < \varepsilon$ according to [F4, Remark, p. 51]. Since $\varepsilon > 0$ is arbitrary, we have altogether established (18), and thereby completed the proof that the variational solution is also the PWB-solution. The final assertion of Theorem 3 therefore follows from Theorem 1 because $f \leqslant p$. □

4. The Dirichlet problem for harmonic maps

In this final section we shall complete the results on continuity of the variational solution up to the boundary, recently obtained in [F9, Theorems 1 and 3], for harmonic maps into spaces of nonpositive curvature. By application of Theorem 3 above (and hence of Theorem 1) we show that *the hypothesis of boundedness of the prescribed map*, imposed in the quoted results from [EF], *can be omitted.*

The source space is an *admissible Riemannian polyhedron* (X, g), where the Riemannian metric g is *simplexwise smooth*, i.e., g is defined and smooth on each topdimensional open simplex s of X, with $g_{|s}$ extending smoothly to a nondegenerate Riemannian metric on the affine m-space containing s, see [EF, Chapter 4]. Such a polyhedron (X, g) is a particular case of a hypoelliptic Dirichlet space as specified in Section 2; see [EF, Chapter 7], where the Dirichlet space $L^{1,2}_0(X)$ replaces the domain $\mathscr{D}$ of the Dirichlet form E. The reader may well think of the particular case of a Riemannian *manifold* (X, g), possibly with boundary.

We require that (X, g) satisfy the *Poincaré inequality*

$$\int_X u^2\, d\mu \leqslant C \int_X |\nabla u|^2\, d\mu \qquad \text{for all } u \in W^{1,2}_0(X) \tag{19}$$

and some constant C; here μ denotes the Riemannian volume measure on (X, g). For example, if $(\overline{X}, g)$ denotes a compact admissible Riemannian polyhedron with boundary $b\overline{X} \neq \varnothing$ (cf. [EF, p. 45]) then (X,g) with $X = \overline{X} \setminus b\overline{X}$ is an admissible Riemannian polyhedron satisfying the Poincaré inequality (19), [F8, Lemma 1(c)].

For target we first take a *simply connected complete geodesic space* (Y, d_Y) *of nonpositive curvature* in the sense of A. D. Alexandrov [Al1], [Al2], cf. [EF, Section 2, Geodesic spaces]. Denote by $\mathscr{E}(X,Y)$ the class of all maps $\varphi\colon X \to Y$ of finite *energy* $E(\varphi)$ in the sense of Korevaar and Schoen [KS] (with $p=2$) for a Riemannian manifold (X,g), and extended in [EF, Chapter 9] to the present case of an admissible Riemannian polyhedron. [3] Every map of class $\mathscr{E}_{\mathrm{loc}}(X,Y)$ (i.e., locally of finite energy) has a quasicontinuous version [F7]. Every locally E-minimizing map (of class $\mathscr{E}_{\mathrm{loc}}(X,Y)$) has a (unique) Hölder continuous version, as shown in [F6, Theorem 1] (sharpening [EF, Theorem 10.1]). Accordingly, we define a *harmonic map* $X \to Y$ in the present setting as a continuous locally E-minimizing map of class $\mathscr{E}_{\mathrm{loc}}(X,Y)$. This will be applied with X replaced by (the components of) an open subset of X.

Given a relatively compact open set $\Omega \subset X$ and a map $\psi \in \mathscr{E}(X,Y)$, write

$$\begin{aligned}\mathscr{E}_\psi(\Omega,Y) &= \bigl\{\varphi \in \mathscr{E}(X,Y) : d_Y(\varphi,\psi) \in W_0^{1,2}(\Omega)\bigr\}\\ &= \bigl\{\varphi \in \mathscr{E}(X,Y) : \varphi = \psi \text{ q.e. in } X \setminus \Omega\bigr\},\end{aligned} \tag{20}$$

the equality being a consequence of (19) and the *spectral synthesis* theorem of Beurling and Deny, see [De2, pp. 168, 172], [FOT, Chapter 2]. Much as in Proposition 1 and Theorem 3 (for functions) we have for maps $\varphi\colon X \to Y$

Theorem 4. (a) *With (Y,d_Y) as described, $\mathscr{E}_\psi(\Omega,Y)$ has a unique element φ of least energy, and φ is the only map of class $\mathscr{E}_\psi(\Omega,Y)$ which is harmonic in Ω.*

(b) *If $\psi_{|\partial\Omega}$ is continuous at some regular point $x_0 \in \partial\Omega$, then*

$$\varphi(x) \to \psi(x_0) \qquad \text{for } x \to x_0,\ x \in \Omega.$$

Proof. Part (a) was established in the proof of [F9, Theorem 1], while (b) was proved there under the additional hypothesis that the prescribed map ψ be *bounded*. Here are the changes in the proof of [F9, Theorem 1(b)] which are needed in order to cover the general case of (b). We redefine φ in a polar set so that $\varphi = \psi$ *everywhere* in $\complement\Omega$, cf. (20). The function

$$f(x) = d_Y\bigl(\varphi(x), \varphi(x_0)\bigr), \qquad x \in X,$$

of class $\mathscr{E}(X,\mathbf{R}) = L^{1,2}(X)$ (in view of [EF, Corollaries 9.1 and 9.2] and because $d_Y\bigl(\,\cdot\,,\varphi(x_0)\bigr)$ is Lipschitz) need not be bounded. Choose $g \in \mathrm{Lip}_c(X)$ with $g=1$ in a relatively compact domain $X_0 \supset \overline{\Omega}$, and replace f by $gf \in W_c^{1,2}(X) \subset L_0^{1,2}(X)$. Theorem 3 above then applies to the truncated function f, taking

[3] In [EF], the present $\mathscr{E}(X,Y)$ is denoted $W^{1,2}(X,Y)$.

$\mathscr{D} = L_0^{1,2}(X)$, and shows that the variational solution u in Ω (with prescribed function f) has the property

$$u(x) \to f(x_0) = 0 \qquad \text{for } x \to x_0\,,\ x \in \Omega\,. \tag{21}$$

In the paragraph containing [F9, (3.5)] this was established for *bounded* ψ, and hence bounded f, by application of results from [F4] about the *fine Dirichlet problem* (for functions). These results, however, require that $f \leqslant p$ for some *finite* semibounded potential p, cf. [F4, Section 14.3], and that need not be fulfilled in the present context where f may be unbounded. For that reason we modify the proof of [F9, Theorem 1(b)] so that—rather than drawing on the theory of finely harmonic functions—we employ Theorem 3 above (including the limit property (21)), which was established in Section 3 by application merely of a key to that theory, notably [F4, Theorem 6.5(c)].

Recall from [EF, Lemma 10.2] that f is weakly subharmonic (i.e., an E-subsolution) in Ω. It follows by [EF, Theorem 5.2] that $f \leqslant u$ μ-a.e. in Ω, and indeed everywhere in Ω because u, φ, and hence f are continuous in Ω. Consequently,

$$0 \leqslant d_Y\big(\varphi(x), \varphi(x_0)\big) = f(x) \leqslant u(x) \to 0 \qquad \text{for } x \to x_0,$$

showing that indeed $\varphi(x) \to \varphi(x_0) = \psi(x_0)$ as $x \to x_0$, $x \in \Omega$. □

Remark 3. As it might be expected, the variational solution φ in Theorem 4 depends only on the restriction of the prescribed map ψ: $X \to Y$ to $\partial\Omega$. Indeed, for any $\psi' \in \mathscr{E}(X,Y)$ and corresponding solution $\varphi' \in \mathscr{E}_{\psi'}(\Omega, Y)$, the quasicontinuous function $v := d_Y(\varphi, \varphi')$ of class $\mathscr{E}(X, \mathbf{R}) \subset W^{1,2}_{\mathrm{loc}}(X)$ (because $d_Y \in \mathrm{Lip}(Y \times Y)$) is subharmonic in Ω, [F8, Section 4, Proof of (b)]. After truncation (not affecting v in $\overline{\Omega}$) we may assume that $v \in W_0^{1,2}(X) \subset L_0^{1,2}(X)$. Then $v \leqslant G\lambda$ q.e. in X for some $\lambda \in \mathscr{E}^+$, by Proposition 2; and the potential $G\lambda$ is semibounded, cf. e.g. [F4, Theorem 2.6]. If $\psi = \psi'$ q.e. on $\partial\Omega$ then $v = 0$ q.e. on $\partial\Omega$, and hence $v = 0$ everywhere in Ω according to the fine boundary minimum principle [F4, Theorem 9.1] applied to the superharmonic (hence finely superharmonic [F4, Theorem 8.7]) function $-v$ on the open (hence finely open) set $\Omega \subset X$. Thus $\varphi = \varphi'$ in Ω.

Remark 4. The requirement in Theorem 4(b) that x_0 be regular is necessary, even if $Y = \mathbf{R}$ in view of Theorem 3, taking for ψ a bounded strict potential of class $L^{1,2}(X)$, cf. [CC, Proposition 7.2.1].

Next, let instead (Y, d_Y) be a complete geodesic space of *curvature bounded from above.* By rescaling the metric d_Y we arrange that Y has curvature $\leqslant 1$. Let $B = B_Y(q, R)$ be a closed geodesically convex ball in Y of radius $R < \frac{1}{2}\pi$ (in a sense best possible, cf. [HKW, Section 6], or see [EF, Example 12.3]), satisfying bipoint uniqueness, i.e., there is only one (minimizing) geodesic segment in Y joining two given points of B, and this segment varies continuously with its endpoints (in the uniform topology on paths). It follows that every closed ball in B is convex, and hence satisfies bipoint uniqueness, along with B. For details, see [F8, text following Theorem 3]. Then Theorem 4 above carries over, with Y replaced by the ball B:

Theorem 5. (a) *With (Y, d_Y) and B as described, $\mathscr{E}_\psi(\Omega, B)$ has a unique element φ of least energy, and φ is the only map of class $\mathscr{E}_\psi(\Omega, B)$ which is locally E-minimizing in Ω.*

(b) *If $\psi_{|\partial\Omega}$ is continuous at some regular point $x_0 \in \partial\Omega$ then there is a polar set $Z \subset \Omega$ such that*

$$\varphi(x) \to \psi(x_0) \qquad \text{for } x \to x_0,\ x \in \Omega \setminus Z.$$

The variational solution φ in Theorem 5 has a Hölder continuous and hence harmonic version, at least if $R < \frac{1}{4}\pi$, or if Y is locally compact, [F6, Theorem 2]. In (b) we may then take $Z = \varnothing$, as in Theorem 4.

This companion to Theorem 4 was established in [F9, Theorem 2]. We bring a variant proof of (b).

Proof of Theorem 5(b). Again we may assume that $\varphi = \psi$ *everywhere* in $X \setminus \Omega$. Fix $\varrho > 0$ with $R + \varrho < \frac{1}{2}\pi$; if $R < \frac{1}{4}\pi$ take $\varrho = \frac{1}{4}\pi$, hence $R < \varrho$. Consider a point $y \in B_Y(q, \varrho)$, and note that $d_Y\big(\varphi(x), y\big) \leqslant R + \varrho < \frac{1}{2}\pi$ for $x \in X$. The function $f\colon X \to \mathbf{R}$ defined by

$$f(x) = \cos d_Y\big(\varphi(x_0), y\big) - \cos d_Y\big(\varphi(x), y\big)$$

is of class $W^{1,2}(X)$ by [EF, Corollaries 9.1 and 9.2] because $\cos d_Y(\,\cdot\,, y) \in \mathrm{Lip}(B)$; and $f_{|\partial\Omega}$ is continuous at x_0 with the value 0. Theorem 3 applies to f (after truncation, as in the proof of Theorem 4(b)), and the corresponding variational solution u has the properties $u - f \in W_0^{1,2}(\Omega)$, u continuous in Ω, and

$$u(x) \to f(x_0) \qquad \text{for } x \to x_0,\ x \in \Omega. \tag{22}$$

According to [EF, (10.19)] with q replaced by y (noting that $B \subset B_Y(y, R + \varrho)$ with $R + \varrho < \frac{1}{2}\pi$), f is weakly subharmonic in Ω, hence $f \leqslant u$ in $\Omega \setminus Z_y$ for some polar set $Z_y \subset \Omega$, [EF, Theorem 5.2], [De1, Théorème 5]. It follows by (22) that

$$\cos d_Y\big(\varphi(x_0), y\big) - \cos d_Y\big(\varphi(x), y\big) = f(x) \leqslant u(x) \to f(x_0) = 0$$

for $x \in \Omega \setminus Z_y$. Because cos is continuous and decreasing on $[0, R + \varrho]$ it follows that

$$\limsup_{\Omega \setminus Z_y \ni x \to x_0} d_Y\big(\varphi(x), y\big) \leqslant d_Y\big(\varphi(x_0), y\big). \tag{23}$$

If $\varphi(x_0) \in B_Y(q, \varrho)$ we may take $y = \varphi(x_0)$ above, and it follows in that case from (23) that $\varphi(x) \to \varphi(x_0)$ as $x \to x_0$ in $\Omega \setminus Z_y$. This applies in particular if $R \leqslant \varrho$ (e.g. if $R < \frac{1}{4}\pi$), so we may suppose that $R > \varrho$. The same inductive scheme as in [F8, Section 8] then serves to prove that indeed $\varphi(x) \to \varphi(x_0)$ as $x \to x_0$ in $\Omega \setminus Z$ for a certain polar set $Z \subset \Omega$, a finite union of sets like Z_y above. □

Remark 5. In Theorem 5, φ depends only on $\psi_{|\partial\Omega}$. For the proof we refer freely to [F8, Theorem 3 and the paragraph containing (2.7)]. For any $\psi' \in \mathscr{E}(X,Y)$ with solution $\varphi' \in \mathscr{E}_{\psi'}(\Omega,Y)$, the function $v = \theta(\varphi,\varphi') \in W^{1,2}_{\mathrm{loc}}(X)$ is weakly subharmonic in Ω, though for a Dirichlet form obtained by multiplying the previous volume measure μ by a certain scalar $a > 0$ for which a and $1/a$ are bounded. This substitution causes no problems, and the argument in Remark 3 carries over.

Remark 6. Theorems 4 and 5 have companions in which the geodesic space target (Y, d_Y) is specialized to be a simply connected complete *Riemannian manifold* (Y,h) of nonpositive Alexandrov curvature as above, or equivalently: of nonpositive *sectional* curvature. The energy of maps $\varphi\colon X \to Y$ can then be defined in an alternative and equivalent way (see [EF, Definition 9.2, Lemma 9.3, and Theorem 9.2] [F8, Proposition 2]) which remains meaningful even when we now drop the previous requirement that the Riemannian metric g on the source polyhedron X be simplexwise smooth; instead we merely require that g be bounded and *measurable*, with elliptic bounds on each simplex of dimension $m = \dim X$, cf. [EF, pp. 47–48]. *Theorems* 4 *and* 5 *remain valid* when we keep the hypotheses that the Poincaré inequality (19) holds on (X,g), that $\Omega \Subset X$ is open, and that the prescribed map ψ is of class $\mathscr{E}(X,Y)$, resp. $\mathscr{E}(X,B)$, (now in terms of the alternative concept of energy). For bounded ψ (in the former case of nonpositive curvature) this was proved in [F9, Theorem 3]. The proof given there remains valid for possibly unbounded ψ when the same change is made in the proof of (b), involving the use of the present Theorem 3, as described above for Theorem 4(b). For upper bounded curvature, see [F9, Theorem 4]; alternatively, adapt the above proof of Theorem 5(b). Also note that, under different hypotheses on X and the Riemannian manifold Y, Picard [Pi] obtained by probabilistic methods continuity up to the boundary of any bounded quasicontinuous map $X \to Y$ which is harmonic in Ω as in Remark 7 below, and everywhere continuous relative to $\partial\Omega$.

Remark 7. A particular case of Theorem 4 and of Theorem 5, now with X *compact*, was obtained in [F8, Theorems 1 and 3], where $\Omega = X \setminus bX$, $bX \neq \varnothing$; then every point of $\partial\Omega = bX$ is regular. Similarly, the companions in Remark 6 above (with manifold target) contain [F8, Theorems 4 and 5] (where likewise $\Omega = X \setminus bX \neq X$).

References

[AG] Armitage, D. H., and S. J. Gardiner: Classical Potential Theory. - Springer, New York, 2001.

[Ar] Armitage, D. H.: A strong type of regularity for the PWB solution of the Dirichlet problem. - Proc. Amer. Math. Soc. 61, 1976, 285–289.

[Al1] Alexandrov, A. D.: A theorem on triangles in a metric space and some of its applications. - Trudy Mat. Inst. Steklov 38, 1951, 5–23 (Russian).

[Al2] Alexandrov, A. D.: Über eine Verallgemeinerung der Riemannschen Geometrie. - Schr. Forschungsinst. Math., Berlin 1, 1957, 33–84.

[Bou] Bouligand, G.: Sur le problème de Dirichlet. - Ann. Soc. Polonaise Math. 4, 1926, 59–112.

[Br1] Brelot, M.: Sur la mesure harmonique et le problème de Dirichlet. - Bull. Sci. Math. 69, 1945, 153–156.

[Br2] Brelot, M.: Lectures on Potential Theory. - Tata Institute of Fundamental Research, Bombay, 1960. Reissued 1967.

[Br3] Brelot, M.: Recherches axiomatiques sur un théorème de Choquet concernant l'effilement. - Nagoya Math. J. 30, 1967, 39–46.

[Br4] Brelot, M.: Capacity and balayage for decreasing sets. - In: Proc. Fifth Berkeley Symp. Math. Stat. and Prob. 1965–66, Univ. California Press, Berkeley, 1967, 279–293.

[Br5] Brelot, M.: La topologie fine en théorie du potentiel. - In: Symposium on Probability Methods in Analysis, Loutraki (Greece), Lecture Notes in Math. 31, Springer, Berlin, 1967, 36–47.

[Br6] Brelot, M.: On Topologies and Boundaries in Potential Theory. - Lecture Notes in Math. 175, Springer, Berlin, 1971.

[Ca1] Cartan, H.: Théorie du potentiel newtonien: énergie, capacité, suites de potentiels. - Bull. Soc. Math. France 73, 1945, 74–106.

[Ca2] Cartan, H.: Théorie générale du balayage en potentiel newtonien. - Ann. Univ. Grenoble 22, 1946, 221–280.

[CC] Constantinescu, C., and A. Cornea: Potential Theory on Harmonic Spaces. - Grundlehren Math. Wiss. 158, Springer, Berlin, 1972.

[Ch1] Choquet, G.: Theory of capacities. - Ann. Inst. Fourier (Grenoble) 5, 1953–54, 131–295.

[Ch2] Choquet, G.: Les noyaux réguliers en théorie du potentiel. - C. R. Acad. Sci. Paris 243, 1956, 635–638.

[Ch3] Choquet, G.: Sur les points d'effilement d'un ensemble. - Ann. Inst. Fourier (Grenoble) 9, 1959, 91–101.

[De1] Deny, J.: Théorie de la capacité dans les espaces fonctionnels. - Sém. Théorie du Potentiel, Paris, 9^{e} année, n^{o} 1, 1964–65.

[De2] Deny, J.: Méthodes hilbertiennes en théorie du potentiel. - In: Potential Theory, C.I.M.E, Stresa 1969, Ed. Cremonese, Roma, 1970, 121–201.

[Ed] Edwards, R. E.: Cartan's balayage theory for hyperbolic Riemann surfaces. - Ann. Inst. Fourier (Grenoble) 8, 1958, 263–272.

[EF] Eells, J., and B. Fuglede: Harmonic Maps Between Riemannian Polyhedra. - Cambridge Tracts in Math. 142, Cambridge Univ. Press, 2001.

[ES] Eells, J., and J. H. Sampson: Harmonic mappings of Riemannian manifolds. - Amer. J. Math. 86, 1964, 109–160.

[F1] Fuglede, B.: On the theory of potentials in locally compact spaces. - Acta Math.103, 1960, 139–215.

[F2] Fuglede, B.: The quasi topology associated with a countably subadditive set function. - Ann. Inst. Fourier (Grenoble) 21:1, 1971, 123–169.

[F3] Fuglede, B.: Capacity as a sublinear functional generalizing an integral. - Mat. Fys. Medd. Dan. Vid. Selsk. 38:7, 1971.

[F4] Fuglede, B.: Finely Harmonic Functions. - Lecture Notes in Math. 289, Springer, Berlin, 1972.

[F5] Fuglede, B.: Fonctions BLD et fonctions finement surharmoniques. - Sém. Théorie du Potentiel Paris 6, 1982, Lecture Notes in Math. 906, Springer, Berlin, 126–157.

[F6] Fuglede, B.: Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature. - Calc. Var. 16, 2003, 375–403.

[F7] Fuglede, B.: Finite energy maps from Riemannian polyhedra to metric spaces. - Ann. Acad. Sci. Fenn. Math. 28, 2003, 433–458.

[F8] Fuglede, B.: The Dirichlet problem for harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature. - Trans. Amer. Math. Soc. 357, 2005, 757–792.

[F9] Fuglede, B.: Dirichlet problems for harmonic maps from regular domains. - Proc. London Math. Soc. 91, 2005, 249–272.

[FOT] Fukushima, M., Y. Oshima, and M. Takeda: Dirichlet Forms and Symmetric Markov Processes. - de Gruyter Stud. Math. 19, de Gruyter, Berlin, 1994.

[FP1] Feyel, D., and A. de La Pradelle: Topologies fines et compactifications associées à certains espaces de Dirichlet. - Ann. Inst. Fourier (Grenoble) 27:4, 1977, 121–146.

[FP2] Feyel, D., and A. de La Pradelle: Construction d'un espace harmonique de Brelot associé à un espace de Dirichlet de type local vérifiant une hypothèse d'hypoellipticité. - Invent. Math. 44, 1978, 109–128.

[He] Hervé, R.-M.: Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel - Ann. Inst. Fourier (Grenoble) 12, 1962, 415–571.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Oxford, New York, 1993.

[HKW] Hildebrandt, S., H. Kaul, and K. O. Widman: An existence theorem for harmonic mappings of Riemannian manifolds. - Acta Math. 138, 1977, 1–16.

[JN] Janssen, K., and Nguyen-Xuan-Loc: On the balayage of Green functions on finely open sets. - Z. Wahrscheinlichkeitstheorie Verw. Geb. 31, 1975, 147–155.

[KS] Korevaar, N. J., and R. M. Schoen: Sobolev spaces and harmonic maps for metric space targets. - Comm. Anal. Geom. 1, 1993, 561–659. Reprinted as Chapter X in Lectures on Harmonic Maps, edited by R. Schoen and S. T. Yau, Conf. Proc. Lecture Notes in Geom. Topology, Vol. II (1997), 204–310.

[La] Lassoued, L.: Espaces de Dirichlet et axiomatique Brelot. - In: Sém. Théorie du Potentiel Paris 5. Lecture Notes in Math. 814, Springer, Berlin, 1980, 117–152.

[Oht] Ohtsuka, M.: Sur un espace complet de mesures positives dans la théorie du potentiel. - Proc. Japan Acad. 32, 1956, 311-313.

[Pi] Picard, J.: The manifold-valued Dirichlet problem for symmetric diffusions. - Potential Analysis 14, 2001, 53–72.

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