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# EQUIVALENCE OF WEAK AND VISCOSITY SOLUTIONS TO THE $p$-LAPLACE EQUATION IN THE HEISENBERG GROUP

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**Abstract.** We prove weak and viscosity solutions to the $p$-Laplace equation in the Heisenberg group coincide by showing that the viscosity sub(super-) solutions coincide with the $p$-sub(super-)harmonic functions from potential theory. We are then able to obtain a comparison principle for viscosity solutions to the $p$-Laplace equation.

## 1. Introduction and motivation

In [JLM], Juutinen, Lindqvist, and Manfredi prove the equivalence of viscosity solutions and weak solutions to the $p$-Laplace equation in $\mathbf{R}^n$, given by

$$-\operatorname{div}(\|Du\|^{p-2}Du) = 0$$

for $1 < p < \infty$. Here, $Du$ denotes the gradient of the real-valued function $u$. The $p$-Laplace equation is a well-known example of a larger class of quasi-linear equations of the form

$$-\operatorname{div}\big(\mathscr{A}_p(u, Du)\big) = 0$$

where $\mathscr{A}_p$ satisfies certain structure conditions. (See [HKM] and [HH] for complete details.) This class of equations plays a major role in non-linear potential theory and has been studied in the Euclidean environment [HKM], Carnot groups [HH], and general metric spaces [KM].

In this paper, we prove the equivalence of the potential-theoretic $p$-harmonic functions and viscosity solutions to the $p$-Laplace equation in the Heisenberg group. (See Section 3 for relevant definitions.) It should be noted that due to the geometry of the Heisenberg group, the method of proof in [JLM] cannot be used, for it relies upon the well-known $C^{1,\alpha}$ regularity of the weak solutions, which is unknown in the Heisenberg group. (Although, for $p$ near 2, this was proved recently by Domokos and Manfredi [DM].) We therefore adopt a different strategy to obtain our results. We begin with a brief review of the Heisenberg group in Section 2, followed by the relevant definitions in Section 3. It is noted that this section highlights the nonlinear potential theory found in [HKM] and its extension to Carnot groups in [HH]. In Section 4, we prove the equivalence of viscosity solutions and $p$-harmonic functions and obtain a comparison principle for viscosity solutions to the $p$-Laplacian.

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## 2. The Heisenberg group

We begin with $\mathbf{R}^{2n+1}$ using the coordinates $(x_1, x_2, \dots, x_{2n}, z)$ and consider the linearly independent vector fields $\{X_i, Z\}$, where the index $i$ ranges from 1 to $2n$, defined by

$$X_i = \begin{cases} \dfrac{\partial}{\partial x_i} - \dfrac{x_{n+i}}{2}\dfrac{\partial}{\partial z}, & \text{if } 1 \le i \le n, \\ \dfrac{\partial}{\partial x_i} + \dfrac{x_{i-n}}{2}\dfrac{\partial}{\partial z}, & \text{if } n < i \le 2n, \end{cases}$$
$$Z = \frac{\partial}{\partial z}\cdot$$

For $i \le j$, these vector fields obey the relations

$$[X_i, X_j] = \begin{cases} Z, & \text{if } j = i+n, \\ 0, & \text{otherwise,} \end{cases}$$

and for all $i$,

$$[X_i, Z] = 0.$$

We then have a Lie algebra denoted $h_n$ that decomposes as a direct sum

$$h_n = V_1 \oplus V_2$$

where $V_1$ is spanned by the $X_i$'s and $V_2$ is spanned by $Z$. We endow $h_n$ with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. The corresponding Lie group is called the general Heisenberg group of dimension $n$ and is denoted by $\mathbf{H}_n$. With this choice of vector fields the exponential map can be used to identify elements of $h_n$ and $\mathbf{H}_n$ with each other via

$$\sum_{i=1}^{2n} x_i X_i + zZ \in h_n \leftrightarrow (x_1, x_2, \dots, x_{2n}, z) \in \mathbf{H}_n.$$

In particular, for any $P$, $Q$ in $\mathbf{H}_n$, written as $P = (x_1, x_2, \dots, x_{2n}, z_1)$ and $Q = (y_1, y_2, \dots, y_{2n}, z_2)$ the group multiplication law is given by

$$P \cdot Q = \left( x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, z_1 + z_2 + \frac{1}{2}\sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \right).$$

The natural metric on $\mathbf{H}_n$ is the Carnot–Carathéodory metric given by

$$d_C(P, Q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt$$

where the set $\Gamma$ is the set of all curves $\gamma$ such that $\gamma(0)=P$, $\gamma(1)=Q$ and $\gamma'(t)\in V_1$. By Chow's theorem (see, for example, [Be]) any two points can be connected by such a curve, which makes $d_C(P,Q)$ a left-invariant metric on $\mathbf{H}_n$. This metric induces a homogeneous norm on $\mathbf{H}_n$, denoted $|\cdot|$, by

$$|P| = d_C(0,P)$$

and we have the estimate

$$|P| \sim \sum_{i=1}^{2n} |x_i| + |z|^{1/2}.$$

This estimate leads us to define the left-invariant gauge $\mathscr{N}$ that is comparable to the Carnot–Carathéodory metric and is given by

$$\mathscr{N}(P) = \left(\left(\sum_{i=1}^{2n} x_i^2\right)^2 + 16z^2\right)^{1/4}.$$

We define the Carnot–Carathéodory balls $B(P,r)$ and the gauge balls $B_{\mathscr{N}}(P,r)$ in the obvious way.

Given a smooth function $u\colon \mathbf{H}_n \mapsto \mathbf{R}$, we define the horizontal gradient by

$$\nabla_0 u = (X_1u, X_2u, \ldots, X_{2n}u),$$

the full gradient by

$$\nabla u = (X_1u, X_2u, \ldots, X_{2n}u, Zu),$$

and the symmetrized horizontal second derivative matrix $(D^2u)^\star$ by

$$\big((D^2u)^\star\big)_{ij} = \tfrac{1}{2}(X_iX_ju + X_jX_iu).$$

Additionally, given a vector field $F=\sum_{i=1}^{2n} f_iX_i + f_{2n+1}Z$, we define the Heisenberg divergence of $F$, denoted $\operatorname{div}_{\mathscr{H}} F$, by

$$\operatorname{div}_{\mathscr{H}} F = \sum_{i=1}^{2n} X_i f_i.$$

A quick calculation shows that when $f_{2n+1}=0$, we have

$$\operatorname{div}_{\mathscr{H}} F = \operatorname{div}_{\mathrm{eucl}} F$$

where $\operatorname{div}_{\mathrm{eucl}}$ is the standard Euclidean divergence. The main operator we are concerned with is the horizontal $p$-Laplacian for $1<p<\infty$ defined by

$$\Delta_p f = \operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2}\nabla_0 f)$$

which is a specific type of operator in an important class of operators in potential theory as detailed in [HH] and [HKM].

A function $f$ is $\mathscr{C}^1_{\mathrm{sub}}$ if $X_if$ is continuous for all $i$ and $f$ is $\mathscr{C}^2_{\mathrm{sub}}$ if $f$ is $\mathscr{C}^1_{\mathrm{sub}}$ and $X_iX_jf$ is continuous for all $i$ and $j$. Using the horizontal gradient, we may also define the Sobolev spaces $\mathrm{W}^{1,\mathrm{p}}$, $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$, etc. in the obvious way. For a more complete treatment of the Heisenberg group, the interested reader is directed to [Be], [B], [F], [FS], [G], [H], [K], [S] and the references therein.

## 3. Notions of solution

In this section, we highlight some results from nonlinear potential theory as detailed in [HKM]. In [HH], many of the Euclidean results were extended into the general setting of Carnot groups. A more complete treatment of viscosity solutions in the Euclidean environment can be found in [CIL] and in the Heisenberg group in [B]. All the results below can be extended into general Carnot groups.

Our main goal is to relate three different notions of solutions to the equation

$$-\Delta_p f = -\operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2}\nabla_0 f) = 0 \tag{3.1}$$

in a bounded domain $\Omega$.

**3.1. Weak solutions.** We begin by defining the concept of weak solutions to equation (3.1). We will actually do more, for we shall define weak solutions to a wider class of equations. Letting $\varepsilon \geq 0$ be a real parameter, we consider equations of the form

$$-\Delta_p f = -\operatorname{div}_{\mathscr{H}}(\|\nabla_0 f\|^{p-2}\nabla_0 f) = \varepsilon \tag{3.2}$$

in a bounded domain $\Omega$. Note that equation (3.1) corresponds to equation (3.2) with $\varepsilon = 0$. We now give the definition of weak solutions.

**Definition 1.** The function $u \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$ is an $\varepsilon$*-weak solution* to equation (3.2) if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle = \varepsilon \int_\Omega \phi \tag{3.3}$$

for all $\phi \in \mathrm{W}^{1,\mathrm{p}}_0(\Omega)$.

A weak solution to equation (3.1) (i.e., a 0-weak solution) is called *p-harmonic*. It is well known that a $p$-harmonic function $u$ has a continuous representative that satisfies

$$\operatorname{osc}_{B_r} u \leq C\left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}_{B_R}$$

when $B_R \subset \Omega$ and $r \leq R$ ([HH, Theorem 4.2]). We note that the constants $C > 0$ and $\alpha > 0$ depend only on the group $\mathbf{H}_n$. We therefore identify $p$-harmonic functions with their continuous representative.

In addition to weak solutions we may define weak supersolutions and weak subsolutions using the following definition (cf. [HKM]).

**Definition 2.** The function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is an *$\varepsilon$-weak supersolution* to equation (3.2) if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \geq \varepsilon \int_\Omega \phi$$

for all non-negative $\phi \in W^{1,p}_0(\Omega)$.

The function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is an *$\varepsilon$-weak subsolution* to equation (3.2) if $-u$ is an $\varepsilon$-weak supersolution, that is, if

$$\int_\Omega \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \leq \varepsilon \int_\Omega \phi$$

for all non-negative $\phi \in W^{1,p}_0(\Omega)$.

Using these definitions for $\varepsilon_1 > \varepsilon_2 \geq 0$, we observe that an $\varepsilon_1$-weak solution is a $\varepsilon_2$-weak supersolution and an $\varepsilon_2$-weak solution is a $\varepsilon_1$-weak subsolution.

It is also well known that 0-weak subsolutions and supersolutions satisfy the following comparison principle

**Lemma 3.1** ([HKM, Lemma 3.18]). *Let $u \in W^{1,p}(\Omega)$ be a weak subsolution to equation* (3.1) *and let $v \in W^{1,p}(\Omega)$ be a weak supersolution to equation* (3.1) *in $\Omega$. If $\gamma \equiv \min\{v-u, 0\} \in W^{1,p}_0(\Omega)$ then $u \leq v$ almost everywhere in $\Omega$.*

We are then able to formulate the existence-uniqueness of $p$-harmonic functions (cf. [HKM, Theorem 3.17], [HH, Section 4.10]).

**Theorem 3.2.** *Given a bounded domain $\Omega$ with boundary data $\Theta \in W^{1,p}(\Omega)$, there is a unique $p$-harmonic function $u$ that satisfies $u - \Theta \in W^{1,p}_0(\Omega)$.*

Using standard techniques in calculus of variations, one can show that $\varepsilon$-weak solutions exist and Lemma 3.1 can be extended to $\varepsilon$-weak solutions. In addition, $\varepsilon$-weak solutions have a continuous representative [CDG] and therefore such solutions will be identified with that representative.

**3.2. $p$-superharmonic functions.** The next class of solutions we wish to consider are $p$-superharmonic functions and $p$-subharmonic functions defined via the following definition.

**Definition 3.** The function $u\colon \Omega \mapsto \mathbf{R}^N \cup \{\infty\}$ is *$p$-superharmonic* if the following hold:

(1) $u$ is lower semicontinuous.
(2) $u$ is not identically infinity in each component of $\Omega$.
(3) For each subdomain $D \subset\subset \Omega$, a $p$-harmonic function $g$ in $D$ that is continuous in $\overline{D}$ with $g \leq u$ on $\partial D$ implies $g \leq u$ in $D$.

A function $u$ is *$p$-subharmonic* if $-u$ is $p$-superharmonic.

The key points of these definitions are that they are based on comparison with $p$-harmonic functions. We then are able to obtain the following comparison principle [KM, Theorem 7.2].

**Lemma 3.3.** *Let $\Omega$ be a bounded domain in $\mathbf{H}_n$. Let $v$ be an $p$-superharmonic function and $u$ be a $p$-subharmonic function in $\Omega$ so that*

$$\limsup_{Q\to P} u(Q) \leq \liminf_{Q\to P} v(Q)$$

*for all $P\in\partial\Omega$ with both sides not simultaneously $-\infty$ or $\infty$. Then $u\leq v$ in $\Omega$.*

We are then able to conclude the following lemma ([HKM, Lemma 7.8]).

**Lemma 3.4.** *A function is $p$-harmonic if an only if it is both $p$-subharmonic and $p$-superharmonic.*

**3.3. Viscosity solutions.** In this subsection, we review the concept of viscosity solution and relate the first two notions of solution to viscosity solutions. Before we begin, we consider equation (3.2) in non-divergence form, namely,

$$(3.4)\qquad -\Big(\|\nabla_0 u\|^{p-2}\operatorname{tr}\big((D^2u)^\star\big) + (p-2)\|\nabla_0 u\|^{p-4}\langle (D^2u)^\star\nabla_0 u, \nabla_0 u\rangle\Big) = \varepsilon.$$

We note that equation (3.4) is degenerate elliptic and proper in the sense of [CIL]. Given the function $u$, we consider the set of functions $\phi$ that touch from below at the point $P_0$. That is, the set $\mathscr{T}\mathscr{B}(u,P_0)$ given by

$$\begin{aligned}\mathscr{T}\mathscr{B}(u,P_0) = \big\{\phi\in\mathscr{C}^2_{\mathrm{sub}}(\Omega) : u(P_0)=\phi(P_0),\\ u(P)>\phi(P) \text{ for } P\neq P_0,\ \nabla_0\phi(P_0)\neq 0\big\}.\end{aligned}$$

We are now able to define the concept of viscosity solutions to equation (3.4).

**Definition 4.** The function $u\colon \Omega\mapsto \mathbf{R}^N\cup\{\infty\}$ is an *$\varepsilon$-viscosity supersolution* to equation (3.4) if the following hold:

(1) $u$ is lower semicontinuous.
(2) $u$ is not identically infinity in each component of $\Omega$.
(3) For $P_0\in\Omega$, $\phi\in\mathscr{T}\mathscr{B}(u,P_0)$ satisfies

$$-\Delta_p\phi(P_0)\geq\varepsilon.$$

A function $u$ is an *$\varepsilon$-viscosity subsolution* to equation (3.4) if $-u$ is an $\varepsilon$-viscosity supersolution. A function $u$ is an *$\varepsilon$-viscosity solution* if it is both an $\varepsilon$-viscosity supersolution and an $\varepsilon$-viscosity subsolution.

We call the collection

$$\big\{\big(\nabla\phi(P_0),(D^2\phi)^\star(P_0)\big):\phi\in\mathscr{TB}(u,P_0)\big\}$$

the subjet of $u$ at $P_0$ and denote it $J^{2,-}u(P_0)$. We define the superjet of $u$ at $P_0$ by $J^{2,+}u(P_0)=-J^{2,-}(-u)(P_0)$. The set-theoretic closure $\bar{J}^{2,-}u(P_0)$ is defined by all pairs $\{(\eta,X)\}$ so that there is a sequence $\{(P_n,\phi_n)\}$ with $\big(\nabla\phi_n(P_n),(D^2\phi_n)^\star(P_n)\big)\in J^{2,-}u(P_n)$ so that $P_n\to P_0, u(P_n)\to u(P_0), \nabla\phi_n(P_n)\to\eta$ and $(D^2\phi_n)^\star(P_n)\to X$. For a more complete discussion of jets, the interested reader is directed to [CIL] for the Euclidean environment and [B] for Heisenberg groups.

Given the viscosity solutions, it is natural to ask how they relate to the previous notions of solutions. It was shown via Lemma 4.1 in [B] that upper(lower) semicontinuous $\varepsilon$-weak sub(super-)solutions are $\varepsilon$-viscosity sub(super-)solutions. In addition, we have the following lemma.

**Lemma 3.5.** *A $p$-sub(super-)harmonic function is a $0$-viscosity sub(super-)solution. Hence, a $p$-harmonic function is a $0$-viscosity solution.*

*Proof.* We will do only the $p$-superharmonic case. We let $u$ be a $p$-superharmonic function in $\Omega$ and choose $P_0\in\Omega$. We let $\phi\in\mathscr{C}^2_{\text{sub}}(\Omega)$ be a function so that $\phi(P_0)=u(P_0)$, $\nabla_0\phi(P_0)\neq 0$, and $u(P)>\phi(P)$ for $P\neq P_0$. If $-\Delta_p\phi(P_0)<0$ then by continuity, there is a small $r>0$ so that $-\Delta_p\phi<0$ in the ball $B(P_0,r)$. Note we also have $\nabla_0\phi(P)\neq 0$ in $B(P_0,r)$. Define the strictly positive number $m\equiv\inf\{u(P)-\phi(P):d_C(P,P_0)=r\}$ and the $\mathscr{C}^2_{\text{sub}}$ function $\Phi\equiv\phi+\frac{1}{2}m$. We therefore have $\Phi$ is a 0-weak subsolution in $B(P_0,r)$. Let $g$ be the unique (continuous) $p$-harmonic function equal to $\Phi$ on $\partial B(P_0,r)$ whose existence is guaranteed by Theorem 3.2. Using the comparison principle (Lemma 3.1), we conclude $\Phi\le g\le u$ in $B(P_0,r)$, contrary to $\Phi(P_0)>u(P_0)$. ▫

We now have existence of all three notions of solutions, but a comparison principle only for the first two. In the next section, we correct this deficiency.

## 4. Equivalence of notions of solution

In this section, our goal is to prove the equivalence of the various notions of solution. As a consequence, we obtain a comparison principle for viscosity solutions to equation (3.4) when $\varepsilon=0$. We recall that $\Omega$ is a bounded domain. Our proof will rely heavily on the Heisenberg geometry.

We begin with a technical lemma whose Euclidean version is Lemma 3.2 in [JLM].

**Lemma 4.1.** *Let $v\in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}$ be a continuous $\varepsilon$-weak solution. Let $P_0\in\Omega$ and let $\phi\in\mathscr{C}^2_{\text{sub}}(\Omega)$ be a function such that $v-\phi$ has a strict local minimum at $P_0$. Then*

$$\limsup_{\substack{P\to P_0\\ P\neq P_0}}\big(-\Delta_p\phi(P)\big)\ge\varepsilon$$

*provided that* $\nabla_0\phi(P_0) \neq 0$ *or* $P_0$ *is an isolated critical point.*

*Proof.* By left translation we may assume $P_0 = 0$ and by adding constants, $\phi(0) = v(0)$. If the conclusion is false, then there is an $r_1 > 0$ so that $\nabla_0\phi(P) \neq 0$ and $-\Delta_p\phi(P) < \varepsilon$ when $0 < \mathscr{N}(P) < r_1$. Because $v - \phi$ has a strict local minimum at $0$, there is an $r_2 > 0$ so that $v(P) > \phi(P)$ when $0 < \mathscr{N}(P) < r_2$. Let $r = \min\{\frac{1}{2}r_1, \frac{1}{2}r_2\}$. Then $\nabla_0\phi(P) \neq 0$, $-\Delta_p\phi(P) < \varepsilon$ and $v > \phi$ when $0 < \mathscr{N}(P) \leq r$. We next define the strictly positive constant $m = \min\{v(P) - \phi(P) : \mathscr{N}(P) = r\}$ and consider the function $\tilde{\phi} = \phi + \frac{1}{2}m$. By construction, $\tilde{\phi} \in \mathscr{C}^2_{\mathrm{sub}}(\Omega)$ and

$$\big(-\Delta_p\tilde{\phi}(P)\big) = \big(-\Delta_p\phi(P)\big).$$

Let $\psi \in C_0^\infty$ be a non-negative test function with compact support contained in the ball $B_{\mathscr{N}}(r)$. Let $0 < \varrho < r$ and consider the annulus $A \equiv B_{\mathscr{N}}(r) \setminus B_{\mathscr{N}}(\varrho)$. Using the identity

$$\operatorname{div}_{\mathscr{H}}(\psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}) = \psi\Delta_p\tilde{\phi} + \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle$$

we obtain

$$\begin{aligned}
\int_A \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle\, dV &= -\int_A \psi\Delta_p\tilde{\phi}\, dV + \int_A \operatorname{div}_{\mathscr{H}}(\psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi})\, dV \\
&\leq \varepsilon\int_A \psi\, dV + \int_{\partial A} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS \\
&\leq \varepsilon\int_{B_{\mathscr{N}}(r)} \psi\, dV - \int_{B_{\mathscr{N}}(\varrho)} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS
\end{aligned}$$

where $\nu$ is the outward unit (Euclidean) normal. We now estimate the last integral.

$$\left|\int_{B_{\mathscr{N}}(\varrho)} \psi\|\nabla_0\tilde{\phi}\|^{p-2}\nabla_0\tilde{\phi}\cdot\nu\, dS\right| \leq \|\psi\|_\infty\|\nabla_0\tilde{\phi}\|_\infty^{p-1}\int_{B_{\mathscr{N}}(\varrho)} dS \lesssim \varrho^{2n+1}.$$

Letting $\varrho \to 0$, we have

$$\int_{B_{\mathscr{N}}(r)} \|\nabla_0\tilde{\phi}\|^{p-2}\langle\nabla_0\tilde{\phi}, \nabla_0\psi\rangle\, dV \leq \varepsilon\int_{B_{\mathscr{N}}(r)} \psi\, dV.$$

Thus, $\tilde{\phi}$ is an $\varepsilon$-weak subsolution.

By the comparison principle for $\varepsilon$-weak solutions, $\tilde{\phi} \leq v$ in $B_{\mathscr{N}}(r)$ because $\tilde{\phi} \leq v$ on $\partial B_{\mathscr{N}}(r)$ by construction. However, we have

$$\tilde{\phi}(0) = \phi(0) + \tfrac{1}{2}m = v(0) + \tfrac{1}{2}m > v(0).$$

This contradiction finishes the proof. □

Note that in the case when $p \geq 2$, by continuity we have $-\Delta_p \phi(P_0) \geq \varepsilon$ and so $\nabla_0 \phi(P) \neq 0$ near $P_0$.

We next consider the function $\varphi$: $\mathbf{H}_n \times \mathbf{H}_n \mapsto \mathbf{R}$ given by

$$\begin{aligned}\varphi(P,Q) &= \frac{1}{m}\sum_{i=1}^{2n}|x_i - y_i|^m + \frac{1}{m}\left|z_1 - z_2 + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})\right|^m \\ &\stackrel{\text{def}}{\equiv} \frac{1}{m}\sum_{i=1}^{2n}|x_i - y_i|^m + \frac{1}{m}|\zeta(P,Q)|^m\end{aligned}$$

for some large positive integer $m \geq 4$. The important properties of $\varphi$ are found in the following lemma.

**Lemma 4.2.** *As above, let $m \geq 4$. Let the vector $\eta$ be given by*

$$\eta = \begin{pmatrix} |x_1 - y_1|^{m-2}(x_1 - y_1) \\ |x_2 - y_2|^{m-2}(x_2 - y_2) \\ \vdots \\ |x_{2n} - y_{2n}|^{m-2}(x_{2n} - y_{2n}) \\ |\zeta(P,Q)|^{m-2}\zeta(P,Q) \end{pmatrix}.$$

*Recall that the differential of left multiplication with respect to $P$, denoted $DL_P$, is given by*

$$\begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}$$

*where the $2n \times 1$ vector $\mathscr{P}$ is given by*

$$\left(-\tfrac{1}{2}x_{n+1}, -\tfrac{1}{2}x_{n+2}, \dots, -\tfrac{1}{2}x_{2n}, \tfrac{1}{2}x_1, \tfrac{1}{2}x_2, \dots, \tfrac{1}{2}x_n\right)^T$$

*with a similar definition for $DL_Q$ using the $2n \times 1$ vector $\mathscr{Q}$ given by*

$$\left(-\tfrac{1}{2}y_{n+1}, -\tfrac{1}{2}y_{n+2}, \dots, -\tfrac{1}{2}y_{2n}, \tfrac{1}{2}y_1, \tfrac{1}{2}y_2, \dots, \tfrac{1}{2}y_n\right)^T.$$

*Define the $(2n+1)\times(2n+1)$ matrix $\mathscr{M}$ by*

$$\mathscr{M}_{ij} = \begin{cases} (m-1)|x_i - y_i|^{m-2}, & i = j,\ 1 \leq i \leq 2n, \\ (m-1)|\zeta(P,Q)|^{m-2}, & i = j = 2n+1, \\ 0, & i \neq j, \end{cases}$$

*and denote Euclidean differentiation with respect to the point $R$ by $D_R$. We then have the following properties:*

(1) $D_P\varphi(P,Q) = DL_Q\eta$, $D_Q\varphi(P,Q) = -DL_P\eta$,

(2) $D_P\eta = \mathscr{M} DL_Q^T,\ D_Q\eta = -\mathscr{M} DL_P^T$,
(3) $DL_P DL_Q = DL_Q DL_P$,
(4) $DL_P D_P\varphi(P,Q) = -DL_Q D_Q\varphi(P,Q) \equiv \Upsilon(P,Q)$,

(5)
$$\begin{aligned} &DL_P(D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T) \\ &\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0cr \end{pmatrix}, \end{aligned}$$

(6)
$$\begin{aligned} &DL_Q(D_{QQ}\varphi(P,Q)DL_Q^T + D_{QP}\varphi(P,Q)DL_P^T) \\ &\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1} \\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}. \end{aligned}$$

(7) *Let* $\xi \in h_n$ *be a vector. Then*

$$\begin{aligned} &\langle D_{PP}\varphi(P,Q)DL_P^T\xi, DL_P^T\xi\rangle + \langle D_{PQ}\varphi(P,Q)DL_Q^T\xi, DL_P^T\xi\rangle \\ &\quad + \langle D_{QP}\varphi(P,Q)DL_P^T\xi, DL_Q^T\xi\rangle + \langle D_{QQ}\varphi(P,Q)DL_Q^T\xi, DL_Q^T\xi\rangle = 0 \end{aligned}$$

(8) *Let* $\xi \in h_n$ *be a vector. We define* $\bar{\xi}$ *to be the projection of* $\xi$ *onto* $V_1$. *That is, if* $\xi = (\xi_1, \xi_2, \dots, \xi_{2n+1})$, *then* $\bar{\xi} = (\xi_1, \xi_2, \dots, \xi_{2n})$. *We then have*

$$\left\| \begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q) \\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q) \end{pmatrix} \begin{pmatrix} DL_P^T\xi \\ DL_Q^T\xi \end{pmatrix} \right\|^2 = \frac{1}{2}\|\bar{\xi}\|^2|\zeta(P,Q)|^{2m-2}.$$

*Proof.* The first three properties are elementary calculations and left to the reader. The fourth follows from the first three. We therefore turn our attention to the last four. Let $M_{PQ}$ be the left-hand side of Property (5). Then,

$$\begin{aligned} M_{PQ} &= DL_P\big(D_P(DL_Q\eta)DL_P^T + D_Q(DL_Q\eta)DL_Q^T\big) \\ &= DL_P\big(DL_Q D_P\eta DL_P^T + D_Q(DL_Q)\eta DL_Q^T + DL_Q D_Q\eta DL_Q^T\big) \\ &= DL_P\big(DL_Q\mathscr{M} DL_Q^T DL_P^T + D_Q(DL_Q)\eta DL_Q^T - DL_Q\mathscr{M} DL_P^T DL_Q^T\big) \\ &= DL_P\big(D_Q(DL_Q)\eta DL_Q^T\big) \end{aligned}$$

and so we are left to compute only the derivative of the matrix $DL_Q$. Knowing the definition of $\eta$ above and the formula for $DL_Q$ as given above, we see that when $1 \le i \le n$, we have $D_{y_i}(DL_Q)$ is a matrix with every entry 0 except for the $(i+n, 2n+1)$ entry, which is $\frac{1}{2}$. When $n < i < 2n$, we have $D_{y_i}(DL_Q)$ has all entries 0 except for the $(i-n, 2n+1)$ entry, which is $-\frac{1}{2}$. Clearly, $D_{z_2}(DL_Q)$ is the 0 matrix. We then compute

$$D_Q(DL_Q)\eta = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q)\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}.$$

We then have

$$\begin{aligned}
&\begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\begin{pmatrix} I_{2n\times 2n} & 0_{2n\times 1} \\ \mathscr{Q}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} I_{2n\times 2n} & \mathscr{P} \\ 0_{1\times 2n} & 1 \end{pmatrix}\begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}
\end{aligned}$$

and Property (5) follows. To prove Property (6), we let $M_{PQ}$ be the left-hand side of the identity. Then,

$$\begin{aligned}
M_{PQ} &= DL_Q\big(-D_Q(DL_P\eta)DL_Q^T + D_P(-DL_P\eta)DL_P^T\big) \\
&= DL_Q\big(-DL_P D_Q\eta DL_Q^T - D_P(DL_P)\eta DL_P^T - DL_P D_P\eta DL_P^T\big) \\
&= DL_Q\big(DL_P\mathscr{M}DL_P^T DL_Q^T - D_P(DL_P)\eta DL_P^T - DL_P\mathscr{M}DL_Q^T DL_P^T\big) \\
&= -DL_Q\big(D_P(DL_P)\eta DL_P^T\big)
\end{aligned}$$

and we compute $D_P(DL_P)\eta$ in the same way as the above computation for $D_Q(DL_Q)\eta$ and arrive at Property (6).

To prove Property (7), we note that the right-hand side can be written as

$$\begin{aligned}
&\langle DL_P(D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T)\xi,\xi\rangle \\
&\quad + \langle DL_Q(D_{QQ}\varphi(P,Q)DL_Q^T + D_{QP}\varphi(P,Q)DL_P^T)\xi,\xi\rangle.
\end{aligned}$$

Using Properties (5) and (6) we see this is zero. Property (7) then follows.

Using the proofs of Properties (5) and (6), we have

$$\begin{aligned}
&\begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q) \\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q) \end{pmatrix} \begin{pmatrix} DL_P^T\xi \\ DL_Q^T\xi \end{pmatrix} \\
&\quad = \begin{pmatrix} (D_{PP}\varphi(P,Q)DL_P^T + D_{PQ}\varphi(P,Q)DL_Q^T)\xi \\ (D_{QP}\varphi(P,Q)DL_P^T + D_{QQ}\varphi(P,Q)DL_Q^T)\xi \end{pmatrix} \\
&\quad = \frac{1}{2}|\zeta(P,Q)|^{m-2}\zeta(P,Q) \begin{bmatrix} \begin{pmatrix} 0_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\xi \\ \begin{pmatrix} 0_{n\times n} & I_{n\times n} & 0_{n\times 1} \\ -I_{n\times n} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 0 \end{pmatrix}\xi \end{bmatrix}.
\end{aligned}$$

We then see that

$$\left\| \begin{pmatrix} D_{PP}\varphi(P,Q) & D_{PQ}\varphi(P,Q) \\ D_{QP}\varphi(P,Q) & D_{QQ}\varphi(P,Q) \end{pmatrix} \begin{pmatrix} DL_P^T\xi \\ DL_Q^T\xi \end{pmatrix} \right\|^2 = \frac{1}{2}\|\bar{\xi}\|^2|\zeta(P,Q)|^{2m-2}$$

and Property (8) is proved. □

We will use the function $\varphi$ above as a penalty function in our proof of a preliminary comparison principle. A key step in the proof will be the twisting of the Euclidean jets into Heisenberg jets.

**Lemma 4.3** ([B, Lemma 3.4]). *Let $DL_{P_0}$ be the differential of the left multiplication map at the point $P_0$, let $J^{2,+}_{\mathrm{eucl}}u(P_0)$ be the traditional Euclidean superjet of $u$ at the point $P_0$ and let $(\eta, X) \in \mathbf{R}^{2n+1} \times S^{2n+1}$. Then,*

$$(\eta, X) \in \bar{J}^{2,+}_{\mathrm{eucl}}u(P_0)$$

*gives the element*

$$\big(DL_{P_0}\eta, (DL_{P_0}\, X\, (DL_{P_0})^T)_{2n}\big) \in \bar{J}^{2,+}u(P_0)$$

*with the convention that for any matrix $M$, $M_n$ is the $n \times n$ principal minor.*

We now prove a preliminary comparison principle.

**Theorem 4.4.** *Fix $\varepsilon > 0$ and $1 < p < \infty$. Let $v$ be a continuous $\varepsilon$-weak solution and let $u$ be a $0$-viscosity subsolution so that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\Omega$.*

*Proof.* Suppose that $\sup(u-v)>0$ occurs at the interior point $P_0$. For each positive integer $j$, we consider the function $\psi_j\colon \mathbf{H}_n\times\mathbf{H}_n\mapsto\mathbf{R}$ defined by

$$\psi_j(P,Q)=u(P)-v(Q)-j\varphi(P,Q)$$

where $\varphi(P,Q)$ is the function from Lemma 4.2, with $m$ chosen so that $m>\max\{4,p/(p-1),p\}$. Following the scheme of [B] and [CIL], we let the maximum of $\psi_j$ occur at $(P_j,Q_j)$ and observe for large $j$, these are interior points. In addition, these points tend to $P_0$ as $j\to\infty$. Using the Euclidean results of [CIL] and the above twisting lemma, we have

$$\big(j\Upsilon(P_j,Q_j),\mathscr{X}_j\big)\in\bar{J}^{2,+}u(P_j)\quad\text{and}\quad\big(j\Upsilon(P_j,Q_j),\mathscr{Y}_j\big)\in\bar{J}^{2,-}v(Q_j)$$

where $\Upsilon(P,Q)=DL_PD_P\varphi(P,Q)=-DL_QD_Q\varphi(P,Q)$ as detailed in Lemma 4.2.

**Claim 4.5.** *By passing to a subsequence if needed, we may assume $P_j\neq Q_j$.*

*Proof.* Fix $j>0$. By definition, we have for any $P$ and $Q$,

$$u(P)-v(Q)-j\varphi(P,Q)\le u(P_j)-v(Q_j)-j\varphi(P_j,Q_j)$$

and so when $P_j=P$, we have

$$v(Q)\ge v(Q_j)+j\varphi(P_j,Q_j)-j\varphi(P_j,Q).$$

Defining the function $\beta(Q)$ by

$$\beta(Q)=v(Q_j)+j\varphi(P_j,Q_j)-j\varphi(P_j,Q)-\varphi(Q_j,Q)$$

we see that $v-\beta$ has a strict local minimum at $Q_j$ and $Q_j$ is an isolated critical point. Applying Lemma 4.1, we have

$$\limsup_{Q\to Q_j}\big(-\Delta_p\beta(Q)\big)\ge\varepsilon. \tag{4.1}$$

Suppose now that $P_j=Q_j$. Then $\beta(Q)=v(Q_j)-(j+1)\varphi(Q_j,Q)$. We then need to estimate $\Delta_p\beta(Q)$. Using the non-divergence form of the $p$-Laplacian (equation (3.4)) and the definition of $\beta(Q)$, we have

$$|\Delta_p\beta(Q)|\lesssim\|\nabla_0\varphi(Q_j,Q)\|^{p-2}\big|\mathrm{tr}(D^2\varphi)^\star(Q_j,Q)+\|(D^2\varphi)^\star(Q_j,Q)\|\,\big|.$$

Using Lemma 4.2, we have

$$\|\nabla_0\varphi(Q_j,Q)\|\sim\|\eta\|\sim\varphi(Q_j,Q)^{(m-1)/m}.$$

We note that given the standard vectors $e_k$ with every entry $0$ except for the $k$th entry which is equal to $1$, we see that for any matrix $A$,

$$\operatorname{tr}(A) = \sum \langle Ae_k, e_k \rangle$$

and so

$$|\operatorname{tr}(D^2\varphi)^\star(Q_j, Q)| \lesssim \|(D^2\varphi)^\star(Q_j, Q)\|.$$

We then conclude via Lemma 4.2

$$\big|\operatorname{tr}(D^2\varphi)^\star(Q_j, Q) + \|(D^2\varphi)^\star(Q_j, Q)\|\,\big| \lesssim \|\mathscr{M}\| \sim \varphi(Q_j, Q)^{(m-2)/m}$$

so that

$$|\Delta_p \beta(Q)| \lesssim (\varphi(Q_j, Q)^{1/m})^{(m-1)(p-2)+(m-2)}.$$

Since $m > p/(p-1)$, we would have

$$\lim_{\substack{Q \to Q_j \\ Q \neq Q_j}} \big(-\Delta_p \beta(Q)\big) = 0.$$

This contradicts equation (4.1). ▫

Proceeding as in [B], $u$ is a viscosity subsolution to equation (3.4) with $\varepsilon = 0$. That is,

$$\begin{aligned} 0 \geq -\Big( &\|j\Upsilon(P_j, Q_j)\|^{p-2} \operatorname{tr}(\mathscr{X}_j)^\star \\ &+ (p-2)\|j\Upsilon(P_j, Q_j)\|^{p-4} \langle \mathscr{X}_j\, j\Upsilon(P_j, Q_j), j\Upsilon(P_j, Q_j)\rangle \Big). \end{aligned}$$

Using Lemmas 3.5 and 4.1 along with the definition of $\bar{J}^{2,-}$, we have

$$\begin{aligned} \varepsilon \leq -\Big( &\|j\Upsilon(P_j, Q_j)\|^{p-2} \operatorname{tr}(\mathscr{Y}_j)^\star \\ &+ (p-2)\|j\Upsilon(P_j, Q_j)\|^{p-4} \langle \mathscr{Y}_j\, j\Upsilon(P_j, Q_j), j\Upsilon(P_j, Q_j)\rangle \Big). \end{aligned}$$

Subtracting these two inequalities, we have

$$\begin{aligned} 0 < \varepsilon < {} & j^{p-2}\|\Upsilon(P_j, Q_j)\|^{p-2}\big(\operatorname{tr}(\mathscr{X}_j) - \operatorname{tr}(\mathscr{Y}_j)\big) \\ & + (p-2) j^{p-2} \|\Upsilon(P_j, Q_j)\|^{p-4} \Big( \langle \mathscr{X}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \\ & \qquad - \langle \mathscr{Y}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \Big). \end{aligned} \tag{4.2}$$

As in the proof of the above claim, we have

$$\|\Upsilon(P_j, Q_j)\| \sim \varphi(P_j, Q_j)^{(m-1)/m}.$$

Given a vector $\eta \in V_1$, we denote its extension to $h_n$ by $\tilde{\eta}$. That is, $\eta = (\eta_1, \eta_2, \ldots, \eta_{2n})$ yields $\tilde{\eta} = (\eta_1, \eta_2, \ldots, \eta_{2n}, 0)$. Using the formulas for the matrices $\mathscr{X}_j$ and $\mathscr{Y}_j$ given by Lemma 4.3 and the standard estimate on the matrix ordering ([CIL, Theorem 3.2]) produces

$$\begin{aligned}
&\langle \mathscr{X}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle - \langle \mathscr{Y}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \\
&\quad = \langle X(DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j)), DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j)\rangle \\
&\qquad - \langle Y(DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)), DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)\rangle \\
&\quad \leq j\langle \mathscr{D}\xi, \xi\rangle
\end{aligned}$$

where the matrix $\mathscr{D}$ is the Euclidean second derivative of $\varphi$ given by

$$\mathscr{D} = \begin{bmatrix} D_{PP}\varphi(P_j, Q_j) & D_{PQ}\varphi(P_j, Q_j) \\ D_{QP}\varphi(P_j, Q_j) & D_{QQ}\varphi(P_j, Q_j) \end{bmatrix}$$

and the vector

$$\xi = \big(DL_{P_j}^T \widetilde{\Upsilon}(P_j, Q_j) \oplus DL_{Q_j}^T \widetilde{\Upsilon}(P_j, Q_j)\big).$$

We then conclude via Properties (7) and (8) that

$$\begin{aligned}
&\langle \mathscr{X}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle - \langle \mathscr{Y}_j \Upsilon(P_j, Q_j), \Upsilon(P_j, Q_j)\rangle \\
&\quad \lesssim j\|\Upsilon(P_j, Q_j)\|^2 \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad \lesssim j\varphi(P_j, Q_j)^{(2m-2)/m} \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad = j\varphi(P_j, Q_j)^{(4m-4)/m}.
\end{aligned}$$

As in the proof of the claim, we write the trace difference as

$$\operatorname{tr}(\mathscr{X}_j) - \operatorname{tr}(\mathscr{Y}_j) = \sum_{k=1}^{2n} \langle \mathscr{X}_j e_k, e_k\rangle - \langle \mathscr{Y}_j e_k, e_k\rangle$$

and following the previous calculation, we obtain

$$\operatorname{tr}(\mathscr{X}_j) - \operatorname{tr}(\mathscr{Y}_j) \lesssim j\big(\varphi(P_j, Q_j)^{(2m-2)/m}\big)$$

so that with equation (4.2), we obtain

$$\begin{aligned}
0 < \varepsilon &\lesssim j^{p-1}\big(\varphi(P_j, Q_j)^{(m-1)/m}\big)^{p-2} \varphi(P_j, Q_j)^{(2m-2)/m} \\
&\quad + j^{p-1}\big(\varphi(P_j, Q_j)^{(m-1)/m}\big)^{p-4}\big(\varphi(P_j, Q_j)^{(4m-4)/m}\big) \\
&\sim j^{p-1}\big(\varphi(P_j, Q_j)^{1/m}\big)^{p(m-1)}.
\end{aligned}$$

Since $m > p$, we have $\big(p(m-1)\big)(1/m) > p - 1$. We arrive at a contradiction as $j \to \infty$. □

It is here where we stray significantly from the Euclidean proof in [JLM]. That proof relies on the $C^{1,\alpha}$ regularity of the solutions. This is not known for the Heisenberg group, although a recent result [DM] has proved this regularity for $p$ near $2$. We therefore adopt a completely different approach beginning with the next lemma. The first difference is that we only have a weaker version of Lemma 3.1 in [JLM], in that our sequence converges pointwise instead of locally uniformly.

**Lemma 4.6.** *Let $v$ be a $p$-harmonic function in $\Omega$. For each $\varepsilon \geq 0$, let $v_\varepsilon$ be the continuous $\varepsilon$-weak solution equal to $v$ on the boundary. Then $v_\varepsilon \to v$ pointwise as $\varepsilon \to 0$.*

*Proof.* Arguing as in [JLM], we see that $v_\varepsilon \to v$ in $L^p$. We may assume that $\varepsilon \leq 1$ and observe that as noted above, if $\varepsilon_1 > \varepsilon_2$, then $v_{\varepsilon_2}$ is a weak subsolution to equation (3.2) for $\varepsilon_1$. By the comparison principle (Lemma 3.1), we have that $v_{\varepsilon_2} \leq v_{\varepsilon_1}$ when $\varepsilon_1 > \varepsilon_2$. In particular for all $\varepsilon > 0$, $v \leq v_\varepsilon$. We then conclude that

$$w = \lim_{\varepsilon \to 0} v_\varepsilon = \inf_{\varepsilon > 0} \{v_\varepsilon\}$$

exists and $v \leq w$. Since $v_\varepsilon \to w$ pointwise, we have $|v_\varepsilon|^p \to |w|^p$. By the Lebesque dominated convergence theorem, (using $v_1$ as dominator) we have $v_\varepsilon \to w$ in $L^p$ so that actually, $v = w$. □

Combining the previous theorem and lemma, we obtain the following consequence.

**Lemma 4.7.** *Let $1 < p < \infty$. $0$-viscosity subsolutions are $p$-subharmonic. $0$-viscosity supersolutions are $p$-superharmonic and $0$-viscosity solutions are $p$-harmonic.*

*Proof.* The last statement follows from the first two and the second follows from the first by replacing $u$ with $-u$. We let $u$ be a $0$-viscosity subsolution that is not $p$-subharmonic. Then there is a $p$-harmonic function $v$ so that $u \leq v$ on $\partial\Omega$ but for some $P \in \Omega$, we have $u(P) > v(P)$. For $\varepsilon \leq 1$, we let $v_\varepsilon$ be $\varepsilon$-weak solutions equal to $v$ on $\partial\Omega$ so that $u \leq v_\varepsilon$ on $\partial\Omega$. By Lemma 4.6 we conclude for some $\varepsilon$ near $0$, $u(P) > v_\varepsilon(P)$, contrary to Theorem 4.4. □

Combining Lemmas 3.5 and 4.7, we have the following corollary.

**Corollary 4.8.** *Let $1 < p < \infty$. Then $0$-viscosity sub(super-)solutions to equation* (3.4) *and $p$-sub(super-)harmonic functions coincide. In particular, for $1 < p < \infty$, a function is $p$-harmonic if and only if it is a $0$-viscosity solution to equation* (3.4).

We are then able to conclude the following comparison principle.

**Corollary 4.9.** *Let $\varepsilon = 0$. Let $v$ be a viscosity supersolution to equation* (3.4) *and let $u$ be a viscosity subsolution of equation* (3.4) *so that $u \le v$ on $\partial\Omega$. Then $u \le v$ in $\Omega$.*

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