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ARBITRARY COMPLEX POWERS OF THE DIRAC OPERATOR ON THE HYPERBOLIC UNIT BALL

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Abstract. In this paper a definition for arbitrary complex powers of the Dirac operator on the m-dimensional hyperbolic unit ball is given and with the aid of Riesz's distributions a fundamental solution for these operators is determined. This fundamental solution is expressed in terms of the Gegenbauer function of the second kind.

1. Introduction

In this paper, Clifford analysis techniques are used to introduce the notion of an arbitrary complex power of the Dirac operator on the hyperbolic unit ball. Clifford analysis offers a nice and elegant way to generalize the Cauchy–Riemann system in the complex plane to higher dimensions, the Dirac operator being the higher-dimensional analogue of the Cauchy–Riemann operator, and an extension of multivariable calculus. In Clifford analysis one studies vector differential operators and functional analysis.

Standard reference books on Clifford analysis on the flat Euclidean space $\mathbf{R}^m$ are [2], [9] and [15] and a nice overview of the most essential results is given in [8]. For the more general case of Dirac operators on manifolds we refer e.g. to [5] and [15]. In this paper we consider the Dirac operator on the hyperbolic unit ball, which is a canonical example of a so-called Riemannian manifold of constant negative curvature. This has already been studied, e.g. in references [13] and [19]. However, as was already noticed in reference [3], the Dirac operator as it was defined in [13] acts on $\mathrm{Spin}(1)$-fields, whereas we define a Dirac operator acting on $\mathrm{Spin}\left(\frac{1}{2}\right)$-fields, hereby following the approach of reference [4]. This Dirac operator is invariant under the group of $\mathrm{Spin}(1,m)$ transformations, the automorphism group of the hyperbolic unit ball, whereas the Dirac operator considered in [19] is the conformal invariant hyperbolic Dirac operator, invariant under the larger conformal group. Following the present approach we thus find a larger class of solutions, including the conformal case as a special case.

In Section 2 we introduce a model for the m-dimensional hyperbolic unit ball and in Section 3 we give a short introduction to Clifford algebras. In Section 4

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we define two distributions by means of a divergent integral, the distribution x_+^λ on the real line and the so-called Riesz distribution Z_λ on the real orthogonal space $\mathbf{R}^{1,m}$, the latter being essential for what follows. In Section 5 we define the Gegenbauer and Legendre functions in the complex plane and in Sections 6, 7 and 8 we define arbitrary powers of the Dirac operator on the hyperbolic unit ball, using a similar technique as in reference [1], and we calculate a fundamental solution for this operator.

2. Hyperbolic spaces

In this section a model for the m-dimensional hyperbolic unit ball will be introduced. For that purpose, consider the real orthogonal space $\mathbf{R}^{1,m}$ of signature $(1,m)$ with an orthonormal basis $(\varepsilon, e_1, \ldots, e_m)$. Space-time vectors will be denoted by $X = \varepsilon T + \vec{X}$, making a clear distinction between the spatial coordinates $(X_1, \ldots, X_m)$ and the time coordinate T. The quadratic form associated with the real orthogonal space $\mathbf{R}^{1,m}$ is given by

$$Q(X) = T^2 - |\vec{X}|^2 \quad \text{for all } X \in \mathbf{R}^{1,m}.$$

The null cone NC is then defined as the set of all space-time vectors X satisfying $Q(X) = 0$, and this NC separates the *time-like* region (space-time vectors X for which $Q(X) > 0$) from the *space-like* region (space-time vectors X for which $Q(X) < 0$). The time-like region is the union of the future cone $FC = \{X : Q(X) > 0,\ T > 0\}$ and the past cone $PC = \{X : Q(X) > 0,\ T < 0\}$.

For those space-time vectors X belonging to the time-like region we define the norm $|X|$ as $Q(X)^{1/2} = (T^2 - |\vec{X}|^2)^{1/2}$ and an associated unit space-time vector ξ, as

$$\xi = \frac{X}{|X|} = \frac{\varepsilon T + \vec{X}}{(T^2 - |\vec{X}|^2)^{1/2}}.$$

A *projective* model for the m-dimensional hyperbolic unit ball is obtained by identifying the rays inside FC with points on the hyperbolic unit ball. Other models for the m-dimensional hyperbolic unit ball are then readily obtained by intersecting the manifold of rays inside FC with any surface Σ inside FC, such that each ray intersects Σ in a unique point.

3. The Clifford setting

The universal Clifford algebra $\mathbf{R}_{1,m}$ is defined as the real linear associative, but non-commutative, algebra generated by the orthonormal basis $(\varepsilon, e_1, \ldots, e_m)$ of $\mathbf{R}^{1,m}$ and the following multiplication rules:

$$\begin{aligned} e_i e_j + e_j e_i &= -2\delta_{ij}, \quad && i,j = 1, \ldots, m, \\ \varepsilon e_i + e_i \varepsilon &= 0, && i = 1, \ldots, m, \\ \varepsilon^2 &= 1. && \end{aligned}$$

Elements of $\mathbf{R}_{1,m}$ are called Clifford numbers and have the form

$$a = \sum_{A \subset M} a_A e_A, \quad a_A \in \mathbf{R},$$

with $A = \{i_1, \dots, i_k\} \subset M = \{0, \dots, m\}$, where $i_1 < \dots < i_k$ and $e_A = e_{i_1} \cdots e_{i_k}$ (here e_0 is to be replaced by ε). For $A = \emptyset$ we put $e_\emptyset = 1$. If A has k elements, e_A is a so-called k-vector and the subspace of k-vectors is denoted as $\mathbf{R}_{1,m}^{(k)}$. Denoting the projection of a Clifford number a onto its k-vector part as $[a]_k$, we have

$$a = \sum_{k=0}^{1+m} [a]_k.$$

The subspace $\mathbf{R}_{1,m}^{(+)} = \sum_{k \text{ even}} \oplus \mathbf{R}_{1,m}^{(k)}$ is a subalgebra of $\mathbf{R}_{1,m}$, called the *even subalgebra* and it is generated by the elements $\varepsilon_j = e_j \varepsilon$, $j = 1, \dots, m$. These generators satisfy $\varepsilon_j^2 = 1$ and $\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$, $i \neq j$, whence the set $\{\varepsilon_j : j = 1, \dots, m\}$ may be regarded as an orthonormal basis for $\mathbf{R}^{m,0}$. This means that the even subalgebra $\mathbf{R}_{1,m}^{(+)}$ is isomorphic to the Clifford algebra $\mathbf{R}_{m,0}$. Notice that space-time vectors X in $\mathbf{R}^{1,m}$ may be identified with 1-vectors in $\mathbf{R}_{1,m}$, but we keep the notation X.

For two space-time vectors X and Y in $\mathbf{R}_{1,m}^{(1)}$, we have

$$XY = X \cdot Y + X \wedge Y$$

where the inner product is defined as

$$X \cdot Y = \frac{XY + YX}{2}$$

and the outer product as

$$X \wedge Y = \frac{XY - YX}{2}.$$

On $\mathbf{R}_{1,m}$, the following involutory (anti-)automorphisms are of importance (in the following formulae e_0 is again to be replaced by ε and $a, b \in \mathbf{R}_{1,m}$, $\lambda \in \mathbf{R}$):

(1) the main involution $a \mapsto \tilde{a}$

$$\tilde{e}_i = -e_i, \quad (a + \lambda b)^\sim = \tilde{a} + \lambda \tilde{b}, \quad (ab)^\sim = \tilde{a}\tilde{b};$$

(2) the reversion $a \mapsto a^*$

$$e_i^* = e_i, \quad (a + \lambda b)^* = a^* + \lambda b^*, \quad (ab)^* = b^* a^*;$$

(3) the conjugation (also known as bar-map) $a \mapsto \bar{a}$

$$\bar{e}_i = -e_i, \quad \overline{(a + \lambda b)} = \bar{a} + \lambda \bar{b}, \quad \overline{(ab)} = \bar{b}\bar{a}.$$

Also, the following subgroups of the real Clifford algebra $\mathbf{R}_{1,m}$ are of interest: the Clifford group $\Gamma(1,m)$, the Pin group $\mathrm{Pin}(1,m)$ and the Spin group $\mathrm{Spin}(1,m)$. $\Gamma(1,m)$ is defined as the set of all invertible elements $g \in \mathbf{R}_{1,m}$ such that for all $X \in \mathbf{R}^{(1)}_{1,m}$ we have $gX\tilde{g}^{-1} \in \mathbf{R}^{(1)}_{1,m}$. The Pin group $\mathrm{Pin}(1,m)$ is the quotient group $\Gamma(1,m)/\mathbf{R}^+$ and the Spin group $\mathrm{Spin}(1,m) = \mathrm{Pin}(1,m) \cap \mathbf{R}^{(+)}_{1,m}$.

For each element $s \in \mathrm{Pin}(1,m)$ the map $\chi(s)\colon \mathbf{R}^{1,m} \mapsto \mathbf{R}^{1,m}\colon X \mapsto sX\bar{s}$ induces a map from $\mathbf{R}^{1,m}$ into itself. In this way, $\mathrm{Pin}(1,m)$ defines a double covering of the orthogonal group $O(1,m)$ whereas $\mathrm{Spin}(1,m)$ defines a double covering of the orthogonal group $SO(1,m)$. For more information we refer the reader to [9] and [17].

4. Distributions defined by divergent integrals

In this section we introduce two distributions that will be used in this paper. Let us start with the distribution x_+^λ on the real line, where λ is an arbitrary complex number (see references [7] and [13]).

Since the function $x_+^\lambda = x^\lambda H(x)$, where $H(x)$ stands for the Heaviside step-function on the real line, defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases}$$

is locally integrable for $\mathrm{Re}(\lambda) > -1$, it defines a regular distribution

$$\langle x_+^\lambda, \varphi \rangle = \int_0^\infty x^\lambda \varphi(x)\, dx, \quad \varphi \in \mathscr{D}(\mathbf{R}),$$

for $\mathrm{Re}(\lambda) > -1$. However, x_+^λ can analytically be continued to the strip $-n-1 < \mathrm{Re}(\lambda) < -n$ as follows:

$$\langle x_+^\lambda, \varphi \rangle = \frac{\left\langle \frac{d^n}{dx^n} x_+^{\lambda+n}, \varphi \right\rangle}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)},$$

where the derivatives with respect to x must be interpreted in distributional sense. Hence, if $-n-1 < \mathrm{Re}(\lambda) < -n$ one defines

$$\langle x_+^\lambda, \varphi \rangle = (-1)^n \frac{\langle x_+^{\lambda+n}, \varphi^{(n)} \rangle}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}, \quad \varphi \in \mathscr{D}(\mathbf{R}).$$

This means that for each test function $\varphi \in \mathscr{D}(\mathbf{R})$, the function $\langle x_+^\lambda, \varphi\rangle$ defines a meromorphic function of λ with simple poles at $\lambda = -1-n$, $n \in \mathbf{N}$.

The residue at $\lambda = -1-n$ is

$$\frac{\varphi^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}\langle \delta^{(n)}, \varphi\rangle,$$

and we can thus say that

$$\mathrm{Res}(x_+^\lambda, \lambda = -1-n) = \frac{(-1)^n}{n!}\delta^{(n)}(x).$$

In order to remove the simple poles of x_+^λ we divide by $\Gamma(1+\lambda)$, and so the distribution $x_+^\lambda/\Gamma(\lambda+1)$ is well-defined on $\mathscr{D}(\mathbf{R})$ for all $\lambda \in \mathbf{C}$ with $\langle x_+^\lambda/\Gamma(\lambda+1), \varphi\rangle$ a holomorphic function of λ for all $\varphi \in \mathscr{D}(\mathbf{R})$.

Next, we introduce the distributions ϱ^λ on $\mathscr{D}(\mathbf{R}^{1,m})$, with λ again an arbitrary complex number. As a general reference to the rest of this section, we refer to [7], [16] and [18]. The function $\varrho(X)$ is defined for space-time vectors $X \in \mathbf{R}^{1,m}$ as

$$\varrho(X) = \begin{cases} Q(X)^{1/2} & \text{in the } FC, \\ 0 & \text{otherwise.} \end{cases}$$

In the half-plane $\mathrm{Re}(\lambda) > -2$, the function ϱ^λ defines a regular distribution since ϱ^λ is locally integrable for these values of λ. Indeed,

$$\langle \varrho^\lambda, \varphi\rangle = \iint Q^{\lambda/2}(T, \vec{X})\varphi(T, \vec{X})\, dT\, d\vec{X}$$

defines an analytic function when $\mathrm{Re}(\lambda) > -2$ for each test function $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$. Using analytic continuation $\langle \varrho^\lambda, \varphi\rangle$ can be extended to a meromorphic function in the whole complex plane.

For that purpose we introduce the wave-operator $\Box$ on $\mathbf{R}^{1,m}$:

$$\Box = \partial_T^2 - \sum_{i=1}^m \partial_{X_i}^2 = \partial_T^2 - \Delta_m.$$

This operator has a decomposition which is similar to that of the Laplace operator on $\mathbf{R}^m$:

$$\Box = \frac{\partial^2}{\partial |X|^2} + \frac{m}{|X|}\frac{\partial}{\partial |X|} + \frac{1}{|X|^2}\Delta_H,$$

Δ_H being the Laplace–Beltrami operator on the hyperboloid $H_+ = \{\xi \in FC : |\xi| = 1\}$ (see e.g. reference [6]).

Letting the wave operator act on ϱ^λ we get

$$\Box\varrho^\lambda = \lambda(\lambda+m-1)\varrho^{\lambda-2}.$$

This suggests the following definition for the distribution ϱ^λ in the strip $-2n-2 < \lambda < -2n$:

$$\langle \varrho^\lambda, \varphi \rangle = \frac{\langle \Box^n \varrho^{\lambda+2n}, \varphi \rangle}{(\lambda+2)(\lambda+4)\cdots(\lambda+2n)(\lambda+m+1)\cdots(\lambda+m+2n-1)}.$$

From this relation it follows that the distribution ϱ^λ has poles at $\lambda = -2-2n$, $n \in \mathbf{N}$ and at $\lambda = -1-m-2n$, $n \in \mathbf{N}$. For m even all the poles are simple, while for m odd the points $-2, -4, \ldots, 1-m$ are simple poles and the points $-m-1, -m-3, \ldots$ are double poles.

The distributions ϱ^λ are normalized by introducing suitable factors. Putting

$$Z_\mu = \frac{\varrho^{\mu-m-1}}{\pi^{(m-1)/2} 2^{\mu-1} \Gamma\big(\frac{1}{2}\mu\big) \Gamma\big(\frac{1}{2}(\mu+1-m)\big)}, \tag{1}$$

the functional $\langle Z_\mu, \varphi \rangle$ becomes an entire function of the complex variable μ for each test function $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$. These so-called Riesz-distributions Z_μ enjoy remarkable properties, a few of which will be listed here (see e.g. [7]):

(1) The support of Z_μ is contained in the set $\overline{FC} = \{X \in \mathbf{R}^{1,m} : T \geq |\vec{X}|\}$.

(2) The distributions Z_μ satisfy the following convolution property: $Z_\mu * Z_\nu = Z_{\mu+\nu}$.

(3) For all $k \in \mathbf{N}$, we have $Z_{-2k} = \Box^k \delta(X)$, with $\delta(X) = \delta(T)\delta(\vec{X})$ the delta-function in space-time coordinates. This is the distribution in $\mathscr{D}'(\mathbf{R})$ acting on test functions $\varphi(T, \vec{X}) \in \mathscr{D}(\mathbf{R})$ as follows:

$$\langle \delta(X), \varphi(T, \vec{X}) \rangle = \varphi(0, \vec{0}).$$

(4) For all $\mu \in \mathbf{C}$ and $k \in \mathbf{N}$, $\Box^k Z_\mu = Z_{\mu-2k}$. In particular, we get $\Box^k Z_{2k} = \delta(X)$.

Let us now introduce $\mathscr{D}'_+(\mathbf{R}^{1,m})$ as the set of all distributions $f \in \mathscr{D}'(\mathbf{R}^{1,m})$ such that their supports are contained in $\overline{FC}$. Taking the convolution of two elements of $\mathscr{D}'_+(\mathbf{R}^{1,m})$, the result is again in $\mathscr{D}'_+(\mathbf{R}^{1,m})$ and hence $\mathscr{D}'_+(\mathbf{R}^{1,m})$ is a convolution algebra. The distributions Z_μ belong to $\mathscr{D}'_+(\mathbf{R}^{1,m})$, and their *uniquely* determined inverses in $\mathscr{D}'_+(\mathbf{R}^{1,m})$ are the distributions $\mathbf{Z}_{-\mu}$:

$$Z_\mu * Z_{-\mu} = \delta(X), \quad \mu \in \mathbf{C}.$$

It follows that the differential equation

$$\Box^k f = g,$$

with f and g belonging to $\mathscr{D}'_+(\mathbf{R}^{1,m})$ has a unique solution

$$f = Z_{2k} * g.$$

5. Gegenbauer and Legendre functions in the complex plane

In this section we introduce the Gegenbauer and Legendre functions in the complex plane for future purposes.

The Legendre functions are solutions of Legendre's differential equation

$$(1-z^2)\frac{d^2 f}{dz^2} - 2z\frac{df}{dz} + \big[\nu(\nu+1) - \mu^2(1-z^2)^{-1}\big] f = 0, \tag{2}$$

with ν and μ unrestricted complex parameters. The solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, defined in terms of the hypergeometric function by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right), \quad |1-z| < 2, \tag{3}$$

$$\begin{aligned} Q_\nu^\mu(z) = {}& \frac{e^{i\mu\pi}\pi^{1/2}}{2^{1+\nu}} \frac{\Gamma(\nu+\mu+1)}{\Gamma\big(\nu+\frac{3}{2}\big)}(z^2-1)^{\mu/2} z^{-1-\nu-\mu} \\ & \times F\left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \quad |z| > 1, \end{aligned} \tag{4}$$

are known as the associated Legendre functions of the first and second kind, respectively. They can be analytically extended to the whole complex plane supposed cut along the real axis from $-\infty$ to 1. By means of the transformation formulas of the hypergeometric function, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are expressible in several ways in the forms

$$\begin{aligned} P_\nu^\mu(z) &= A_1 F(a_1, b_1; c_1; \zeta) + A_2 F(a_2, b_2; c_2; \zeta), \\ Q_\nu^\mu(z) &= e^{i\mu\pi}\big(A_3 F(a_3, b_3; c_3; \zeta) + A_4 F(a_4, b_4; c_4; \zeta)\big), \end{aligned}$$

where ζ is a function of z, such that $|\zeta| < 1$. The various expansions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ can be found e.g. in [12]. One of these expansions for the Legendre function $Q_\nu^\mu(z)$ is the following:

$$\begin{aligned} Q_\nu^\mu(z) = {}& e^{i\mu\pi}\frac{\pi^{1/2}2^\mu\Gamma(1+\mu+\nu)}{\Gamma\big(\nu+\frac{3}{2}\big)} \frac{(z^2-1)^{\mu/2}}{\big(z+(z^2-1)^{1/2}\big)^{1+\mu+\nu}} \\ & \times F\left(\frac{1}{2}+\mu, 1+\mu+\nu; \nu+\frac{3}{2}; \frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right). \end{aligned} \tag{5}$$

We will also need the following relation:

$$e^{-i\mu\pi}\Gamma(1-\mu+\nu) Q_\nu^\mu(z) = e^{i\mu\pi}\Gamma(1+\mu+\nu) Q_\nu^{-\mu}(z). \tag{6}$$

The Gegenbauer functions $C_\nu^\mu(z)$ and $D_\nu^\mu(z)$ are holomorphic functions in the z-plane cut along the real axis from $-\infty$ to 1, and solution in this region of Gegenbauer's differential equation

$$(1-z^2)\frac{d^2f}{dz^2}-(2\mu+1)z\frac{df}{dz}+\nu(\nu+2\mu)f=0.$$

The Gegenbauer functions are defined in terms of the associated Legendre functions as follows:

$$C_\nu^\mu(z)=\pi^{1/2}2^{-\mu+1/2}\frac{\Gamma(\nu+2\mu)}{\Gamma(\mu)\Gamma(1+\nu)}(z^2-1)^{(1/4)-(\mu/2)}P_{\nu+\mu-1/2}^{-\mu+1/2}(z), \tag{7}$$

$$\begin{aligned} D_\nu^\mu(z)&=\pi^{-1/2}e^{2i\pi(\mu-1/4)}2^{-\mu+1/2}\\ &\quad\times\frac{\Gamma(\nu+2\mu)}{\Gamma(\mu)\Gamma(1+\nu)}(z^2-1)^{(1/4)-(\mu/2)}Q_{\nu+\mu-1/2}^{-\mu+1/2}(z). \end{aligned} \tag{8}$$

Note that the Gegenbauer function $D_\nu^\mu(z)$ has zeroes for $\mu\in-\mathbf{N}$ and poles for $\nu+2\mu\in-\mathbf{N}$ (see e.g. [10]). To calculate the residue in $\nu=-2\mu-k$, where k is an arbitrary integer and μ is being held fixed, we use the following hypergeometric representation of the Gegenbauer function $D_\nu^\mu(z)$ (combining definitions (4) and (8)):

$$D_\nu^\mu(z)=\frac{e^{i\pi\mu}}{2^{2\mu+\nu}}\frac{\Gamma(\nu+2\mu)}{\Gamma(\mu)\Gamma(1+\nu+\mu)}\frac{(z^2-1)^{(1/2)-\mu}}{z^{1+\nu}}F\left(\frac{2+\nu}{2},\frac{1+\nu}{2};1+\nu+\mu;\frac{1}{z^2}\right).$$

Together with $\mathrm{Res}[\Gamma(z),z=-k]=(-1)^k/k!$ and $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, we find

$$\mathrm{Res}_\nu\big[D_\nu^\mu(z),\nu=-2\mu-k\big]=(-1)^{k+1}\frac{\sin(k\pi)}{\pi}D_{-k-2\mu}^\mu(z). \tag{9}$$

The Gegenbauer function also satisfies

$$\frac{d}{dz}D_\nu^\mu(z)=2\mu D_{\nu-1}^{\mu+1}(z) \tag{10}$$

and

$$\nu D_\nu^\mu(z)=2\mu\big[zD_{\nu-1}^{\mu+1}(z)-D_{\nu-2}^{\mu+1}(z)\big]. \tag{11}$$

6. Arbitrary powers of the Dirac operator on $\mathbf{R}^{1,m}$

In this section we introduce the Dirac operator ∂_X on $\mathbf{R}^{1,m}$ and we define the notion of the Dirac operator raised to an arbitrary power $\mu \in \mathbf{C}$.

Let $\partial_X = \varepsilon\partial_T - \partial_{\vec{X}}$ be the Dirac operator on $\mathbf{R}^{1,m}$, where $\partial_{\vec{X}} = \sum_{j=1}^m e_j \partial_{X_j}$ stands for the Dirac operator on $\mathbf{R}^m$ in coordinates $\vec{X} \in \mathbf{R}^m$ (see e.g. [2] and [15]). As

$$\begin{aligned} X\partial_X &= (\varepsilon T + \vec{X})(\varepsilon\partial_T - \partial_{\vec{X}}) \\ &= T\partial_T + \sum_{j=1}^m X_j \partial_{X_j} + \vec{X}\varepsilon\partial_T - T\varepsilon\partial_{\vec{X}} - \sum_{i<j} e_{ij}(X_i\partial_{X_j} - X_j\partial_{X_i}), \end{aligned}$$

it is clear that we have the following decomposition for the operator ∂_X in the FC:

$$\partial_X = \xi\left(\partial_{|X|} + \frac{1}{|X|}\Gamma\right) = \frac{\xi}{|X|}(\mathbf{E}_{|X|} + \Gamma), \tag{12}$$

with $\xi = X/|X|$ the unit space-time vector associated to $X \in FC$, with

$$\mathbf{E}_{|X|} = T\partial_T + \sum_{j=1}^m X_j\partial_{X_j}$$

the Euler operator in space-time coordinates and with $\Gamma = X \wedge \partial_X$ the angular hyperbolic operator, tangent to the hyperboloid $H_+ = \{\xi \in FC : |\xi| = 1\}$, in explicit space-time coordinates given by

$$\Gamma = X \wedge \partial_X = \vec{X}\varepsilon\partial_T - T\varepsilon\partial_{\vec{X}} - \sum_{i<j} e_{ij}(X_i\partial_{X_j} - X_j\partial_{X_i}).$$

For two space-time vectors X and Y in $\mathbf{R}^{1,m}$ the angular operator Γ acting on their inner product yields

$$\Gamma(X \cdot Y) = X \wedge Y.$$

Furthermore, using the fact that

$$\partial_X^2 = \Box = \frac{\partial^2}{\partial|X|^2} + \frac{1}{|X|}(\Gamma + \xi\Gamma\xi)\frac{\partial}{\partial|X|} + \frac{1}{|X|^2}(\xi\Gamma\xi\Gamma - \Gamma)$$

and recalling the decomposition of the wave operator $\Box$ as given in the third section, we have

$$\Gamma + \xi\Gamma\xi = m \quad \Longrightarrow \quad \Gamma\xi = m\xi$$

and the following decomposition of the Laplace–Beltrami operator on the hyperboloid H_+:

$$\Delta_H = (m-1-\Gamma)\Gamma.$$

Before we define ∂_X^μ for general powers $\mu \in \mathbf{C}$, we first note that for $\mu \in 2\mathbf{N}$ we get $\partial_X^{2k} f = \Box^k f$. Since $Z_{-2k} = \Box^k \delta(X)$, this leads immediately to the following definition for ∂_X^{2k} as a convolution operator on $\mathscr{D}'_+(\mathbf{R}^{1,m})$:

$$\partial_X^{2k} f = Z_{-2k} * f \quad \text{for all } f \in \mathscr{D}'_+(\mathbf{R}^{1,m}). \tag{13}$$

When considering odd powers $\mu \in 2\mathbf{N}+1$, we have $\partial_X^{2k+1} f = \Box^k(\partial_X f) = (\partial_X Z_{-2k}) * f$. This leads to

$$\partial_X^{2k+1} f = \partial_X Z_{-2k} * f \quad \text{for all } f \in \mathscr{D}'_+(\mathbf{R}^{1,m}). \tag{14}$$

In the following lemma, we try to rewrite this distribution $\partial_X Z_{-2k}$.

Lemma 1. *For all $\mu \in \mathbf{C}$ and for all $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$ we have*

$$\langle \partial_X Z_\mu, \varphi\rangle = \left\langle \frac{X Z_{\mu-2}}{\mu-2}, \varphi \right\rangle.$$

Proof. Consider an arbitrary $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$. By definition we have

$$\langle \partial_X Z_\mu, \varphi\rangle = -\varepsilon\langle Z_\mu, \partial_T\varphi\rangle + \sum_{i=1}^m e_i \langle Z_\mu, \partial_{X_i}\varphi\rangle.$$

Let us first consider μ such that $\mathrm{Re}(\mu) > 1+m$.

Putting $c(\mu,m) = \pi^{(m-1)/2} 2^{\mu-1}\Gamma\big(\frac{1}{2}\mu\big)\Gamma\big(\frac{1}{2}(\mu+1-m)\big)$, and using partial integration we get for the first term

$$\begin{aligned}\langle Z_\mu, \partial_T\varphi\rangle &= \frac{1}{c(\mu,m)}\int_{\mathbf{R}^m} d\vec{X}\int_{|\vec{X}|}^\infty (T^2-|\vec{X}|^2)^{(\mu-m-1)/2}\partial_T\varphi(T,\vec{X})\\ &= \frac{1+m-\mu}{c(\mu,m)}\int_{\mathbf{R}^m} d\vec{X}\int_{|\vec{X}|}^\infty T(T^2-|\vec{X}|^2)^{(\mu-m-3)/2}\varphi(T,\vec{X}),\end{aligned}$$

where we have used the fact that φ has a compact support and that $\mathrm{Re}(\mu) > 1+m$. Using the definition of the Riesz distribution $Z_{\mu-2}$, this can also be written as

$$\langle Z_\mu, \partial_T\varphi\rangle = -\frac{1}{\mu-2}\langle T Z_{\mu-2}, \varphi\rangle.$$

The same argument can be used to obtain

$$\begin{aligned}\langle Z_\mu, \partial_{X_i}\varphi\rangle &= \frac{1}{c(\mu,m)}\int_0^\infty dT \int_{B(0,T)} d\vec{X}(T^2-|\vec{X}|^2)^{(\mu-m-1)/2}\partial_{X_i}\varphi(T,\vec{X})\\ &= \frac{1}{\mu-2}\langle X_i Z_{\mu-2},\varphi\rangle.\end{aligned}$$

This means that for all $\mu \in \mathbf{C}$ such that $\mathrm{Re}(\mu) > 1+m$ and for all $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$ we have

$$\langle \partial_X Z_\mu, \varphi\rangle = \frac{1}{\mu-2}\langle X Z_{\mu-2},\varphi\rangle. \tag{15}$$

Note that the distribution at the right-hand side does *not* have a pole at $\mu = 2$ since

$$\lim_{\mu\to 2} X Z_{\mu-2} = X\delta(X) = 0,$$

whence $XZ_{\mu-2}/(\mu-2)$ is well-defined for $\mu = 2$ by putting

$$\lim_{\mu\to 2}\left\langle \frac{XZ_{\mu-2}}{\mu-2},\varphi\right\rangle = \langle E(X),\varphi\rangle,$$

with $E(X) = \partial_X Z_2$ the fundamental solution for the Dirac operator ∂_X on $\mathbf{R}^{1,m}$. This means that both sides of equation (15) define a holomorphic function of μ for all $\varphi \in \mathscr{D}(\mathbf{R}^{1,m})$. Since those functions coincide in the region where $\mathrm{Re}(\mu) > 1+m$, they are equal. As φ was chosen arbitrarily, this proves the lemma. □

So far we thus have, for all $k \in \mathbf{N}$ and for all $f \in \mathscr{D}'_+(\mathbf{R}^{1,m})$,

$$\begin{aligned}\partial_X^{2k} f &= Z_{-2k} * f,\\ \partial_X^{2k+1} f &= \partial_X Z_{-2k} * f = -\frac{XZ_{-2k-2} * f}{2k+2}.\end{aligned}$$

By analogy with what was done in [1] we thus define $\partial_X^\mu f$ as

$$\begin{aligned}\partial_X^\mu f &= \left(\frac{1+e^{i\pi\mu}}{2}Z_{-\mu} - \frac{1-e^{i\pi\mu}}{2}\frac{Z_{-\mu-1}}{1+\mu}\right) * f\\ &= \left(\frac{1+e^{i\pi\mu}}{2}Z_{-\mu} + \frac{1-e^{i\pi\mu}}{2}\frac{\Gamma\left(-\frac{\mu}{2}\right)\Gamma\left(\frac{1-m-\mu}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{-m-\mu}{2}\right)}\xi Z_{-\mu}\right) * f.\end{aligned}$$

Introducing $c_\pm = \frac{1}{2}(1 \pm e^{i\pi\mu})$, we will often write $\partial_X^\mu f$ as

$$\partial_X^\mu f = \left(c_+ Z_{-\mu} + c_- \partial_X Z_{1-\mu}\right) * f.$$

7. The fundamental solution for the operator ∂_X^μ

In this section we will construct a distribution $E_\mu(X) \in \mathscr{D}'_+(\mathbf{R}^{1,m})$ such that

$$\partial_X^\mu E_\mu(X) = (c_+ Z_{-\mu} + c_- \partial_X Z_{1-\mu}) * E_\mu(X) = \delta(X).$$

Since $Z_\mu * Z_{-\mu} = Z_0 = \delta(X)$ and $\partial_X Z_{1-\mu} * \partial_X Z_{1+\mu} = \Box Z_2 = \delta(X)$, it seems natural to look for a fundamental solution which has the form

$$E_\mu(X) = a Z_\mu + b \partial_X Z_{1+\mu} = a Z_\mu + b \frac{X Z_{\mu-1}}{\mu - 1},$$

with a and b two complex constants that still need to be determined. Letting the operator ∂_X^μ act on $E_\mu(X)$, one finds four terms

$$\begin{aligned} a c_+ Z_{-\mu} * Z_\mu &= a c_+ \delta(X), \\ b c_+ Z_{-\mu} * \partial_X Z_{\mu+1} &= b c_+ \partial_X Z_1, \\ a c_- \partial_X Z_{1-\mu} * Z_\mu &= a c_- \partial_X Z_1, \\ b c_- \partial_X Z_{1-\mu} * \partial_X Z_{\mu+1} &= b c_- \delta(X) \end{aligned}$$

so that in order to obtain a fundamental solution, we choose $a = c_+$ and $b = -c_-$ such that

$$\partial_X^\mu E_\mu(X) = (c_+^2 - c_-^2)\delta(X) = e^{i\pi\mu}\delta(X).$$

Let us therefore define the fundamental solution for the operator ∂_X^μ, for all $\mu \in \mathbf{C}$, as

$$\begin{aligned} E_\mu(X) &= \frac{1 + e^{-i\pi\mu}}{2} Z_\mu + \frac{1 - e^{-i\pi\mu}}{2} \partial_X Z_{1+\mu} \\ &= \frac{1 + e^{-i\pi\mu}}{2} Z_\mu + \frac{1 - e^{-i\pi\mu}}{2} \frac{X Z_{\mu-1}}{\mu - 1}. \end{aligned}$$

8. Arbitrary powers of the Dirac operator on the hyperbolic unit ball

In this section, we determine the fundamental solution for an arbitrary complex power of the Dirac operator on the hyperbolic unit ball. For that purpose, we have to solve the equation

$$\partial_X^\mu E_{\mu,\alpha}(X) = T_+^{\alpha+m-\mu}\delta(\vec{X}). \tag{16}$$

This can be understood as follows. Because our model for the hyperbolic unit ball is projective, each object we introduce—such as a fundamental solution—has to be defined on the manifold of rays, our true hyperbolic space. This can be done by considering the homogeneous Clifford line-bundle, defined as couples $(X, c) \in \mathbf{R}_0^{1,m} \times \mathbf{R}_{1,m}$ together with the equivalence relation $(X, c) \propto (\lambda X, \lambda^\alpha c)$,

α being an arbitrary complex number. Each function on the hyperbolic unit ball is then defined as a section of this homogeneous bundle, i.e. a homogeneous function in space-time co-ordinates $(T, \vec{X})$:

$$F(\lambda X) = \lambda^\alpha F(X).$$

The right-hand side of equation (16) expresses the fact that we are looking for a fundamental solution which is homogeneous of degree α, having singularities on the time-axis. For the case $\mu = k \in \mathbf{N}_0$ this was already explained in reference [11], and (16) is the generalization to arbitrary powers $\mu \in \mathbf{C}$.

Since $E_\mu(X)$ is the fundamental solution for the operator ∂_X^μ, we have

$$\begin{aligned} E_{\mu,\alpha}(X) &= E_\mu(X) * T_+^{\alpha+m-\mu}\delta(\vec{X}) \\ &= \left(\frac{1+e^{-i\pi\mu}}{2} Z_\mu + \frac{1-e^{-i\pi\mu}}{2} \partial_X Z_{1+\mu}\right) * T_+^{\alpha+m-\mu}\delta(\vec{X}). \end{aligned}$$

Let us therefore calculate $Z_\sigma * T_+^{\alpha+m-\mu}\delta(\vec{X})$, σ being an arbitrary complex number. Denoting $R = |\vec{X}|$, we get

$$\begin{aligned} Z_\sigma * T_+^{\alpha+m-\mu}\delta(\vec{X}) &= H(T-R)\frac{\int_0^{T-R}\big((T-S)^2 - R^2\big)^{(\sigma-m-1)/2} S^{\alpha+m-\mu} dS}{\pi^{(m-1)/2} 2^{\sigma-1}\Gamma\big(\frac{1}{2}\sigma\big)\Gamma\big(\frac{1}{2}(\sigma+1-m)\big)} \\ &= H(T-R)\frac{|X|^{\sigma-m-1}(T-R)^{1+\alpha+m-\mu}}{\pi^{(m-1)/2} 2^{\sigma-1}\Gamma\big(\frac{1}{2}\sigma\big)\Gamma\big(\frac{1}{2}(\sigma+1-m)\big)} \\ &\quad \times \int_0^1 \big((1-t)(1-zt)\big)^{(\sigma-m-1)/2} t^{\alpha+m-\mu}\, dt \end{aligned}$$

where we have put $z = (T-R)/(T+R)$. Using Euler's representation formula for the hypergeometric function, the integral can be written as

$$\frac{\Gamma(1+\alpha+m-\mu)\Gamma\big(\frac{1}{2}(\sigma+1-m)\big)}{\Gamma\big(\alpha-\mu+\frac{1}{2}(\sigma+3+m)\big)} F\left(\frac{1+m-\sigma}{2}, 1+\alpha+m-\mu; \alpha-\mu+\frac{\sigma+3+m}{2}; z\right),$$

if we assume that $\mathrm{Re}(\sigma) > m-1$. Since

$$z = \frac{T-R}{T+R} = \frac{\tau - (\tau^2-1)^{1/2}}{\tau + (\tau^2-1)^{1/2}} \quad \text{for } \tau = \frac{T}{|X|},$$

we find with the aid of (5) that the hypergeometric function is equal to an associated Legendre function of the second kind

$$\begin{aligned} & e^{-i(m-\sigma)\pi/2}\frac{\Gamma\big(\alpha-\mu+\frac{1}{2}(\sigma+3+m)\big)}{\sqrt{\pi}\, 2^{(m-\sigma)/2}\Gamma(1+\alpha+m-\mu)} \\ & \qquad \times \frac{\big(\tau+(\tau^2-1)^{1/2}\big)^{1+\alpha+m-\mu}}{(\tau^2-1)^{(m-\sigma)/4}} Q^{(m-\sigma)/2}_{\alpha-\mu+(\sigma+m)/2}(\tau). \end{aligned}$$

With the aid of (6), we will eventually find that

$$Z_\sigma * T_+^{\alpha+m-\mu}\delta(\vec{X}) = H(T-R)e^{i\pi(\sigma-m-1/2}|X|^{\alpha+\sigma-\mu} \times \frac{\Gamma\big(\frac{1}{2}(1+m-\sigma)\big)}{2^{\sigma-1}\pi^{(m-1)/2}\Gamma\big(\frac{1}{2}\sigma\big)} D_{\alpha+\sigma-\mu}^{(1+m-\sigma)/2}(\tau).$$

Because the Gegenbauer functions are defined in the complex plane cut along $]-\infty,1]$, the factor $H(T-R)$ may be omitted. Indeed, as $\tau \in \mathbf{R}^+$ the condition $|\arg(\tau-1)| < \pi$ is equivalent to $\tau > 1 \Leftrightarrow T > R$. The Gegenbauer function has zeroes for $\frac{1}{2}(1+m-\sigma) \in -\mathbf{N}$, cancelling the poles of the Gamma function $\Gamma\big(\frac{1}{2}(1+m-\sigma)\big)$, and poles at $(\alpha-\mu) = -k-m$ with $k \in \mathbf{N}_0$. Note that these poles were to be expected since the distribution $T_+^{\alpha+m-\mu}$ also has poles at these values.

We thus have

$$Z_\mu * T_+^{\alpha+m-\mu}\delta(\vec{X}) = |X|^\alpha \frac{e^{i\pi(\mu-m-1)/2}}{2^{\mu-1}\pi^{(m-1)/2}} \frac{\Gamma\big(\frac{1}{2}(1+m-\mu)\big)}{\Gamma\big(\frac{1}{2}\mu\big)} D_\alpha^{(1+m-\mu)/2}(\tau)$$

and

$$\partial_X Z_{1+\mu} * T_+^{\alpha+m-\mu}\delta(\vec{X}) = \partial_X \left[|X|^{1+\alpha} \frac{e^{i\pi(\mu-m)/2}}{2^\mu \pi^{(m-1)/2}} \frac{\Gamma\big(\frac{1}{2}(m-\mu)\big)}{\Gamma\big(\frac{1}{2}(1+\mu)\big)} D_{1+\alpha}^{(m-\mu)/2}(\tau)\right].$$

Since $\partial_X = \xi\big(\partial_{|X|} + (|X|\Gamma)^{-1}\big)$ and $\Gamma(\tau) = \Gamma(\xi\cdot\varepsilon) = \xi\wedge\varepsilon$, we get

$$\partial_X |X|^{1+\alpha} D_{1+\alpha}^{(m-\mu)/2}(\tau) = \xi|X|^\alpha\big((m-\mu)D_\alpha^{((m-\mu)/2)+1}(\tau)\xi\wedge\varepsilon + (1+\alpha)D_{1+\alpha}^{(m-\mu)/2}(\tau)\big).$$

Writing $\xi(\xi\wedge\varepsilon)$ as $\varepsilon - \tau\xi$ and using (1), we will eventually find

$$\partial_X |X|^{1+\alpha} D_{1+\alpha}^{(m-\mu)/2}(\tau) = (\mu-m)|X|^\alpha\big(D_{\alpha-1}^{((m-\mu)/2)+1}(\tau)\xi - D_\alpha^{((m-\mu)/2)+1}(\tau)\varepsilon\big).$$

This means that we have now found the fundamental solution for the operator ∂_X^μ on the hyperbolic unit ball, for all $\mu \in \mathbf{C}$ and $\alpha \neq \mu - m - k$, $k \in \mathbf{N}_0$:

$$E_{\mu,\alpha}(X) = \frac{1+e^{-i\pi\mu}}{2}|X|^\alpha \frac{e^{-i\pi(m+1-\mu)/2}}{2^{\mu-1}\pi^{(m-1)/2}} \frac{\Gamma\big(\frac{1}{2}(1+m-\mu)\big)}{\Gamma\big(\frac{1}{2}\mu\big)} D_\alpha^{(1+m-\mu)/2}(\tau) - \frac{1-e^{-i\pi\mu}}{2}|X|^\alpha \frac{e^{-i\pi(m-\mu)/2}}{2^{\mu-1}\pi^{(m-1)/2}} \frac{\Gamma\big(1+\frac{1}{2}(m-\mu)\big)}{\Gamma\big(\frac{1}{2}(1+\mu)\big)} \times \big(D_{\alpha-1}^{((m-\mu)/2)+1}(\tau)\xi - D_\alpha^{((m-\mu)/2)+1}(\tau)\varepsilon\big).$$

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