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PERTURBATION THEORY FOR NONLINEAR DIRICHLET PROBLEMS

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Abstract. We consider Sobolev–Dirichlet problems as well as Dirichlet problems in the PWB-method for quasi-linear second order elliptic differential equations on a euclidean domain. We discuss boundedness of solutions of these problems and convergence of solutions under perturbation of the 0-th order term.

Introduction

In the previous papers [MO1] and [MO2], we developed a potential theory and discussed Dirichlet problems for elliptic quasi-linear equations of the form

$$(\mathrm{E}_{\mathscr{A},\mathscr{B}}) \qquad -\operatorname{div}\mathscr{A}\big(x,\nabla u(x)\big)+\mathscr{B}\big(x,u(x)\big)=0$$

on a domain Ω in $\mathbf{R}^N$ $(N\ge 2)$, where $\mathscr{A}(x,\xi)\colon \Omega\times\mathbf{R}^N\to\mathbf{R}^N$ satisfies weighted structure conditions of p-th order with a weight $w(x)$ and $\mathscr{B}(x,t)\colon \Omega\times\mathbf{R}\to\mathbf{R}$ is nondecreasing in t (see Section 1 below for more details).

The purpose of the present paper is to investigate how the solution of a Dirichlet problem varies under perturbation of the term $\mathscr{B}$.

As in [MO1] and [MO2], we consider the space $\mathscr{D}^p(\Omega;\mu)$ of bounded continuous functions with finite (p,μ)-Dirichlet integrals on Ω $(d\mu=w\,dx)$ and its subspace $\mathscr{D}^p_0(\Omega;\mu)$ consisting of $f\in\mathscr{D}^p(\Omega;\mu)$ which are uniformly bounded limits of $\varphi_n\in C_0^\infty(\Omega)$ such that $\nabla\varphi_n\to\nabla f$ in $L^p(\Omega;\mu)$. Throughout this paper we assume that Ω is (p,μ)-hyperbolic, namely $1\notin\mathscr{D}^p_0(\Omega;\mu)$. Given $\theta\in\mathscr{D}^p(\Omega;\mu)$, the solution u of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ satifying $u-\theta\in\mathscr{D}^p_0(\Omega;\mu)$ may be called the solution of the Sobolev–Dirichlet problem with the boundary data θ and is denoted by $u_{(\mathscr{A},\mathscr{B},\theta)}$.

We first introduce a class of L^1-functions on Ω, and using functions in this class, we present a condition on $\mathscr{B}$ (condition (B.4) in Section 3) which assures the boundedness of $u_{(\mathscr{A},\mathscr{B},\theta)}$.

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As our main theorem (Theorem 4.1), we shall show that if $\mathscr{B}_n$, $n = 1, 2, \dots$ satisfy (B.4) with the same data, they are uniformly bounded in a certain sense and if $\sup_{-M_2 \le t \le M_1} |\mathscr{B}_n(x,t) - \mathscr{B}(x,t)| \to 0$ in $L^1(\Omega)$, then $u_{(\mathscr{A},\mathscr{B}_n,\theta)} \to u_{(\mathscr{A},\mathscr{B},\theta)}$ locally uniformly on Ω, where M_1 and M_2 are constants determined by the data given in (B.4) and θ. We shall also give the convergence of $\nabla u_{(\mathscr{A},\mathscr{B}_n,\theta)}$ to $\nabla u_{(\mathscr{A},\mathscr{B},\theta)}$ in $L^p(\Omega;\mu)$ (Theorem 4.2).

In [MO1], we have studied Dirichlet problems in the PWB-method with respect to a Royden type ideal boundary for our equation $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$. In Section 5, under the same conditions as in Theorem 4.1, we give local uniform convergence of such Dirichlet solutions.

1. Preliminaries

As in [MO1] and [MO2] we assume that $\mathscr{A}\colon \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathscr{B}\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < \infty$ and a *weight* w which is p-*admissible* in the sense of [HKM]:

(A.1) $x \mapsto \mathscr{A}(x,\xi)$ is measurable on Ω for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathscr{A}(x,\xi)$ is continuous for a.e. $x \in \Omega$;

(A.2) $\mathscr{A}(x,\xi)\cdot\xi \ge \alpha_1 w(x)|\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathscr{A}(x,\xi)| \le \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_2 > 0$;

(A.4) $\big(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)\big)\cdot\big(\xi_1 - \xi_2\big) > 0$ whenever $\xi_1, \xi_2 \in \mathbf{R}^N$, $\xi_1 \ne \xi_2$, for a.e. $x \in \Omega$;

(B.1) $x \mapsto \mathscr{B}(x,t)$ is measurable on Ω for every $t \in \mathbf{R}$ and $t \mapsto \mathscr{B}(x,t)$ is continuous for a.e. $x \in \Omega$;

(B.2) For any open set $D \Subset \Omega$, there is a constant $\alpha_3(D) \ge 0$ such that $|\mathscr{B}(x,t)| \le \alpha_3(D)w(x)(|t|^{p-1}+1)$ for all $t \in \mathbf{R}$ and a.e. $x \in D$;

(B.3) $t \mapsto \mathscr{B}(x,t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \Omega$.

For the nonnegative measure $\mu : d\mu(x) = w(x)\,dx$ and an open set D, we consider the weighted Sobolev spaces $H^{1,p}(D;\mu)$, $H_0^{1,p}(D;\mu)$ and $H_{\mathrm{loc}}^{1,p}(D;\mu)$ (see [HKM] for details).

Let D be an open subset of Ω. Then $u \in H_{\mathrm{loc}}^{1,p}(D;\mu)$ is said to be a (weak) *solution* of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ in D if

$$\int_D \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_D \mathscr{B}(x,u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(D)$. $u \in H_{\mathrm{loc}}^{1,p}(D;\mu)$ is said to be a *supersolution* (respectively *subsolution*) of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ in D if

$$\int_D \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_D \mathscr{B}(x,u)\varphi\,dx \ge 0 \qquad (\text{respectively} \le 0)$$

for all nonnegative $\varphi \in C_0^\infty(D)$.

A continuous solution of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ in an open set $D \subset \Omega$ is called $(\mathscr{A},\mathscr{B})$-*harmonic* in D.

We say that an open set D in Ω is $(\mathscr{A},\mathscr{B})$-*regular*, if $D \Subset \Omega$ and for any $\theta \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ which is continuous at each point of ∂D, there exists a unique $h \in C(\overline{D}) \cap H^{1,p}(D;\mu)$ such that $h = \theta$ on ∂D and h is $(\mathscr{A},\mathscr{B})$-harmonic in D.

Proposition A ([MO1; Theorem 1.4] and [HKM; Theorem 6.31]). *Any ball $B \Subset \Omega$ is $(\mathscr{A},\mathscr{B})$-regular.*

A function $u\colon D \to \mathbf{R} \cup \{\infty\}$ is said to be $(\mathscr{A},\mathscr{B})$-*superharmonic* in D if it is lower semicontinuous, finite on a dense set in D and, for each open set $G \Subset D$ and for $h \in C(\overline{G})$ which is $(\mathscr{A},\mathscr{B})$-harmonic in G, $u \ge h$ on ∂G implies $u \ge h$ in G. $(\mathscr{A},\mathscr{B})$-*subharmonic* functions are similarly defined. Note that a continuous supersolution of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ is $(\mathscr{A},\mathscr{B})$-superharmonic (cf. [MO1; Section 2]).

We recall the following two spaces which are defined in [MO1] (cf. Introduction):

$$\mathscr{D}^p(\Omega;\mu) = \big\{ f \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu) \mid |\nabla f| \in L^p(\Omega;\mu),\ f \text{ is bounded continuous} \big\},$$
$$\mathscr{D}^p_0(\Omega;\mu) = \big\{ f \in \mathscr{D}^p(\Omega;\mu) \mid \text{there exist } \varphi_n \in C_0^\infty(\Omega) \text{ such that } \varphi_n \to f \text{ a.e.,}$$
$$\{\varphi_n\} \text{ is uniformly bounded, } \nabla\varphi_n \to \nabla f \text{ in } L^p(\Omega;\mu) \big\}.$$

Note that $H_0^{1,p}(\Omega;\mu) \cap \mathscr{D}^p(\Omega;\mu) \subset \mathscr{D}^p_0(\Omega;\mu)$ and the inclusion becomes equality if Ω is bounded.

Lemma 1.1. *Suppose $\mathscr{B}$ satisfies*

$$\text{(B.5)} \qquad \int_\Omega |\mathscr{B}(x,t)|\,dx < \infty \quad \textit{for any } t \in \mathbf{R}.$$

For $u \in \mathscr{D}^p(\Omega;\mu)$, if u is a solution (respectively supersolution, subsolution) of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$, then

$$\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega \mathscr{B}(x,u)\varphi\,dx = 0 \quad (\textit{respectively } \ge 0,\ \le 0)$$

for all $\varphi \in \mathscr{D}^p_0(\Omega;\mu)$ (respectively for all nonnegative $\varphi \in \mathscr{D}^p_0(\Omega;\mu)$).

Proof. Choose $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n \to \varphi$ a.e., $\{\varphi_n\}$ is uniformly bounded and $\nabla\varphi_n \to \nabla\varphi$ in $L^p(\Omega;\mu)$ (respectively and further $\varphi_n \ge 0$). Since u is a solution (respectively supersolution, subsolution) in Ω, we have

$$\text{(1.1)} \quad \int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi_n\,dx + \int_\Omega \mathscr{B}(x,u)\varphi_n\,dx = 0 \quad (\text{respectively } \ge 0,\ \le 0).$$

Then by (A.3)

$$\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi_n\,dx \to \int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx$$

(cf. the proof of [HKM; Lemma 3.11]). By (B.3) and (B.5), $\int_\Omega |\mathscr{B}(x,u)|\,dx<\infty$. Hence, by Lebesgue's convergence theorem

$$\int_\Omega \mathscr{B}(x,u)\varphi_n\,dx \to \int_\Omega \mathscr{B}(x,u)\varphi\,dx.$$

Therefore, letting $n\to\infty$ in (1.1), we have

$$\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega \mathscr{B}(x,u)\varphi\,dx = 0 \quad (\text{respectively } \ge 0,\ \le 0).$$

We have the following variant of the comparison principle (cf. [MO1; Proposition 1.1]):

Lemma 1.2. *Suppose* Ω *is* (p,μ)*-hyperbolic and* $\mathscr{B}$ *satisfies* (B.5). *For* u, $v\in\mathscr{D}^p(\Omega;\mu)$, *if* u *is a supersolution of* $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$, v *is a subsolution of* $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$ *and* $\min(u-v,0)\in\mathscr{D}^p_0(\Omega;\mu)$, *then* $u\ge v$ *on* Ω.

Proof. Set $\eta=\min(u-v,0)$. Since $\eta\in\mathscr{D}^p_0(\Omega;\mu)$ and $\eta\le 0$, by Lemma 1.1 we have

$$\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\eta\,dx + \int_\Omega \mathscr{B}(x,u)\eta\,dx \le 0$$

and

$$\int_\Omega \mathscr{A}(x,\nabla v)\cdot\nabla\eta\,dx + \int_\Omega \mathscr{B}(x,v)\eta\,dx \ge 0.$$

By (A.4) and (B.3),

$$\begin{aligned}\int_\Omega \big(\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u)\big)\cdot\nabla\eta\,dx &\\ &= -\int_{\{u<v\}}\big(\mathscr{A}(x,\nabla v)-\mathscr{A}(x,\nabla u)\big)\cdot(\nabla v-\nabla u)\,dx\\ &\le 0\end{aligned}$$

and

$$\int_\Omega \big(\mathscr{B}(x,v)-\mathscr{B}(x,u)\big)\eta\,dx = -\int_{\{u<v\}}\big(\mathscr{B}(x,v)-\mathscr{B}(x,u)\big)(v-u)\,dx \le 0.$$

Thus

$$\begin{aligned}0 &\le \left(\int_\Omega \mathscr{A}(x,\nabla v)\cdot\nabla\eta\,dx + \int_\Omega \mathscr{B}(x,v)\eta\,dx\right)\\ &\quad - \left(\int_\Omega \mathscr{A}(x,\nabla u)\cdot\nabla\eta\,dx + \int_\Omega \mathscr{B}(x,u)\eta\,dx\right)\\ &\le 0,\end{aligned}$$

and hence

$$\int_{\{u<v\}} \big(\mathscr{A}(x,\nabla v) - \mathscr{A}(x,\nabla u)\big)\cdot(\nabla v - \nabla u)\,dx = 0.$$

Therefore again by (A.4), $\nabla v - \nabla u = 0$ a.e. on $\{u < v\}$. Thus $\nabla\eta = 0$ a.e., so that $\eta = c$. Since Ω is (p,μ)-hyperbolic, we see that $c = 0$, and the lemma follows.

The next lemma will be used in the proof of Theorem 3.2.

Lemma 1.3. *Suppose K_1 and K_2 are constants such that $K_1 \le K_2$. Given $\theta \in \mathscr{D}^p(\Omega;\mu)$, let $\theta^* = \max\big(\min(\theta, K_2), K_1\big)$. If $u - \theta \in \mathscr{D}^p_0(\Omega;\mu)$ and $K_1 \le u \le K_2$ in Ω, then $u - \theta^* \in \mathscr{D}^p_0(\Omega;\mu)$.*

Proof. Choose $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n \to u - \theta$ a.e., $\{\varphi_n\}$ is uniformly bounded and $\nabla\varphi_n \to \nabla(u-\theta)$ in $L^p(\Omega;\mu)$. Setting

$$\varphi_n^* = \max\big(\min(\varphi_n, u - K_1), u - K_2\big),$$

we have $\varphi_n^* \to \max\big(\min(u-\theta, u-K_1), u-K_2\big) = u - \theta^*$ a.e. in Ω. Also, by [HKM; Lemma 1.2.2], $\nabla\varphi_n^* \to \nabla(u-\theta^*)$ in $L^p(\Omega;\mu)$. Since $u - K_1 \ge 0$ and $u - K_2 \le 0$, $\operatorname{supp}\varphi_n^*$ is compact in Ω for each n. Thus, considering mollified functions, we obtain approximating functions for $u - \theta^*$.

We recall the following condition, which has been considered in [MO1] and [MO2] for the discussion of resolutivity and harmonizability.

(C_1) There exist a bounded supersolution of ($\mathrm{E}_{\mathscr{A},\mathscr{B}}$) in Ω and a bounded subsolution of ($\mathrm{E}_{\mathscr{A},\mathscr{B}}$) in Ω.

The following theorem which asserts the existence and uniqueness of the Sobolev–Dirichlet problem is shown by [MO2; Theorem 2.2 and Proposition 1.5].

Theorem A. *Suppose that Ω is (p,μ)-hyperbolic and suppose that conditions* (C_1) *and* (B.5) *are satisfied. If $\theta \in \mathscr{D}^p(\Omega;\mu)$, then there exists a unique $(\mathscr{A},\mathscr{B})$-harmonic function $u_{(\mathscr{A},\mathscr{B},\theta)}$ on Ω such that $u_{(\mathscr{A},\mathscr{B},\theta)} - \theta \in \mathscr{D}^p_0(\Omega;\mu)$.*

2. A class of L^1-functions on Ω

Hereafter, we always assume that Ω *is* (p,μ)*-hyperbolic*, namely, $1 \notin \mathscr{D}_0^p(\Omega;\mu)$. Note that any bounded domain is (p,μ)-hyperbolic.

We consider the following function spaces:

$$\mathscr{F}(\mathscr{A}) = \{f \in L^1(\Omega) \mid f/w \text{ is locally bounded in } \Omega \text{ and } -\operatorname{div}\mathscr{A}(x,\nabla u) = f \text{ has a solution in } \mathscr{D}_0^p(\Omega;\mu)\}$$

$$\mathscr{F}^+(\mathscr{A}) = \{f \in \mathscr{F}(\mathscr{A}) \mid f \geq 0\} \quad \text{and} \quad \mathscr{F}^-(\mathscr{A}) = \{f \in \mathscr{F}(\mathscr{A}) \mid f \leq 0\}.$$

For $f \in \mathscr{F}(\mathscr{A})$, the solution of $-\operatorname{div}\mathscr{A}(x,\nabla u) = f$ in $\mathscr{D}_0^p(\Omega;\mu)$ will be denoted by U^f. Obviously, $0 \in \mathscr{F}(\mathscr{A})$ and $U^0 = 0$. If $f \in L^1(\Omega)$, f/w is locally bounded on Ω and (C_1) is satisfied for $(\mathrm{E}_{\mathscr{A},-f})$, then $f \in \mathscr{F}$ and $U^f = u_{(\mathscr{A},-f,0)}$.

Proposition 2.1. *If* f_1, $f_2 \in \mathscr{F}(\mathscr{A})$ *and* $f_1 \leq f_2$, *then* $U^{f_1} \leq U^{f_2}$.

Proof. Since U^{f_2} is a supersolution of $(\mathrm{E}_{\mathscr{A},-f_1})$, U^{f_1} is a solution of $(\mathrm{E}_{\mathscr{A},-f_1})$ and $\min(U^{f_2}-U^{f_1},0) \in \mathscr{D}_0^p(\Omega;\mu)$, the proposition follows from Lemma 1.2.

Corollary 2.1. $U^f \geq 0$ *for* $f \in \mathscr{F}^+(\mathscr{A})$ *and* $U^f \leq 0$ *for* $f \in \mathscr{F}^-(\mathscr{A})$.

We can easily show

Proposition 2.2. *If* $\mathscr{A}$ *satisfies the homogeneity condition*

$$\text{(A.5)} \qquad \mathscr{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi) \quad \text{for any } \lambda \in \mathbf{R},$$

then, $\lambda f \in \mathscr{F}(\mathscr{A})$ *for* $f \in \mathscr{F}(\mathscr{A})$ *and* $\lambda \in \mathbf{R}$, *and* $U^{\lambda f} = \lambda|\lambda|^{(2-p)/(p-1)}U^f$.

Proposition 2.3. *If* f *is measurable and* $g_1 \leq f \leq g_2$ *for some* $g_1 \in \mathscr{F}^-(\mathscr{A})$ *and* $g_2 \in \mathscr{F}^+(\mathscr{A})$, *then* $f \in \mathscr{F}(\mathscr{A})$.

Proof. Since U^{g_1} is a bounded subsolution of $(\mathrm{E}_{\mathscr{A},-f})$ and U^{g_2} is a bounded supersolution of $(\mathrm{E}_{\mathscr{A},-f})$, Theorem A asserts the existence of $u_{(\mathscr{A},-f,0)}$, which is U^f.

Proposition 2.4. *In case* Ω *is bounded, any measurable function* f *satisfying* $0 \leq f(x) \leq \beta w(x)$ *(respectively* $-\beta w(x) \leq f(x) \leq 0$*) for some* $\beta > 0$ *belongs to* $\mathscr{F}^+(\mathscr{A})$ *(respectively* $\mathscr{F}^-(\mathscr{A})$*).*

Proof. By the above proposition, it is enough to show that $\beta w \in \mathscr{F}^+(\mathscr{A})$ and $-\beta w \in \mathscr{F}^-(\mathscr{A})$. We consider $\mathscr{A}_1$: $\mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ defined by

$$\mathscr{A}_1(x,\xi) = \begin{cases} \mathscr{A}(x,\xi), & x \in \Omega, \\ w(x)|\xi|^{p-2}\xi, & x \in \mathbf{R}^N \setminus \Omega, \end{cases}$$

and take an open ball $B \supset \Omega$. By Proposition A, there exists $u \in C(\overline{B}) \cap H^{1,p}(B;\mu)$ such that $u = 0$ on ∂B and u is $(\mathscr{A}_1, -\beta w)$-harmonic in B. Then u is bounded and $(\mathscr{A}, -\beta w)$-harmonic in Ω. It follows from Theorem A that $U^{\beta w}$ exists and hence $\beta w \in \mathscr{F}^+(\mathscr{A})$. Similarly, we see that $-\beta w \in \mathscr{F}^-(\mathscr{A})$.

Example 2.1. Let $\Omega = B(0,R) = \{|x| < R\}$ for $0 < R < \infty$, $w(x) = |x|^\delta$ with $\delta > -N$ and $\mathscr{A}(x,\xi) = |x|^\delta |\xi|^{p-2}\xi$. Let g be a non-negative L^1-function on $[0,R)$ which is bounded on $[0,\varrho]$ for any $0 < \varrho < R$ and let $f(x) = |x|^\delta g(|x|)$. Then $f \in \mathscr{F}^+(\mathscr{A})$ and

$$U^f(x) = \int_{|x|}^R \left(\frac{1}{r^{N+\delta-1}} \int_0^r g(t) t^{N+\delta-1}\, dt \right)^{1/(p-1)} dr.$$

If we take $g(t) \equiv 1$, i.e., $f(x) = |x|^\delta$, then

$$U^f(x) = (N+\delta)^{-1/(p-1)} p'^{-1} (R^{p'} - |x|^{p'}),$$

where $1/p + 1/p' = 1$.

We can take unbounded f, e.g., $f(x) = |x|^\delta (R - |x|)^{-\alpha}$ with $0 < \alpha < 1$.

Example 2.2. Let $\Omega = \mathbf{R}^N$, $w(x) = |x|^\delta$ with $p < N + \delta$ and $\mathscr{A}(x,\xi) = |x|^\delta |\xi|^{p-2}\xi$. Note that Ω is (p,μ)-hyperbolic (see [MO1; p. 570]). Let g be a non-negative function on $[0,\infty)$ which is bounded on $[0,\varrho]$ for any $0 < \varrho < \infty$ and for which $\int_0^\infty g(t) t^{N+\delta-1}\, dt < \infty$. Then $f(x) = |x|^\delta g(|x|)$ belongs to $\mathscr{F}^+(\mathscr{A})$ and

$$U^f(x) = \int_{|x|}^\infty \left(\frac{1}{r^{N+\delta-1}} \int_0^r g(t) t^{N+\delta-1}\, dt \right)^{1/(p-1)} dr.$$

Proof of Examples 2.1 *and* 2.2. Obviously, $f \geq 0$ and $f/|x|^\delta$ is locally bounded on Ω. Also, it is easily verified that $f \in L^1(\Omega)$. Set

$$G(r) = \int_0^r g(t) t^{N+\delta-1}\, dt \quad \text{and} \quad u(x) = \int_{|x|}^R \left(\frac{1}{r^{N+\delta-1}} G(r) \right)^{1/(p-1)} dr,$$

where we set $R = \infty$ for Example 2.2. Then $G(r)$ is bounded for $0 < r < R$ and $G(r) \leq c_\varrho r^{N+\delta}$ for $0 < r < \varrho$, $\varrho < R$. From these it follows that $u(x) \in C^1(\Omega)$ and

$$\nabla u(x) = -\frac{x}{|x|} \left(\frac{1}{|x|^{N+\delta-1}} G(|x|) \right)^{1/(p-1)}.$$

In case $R < \infty$, since $|\nabla u(x)|$ is bounded, $\int_\Omega |\nabla u(x)|^p |x|^\delta\, dx < \infty$. In case $R = \infty$, since

$$\int_{|x|\geq 1} |\nabla u(x)|^p |x|^\delta\, dx \leq c \int_{|x| \geq 1} |x|^{-(pN+\delta-p)/(p-1)}\, dx$$

and $p < N + \delta$, we have $\int_\Omega |\nabla u(x)|^p |x|^\delta\, dx < \infty$. Moreover, if $|x| \to R$, then $u(x) \to 0$. Thus, setting $\varphi_n(x) = \max\big(u(x) - 1/n, 0\big)$, we have $\varphi_n \in \mathscr{D}^p(\Omega; |x|^\delta\, dx)$

and $\operatorname{supp}\varphi_n$ is compact. Since $\{\varphi_n\}$ is uniformly bounded, $\varphi_n \to u$ in Ω and $\nabla\varphi_n \to \nabla u$ in $L^p(\Omega; |x|^\delta\,dx)$, it follows that $u \in \mathscr{D}^p_0(\Omega; |x|^\delta\,dx)$. Finally, we have

$$\begin{aligned}-\operatorname{div}\big(|x|^\delta|\nabla u(x)|^{p-2}\nabla u(x)\big) &= \operatorname{div}\big(x|x|^{-N}G(|x|)\big)\\ &= -N|x|^{-N}G(|x|) + |x|^\delta g(|x|) + N|x|^{-N}G(|x|)\\ &= |x|^\delta g(|x|) = f(x),\end{aligned}$$

in the weak sense, so that u is a solution of $-\operatorname{div}(|x|^\delta|\nabla u|^{p-2}\nabla u) = f$.

3. Boundedness of solutions of Sobolev–Dirichlet problems

In addition to (B.1), (B.2) and (B.3), we shall always assume that $\mathscr{B}$ satisfies condition (B.5). Further we consider the following condition on $\mathscr{B}$:

(B.4) There exist nonnegative numbers T_1, T_2, functions $f_1 \in \mathscr{F}^+(\mathscr{A})$ and $f_2 \in \mathscr{F}^-(\mathscr{A})$ such that

$$\mathscr{B}^-(x, T_1) \le f_1(x) \quad \text{and} \quad \mathscr{B}^+(x, -T_2) \le -f_2(x) \quad \text{a.e. in } \Omega.$$

Example 3.1. Let $\zeta(t)$ be a nondecreasing continuous function on $\mathbf{R}$ such that $|\zeta(t)| \le c|t|^{p-1}$ for $|t| \ge 1$ and $\zeta(t_0) = 0$ for some $t_0 \in \mathbf{R}$. We set $\mathscr{B}(x,t) = b(x)\zeta(t)$ with $b \in L^1(\Omega)$ such that $b \ge 0$ and b/w is locally bounded on Ω. Then $\mathscr{B}$ satisfies (B.4) with $T_1 = t_0^+$, $T_2 = t_0^-$ and $f_1 = f_2 = 0$.

If $\mathscr{B}$ satisfies (B.4), then $T_1 + U^{f_1}$ is a supersolution and $-T_2 + U^{f_2}$ is a subsolution of $(\mathrm{E}_{\mathscr{A},\mathscr{B}})$. Thus, condition (C_1) is satisfied.

For $\theta \in \mathscr{D}^p(\Omega; \mu)$, we define

$$\sup_{\partial\Omega}\theta = \inf\{k \mid (\theta - k)^+ \in \mathscr{D}^p_0(\Omega;\mu)\} \quad \text{and} \quad \inf_{\partial\Omega}\theta = \sup\{k \mid (\theta - k)^- \in \mathscr{D}^p_0(\Omega;\mu)\}.$$

Theorem 3.1. *Suppose $\mathscr{B}$ satisfies* (B.4). *Then for any $\theta \in \mathscr{D}^p(\Omega;\mu)$, there exists a unique $(\mathscr{A},\mathscr{B})$-harmonic function $u_{(\mathscr{A},\mathscr{B},\theta)}$ on Ω such that $u_{(\mathscr{A},\mathscr{B},\theta)} - \theta \in \mathscr{D}^p_0(\Omega;\mu)$. Further it satisfies*

$$\min\Big(-T_2, \inf_{\partial\Omega}\theta\Big) + U^{f_2}(x) \le u_{(\mathscr{A},\mathscr{B},\theta)}(x) \le \max\Big(T_1, \sup_{\partial\Omega}\theta\Big) + U^{f_1}(x)$$

on Ω.

Proof. Since condition (B.4) implies condition (C_1), the existence and the uniqueness follow from Theorem A, we show only the inequalities. Fix $\varepsilon > 0$ and let $v = \max(T_1, \sup_{\partial\Omega}\theta) + U^{f_1} + \varepsilon$. Since $\mathscr{B}(x,v) \ge \mathscr{B}(x,T_1) \ge -f_1$ by (B.4), v is a supersolution of $(E_{\mathscr{A},\mathscr{B}})$. Also, since $v \ge \sup_{\partial\Omega}\theta + \varepsilon$, we see that $\max(\theta - v, 0) \in \mathscr{D}^p_0(\Omega;\mu)$, and hence that $\min(v - u_{(\mathscr{A},\mathscr{B},\theta)}, 0) \in \mathscr{D}^p_0(\Omega;\mu)$. Hence by Lemma 1.2, $v \ge u_{(\mathscr{A},\mathscr{B},\theta)}$ in Ω. Since $\varepsilon > 0$ is arbitrary, $\max(T_1, \sup_{\partial\Omega}\theta) + U^{f_1} \ge u_{(\mathscr{A},\mathscr{B},\theta)}$ in Ω. The other inequality follows similarly.

In view of Example 3.1, we can state

Corollary 3.1. *Let* $\mathscr{B}(x,t) = b(x)\zeta(t)$ *be as in Example* 3.1. *Then*

$$\min\Big(-t_0^-, \inf_{\partial\Omega} \theta\Big) \le u_{(\mathscr{A},\mathscr{B},\theta)}(x) \le \max\Big(t_0^+, \sup_{\partial\Omega} \theta\Big)$$

on Ω.

In view of Propositions 2.4 and 2.2, we also have

Corollary 3.2. *Suppose* Ω *is bounded,* $\mathscr{A}$ *satisfies* (A.5) (*in Proposition* 2.2) *and* $|\mathscr{B}(x,0)| \le \beta w(x)$ *a.e. in* Ω. *Then*

$$\min\Big(\inf_{\partial\Omega} \theta, 0\Big) - \beta^{1/(p-1)} U^w(x) \le u_{(\mathscr{A},\mathscr{B},\theta)}(x) \le \max\Big(\sup_{\partial\Omega} \theta, 0\Big) + \beta^{1/(p-1)} U^w(x)$$

on Ω.

Given nonnegative numbers T_1, T_2, functions $f_1 \in \mathscr{F}^+(\mathscr{A})$, $f_2 \in \mathscr{F}^-(\mathscr{A})$ and given $\theta \in \mathscr{D}^p(\Omega;\mu)$, let

$$M^+(T_1, f_1, \theta) = \max\Big(T_1, \sup_{\partial\Omega} \theta\Big) + \sup_{\Omega} U^{f_1},$$
$$M^-(T_2, f_2, \theta) = \max\Big(T_2, -\inf_{\partial\Omega} \theta\Big) - \inf_{\Omega} U^{f_2}.$$

Theorem 3.1 asserts that

$$-M^-(T_2, f_2, \theta) \le u_{(\mathscr{A},\mathscr{B},\theta)} \le M^+(T_1, f_1, \theta).$$

Theorem 3.2. *Suppose* $\mathscr{B}$ *satisfies* (B.4). *Then for* $\theta \in \mathscr{D}^p(\Omega;\mu)$,

$$\int_\Omega \big|\nabla u_{(\mathscr{A},\mathscr{B},\theta)}(x)\big|^p \, d\mu \le \left(\frac{\alpha_2}{\alpha_1}\right)^p \int_\Omega |\nabla\theta(x)|^p \, d\mu + \frac{pM}{\alpha_1} \int_\Omega \big|\mathscr{B}\big(x, \theta^*(x)\big)\big| \, dx,$$

where

$$M = M^+(T_1, f_1, \theta) + M^-(T_2, f_2, \theta)$$

and

$$\theta^* = \max\big(\min\big(\theta, M^+(T_1, f_1, \theta)\big), -M^-(T_1, f_1, \theta)\big).$$

Proof. Since $u_{(\mathscr{A},\mathscr{B},\theta)} - \theta^* \in \mathscr{D}_0^p(\Omega;\mu)$ by Lemma 1.3, we have

$$\begin{aligned} \int_\Omega \mathscr{A}\big(x, \nabla u_{(\mathscr{A},\mathscr{B},\theta)}\big) \cdot \big(\nabla u_{(\mathscr{A},\mathscr{B},\theta)} - \nabla\theta^*\big) \, dx & \\ + \int_\Omega \mathscr{B}\big(x, u_{(\mathscr{A},\mathscr{B},\theta)}\big)\big(u_{(\mathscr{A},\mathscr{B},\theta)} - \theta^*\big) \, dx & \\ = 0. & \end{aligned}$$

By Theorem 3.1, $|u_{(\mathscr{A},\mathscr{B},\theta)} - \theta^*| \le M$. Hence, using (A.2), (A.3) and (B.3), we have

$$\begin{aligned}
\alpha_1 \int_\Omega |\nabla u_{(\mathscr{A},\mathscr{B},\theta)}|^p \, d\mu &\le \alpha_2 \int_\Omega |\nabla u_{(\mathscr{A},\mathscr{B},\theta)}|^{p-1} |\nabla \theta^*| \, d\mu \\
&\qquad - \int_\Omega \mathscr{B}(x, u_{(\mathscr{A},\mathscr{B},\theta)})(u_{(\mathscr{A},\mathscr{B},\theta)} - \theta^*) \, dx \\
&\le \alpha_2 \int_\Omega |\nabla u_{(\mathscr{A},\mathscr{B},\theta)}|^{p-1} |\nabla \theta| \, d\mu \\
&\qquad - \int_\Omega \mathscr{B}(x, \theta^*)(u_{(\mathscr{A},\mathscr{B},\theta)} - \theta^*) \, dx \\
&\le \alpha_2 \left(\int_\Omega |\nabla u_{(\mathscr{A},\mathscr{B},\theta)}|^p \, d\mu \right)^{(p-1)/p} \left(\int_\Omega |\nabla \theta|^p \, d\mu \right)^{1/p} \\
&\qquad + M \int_\Omega |\mathscr{B}(x, \theta^*)| \, dx,
\end{aligned} \tag{3.1}$$

where in the last inequality we have used Hölder's inequality. An application of Young's inequality yields that $X \le AX^{(p-1)/p} + C$ implies $X \le A^p + pC$ for $X \ge 0$, $A \ge 0$ and $C \ge 0$. Hence, from (3.1) we obtain the desired estimate.

Theorem 3.3. *Suppose $\mathscr{B}_1$ and $\mathscr{B}_2$ satisfy* (B.4) *with the same T_1, T_2, f_1 and f_2. Let $\theta \in \mathscr{D}^p(\Omega; \mu)$ and set $u_j = u_{(\mathscr{A},\mathscr{B}_j,\theta)}$, $j = 1, 2$. Then*

$$\int_\Omega \big(\mathscr{A}(x, \nabla u_1) - \mathscr{A}(x, \nabla u_2) \big) \cdot (\nabla u_1 - \nabla u_2) \, dx \le M \int_\Omega \sup_{-M_2 \le t \le M_1} \big| \mathscr{B}_1(x,t) - \mathscr{B}_2(x,t) \big| \, dx,$$

where $M_1 = M^+(T_1, f_1, \theta)$, $M_2 = M^-(T_2, f_2, \theta)$ and $M = M_1 + M_2$.

Proof. Since $u_1 - u_2 \in \mathscr{D}_0^p(\Omega; \mu)$,

$$\int_\Omega \big(\mathscr{A}(x, \nabla u_1) - \mathscr{A}(x, \nabla u_2) \big) \cdot (\nabla u_1 - \nabla u_2) \, dx + \int_\Omega \big(\mathscr{B}_1(x, u_1) - \mathscr{B}_2(x, u_2) \big)(u_1 - u_2) \, dx = 0.$$

Hence using (B.3) we have

$$\int_\Omega \big(\mathscr{A}(x, \nabla u_1) - \mathscr{A}(x, \nabla u_2) \big) \cdot (\nabla u_1 - \nabla u_2) \, dx \le \int_\Omega \big(\mathscr{B}_1(x, u_2) - \mathscr{B}_2(x, u_2) \big)(u_2 - u_1) \, dx.$$

Since $-M_2 \le u_j \le M_1$, $j = 1, 2$, by Theorem 3.1, we obtain the desired estimate.

4. Convergence theorems

The following lemma can be shown in the same manner as [MO1; Lemma 5.1].

Lemma 4.1. *Let $\{u_n\}$ be a uniformly bounded sequence of functions in $\mathscr{D}_0^p(\Omega;\mu)$ such that $\left\{\int_\Omega |\nabla u_n|^p\, d\mu\right\}$ is bounded and $u_n \to u$ a.e. in Ω. If u is continuous, then $u \in \mathscr{D}_0^p(\Omega;\mu)$.*

The next lemma will be used to show Theorem 4.1.

Lemma 4.2 ([O; Theorem 4.7]). *Let u be an $(\mathscr{A},\mathscr{B})$-harmonic function in Ω and x_0 be any point of Ω. If $0 < R < \infty$ is such that $\overline{B}(x_0,R) \subset \Omega$ and if $|u| \le L$ in $B(x_0,R)$, then there are constants c and $0<\lambda<1$ such that*

$$\sup_{B(x_0,\varrho)} u - \inf_{B(x_0,\varrho)} u \le c\left(\frac{\varrho}{R}\right)^{\lambda},$$

whenever $0<\varrho<R$. Here c and λ depend only on N, p, α_1, α_2, $\alpha_3\big(B(x_0,R)\big)$, μ, R and L.

Theorem 4.1. *Suppose $\mathscr{B}_n$, $n=1,2,\ldots$, and $\mathscr{B}$ all satisfy* (B.4) *with the same T_1, T_2, $f_1 \in \mathscr{F}^+(\mathscr{A})$, $f_2 \in \mathscr{F}^-(\mathscr{A})$. Let $\theta \in \mathscr{D}^p(\Omega;\mu)$. Assume further that there exists a nonnegative function b on Ω such that b/w is locally bounded in Ω and*

$$\mathscr{B}_n^+(x,M_1) + \mathscr{B}_n^-(x,-M_2) \le b(x) \quad \text{a.e. on } \Omega \tag{4.1}$$

for all n, where M_1, M_2 are as in Theorem 3.3. If

$$\int_\Omega \sup_{-M_2\le t\le M_1} \big|\mathscr{B}_n(x,t) - \mathscr{B}(x,t)\big|\, dx \to 0 \quad (n\to\infty), \tag{4.2}$$

then $u_{(\mathscr{A},\mathscr{B}_n,\theta)} \to u_{(\mathscr{A},\mathscr{B},\theta)}$ as $n\to\infty$ locally uniformly on Ω.

Proof. If we set

$$\mathscr{B}_n^*(x,t) = \begin{cases} \mathscr{B}_n(x,M_1), & t \ge M_1,\\ \mathscr{B}_n(x,t), & -M_2 < t < M_1,\\ \mathscr{B}_n(x,-M_2), & t \le -M_2,\end{cases}$$

then $u_{(\mathscr{A},\mathscr{B}_n,\theta)}$ is a solution of $(\mathrm{E}_{\mathscr{A},\mathscr{B}_n^*})$, and hence $u_{(\mathscr{A},\mathscr{B}_n,\theta)} = u_{(\mathscr{A},\mathscr{B}_n^*,\theta)}$. Thus by (4.1), we may assume that $|\mathscr{B}_n(x,t)| \le b(x)$ $(n=1,2,\ldots)$ for any $t\in\mathbf{R}$. Then, for any $D \Subset \Omega$, we can take $\alpha_3(D) = \sup_D b/w$ in (B.2) for $\mathscr{B}_n$ for all n.

Let $u_n = u_{(\mathscr{A},\mathscr{B}_n,\theta)}$ and $u = u_{(\mathscr{A},\mathscr{B},\theta)}$. By Theorem 3.1, $\{u_n\}$ is uniformly bounded in Ω. Hence, by Lemma 4.2, we see that $\{u_n\}$ is equi-continuous at each point of Ω. Hence it follows from Ascoli–Arzela's theorem that any subsequence of $\{u_n\}$ has a locally uniformly convergent subsequence.

Let $\{u_{n_k}\}$ be any subsequence of $\{u_n\}$ which converges locally uniformly to u^*. Since $\{\sup_{-M_2 \le t \le M_1} |\mathscr{B}_n(x,t)|\}$ is bounded in $L^1(\Omega)$ by (4.2), Theorem 3.2 yields that $\{\nabla u_n\}$ is bounded in $L^p(\Omega;\mu)$. Thus, by Lemma 4.1 and [HMK; Lemma 1.3.3], we see that $u^* - u \in \mathscr{D}_0^p(\Omega;\mu)$ and $\nabla u_{n_k} \to \nabla u^*$ weakly in $L^p(\Omega;\mu)$. On the other hand, by Theorem 3.3 and [HKM; Lemma 3.73], $\nabla u_n \to \nabla u$ weakly in $L^p(\Omega;\mu)$, and hence $\nabla u^* = \nabla u$. Since Ω is (p,μ)-hyperbolic, it follows that $u^* = u$. Therefore, $u_n \to u$ locally uniformly in Ω.

In view of Example 3.1, we can state

Corollary 4.1. *Let $\zeta(t)$ be as in Example 3.1 and b_n, $n = 1, 2, \ldots$, and b be nonnegative measurable functions such that*

$$b_n(x) \le b_0(x) \quad \text{a.e.} \quad \text{on } \Omega \tag{4.3}$$

for some function b_0 such that b_0/w is locally bounded in Ω and

$$\int_\Omega |b_n(x) - b(x)|\, dx \to 0 \quad (n \to \infty). \tag{4.4}$$

Then, for $\mathscr{B}_n(x,t) = b_n(x)\zeta(t)$, $n = 1, 2, \ldots$, and $\mathscr{B}(x,t) = b(x)\zeta(t)$, $u_{(\mathscr{A},\mathscr{B}_n,\theta)} \to u_{(\mathscr{A},\mathscr{B},\theta)}$ locally uniformly on Ω.

The following example shows that we cannot assert the uniform convergence on Ω in the above theorem and corollary:

Example 4.1. Let $\Omega = B(0,1)$, $1 < p \le N$, $w(x) = 1$ and $\mathscr{A}(x,\xi) = |\xi|^{p-2}\xi$. Let $\{a_n\}$ be a sequence of points in $B(0,1)$ such that $|a_n| \to 1$ and set $b_n(x) = 2^{-n} r_n^{-N} \chi_{B(a_n,r_n)}(x)$ with $0 < r_n < 1 - |a_n|$, $n = 1, 2, \ldots$. For a nondecreasing continuous function ζ on $\mathbf{R}$ such that $\zeta(0) = 0$, $\zeta(1) > 0$ and $|\zeta(t)| \le c|t|^{p-1}$ for $|t| \ge 1$, we set

$$\mathscr{B}_n(x,t) = \sum_{k=1}^{n} b_k(x)\zeta(t) \quad \text{and} \quad \mathscr{B}(x,t) = \sum_{n=1}^{\infty} b_n(x)\zeta(t).$$

Then, for $\theta = 1$, $\mathscr{B}_n$ and $\mathscr{B}$ satisfy the conditions in Theorem 4.1. If we choose $\{r_n\}$ suitably, then $\{u_{(\mathscr{A},\mathscr{B}_n,1)}\}$ does not converge uniformly on $B(0,1)$.

Proof. Let $B_n = B(a_n, 1 - |a_n|)$ and v_n be the solution of the equation $-\operatorname{div} \mathscr{A}(x, \nabla u) = b_n$ on B_n which belongs to $\mathscr{D}_0^p(B_n; dx)$. By Lemma 1.2, we see that $0 \le v_n \le U^{b_n}$ in B_n. Noting that $\mathscr{A}(x,\xi) = |\xi|^{p-2}\xi$ is translation invariant, by Example 2.1 we have

$$v_n(a_n) = (2^{-n}r_n^{-N})^{1/(p-1)}\left\{\int_0^{r_n}\left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}\,dt\right)^{1/(p-1)} dr \right.$$
$$\left. + \int_{r_n}^{1-|a_n|}\left(\frac{1}{r^{N-1}}\int_0^{r_n} t^{N-1}\,dt\right)^{1/(p-1)} dr\right\}$$
$$= \begin{cases} (2^n N)^{1/(1-N)}\left\{1-\dfrac{1}{N}+\log\dfrac{1-|a_n|}{r_n}\right\} \text{ if } p=N, \\ (2^n N)^{1/(1-p)}\left\{\dfrac{N(p-1)}{p(N-p)}r_n^{(p-N)/(p-1)} - \dfrac{p-1}{N-p}(1-|a_n|)^{(p-N)/(p-1)}\right\} \\ \qquad\qquad\qquad \text{if } p<N. \end{cases}$$

Thus, $U^{b_n}(a_n) \ge v_n(a_n) \ge 1$ for sufficiently small r_n.

Let $u_n = u_{(\mathscr{A},\mathscr{B}_n,1)}$ and $u = u_{(\mathscr{A},\mathscr{B},1)}$ for simplicity. By considering the extension of $\mathscr{B}_n$ by 0 outside $B(0,1)$ and using Proposition A, we see that $u_n(x) \to 1$ as $|x| \to 1$.

Now suppose that $\{u_n\}$ converges to u uniformly on $B(0,1)$. Then $u(x) \to 1$ as $|x| \to 1$. Choose $\varepsilon > 0$ such that $\varepsilon^{p-1} < \zeta(1-\varepsilon)$. Then there is $r_0 < 1$ such that $u(x) \ge 1-\varepsilon$ for $|x| \ge r_0$. For large n, $B(a_n, r_n) \cap B(0, r_0) = \emptyset$ and

$$-\mathrm{div}\,(|\nabla(1-u)|^{p-2}\nabla(1-u)) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$
$$= \sum_{n=1}^{\infty} b_n(x)\zeta\big(u(x)\big) \ge \zeta(1-\varepsilon)b_n(x).$$

Thus, by Lemma 1.2, $1-u \ge \zeta(1-\varepsilon)^{1/(p-1)}U^{b_n}$, so that

$$u(a_n) \le 1 - \zeta(1-\varepsilon)^{1/(p-1)}U^{b_n}(a_n) < 1-\varepsilon$$

for large n. This contradicts our assumption that $u(x) \ge 1-\varepsilon$ for $|x| \ge r_0$.

Theorem 4.2. *Suppose $\mathscr{B}_n$, $n = 1, 2, \ldots$, and $\mathscr{B}$ all satisfy* (B.4) *with the same T_1, T_2, $f_1 \in \mathscr{F}^+(\mathscr{A})$, $f_2 \in \mathscr{F}^-(\mathscr{A})$. Let $\theta \in \mathscr{D}^p(\Omega;\mu)$. Assume further that $\mathscr{A}$ satisfies*

$$\text{(A.4s)} \qquad \big(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)\big)\cdot(\xi_1-\xi_2) \ge \alpha_4 w(x)(|\xi_1|+|\xi_2|)^{p-2}|\xi_1-\xi_2|^2$$

with $\alpha_4 > 0$. If (4.2) *holds, then $\nabla u_{(\mathscr{A},\mathscr{B}_n,\theta)} \to \nabla u_{(\mathscr{A},\mathscr{B},\theta)}$ in $L^p(\Omega;\mu)$; further $u_{(\mathscr{A},\mathscr{B}_n,\theta)} \to u_{(\mathscr{A},\mathscr{B},\theta)}$ in $H^{1,p}(\Omega;\mu)$ in case Ω is bounded.*

Proof. If $p \geq 2$, the first assertion is obvious from Theorem 3.3. In case $1 < p < 2$, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n - \nabla u|^p \, d\mu \\
&\quad = \int_\Omega \big\{(|\nabla u_n| + |\nabla u|)^{p-2} |\nabla u_n - \nabla u|^2\big\}^{p/2} (|\nabla u_n| + |\nabla u|)^{p(2-p)/2} \, d\mu \\
&\quad \leq \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2} |\nabla u_n - \nabla u|^2 \, d\mu\right)^{p/2} \\
&\qquad \times \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p \, d\mu\right)^{(2-p)/2}.
\end{aligned}$$

Hence the first assertion in this case also follows from Theorems 3.2 and 3.3.

The second assertion follows from the first assertion and the Poincaré inequality.

5. Boundedness and convergence of solutions for Dirichlet problems

Given a compactification Ω^* of Ω and a bounded function ψ on $\partial^*\Omega = \Omega^* \setminus \Omega$, let

$$\mathscr{U}_\psi = \Big\{u : (\mathscr{A}, \mathscr{B})\text{-superharmonic in } \Omega \text{ and } \liminf_{x \to \xi} u(x) \geq \psi(\xi) \text{ for all } \xi \in \partial^*\Omega\Big\}$$

and

$$\mathscr{L}_\psi = \Big\{v : (\mathscr{A}, \mathscr{B})\text{-subharmonic in } \Omega \text{ and } \limsup_{x \to \xi} v(x) \leq \psi(\xi) \text{ for all } \xi \in \partial^*\Omega\Big\}.$$

If both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty, then

$$\overline{H}(\psi; \Omega^*) = \overline{H}^{(\mathscr{A}, \mathscr{B})}(\psi; \Omega^*) := \inf \mathscr{U}_\psi$$

and

$$\underline{H}(\psi; \Omega^*) = \underline{H}^{(\mathscr{A}, \mathscr{B})}(\psi; \Omega^*) := \sup \mathscr{L}_\psi$$

are $(\mathscr{A}, \mathscr{B})$-harmonic in Ω and $\underline{H}(\psi; \Omega^*) \leq \overline{H}(\psi; \Omega^*)$ ([MO1; Theorem 3.1]). We say that ψ is $(\mathscr{A}, \mathscr{B})$-*resolutive* if both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty and $\underline{H}(\psi; \Omega^*) = \overline{H}(\psi; \Omega^*)$. In this case we write $H(\psi; \Omega^*) = H^{(\mathscr{A}, \mathscr{B})}(\psi; \Omega^*)$ for $\underline{H}(\psi; \Omega^*) = \overline{H}(\psi; \Omega^*)$. Ω^* is said to be an $(\mathscr{A}, \mathscr{B})$-*resolutive compactification* if every $\psi \in C(\partial^*\Omega)$ is $(\mathscr{A}, \mathscr{B})$-resolutive.

In the proof of [MO1; Proposition 3.1] we have shown

Lemma 5.1. *If ψ_1 and ψ_2 are $(\mathscr{A},\mathscr{B})$-resolutive functions on $\partial^*\Omega$, then*

$$|H(\psi_1;\Omega^*) - H(\psi_2;\Omega^*)| \le \sup_{\partial^*\Omega} |\psi_1 - \psi_2|.$$

Theorem 5.1. *Let Ω^* be a compactification of Ω. If $\mathscr{B}$ satisfies* (B.4) *and if ψ is a bounded function on $\partial^*\Omega$, then both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty and*

$$\min\Big(-T_2, \inf_{\partial^*\Omega} \psi\Big) + U^{f_2} \le \underline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*) \le \overline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*) \le \max\Big(T_1, \sup_{\partial^*\Omega} \psi\Big) + U^{f_1}.$$

Proof. By (B.4) we see that

$$\max\Big(T_1, \sup_{\partial^*\Omega} \psi\Big) + U^{f_1} \in \mathscr{U}_\psi$$

and

$$\min\Big(-T_2, \inf_{\partial^*\Omega} \psi\Big) + U^{f_2} \in \mathscr{L}_\psi.$$

Thus the theorem follows.

Like Corollaries 3.1 and 3.2, we have the following two corollaries.

Corollary 5.1. *Let $\mathscr{B}(x,t) = b(x)\zeta(t)$ be as in Example* 3.1. *If ψ is a bounded function on $\partial^*\Omega$, then both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty and*

$$\min\Big(-t_0^-, \inf_{\partial^*\Omega} \psi\Big) \le \underline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*) \le \overline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*) \le \max\Big(t_0^+, \sup_{\partial^*\Omega} \psi\Big).$$

Corollary 5.2. *Suppose Ω is bounded and let Ω^* be a compactification of Ω. If $\mathscr{A}$ satisfies* (A.5) *and if $|\mathscr{B}(x,0)| \le \beta w(x)$ for a.e. $x \in \Omega$, then*

$$|H^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*)| \le \sup_{\partial^*\Omega} |\psi| + \beta^{1/(p-1)} U^w$$

for any bounded $(\mathscr{A},\mathscr{B})$-resolutive function ψ on $\partial^*\Omega$.

We recall ([MO1; Theorem 3.2]) that under conditions (C_1) and (B.5), the Q-compactification Ω_Q^* of Ω (see [CC]) is an $(\mathscr{A},\mathscr{B})$-resolutive compactification if $Q \subset \mathscr{D}^p(\Omega;\mu)$.

Theorem 5.2. *Let $Q \subset \mathscr{D}^p(\Omega;\mu)$, Ω_Q^* be the Q-compactification of Ω, $\Gamma = \Omega_Q^* \setminus \Omega$ and let $\psi \in C(\Gamma)$. Suppose $\mathscr{B}_n$, $n = 1,2,\dots$, and $\mathscr{B}$ all satisfy* (B.4) *with the same T_1, T_2, $f_1 \in \mathscr{F}^+(\mathscr{A})$, $f_2 \in \mathscr{F}^-(\mathscr{A})$. Set*

$$M_1 = \max\Big(T_1, \max_\Gamma \psi\Big) + \sup_\Omega U^{f_1} \quad \text{and} \quad M_2 = \max\Big(T_2, -\min_\Gamma \psi\Big) - \inf_\Omega U^{f_2}.$$

Assume further that there exists a nonnegative function $b(x)$ on Ω such that $b(x)/w(x)$ is locally bounded in Ω and $\mathscr{B}_n$ satisfy (4.1) *for all n. If* (4.2) *holds, then $H^{(\mathscr{A},\mathscr{B}_n)}(\psi;\Omega_Q^*) \to H^{(\mathscr{A},\mathscr{B})}(\psi;\Omega_Q^*)$ locally uniformly on Ω.*

Proof. Since the set of continuous extensions of functions in Q is dense in $C(\Gamma)$ with respect to the uniform convergence, given $\varepsilon > 0$ there exists $\theta \in Q$ such that $\sup_{\xi\in\Gamma} |\theta^*(\xi) - \psi(\xi)| \le \varepsilon$ and $\inf\psi \le \theta^* \le \sup\psi$ on Γ, where θ^* is the continuous extension of θ to Γ. Note that ψ and θ^* are $(\mathscr{A},\mathscr{B})$-resolutive as well as $(\mathscr{A},\mathscr{B}_n)$-resolutive for all n ([MO1; Theorem 3.2]). For simplicity, let $H_n(\psi) = H^{(\mathscr{A},\mathscr{B}_n)}(\psi;\Omega_Q^*)$, $H_n(\theta) = H^{(\mathscr{A},\mathscr{B}_n)}(\theta^*;\Omega_Q^*)$, $H(\psi) = H^{(\mathscr{A},\mathscr{B})}(\psi;\Omega_Q^*)$ and $H(\theta^*) = H^{(\mathscr{A},\mathscr{B})}(\theta^*;\Omega_Q^*)$. By Lemma 5.1, we see that $|H(\theta^*) - H(\psi)| \le \varepsilon$ and $|H_n(\theta^*) - H_n(\psi)| \le \varepsilon$ for all n. On the other hand, by [MO2; Proposition 2.2], $H_n(\theta^*) = u_{(\mathscr{A},\mathscr{B}_n,\theta)}$ and $H(\theta^*) = u_{(\mathscr{A},\mathscr{B},\theta)}$. Also, since $\mathscr{B}_n^+\big(x, M^+(T_1,f_1,\theta)\big) \le \mathscr{B}_n^+(x,M_1)$ and $\mathscr{B}_n^-\big(x,-M^-(T_2,f_2,\theta)\big) \le \mathscr{B}_n^-(x,-M_2)$, by Theorem 4.1, for any open $G \Subset \Omega$, there is n_0 such that

$$\sup_G \big| u_{(\mathscr{A},\mathscr{B}_n,\theta)} - u_{(\mathscr{A},\mathscr{B},\theta)}\big| \le \varepsilon$$

for $n \ge n_0$. Thus, for $n \ge n_0$

$$\begin{aligned}
&\sup_G |H_n(\psi) - H(\psi)| \\
&\qquad \le \sup_G \big\{ |H_n(\psi) - H_n(\theta^*)| + |H_n(\theta^*) - H(\theta^*)| + |H(\theta^*) - H(\psi)| \big\} \\
&\qquad \le 3\varepsilon.
\end{aligned}$$

Hence the theorem follows.

Like Corollary 4.1, we obtain

Corollary 5.3. *Let Ω_Q^* and Γ be as in Theorem* 5.2 *and let $\mathscr{B}_n(x,t) = b_n(x)\zeta(t)$ and $\mathscr{B}(x,t) = b(x)\zeta(t)$ with nonnegative measurable functions b_n, b on Ω and ζ as in Example* 3.1. *If* (4.3) *and* (4.4) *are satisfied then $H^{(\mathscr{A},\mathscr{B}_n)}(\psi;\Omega_Q^*) \to H^{(\mathscr{A},\mathscr{B})}(\psi;\Omega_Q^*)$ locally uniformly on Ω for any $\psi \in C(\Gamma)$.*

References

[CC] Constantinescu, C., and A. Cornea: Ideale Ränder Riemannscher Flächen. - Springer-Verlag, 1963.

[HKM] Heinonen, J., T. Kilpeläinen and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Clarendon Press, 1993.

[MO1] Maeda, F.-Y., and T. Ono: Resolutivity of ideal boundary for nonlinear Dirichlet problems. - J. Math. Soc. Japan 52, 2000, 561–581.

[MO2] Maeda, F.-Y., and T. Ono: Properties of harmonic boundary in nonlinear potential theory. - Hiroshima Math. J. 30, 2000, 513–523.

[O] Ono, T.: On solutions of quasi-linear partial differential equations $-\operatorname{div}\mathscr{A}(x,\nabla u) + \mathscr{B}(x,u) = 0$. - RIMS Kokyuroku 1016, 1997, 146–165.

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