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# FINITE ENERGY MAPS FROM RIEMANNIAN POLYHEDRA TO METRIC SPACES

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*Dedicated to the memory of Professor Heinz Bauer*

**Abstract.** It is shown that every map of finite energy in the sense of Korevaar and Schoen into a complete metric space $Y$ (not necessarily locally compact) is quasicontinuous, the domain space being an admissible Riemannian polyhedron. Assuming that $Y$ is a geodesic space of upper bounded Alexandrov curvature, two inequalities are obtained for the energy of certain maps associated with a given pair of maps. One of these inequalities is due to T. Serbinowski (unpublished) and applied to establish existence and uniqueness of the solution to the variational Dirichlet problem for harmonic maps into $Y$.

## 1. Introduction and preliminaries

In this article the hypothesis of local compactness of the target space is omitted in certain results about maps of finite energy established in [EF], [F1], and [F2].

We first show (Theorem 1) that every finite energy map $\varphi\colon X \to Y$ is *quasicontinuous*, i.e., continuous relative to the complement of open subsets of $X$ of arbitrarily small capacity, cf. [EF, p. 153]. The domain space is a Riemannian manifold, or more generally an *admissible Riemannian polyhedron* $(X, g)$, $\dim X = m$. The target is a complete metric space $(Y, d_Y)$. Under the extra hypothesis that closed balls in $Y$ be compact, the result was obtained in [EF, Theorem 9.1] by a non-constructive compactness argument.

Basically following Serbinowski [Se] (Thesis, unpublished) we next establish existence and uniqueness of the solution to the variational Dirichlet problem for harmonic maps of $X$ into suitable balls in $Y$, assuming that $Y$ has upper bounded Alexandrov curvature (Theorem 2). See Section 2 for a precise formulation.

Referring to [EF, Chapter 4] we recall that a (Lipschitz) polyhedron $X$ is defined as a metric space which is Lip homeomorphic to a connected locally finite simplicial complex. Admissibility means that $X$ is dimensionally homogeneous and that (if $m \geqslant 2$) any two $m$-simplexes of $X$ with a common face $\sigma$ ($\dim\sigma = 0, 1, \ldots, m-2$) can be joined by a chain of $m$-simplexes containing $\sigma$, any two consecutive ones of which have a common $(m-1)$-face containing $\sigma$.

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The polyhedron $X$ becomes a Riemannian polyhedron when endowed with a Riemannian metric $g$, defined by giving on each open $m$-simplex $s$ of $X$ a nondegenerate Riemannian metric $g|_s$. We require that $g$ be *simplexwise smooth*, i.e., each $g|_s$ shall be smooth, and shall extend smoothly to the affine span of $s$, cf. [EF, Remark 4.1]. The associated volume measure on $X$ is denoted by $\mu_g = \mu$, the intrinsic (Riemannian) distance on $X$ by $d_X^g = d_X$, and the closed ball with centre $x \in X$ and radius $r$ by $B_X(x,r)$.

Based on the work of Korevaar and Schoen [KS] a concept of *energy* of a map $\varphi$ of $(X,g)$ into a metric space $(Y,d_Y)$ is developed in [EF, Chapter 9]. The map $\varphi$ is supposed first of all to be measurable with separable essential range, and to be of class $L^2_{\mathrm{loc}}(X,Y)$ in the sense that the distance function $d_Y\big(\varphi(\,\cdot\,),y\big)$ is of class $L^2_{\mathrm{loc}}(X,\mu)$ for some and hence (by the triangle inequality) for any point $y \in Y$. The *approximate energy density* $e_\varepsilon(\varphi) \in L^1_{\mathrm{loc}}(X,\mu)$ is then defined for $\varepsilon > 0$ at every point $x \in X$ by

$$e_\varepsilon(\varphi)(x) = \int_{B_X(x,\varepsilon)} \frac{d_Y^2\big(\varphi(x),\varphi(x')\big)}{\varepsilon^{m+2}}\, d\mu(x'). \tag{1.1}$$

The *energy* of $\varphi\colon (X,g) \to (Y,d_Y)$ is defined as

$$E(\varphi) = \sup_{f \in C_c(X,[0,1])} \Big( \limsup_{\varepsilon\to 0} \int_X f e_\varepsilon(\varphi)\, d\mu \Big) \quad (\leqslant \infty), \tag{1.2}$$

where $C_c$ stands for continuous functions of compact support. $W^{1,2}_{\mathrm{loc}}(X,Y)$ denotes the space of all maps $X \to Y$ for which $E(\varphi|_U) < \infty$ for every relatively compact connected open set $U \subset X$ (equivalently: the above lim sup is finite for every $f$). If $X$ is compact then (1.2) reduces to

$$E(\varphi) = \limsup_{\varepsilon\to 0} \int_X e_\varepsilon(\varphi)\, d\mu.$$

If $\varphi \in W^{1,2}_{\mathrm{loc}}(X,Y)$ (and only then), there exists a non-negative function $e(\varphi) \in L^1_{\mathrm{loc}}(X,\mu)$, called the *energy density* of $\varphi$ (more precisely: the 2-energy density), such that $e_\varepsilon(\varphi) \to e(\varphi)$ as $\varepsilon \to 0$, in the sense of weak convergence as measures:

$$\lim_{\varepsilon\to 0} \int_X f e_\varepsilon(\varphi)\, d\mu = \int_X f e(\varphi)\, d\mu \tag{1.3}$$

for every $f \in C_c(X)$. In the affirmative case it follows from (1.2), (1.3) that

$$E(\varphi) = \int_X e(\varphi)\, d\mu.$$

For the above assertions, see Steps 2, 3, and 4 of the proof of [EF, Theorem 9.1]. These steps are independent of the general requirement in [EF] that also the target of maps $X \to Y$ shall be locally compact. However, the proof of the second part of Step 1, leading to quasicontinuity of $\varphi$, required compactness of closed balls in $Y$.[1] In the present article we show that this extra hypothesis can be omitted; and that appears to be new even when the domain is a manifold.

A function $u\colon X \to \mathbf{R}$ is of class $W^{1,2}_{\mathrm{loc}}(X,\mathbf{R})$ in the above sense (with $Y = \mathbf{R}$) if and only if $u \in W^{1,2}_{\mathrm{loc}}(X)$ as defined in [EF, p. 63] (cf. [F1, footnote 2] for the uniqueness of $\nabla u$). And if that is the case, the energy density of $u$ equals

$$e(u) = c_m |\nabla u|^2 = c_m g^{ij} \partial_i u \, \partial_j u \qquad \text{a.e. in } X, \tag{1.4}$$

with the usual summation convention. Here $c_m = \omega_m/(m+2)$, $\omega_m$ being the volume of the unit ball in $\mathbf{R}^m$. See [EF, Corollary 9.2], which is based on [KS, Theorem 1.6.2] (where $X$ is a Riemannian domain in a Riemannian manifold), and is also a particular case of [EF, Theorem 9.2].

## 2. Formulation of results

A *version* of a $\mu$-measurable map $\varphi$ arises when $\varphi$ is redefined on some $\mu$-nullset.

**Theorem 1.** *Every map $\varphi \in W^{1,2}_{\mathrm{loc}}(X,Y)$ of an admissible Riemannian polyhedron $(X,g)$ into a complete metric space $(Y,d_Y)$ has a quasicontinuous version. When $\dim X = 1$, $\varphi$ has a Hölder continuous version with exponent $\frac{1}{2}$.*

The proof of this theorem, given in Section 3, includes the following explicit description of a quasicontinuous version $\varphi^*\colon X \to Y$ of $\varphi$: For any point $a$ in $X$ less a certain set $P$ (likewise explicitly described) of capacity $0$, *$\varphi^*(a)$ is the essential radial limit of $\varphi(x)$ as $x \to a$ along small rays issuing from $a$ in almost all directions.* In the proof of the theorem (for $m \geqslant 2$) we employ the fine topology of H. Cartan on $\mathbf{R}^m$—the weakest topology in which every subharmonic function in $\mathbf{R}^m$ is continuous. The fine topology is stronger than the metric topology. We shall also use the following lemma on finite energy maps $\mathbf{R}^m \to Y$, likewise established in Section 3, and drawing on Korevaar–Schoen's study of directional energies [KS].

**Lemma 1.** *Let $\varphi$ be a map of finite energy from $\mathbf{R}^m$, $m \geqslant 2$, into a complete metric space $(Y,d_Y)$. In terms of the 2-energy density $e(\varphi) \in L^1(\mathbf{R}^m)$ suppose that*

$$\int_{\mathbf{R}^m} |x|^{1-m} \sqrt{e(\varphi)(x)}\, dx < \infty.$$

[1] At this point in [EF], local compactness of $Y$ should be read as closed balls in $Y$ being compact. Furthermore, the reference to Remark 7.5 should be to Remark 7.6.

*For almost every open ray $R$ in $\mathbf{R}^m$ issuing from $0$, the restriction $\varphi|_R\colon R\to Y$ has finite $1$-energy, and therefore possesses a version of bounded variation up to the point $0$.*

In the proof of Theorem 1 we shall furthermore need the potential theoretic notion of "thinness" of a set, introduced by Brelot in 1939: $A\subset\mathbf{R}^m$ is thin at a point $x_0\in\mathbf{R}^m\setminus A$ if either $x_0\notin\bar{A}$ or there exists a superharmonic function $u\geqslant 0$ in a neighbourhood $\omega$ of $x_0$ in $\mathbf{R}^m$ such that

$$u(x_0)<\liminf_{\omega\cap A\ni x\to x_0}u(x), \tag{2.1}$$

cf. [Br1, Sections 3, 11] or [Br2, p. 2]. It was this concept that led Cartan to introducing the fine topology, having observed that the complements of the sets $A\subset\mathbf{R}^m\backslash\{x_0\}$ which are thin at $x_0$ are precisely the fine neighbourhoods of $x_0$, cf. [Br1, Théorème 5] or [Br2, p. 3].—All this remains in force with $\mathbf{R}^m$ replaced more generally by an admissible Riemannian polyhedron $(X,g)$, cf. [EF, Chapter 7].

In the rest of this section, the *target* $(Y,d_Y)$ *is a complete geodesic space* (again not necessarily locally compact) *of Alexandrov curvature* $\leqslant 1$. (The case of curvature $\leqslant K$ for a constant $K>0$ reduces to the case $K=1$ by rescaling the metric on $Y$.) All maps are assumed to have essential range contained in a closed geodesically convex ball $B=B_Y(q,R)$ in $Y$ of radius $R<\frac{1}{2}\pi$ and satisfying *bipoint uniqueness* (i.e., geodesics in $B$ shall be uniquely determined by their endpoints and shall vary continuously with them). For the above concepts see [EF, Chapter 2], [F1], [F2], and literature quoted there.

By way of preparation to Theorem 2 below we bring, in the following two propositions, two inequalities connected with a pair of finite energy maps $\varphi_0,\varphi_1\colon X\to B$. Their distance function $u(x)=d_Y\big(\varphi_0(x),\varphi_1(x)\big)$, $x\in X$, is of class $W^{1,2}_{\mathrm{loc}}(X)$ because $\varphi_i\in L^2_{\mathrm{loc}}(X,Y)$ (hence $u\in L^2_{\mathrm{loc}}(X)$), and $u$ has finite Dirichlet integral satisfying

$$\int_X|\nabla u|^2\,d\mu\leqslant 2c_m\big(E(\varphi_0)+E(\varphi_1)\big), \tag{2.2}$$

as seen from (1.1), (1.2), and (1.4), by application of the triangle inequality:

$$|u(x)-u(x')|^2\leqslant\Big(\sum_{i=0}^{1}d_Y\big(\varphi_i(x),\varphi_i(x')\big)\Big)^2\leqslant 2\sum_{i=0}^{1}d_Y^2\big(\varphi_i(x),\varphi_i(x')\big).$$

**Proposition 1.** *For any two finite energy maps $\varphi_i\colon X\to B$, $i=0,1$, and any function $\kappa\colon X\to[0,1]$ with finite Dirichlet integral, the map $\varphi_\kappa=(1-\kappa)\varphi_0+\kappa\varphi_1\colon X\to B$ has finite energy satisfying*

$$E\big((1-\kappa)\varphi_0+\kappa\varphi_1\big)\leqslant C\Big(E(\varphi_0)+E(\varphi_1)+\int_X|\nabla\kappa|^2\,d\mu\Big), \tag{2.3}$$

*where $B=B_Y(q,R)$, and where $C$ depends on $R$ and $\dim X=m$ only.*

By abuse of notation, the map $(1-\kappa)\varphi_0+\kappa\varphi_1$ is defined at a point $x \in X$ as the point of the geodesic segment $[\varphi_0(x),\varphi_1(x)]$ at distance $\kappa(x)d_Y\big(\varphi_0(x),\varphi_1(x)\big)$ from $\varphi_0(x)$. The proof of Proposition 1 is given in Section 4.

**Proposition 2** (Serbinowski's inequality, [Se]). *Given two finite energy maps* $\varphi_0,\varphi_1\colon X \to B$ *with midpoint map* $\varphi_{1/2}=\frac{1}{2}\varphi_0+\frac{1}{2}\varphi_1$ *and distance function* $u=d_Y(\varphi_0,\varphi_1)$ $(\in W^{1,2}_{\mathrm{loc}}(X))$, *consider the finite energy map* $\widehat{\varphi}_{1/2}\colon X \to B$ *given by*

$$\widehat{\varphi}_{1/2}=(1-\eta)\varphi_{1/2}+\eta q, \tag{2.4}$$

*where the function* $\eta\colon X \to [0,1[$ *of class* $W^{1,2}_{\mathrm{loc}}(X)$ *is defined in terms of the function* $\varrho := d_Y\big(\varphi_{1/2}(\,\cdot\,),q\big) \in W^{1,2}_{\mathrm{loc}}(X)$ *by*

$$\sin[(1-\eta)\varrho]=\sin\varrho\cos\big(\tfrac{1}{2}u\big) \tag{2.5}$$

*at points* $x \in X$ *where* $\varrho(x)>0$, *and by* $1-\eta=\cos\big(\frac{1}{2}u\big)$ *elsewhere. Then*

$$c_m\cos^8 R\left|\nabla\frac{\tan\big(\frac{1}{2}u\big)}{\cos\varrho}\right|^2 \leqslant \frac{1}{2}e(\varphi_0)+\frac{1}{2}e(\varphi_1)-e(\widehat{\varphi}_{1/2}) \tag{2.6}$$

$\mu$*-a.e. If* $u \in W^{1,2}_0(X\setminus bX)$ *then* $d_Y(\varphi_{1/2},\widehat{\varphi}_{1/2})=\eta\varrho \in W^{1,2}_0(X\setminus bX)$.

By integration, (2.6) yields[2]

$$c_m\cos^8 R\int_X\left|\nabla\frac{\tan\big(\frac{1}{2}u\big)}{\cos\varrho}\right|^2 d\mu \leqslant \frac{1}{2}E(\varphi_0)+\frac{1}{2}E(\varphi_1)-E(\widehat{\varphi}_{1/2}). \tag{2.7}$$

Proposition 2 is an analogue of the inequality [EF, (11.2)], cf. [KS, (2.2iv)], expressing strict convexity of the energy of maps into a simply connected complete geodesic space of nonpositive curvature. The proof of Proposition 2 is based on the theory of directional energies in [KS] and is given in Section 5 below, essentially following [Se] (though of necessity invoking Proposition 1 above).

Theorem 2 is about the *variational Dirichlet problem for energy minimizing maps* $X \to Y$; and the *domain space* $X$ *is therefore required to be compact with nonvoid boundary* $bX$ (the union of all $(m-1)$-simplexes of $X$ contained in only

[2] In [Se], $X$ is a Riemannian domain (in a Riemannian manifold). That leads to (2.6) for the present admissible Riemannian polyhedron $X$, simply by application to each open $m$-simplex of $X$, the $(m-1)$-skeleton being a $\mu$-nullset. In [Se], and hence (sic!) in [EF], the inequality (2.7) is unfortunately mis-stated, the "hat" over $\varphi_{1/2}$ being missing (thereby invalidating the inequality, even in the case of geodesics $\varphi_0,\varphi_1$ on the standard 2-sphere). The proof given in [Se] pertains of course to the correct version as stated above.—The parenthetical statement in [EF, p. 201] about avoiding directional energies if $R<\pi/4$ is dubious in the case of discontinuous maps.

one $m$-simplex). We denote by $W^{1,2}(X,B)$ the class of all maps $X \to B$ of finite energy. The *trace* $\operatorname{tr}_{bX}\varphi$ on $bX$ of a map $\varphi \in W^{1,2}(X,B)$ is defined by collecting the traces $\operatorname{tr}_\sigma \varphi$ of $\varphi$ on the various $(m-1)$-simplexes $\sigma$ of $bX$. (If $s$ denotes the open $m$-simplex of $X$ having $\sigma$ as a face, $\operatorname{tr}_\sigma\varphi$ is defined as in [KS, Section 1.12] applied to $\Omega = s$, $\Gamma = \sigma$.) Then $\operatorname{tr}_{bX}\varphi$ is defined $\mathscr{H}_{m-1}$-a.e. on $bX$ ($\mathscr{H}_{m-1}$ denoting $(m-1)$-dimensional Hausdorff measure). In terms of a quasicontinuous version of $\varphi$ (cf. Theorem 1), we have $\operatorname{tr}_{bX}\varphi = \varphi|_{bX}$ $\mathscr{H}_{m-1}$-a.e. on $bX$, cf. [F2, Section 2] (this will not be used in the present paper).

Given a map $\psi \in W^{1,2}(X,B)$, consider the subclass

$$(2.8)\qquad \begin{aligned} W^{1,2}_\psi(X,B) &= \{\varphi \in W^{1,2}(X,B) : \operatorname{tr}_{bX}\varphi = \operatorname{tr}_{bX}\psi \ \mathscr{H}_{m-1}\text{-a.e.}\} \\ &= \{\varphi \in W^{1,2}(X,B) : d_Y(\varphi,\psi) \in W^{1,2}_0(X \setminus bX)\}. \end{aligned}$$

For the latter equation see [F2, Lemma 1(b)] (applied to $\Gamma = bX$), which of course also shows that, for any two maps $\varphi_0, \varphi_1 \in W^{1,2}_\psi(X,B)$, the distance function $d_Y(\varphi_0,\varphi_1)$ is of class $W^{1,2}_0(X\setminus bX)$ because $\operatorname{tr}_{bX}\varphi_0 = \operatorname{tr}_{bX}\varphi_1$ $\mathscr{H}_{m-1}$-a.e.

**Theorem 2** (Serbinowski [Se]). *For any map $\psi \in W^{1,2}(X,B)$ there exists a unique map $\varphi \in W^{1,2}_\psi(X,B)$ of least energy.*

This map $\varphi$ is called the solution to the variational Dirichlet problem, or the *variational solution* to the Dirichlet problem. Its existence and uniqueness ($\mu$-a.e.) was established in [F2], assuming for existence that $Y$ be locally compact. Serbinowski's proof [Se] of existence and uniqueness of the variational solution, without local compactness of $Y$, is based on Proposition 2 above, but is otherwise quite similar to the proof by Korevaar–Schoen of the analogous result for targets of nonpositive curvature [KS, Theorem 2.2]. The proof in the present setting with a polyhedral domain is also similar, again in view of Proposition 2: Write $d_Y(\varphi_0,\varphi_1) = u$ and

$$E_0 = \inf\{E(\varphi) : \varphi \in W^{1,2}_\psi(X,B)\}.$$

For *uniqueness*, let $\varphi_0,\varphi_1 \in W^{1,2}_\psi(X,B)$. Then $\varphi_{1/2} \in W^{1,2}_\psi(X,B)$, by (2.8) and Proposition 1, because $d_Y(\varphi_0,\varphi_{1/2}) = \frac12 u \in W^{1,2}_0(X\setminus bX)$. It follows similarly that $\widehat{\varphi}_{1/2} \in W^{1,2}_\psi(X,B)$, and so $E(\widehat{\varphi}_{1/2}) \geqslant E_0$, because $d_Y(\varphi_{1/2},\widehat{\varphi}_{1/2}) = \eta\varrho \in W^{1,2}_0(X\setminus bX)$, by the final assertion of Proposition 2. (Alternatively, consider the traces of these functions and maps on $bX$.) When $\varphi_0,\varphi_1$ are minimizers, i.e., $E(\varphi_0) = E(\varphi_1) = E_0$, (2.7) shows that the function $\tan\bigl(\frac12 u\bigr)/\cos\varrho$ of class $W^{1,2}_0(X\setminus bX)$ is constant, hence equals $0$ $\mu$-a.e. because $1 \notin W^{1,2}_0(X\setminus bX)$ by the Poincaré inequality [F2, Lemma 1(c)]. Consequently, $u = d_Y(\varphi_0,\varphi_1) = 0$ $\mu$-a.e., and so $\varphi_0 = \varphi_1$ $\mu$-a.e.

For *existence* consider a minimizing sequence $(\varphi_i)$ in $W^{1,2}_\psi(X,B)$. Write

$$u_{ij} = d_Y(\varphi_i, \varphi_j), \quad \varphi_{ij} = \tfrac{1}{2}\varphi_i + \tfrac{1}{2}\varphi_j, \quad \varrho_{ij} = d_Y(\varphi_{ij}, q),$$

and define $\eta_{ij}$ as in (2.5) (now with $\eta, \varrho, u$ replaced by $\eta_{ij}, \varrho_{ij}, u_{ij}$). Defining $\widehat{\varphi}_{ij} = (1-\eta_{ij})\varphi_{ij} + \eta_{ij} q$, cf. (2.4), we then have $\varphi_{ij}$, $\widehat{\varphi}_{ij} \in W^{1,2}_\psi(X,B)$, by the same argument as above, and hence

$$E(\varphi_i) + E(\varphi_j) - 2E(\widehat{\varphi}_{ij}) \leqslant E(\varphi_i) + E(\varphi_j) - 2E_0,$$

so that the left-hand member converges to $0$ as $i,j \to \infty$. By (2.7) this implies

$$\lim_{i,j\to\infty} \int_X \left| \nabla \frac{\tan\left(\frac{1}{2}u_{ij}\right)}{\cos \varrho_{ij}} \right|^2 d\mu = 0.$$

It follows by the quoted Poincaré inequality that $\tan\left(\frac{1}{2}u_{ij}\right)/\cos\varrho_{ij} \to 0$ in $L^2(X)$. The same therefore applies to $u_{ij}$ itself, and so $(\varphi_i)$ has a limit $\varphi$ in the complete metric space $(L^2(X,B), D)$, where $D^2(\varphi', \varphi'') := \int_X d_Y^2(\varphi', \varphi'')\, d\mu$ for $\varphi', \varphi'' \in L^2(X,B)$, cf. [KS, Section 1.1]. Because the energy functional is lower semicontinuous, [EF, Lemma 9.1] (this does not depend on local compactness of the target), we conclude that $\varphi \in W^{1,2}(X,B)$ and that $E(\varphi) \leqslant E_0$. According to [F2, Lemma 1(a)] (extending [KS, Theorem 1.12.2]) it follows that $\operatorname{tr}_{bX}\varphi_i \to \operatorname{tr}_{bX}\varphi$ in $L^2(bX, B)$ (complete, with metric analogous to $D$ above), and so $\operatorname{tr}_{bX}\varphi = \operatorname{tr}_{bX}\psi$ $(=\operatorname{tr}_{bX}\varphi_i)$. Altogether, $\varphi \in W^{1,2}_\psi(X,B)$, and we conclude that $E(\varphi) = E_0$.

**Remark 1.** The solution to the above variational problem is known to have a Hölder continuous version in $X \setminus bX$ (continuous up to the boundary $bX$ if $\operatorname{tr}_{bX}\psi$ is continuous), provided that either $R < \frac{1}{4}\pi$ (rather than $R < \frac{1}{2}\pi$) or that $Y$ is locally compact, [EF, Theorem 11.4], [F1, Theorem 2], [F2, Theorem 3].

## 3. Proof of Lemma 1 and Theorem 1

*Proof of Lemma* 1. For $0 < \alpha < \beta < \infty$ consider the Euclidean Riemannian domain $\Omega_{\alpha\beta} = \{x \in \mathbf{R}^m : \alpha < |x| < \beta\}$ and the unit vector field

$$\omega(x) = x/|x|, \qquad x \in \Omega_{\alpha\beta},$$

with associated directional 2-energy density ${}^{\omega}e(\varphi) = |\varphi_*(\omega)|^2 \leqslant C(m)^2 e(\varphi)$ Lebesgue a.e. in $\Omega_{\alpha\beta}$, where $|\varphi_*(\omega)|(x)$ denotes the 1-energy density of $\varphi$ in the direction $\omega(x)$, and $C(m)$ depends on $m$ only, [KS, Theorem 1.11]. In particular, $|\varphi_*(\omega)| \in L^2(\Omega_{\alpha\beta})$. Denoting by $S^{m-1}$ the unit sphere in $\mathbf{R}^m$, with surface measure $\sigma$, we therefore have

$$\int_{S^{m-1}} d\sigma(\xi) \int_\alpha^\beta r^{m-1} r^{1-m} |\varphi_*(\omega)|(r\xi)\, dr \leqslant C(m) \int_{\Omega_{\alpha\beta}} |x|^{1-m} \sqrt{e(\varphi)(x)}\, dx.$$

For $\alpha \to 0$, $\beta \to \infty$ this leads, by the hypothesis of the lemma, to

$$\int_0^\infty |\varphi_*(\omega)|(r\xi)\,dr < \infty \qquad \text{for } \sigma\text{-a.e. } \xi \in S^{m-1}. \tag{3.1}$$

We show that (again for $\sigma$-a.e. $\xi \in S^{m-1}$) the map $\varphi_\xi\colon \mathbf{R}_+ \to Y$, defined Lebesgue a.e. by $\varphi_\xi(r) = \varphi(r\xi)$, has finite 1-energy, and its 1-energy density $e(\varphi_\xi)$ satisfies

$$e(\varphi_\xi)(r) \leqslant |\varphi_*(\omega)|(r\xi) \tag{3.2}$$

for Lebesque a.e. $r \in \mathbf{R}_+$. It will then follow that $\varphi_\xi$ possesses a version of bounded variation on every interval $]0,\beta[$ $(0 < \beta < \infty)$, cf. [KS, Lemma 1.9.2], applied in dimension 1.

For $0 < 4\varepsilon < \beta - \alpha$ the approximate 1-energy density ${}^\omega e_\varepsilon(\varphi)(x)$ of $\varphi$ at $x = r\xi$ in the direction $\omega(x) = x/|x| = \xi$ is defined on $\Omega_{\alpha+\varepsilon,\beta-\varepsilon}$ for $\sigma$-a.e. $\xi \in S^{m-1}$ by

$${}^\omega e_\varepsilon(\varphi)(r\xi) = \frac{1}{\varepsilon} d_Y\big(\varphi(r\xi), \varphi\big((r+\varepsilon)\xi\big)\big), \qquad \alpha+\varepsilon < r < \beta-\varepsilon.$$

As shown by Serbinowski [Se, Lemma 2.5] (his proof is reproduced below in the proof of Lemma 2(a) in Section 5),

$$d_Y\big(\varphi(r\xi), \varphi\big((r+\varepsilon)\xi\big)\big) \leqslant \int_0^\varepsilon |\varphi_*(\omega)|\big((r+t)\xi\big)\,dt$$

for $\sigma$-a.e. $\xi \in S^{m-1}$ and for Lebesgue a.e. $r \in ]\alpha+2\varepsilon, \beta-2\varepsilon[$ . For any function $f \in C_c^+(]\alpha,\beta[)$ it follows for small $\varepsilon > 0$ that

$$\begin{aligned} \int_\alpha^\beta {}^\omega e_\varepsilon(\varphi)(r\xi) f(r)\,dr &\leqslant \frac{1}{\varepsilon}\int_0^\varepsilon dt \int_\alpha^\beta |\varphi_*(\omega)|\big((r+t)\xi\big) f(r)\,dr \\ &\to \int_\alpha^\beta |\varphi_*(\omega)|(r\xi) f(r)\,dr \end{aligned} \tag{3.3}$$

as $\varepsilon \to 0$, the inner integral of convolution type on the right of the inequality being continuous in $t$ on $]0,\varepsilon[$ in view of (3.1). Consequently, the lim sup of the left-hand member of (3.3) for $\varepsilon \to 0$ is no bigger than $\int_\alpha^\beta |\varphi_*(\omega)|(r\xi) f(r)\,dr < \infty$, and so $\varphi_\xi$ has indeed (for $\sigma$-a.e. $\xi \in S^{m-1}$) finite energy on $]\alpha,\beta[$ , with energy density $e(\varphi_\xi)$ satisfying (3.2) there. Using (3.1) fully, this shows that $E(\varphi_\xi) < \infty$, and that (3.2) holds for a.e. $r \in \mathbf{R}_+$.

*Proof of Theorem* 1. The assertions are easily reduced to the case where $E(\varphi) < \infty$, i.e., $\sqrt{e(\varphi)} \in L^2(X)$. The case $m = 1$ is contained in [KS, Lemma 1.9.2], so suppose $m \geqslant 2$. Topological notions relative to the Cartan fine topology on $X$ (mentioned early in Section 2) are indicated by adding "fine(ly)". Referring to [EF, Proposition 7.8] we recall that a map $\varphi\colon X \to Y$ is quasicontinuous if and only if $\varphi$ is *finely continuous quasi-eveywhere*, i.e., everywhere except in some *polar* set. A polar set is the same as a finely discrete and hence finely closed set; it is also the same as a set of capacity $0$. A polar set has $\mu$-measure $0$, and a nonvoid finely open set has $\mu$-measure $> 0$, cf. [F1, Section 8, Lemma 4].

*Case* 1. Let $X = \mathbf{R}^m$ with the Euclidean Riemannian metric. The set

$$P' = \left\{ a \in X : \int_X |a-x|^{1-m} \sqrt{e(\varphi)(x)}\, dx = \infty \right\} \tag{3.4}$$

is *polar*, as shown by Deny [De2] with $\sqrt{e(\varphi)}$ replaced by any function $f \in L^2(\mathbf{R}^m)$, $f \geqslant 0$. Consider a point $a \in X \setminus P'$ and a fine neighbourhood $U$ of $a$ in $X$. As shown in [De1], $U$ contains for $\sigma$-a.e. $\xi \in S^{m-1}$ a straight line segment $[a, a+\varrho(\xi)\xi]$, $\varrho(\xi) > 0$. According to Lemma 1 we may assume that the map $r \mapsto \varphi(a + r\xi)$ has a version of bounded variation over $]0, \varrho(\xi)]$, and so there exists

$$\varphi^*(a,\xi) := \operatorname*{ess\,lim}_{0<r\to 0} \varphi(a + r\xi) \in Y \tag{3.5}$$

for $\sigma$-a.e. $\xi \in S^{m-1}$, $(Y, d_Y)$ being complete.

Denote by $\varphi^*(a, S^{m-1})$ the $\sigma$-essential range of the map $\xi \mapsto \varphi^*(a,\xi)$ of $S^{m-1}$ into $Y$; it is independent of $U$.

In order next to prove that $\varphi^*(a, S^{m-1})$ consists of a *single point*, choose a dense sequence $(z_n)$ in $Y$, and write for brevity

$$d_Y\bigl(\varphi(x), z_n\bigr) = v(x, z_n). \tag{3.6}$$

By [EF, Corollaries 9.1, 9.2], $v(\,\cdot\,, z_n) \in W^{1,2}(X, \mathbf{R}) = W^{1,2}(X)$, and hence $v(\,\cdot\,, z_n)$ has a quasicontinuous version $v^*(\,\cdot\,, z_n)$ (see e.g. [EF, proof of Theorem 7.2]). Thus there is a $\mu$-nullset $N$ such that, for all $n$,

$$v(\,\cdot\,, z_n) = v^*(\,\cdot\,, z_n) \qquad \text{in } X \setminus N, \tag{3.7}$$

and a polar set $P'' \subset X$ such that $v^*(\,\cdot\,, z_n)$ is finely continuous in $X \setminus P''$, [EF, Proposition 7.8(c)]; then $P := P' \cup P''$ is likewise polar.

Henceforth, let $a \in X \setminus P$. Given $\varepsilon > 0$ and $y \in \varphi^*(a, S^{m-1})$, choose $i = i(y)$ so that $d_Y(y, z_i) < \varepsilon$; then

$$d_Y\bigl(\varphi^*(a,\xi), z_i\bigr) < \varepsilon \tag{3.8}$$

for points $\xi \in S^{m-1}$ forming a set of $\sigma$-measure $> 0$.

For $U$ above take now any one of the following finely open sets containing $a$:

$$U_{n,\varepsilon} = \{x \in X \setminus P : |v^*(x, z_n) - v^*(a, z_n)| < \varepsilon\}. \tag{3.9}$$

By (3.6), (3.7), and (3.9),

$$\bigl|d_Y\bigl(\varphi(x), z_n\bigr) - v^*(a, z_n)\bigr| < \varepsilon \tag{3.10}$$

for every $x = a + r\xi \in U_{n,\varepsilon} \setminus N$; and hence for $\sigma$-a.e. $\xi \in S^{m-1}$ and for (Lebesgue) a.e. small $r > 0$, by [De1], as noted above (because $a + r\xi \notin N$ for $\sigma$-a.e. $\xi \in S^{m-1}$ and a.e. $r > 0$). For $r \to 0$ this leads by (3.5) for $\sigma$-a.e. $\xi \in S^{m-1}$ to

$$\bigl|d_Y\bigl(\varphi^*(a, \xi), z_n\bigr) - v^*(a, z_n)\bigr| \leqslant \varepsilon. \tag{3.11}$$

Applying (3.11) with $n = i$ from (3.8), and combining with (3.8), gives $v^*(a, z_i) < 2\varepsilon$. For any other point $y' \in \varphi^*(a, S^{m-1})$ choose similarly $j = j(y')$ so that $d_Y(y', z_j) < \varepsilon$ and hence $v^*(a, z_j) < 2\varepsilon$. For $x = a + r\xi \in U_{i,\varepsilon} \cap U_{j,\varepsilon} \setminus N$ $(\neq \varnothing)$ we thus obtain from (3.10)

$$d_Y(z_i, z_j) \leqslant d_Y\bigl(\varphi(x), z_i\bigr) + d_Y\bigl(\varphi(x), z_j\bigr) \leqslant v^*(a, z_i) + v^*(a, z_j) + 2\varepsilon < 6\varepsilon.$$

Consequently, $d_Y(y, y') < d_Y(z_i, z_j) + 2\varepsilon < 8\varepsilon$. This holds for any two $y, y' \in \varphi^*(a, S^{m-1})$ and for any $\varepsilon > 0$. Thus $y = y'$, and $\varphi^*(a, S^{m-1})$, defined for $a \in X \setminus P$, reduces indeed to a point $\varphi^*(a) \in Y$; hence (3.5) holds with $\varphi^*(a, \xi)$ replaced by $\varphi^*(a)$.

Again for $a \in X \setminus P$ and for given $\varepsilon > 0$, choose $i$ so that now $d_Y\bigl(\varphi^*(a), z_i\bigr) < \varepsilon$. Inserting $\varphi^*(a, \xi) = \varphi^*(a)$ in (3.11) we obtain as above $v^*(a, z_i) < 2\varepsilon$. For $x \in U_{i,\varepsilon} \setminus N$, cf. (3.9), we therefore have $d_Y\bigl(\varphi(x), z_i\bigr) < 3\varepsilon$ in view of (3.10). Hence, for such $x$,

$$d_Y\bigl(\varphi(x), \varphi^*(a)\bigr) \leqslant d_Y\bigl(\varphi(x), z_i\bigr) + d_Y\bigl(\varphi^*(a), z_i\bigr) < 4\varepsilon.$$

Consequently, for every $a \in X \setminus P$,

$$\varphi(x) \to \varphi^*(a) \qquad \text{as } x \to a \text{ finely through } X \setminus N. \tag{3.12}$$

It follows by (3.6), (3.7), and (3.12) that, for every $a \in X \setminus P$ and every $n$,

$$v^*(a, z_n) = \operatorname*{fine\,lim}_{X \setminus N \ni x \to a} d_Y\bigl(\varphi(x), z_n\bigr) = d_Y\bigl(\varphi^*(a), z_n\bigr). \tag{3.13}$$

If $a \in X \setminus (N \cup P)$ this implies $\varphi(a) = \varphi^*(a)$ because

$$d_Y\bigl(\varphi(a), z_n\bigr) = v(a, z_n) = v^*(a, z_n) = d_Y\bigl(\varphi^*(a), z_n\bigr)$$

for every $n$, and because a point of $Y$ is uniquely determined by its distances from the points $z_n$ of a dense sequence in $Y$.

For $a \in X \setminus P$ and for given $\varepsilon > 0$ choose $n = n(\varepsilon)$ so that $d_Y\big(\varphi^*(a), z_n\big) < \varepsilon$; and let $x \in U_{n,\varepsilon} \setminus P$. By (3.13) applied with $a$ replaced by $x$, by (3.9), and by (3.13) as it stands, it follows that

$$\begin{aligned} d_Y\big(\varphi^*(x), \varphi^*(a)\big) &\leqslant d_Y\big(\varphi^*(x), z_n\big) + \varepsilon = v^*(x, z_n) + \varepsilon \\ &< v^*(a, z_n) + 2\varepsilon = d_Y\big(\varphi^*(a), z_n\big) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Thus $\varphi^*$ is finely continuous at every point $a$ of the finely open set $X \setminus P$. Regardless of how $\varphi^*$ is defined in $P$ we conclude that $\varphi^*$ *is a quasicontinuous version of* $\varphi$, cf. [EF, Proposition 7.8(c)], because $\varphi^*(a) = \varphi(a)$ for $a$ not in the $\mu$-nullset $N \cup P$.

*Case* 2. Let $X$ be the union of closed halfspaces $X_1, \dots, X_k$ (in copies $\widetilde{X}_1, \dots, \widetilde{X}_k$ of $\mathbf{R}^m$), disjoint save for a common boundary hyperplane—a copy $X_0$ of $\mathbf{R}^{m-1}$. Let $\tau_i$ denote reflection of $\widetilde{X}_i$ in $X_0$. We endow $X$ with the Euclidean Riemannian structure and associated intrinsic distance $d_X$, cf. [EF, Section 4]. Claims:

(i) A set $A \subset X_i$ is *polar* relative to $X$ if and only if $A$ is polar relative to $X_i$, or equivalently: relative to $\widetilde{X}_i = X_i \cup \tau_i(X_i)$ $(= \mathbf{R}^m)$.

(ii) A set $A \subset X_i$ is *thin* at a point $x_0 \in X_i$ if and only if $A$ is thin at $x_0$ relative to $X_i$, or equivalently: relative to $\widetilde{X}_i$.

Note that, in case $k = 2$, $\widetilde{X}_i$ is the same as the space $X$ itself. A permutation $\pi$ of $\{1, \dots, k\}$ induces an isometry $\pi\colon X \to X$ such that $\pi|_{X_j}$ $(j \in \{1, \dots, k\})$ is the obvious identification of $X_j$ with $X_{\pi(j)}$, leaving $X_0$ pointwise fixed. Every neighbourhood in $X$ of a point of $X_0$ contains an open neighbourhood $\omega$ of that point such that $\omega$ is *symmetric*, i.e., $\pi(\omega) = \omega$ for every $\pi$.

Ad (i). Suppose first that $A \subset X_i$ is polar in $X$. Clearly, $A \setminus X_0$ is then polar in $X_i$ as well, so we may assume that $A \subset X_0$. There exists a superharmonic function $u \geqslant 0$ on a symmetric neighbourhood $\omega$ in $X$, as above, such that $u = \infty$ on $A \cap \omega$. For any permutation $\pi$, $u \circ \pi$ has the same properties because $A \subset X_0$, and so has $u^* := \sum_\pi u \circ \pi$ $(\geqslant u)$. It remains to prove that $u^*$, restricted to $\omega_i := \omega \cap X_i$, is superharmonic relative to $X_i$. Any $\lambda \in \mathrm{Lip}_c^+(\omega_i)$ extends by symmetry across $X_0$ to a symmetric (i.e., permutation invariant) function $\lambda^* \in \mathrm{Lip}_c^+(\omega)$, and

$$k \int_{\omega_i} \langle \nabla u^*, \nabla \lambda \rangle \, d\mu = \int_\omega \langle \nabla u^*, \nabla \lambda^* \rangle \, d\mu \geqslant 0$$

because $u^*$ is superharmonic and symmetric in $\omega$ relative to $X$. This shows that $(u^*)|_{\omega_i}$ is weakly superharmonic, and even *superharmonic*. Indeed, there is a (unique) superharmonic function $\tilde{u}$ on $\omega_i$ and a $\mu$-nullset $N \subset \omega_i$ such that $\tilde{u} = u^*$ in $\omega_i \setminus N$. Actually we may take $N \subset \omega_i \cap X_0$ because $u^*$ is superharmonic in the open set $\omega_i \setminus X_0$. According to [EF, Proposition 7.8(d)] it follows for every $x_0 \in \omega \cap N = \omega_i \cap N$ that

$$\tilde{u}(x_0) = \liminf_{\omega_i \setminus N \ni x \to x_0} \tilde{u}(x) = \liminf_{\omega_i \setminus N \ni x \to x_0} u^*(x) = u^*(x_0). \tag{3.14}$$

The first, respectively last, equation (3.14) holds because $\tilde{u}$, respectively $u^*$, is superharmonic in $\omega_i$, respectively $\omega$, and because $\omega_i \setminus N$ is not thin at $x_0$ relative to $X_i$, respectively $X$, cf. (2.1) (for then the nullset $N$ would be a fine neighbourhood of $x_0$ in $\omega_i$, respectively the isometric images $\omega_j \setminus N$ of $\omega_i \setminus N$, $j = 1, \dots, k$, would likewise be thin at $x_0$ relative to $X$, and so would their union $\omega \setminus N$, i.e., $N$ would again be a fine neighbourhood of $x_0$ in $X$). We conclude that $\tilde{u} = u^*$ holds not only in $\omega_i \setminus N$, but also in $\omega_i \cap N$, by (3.14), and so $u^* = \tilde{u}$ is indeed superharmonic in all of $\omega_i$ relative to $X_i$.

Conversely, if $A \subset X_i$ is polar relative to $X_i$ then $A \setminus X_0$ is polar also in $X$, so we may assume again that $A \subset X_0$. There exists a superharmonic function $u_i \geqslant 0$ in some open neighbourhood $\omega_i$ in $X_i$ of a given point of $X_0$ such that $u_i = \infty$ on $A \cap \omega_i$. By symmetry across $X_0$, $\omega_i$ extends to a symmetric open set $\omega \subset X$ with $\omega \cap X_i = \omega_i$, and $u_i$ extends to a symmetric weakly superharmonic function $u \geqslant 0$ on $\omega$ with $u = \infty$ on $A \cap \omega = A \cap \omega_i$. It is shown much like above (replacing now $\omega_i, u^*$ by $\omega, u$ in (3.14) and exploiting the symmetry of $\omega$ and $u$) that $u$ is actually superharmomnic in $\omega$.

Ad (ii). Thinness being a local property, we may assume that $x_0 \in X_0$. Suppose first that $A \subset X_i$ is thin at $x_0$ relative to $X$, and let $u \geqslant 0$ denote a superharmonic function on a symmetric open neighbourhood $\omega$ of $x_0$ in $X$ such that

$$u(x_0) < \liminf_{A \cap \omega \ni x \to x_0} u(x), \tag{3.15}$$

cf. (2.1). Exactly as in the proof of (i), $u^* := \sum_\pi u \circ \pi \geqslant 0$ is likewise superharmonic on $\omega$; and $u^*$, restricted to $\omega_i := \omega \cap X_i$, is weakly superharmonic, and indeed superharmonic. From (3.15) follows the same with $\omega$ replaced by $\omega_i$ and $u$ by $u^*|_{\omega_i}$. Because $A \cap \omega_i = A \cap \omega$ this is seen by adding over all permutations $\pi$ the inequalities

$$(u \circ \pi)(x_0) \leqslant \liminf_{A \cap \omega \ni x \to x_0} (u \circ \pi)(x),$$

valid by lower semicontinuity of $u$ and hence of $u \circ \pi$, noting that there is sharp inequality for $\pi = \mathrm{id}$ (the identity permutation) by (3.15) as it stands. It follows that $A$ is indeed thin at $x_0$ relative to $X_i$.

Conversely, if $A \subset X_i$ is thin at $x_0 \in X_0$ relative to $X_i$, there exists a superharmonic function $u_i \geqslant 0$ on an open neighbourhood $\omega_i$ of $x_0$ in $X_i$ such that (3.15) holds with $\omega, u$ replaced by $\omega_i, u_i$. Exactly as in (i), by symmetry across $X_0$, $\omega_i$ extends to a symmetric open neighbourhood $\omega$ of $x_0$ in $X$ with $\omega \cap X_i = \omega_i$, and $u_i$ extends to a superharmonic function $u \geqslant 0$ on $\omega$. Then (3.15) holds as it stands because $u(x_0) = u_i(x_0)$ and $u = u_i$ on $\omega \cap A = \omega_i \cap A$. Consequently, $A$ is indeed thin at $x_0$ relative to $X$.

Thus prepared, we now establish Theorem 1 in the present Case 2. The set

$$P' = \left\{ a \in X : \int_X d_X(a,x)^{1-m} \sqrt{e(\varphi)(x)}\, dx = \infty \right\} \tag{3.16}$$

(cf. (3.4)) is *polar* (in $X$). Indeed, $P'$ is covered by the sets $P'_i \subset X_i$ obtained by replacing $X$ by $X_i$ on the right-hand side of (3.16), $i \in \{1, \dots, k\}$; and each $P'_i$ is polar in $X_i$ by (i) above, being polar in $\widetilde{X}_i = X_i \cup \tau_i(X_i) = \mathbf{R}^m$ according to Case 1 because $\sqrt{e(\varphi)}|_{X_i} \in L^2(X_i)^+$ can be extended to a function of class $L^2(\widetilde{X}_i)^+$. Hence $P'_i$ is polar in $X$, by (i), and so is therefore $P'$.

For any point $a \in X \setminus P'$, every fine neighbourhood $U$ of $a$ in $X$ contains small segments issuing from $a$ in almost all directions, and the restriction of $\varphi$ to any of these segments has an essential limit at $a$. This follows at once from Case 1 if $a \notin X_0$, so we may assume that $a \in X_0$. For $i \in \{1, \dots, k\}$ write $U \cap X_i = U_i$, and denote $\widetilde{U}_i = U_i \cup \tau_i(U_i)$. Then $X \setminus U$ is thin at $a$, and hence, by (ii) above, $X_i \setminus U_i = X_i \setminus U$ is thin at $a$ relative to $X_i$ and therefore also relative to $\widetilde{X}_i$. It follows by symmetry that $\tau_i(X_i) \setminus \tau_i(U_i) = \tau_i(X_i \setminus U_i)$ is thin at $a$ relative to $\widetilde{X}_i$, and so is therefore $\widetilde{X}_i \setminus \widetilde{U}_i \subset (X_i \setminus U_i) \cup \big(\tau_i(X_i) \setminus \tau_i(U_i)\big)$. This means that $\widetilde{U}_i$ is a fine neighbourhood of $a$ in $\widetilde{X}_i$ ($= \mathbf{R}^m$). Writing $\varphi|_{X_i} = \varphi_i$, we have $e_\varepsilon(\varphi_i)(x) \leqslant e_\varepsilon(\varphi)(x)$ for every $x \in X_i$ (because $B_{X_i}(x,\varepsilon) \subset B_X(x,\varepsilon)$). For $\varepsilon \to 0$ it follows that $e(\varphi_i) \leqslant e(\varphi)$ on $X_i$. Denote $\widetilde{\varphi}_i \colon \widetilde{X}_i \to Y$ the extension of $\varphi_i$ to $\widetilde{X}_i$ by symmetry across $X_0$; then $e(\widetilde{\varphi}_i)$ is the "even" extension of $e(\varphi_i)$ from $X_i$ to $\widetilde{X}_i$ (this clearly holds in $\widetilde{X}_i \setminus X_0$, hence a.e. in $\widetilde{X}_i$). It follows that $E(\widetilde{\varphi}_i) \leqslant 2E(\varphi_i) \leqslant 2E(\varphi)$, and (3.5) therefore holds with $\varphi, U$ replaced by $\widetilde{\varphi}_i, \widetilde{U}_i$, or by $\varphi_i, U_i$ in particular.

The rest of the proof of Theorem 1 in Case 1 now carries over to the present Case 2, including the explicit description of a quasicontinuous version $\varphi^*$ of $\varphi$, whereby $\varphi^*(a)$ (even for $a \in X_0$) is the essential radial limit of $\varphi(x)$ as $x \to a$ in almost every direction (within each $X_i$), provided that $a \in X \setminus P$ for a certain polar set $P = P' \cup P'' \subset X$.

*Case* 3. Let $X$ be any admissible $m$-dimensional polyhedron, embedded in some Euclidean space $V$, with each simplex affinely embedded, cf. [EF, Remark 4.1]. We give $X$ the Euclidean Riemannian metric induced by that of $V$. The $(m-2)$-skeleton $X^{(m-2)}$ is a polar set [EF, Proposition 7.6], and may therefore be disregarded. The assertion of the theorem being local, we are left with the case of the star of an open $(m-1)$-simplex of $X$; and that is covered by Case 2.

*Case* 4. This is the general case, where $(X, g)$ is an arbitrary admissible Riemannian polyhedron ($g$ simplexwise smooth). We may assume that $X$ is compact; then $g$ is equivalent with the Euclidean Riemannian metric $g^e$ induced by that of a Euclidean space $V$ in which $X$ is embedded, as in Case 3. Simple estimates involving the ellipticity constant $\Lambda$ of $g$ allow us to reduce the theorem to Case 3 above, cf. [EF, Section 4 and Corollary 7.1, p. 120].

## 4. Proof of Proposition 1

*Step* 1. For any (ordered) quadruple $PQRS$ in $B = B_Y(q, R)$ (no restriction on the perimeter of the corresponding quadrilateral) with midpoints $M$ and $N$ of $PS$ and $QR$, respectively, we show that

$$MN \leqslant \frac{c}{2}(PQ + RS) \leqslant c\sqrt{PQ^2 + RS^2}, \qquad c = \frac{\pi}{\sqrt{2}\,\cos R}. \tag{4.1}$$

Here and elsewhere we write briefly $PQ$ in place of $d_Y(P, Q)$ for points $P, Q \in Y$.

Consider first the case $PQ, RS < \varrho := \frac{1}{2}\pi - R$. Then [F2, Lemma 2(a)] applies and produces a (convex) *comparison trapezoid* $\widetilde{P}\widetilde{Q}\widetilde{R}\widetilde{S}$ in the unit sphere $S^2$ in $\mathbf{R}^3$, symmetric about a great circle $\widetilde{\gamma}$ in $S^2$, and with side lengths

$$\widetilde{P}\widetilde{S} = PS, \quad \widetilde{Q}\widetilde{R} = QR, \quad \widetilde{P}\widetilde{Q} = \widetilde{R}\widetilde{S} = PQ \diamond RS, \tag{4.2}$$

where the "cosine mean" $a \diamond b \in \left[0, \frac{1}{2}\pi\right]$ of two numbers $a, b \in \left[0, \frac{1}{2}\pi\right]$ is defined by

$$\cos(a \diamond b) = \tfrac{1}{2}(\cos a + \cos b). \tag{4.3}$$

By [F2, Lemma 2(b)], $MN \leqslant \widetilde{M}\,\widetilde{N}$ ($\widetilde{M}$ and $\widetilde{N}$ denoting the midpoint of $\widetilde{P}\widetilde{S}$ and $\widetilde{Q}\widetilde{R}$, respectively). Let $\widetilde{O}$ denote the pole of $\widetilde{\gamma}$ in $S^2$ on the same side of $\widetilde{\gamma}$ as $\widetilde{R}$ and $\widetilde{S}$. The cosine relation for the spherical triangle $\widetilde{P}\widetilde{Q}\widetilde{O}$ with angle $\theta$ at $\widetilde{O}$ may be written

$$\sin^2\bigl(\tfrac{1}{2}\widetilde{P}\widetilde{Q}\bigr) = \sin^2\bigl[\tfrac{1}{2}(\widetilde{O}\widetilde{P} - \widetilde{O}\widetilde{Q})\bigr] + \sin\widetilde{O}\widetilde{P}\sin\widetilde{O}\widetilde{Q}\sin^2\bigl(\tfrac{1}{2}\theta\bigr)$$

(also if the triangle degenerates). Because $MN \leqslant \widetilde{M}\,\widetilde{N} = \theta < \pi$ and $\sin\widetilde{O}\widetilde{P} = \cos\bigl(\frac{1}{2}\widetilde{P}\widetilde{S}\bigr) \geqslant \cos R$, etc., it follows by (4.2), (4.3) that

$$\begin{aligned}\cos^2 R\sin^2\bigl(\tfrac{1}{2}MN\bigr) &\leqslant \sin^2\bigl(\tfrac{1}{2}\widetilde{P}\widetilde{Q}\bigr) = \tfrac{1}{2}\sin^2\bigl(\tfrac{1}{2}PQ\bigr) + \tfrac{1}{2}\sin^2\bigl(\tfrac{1}{2}RS\bigr)\\ &\leqslant \tfrac{1}{8}(PQ^2 + RS^2) \leqslant \tfrac{1}{8}(PQ + RS)^2,\end{aligned}$$

and so indeed $MN\cos R \leqslant \pi\cos R\sin\bigl(\frac{1}{2}MN\bigr) \leqslant \bigl(\pi/\sqrt{8}\,\bigr)(PQ + RS)$.

For an arbitrary quadruple $PQRS$ in $B$, (4.1) now follows by partitioning $PQ$, respectively $SR$, into $n$ equal segments of length $PQ/n, RS/n \leqslant 2R/n < \varrho$. It remains to apply (4.1) to each of the $n$ smaller quadrilaterals thus obtained, and to add up, using the triangle inequality. When applied to the quadrilateral

$$PQRS = \varphi_0(x)\varphi_0(x')\varphi_1(x')\varphi_1(x),$$

and hence $M = \varphi_{1/2}(x)$, $N = \varphi_{1/2}(x')$, it follows from the latter inequality (4.1) in view of (1.2) that

$$E(\varphi_{1/2}) \leqslant c^2\big(E(\varphi_0) + E(\varphi_1)\big). \tag{4.4}$$

*Step* 2. Suppose that $u := d_Y(\varphi_0, \varphi_1) < \varrho$. Note that $u \in W^{1,2}(X)$, by (2.2). For any $x \in X$ and $x' \in B_X(x, \varepsilon)$, [F2, Lemma 2] applies to the quadrilateral $PQRS$ defined in the preceding paragraph. That produces a comparison trapezoid $\widetilde{P}\widetilde{Q}\widetilde{R}\widetilde{S}$ in $S^2$, symmetric about a great circle $\widetilde{\gamma}$ in $S^2$, and with sidelengths as in (4.2). Write

$$\begin{aligned}\varphi_\kappa(x) &= \big(1-\kappa(x)\big)\varphi_0(x) + \kappa(x)\varphi_1(x),\\ \varphi_{\kappa'}(x') &= \big(1-\kappa(x')\big)\varphi_0(x') + \kappa(x')\varphi_1(x');\end{aligned}$$

furthermore,

$$d_Y\big(\varphi_i(x), \varphi_i(x')\big) = d_i \quad (i = 0, 1), \qquad d_Y\big(\varphi_\kappa(x), \varphi_{\kappa'}(x')\big) = d_\kappa, \tag{4.5}$$

and similarly with $\kappa$ replaced by $1-\kappa$. Finally, write $u(x) = u$, $u(x') = u'$, $\kappa(x) = \kappa$, $\kappa(x') = \kappa'$, and consider in $S^2$ the points

$$\widetilde{P}_\kappa = (1-\kappa)\widetilde{P} + \kappa\widetilde{S}, \quad \widetilde{Q}_{\kappa'} = (1-\kappa')\widetilde{Q} + \kappa'\widetilde{R}.$$

According to [F2, Lemma 2(b)], $d_\kappa \diamond d_{1-\kappa} \leqslant \widetilde{P}_\kappa\widetilde{Q}_{\kappa'}$, and since $d_0 \diamond d_1 = \widetilde{P}\widetilde{Q}$ by (4.2), we obtain

$$\tfrac{1}{2}(\cos d_0 - \cos d_\kappa) + \tfrac{1}{2}(\cos d_1 - \cos d_{1-\kappa}) \leqslant \cos\widetilde{P}\widetilde{Q} - \cos\widetilde{P}_\kappa\widetilde{Q}_{\kappa'}. \tag{4.6}$$

Let $\widetilde{O}$ denote the pole of $\widetilde{\gamma}$ in $S^2$ on the same side of $\widetilde{\gamma}$ as $\widetilde{R}$ and $\widetilde{S}$, and write

$$d := d_0 \diamond d_1 = \widetilde{P}\widetilde{Q}, \tag{4.7}$$

$$v := \widetilde{O}\widetilde{P} = \tfrac{1}{2}(\pi + u), \quad v' := \widetilde{O}\widetilde{Q} = \tfrac{1}{2}(\pi + u'). \tag{4.8}$$

Then $v, v' \in \big[\tfrac{1}{2}\pi, \tfrac{1}{2}\pi + R\big]$, and so $0 \leqslant -\cot v, -\cot v' \leqslant \tan R$.

Define a function $\lambda \in W^{1,2}(X)$, $0 \leqslant \lambda \leqslant 1$, by $\lambda = \kappa u/v$, and write $\lambda(x) = \lambda$, $\lambda(x') = \lambda'$. Then $\widetilde{P}\widetilde{P}_\kappa = \kappa u = \lambda v = \lambda \widetilde{O}\widetilde{P}$ and $\widetilde{Q}\widetilde{Q}_{\kappa'} = \lambda' \widetilde{O}\widetilde{Q}$.

By the spherical cosine relation, applied to the triangles $\widetilde{O}\widetilde{P}\widetilde{Q}$ and $\widetilde{O}\widetilde{P}_\kappa\widetilde{Q}_{\kappa'}$, we therefore obtain (cf. [EF, p. 193]), after eliminating the common angle at $\widetilde{O}$,

$$\begin{aligned}\sin v \sin v' \cos \widetilde{P}_\kappa\widetilde{Q}_{\kappa'} = {} & \sin(v - \lambda v)\sin(v' - \lambda' v')\cos d \\ & + \sin(v - \lambda v)\sin(\lambda' v')\cos v \\ & + \sin v' \sin(\lambda v)\cos(v' - \lambda' v').\end{aligned}$$

Insert this expression for $\cos \widetilde{P}_\kappa\widetilde{Q}_{\kappa'}$ in (4.6), together with $\cos \widetilde{P}\widetilde{Q} = \cos d = \frac{1}{2}(\cos d_0 + \cos d_1)$ from (4.5), (4.7). After some manipulations serving to make (1.2) through (1.4) applicable this leads to

$$\tag{4.9} \tfrac{1}{2}(\cos d_0 - \cos d_\kappa) + \tfrac{1}{2}(\cos d_1 - \cos d_{1-\kappa}) \leqslant R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)},$$

cf. [EF, equation (10.17)], [F1, equation (8.2)]. Here

$$\begin{aligned} R^{(1)} &:= -2\sin^2\left(\tfrac{1}{2}d\right)\left(1 - \frac{\sin(v - \lambda v)}{\sin v}\,\frac{\sin(v' - \lambda' v')}{\sin v'}\right) \\ &\leqslant C_1(d_0^2 + d_1^2); \qquad\qquad (4.10) \\ R^{(2)} &:= \cos(\lambda v)\cos(\lambda' v')(\cos v - \cos v')\left(\frac{\tan(\lambda v)}{\sin v} - \frac{\tan(\lambda' v')}{\sin v'}\right) \\ &\leqslant C_2\left[(\cos v - \cos v')^2 + \left(\frac{\tan(\lambda v)}{\sin v} - \frac{\tan(\lambda' v')}{\sin v'}\right)^2\right]; \\ R^{(3)} &:= (\cos v - \cos v')^2\frac{\sin(\lambda v)}{\sin v}\,\frac{\sin(\lambda' v')}{\sin v'} \leqslant (\cos v - \cos v')^2; \\ R^{(4)} &:= 2\sin^2\frac{\lambda v - \lambda' v'}{2} - 2\sin^2\frac{v - v'}{2}\,\frac{\sin(\lambda v)}{\sin v}\,\frac{\sin(\lambda' v')}{\sin v'} \leqslant \tfrac{1}{2}(\lambda v - \lambda' v')^2, \end{aligned}$$

where $C_1, C_2$ and subsequent constants $C_3, \ldots$ depend on $R$ and $\dim X = m$ only.

The power series of $1 - \cos t$ is alternating, with terms that decrease in absolute value $t^{2n}/(2n)!$, $n \geqslant 1$, when $t^2 < 4!/2! = 12$. Since $d_0, d_\kappa \leqslant 2R < \pi < \sqrt{12}$, it follows that $\frac{1}{2}d_0^2 \geqslant 1 - \cos d_0$ and $\frac{1}{2}d_\kappa^2 - \frac{1}{24}d_\kappa^4 \leqslant 1 - \cos d_\kappa$. Inserting $1 - \frac{1}{12}d_\kappa^2 \geqslant 1 - \frac{1}{12}\pi^2 = C_3 > 0$, leads to

$$\tfrac{1}{2}C_3 d_\kappa^2 \leqslant \tfrac{1}{2}d_0^2 + (\cos d_0 - \cos d_\kappa).$$

Adding to this the corresponding inequality with $\kappa$ replaced by $1 - \kappa$, and 0 by 1, we obtain for $f \in C_c(X, [0,1])$ after dividing by $2\varepsilon^{m+2}$ and invoking (1.1), (1.3), (4.3), and (4.9):

$$C_3 \limsup_{\varepsilon\to 0} \int_X \big(e_\varepsilon(\varphi_\kappa) + e_\varepsilon(\varphi_{1-\kappa})\big) f \, d\mu \leqslant \limsup_{\varepsilon\to 0} \int_X \big(e_\varepsilon(\varphi_0) + e_\varepsilon(\varphi_1)\big) f \, d\mu$$

$$+ 4 \limsup_{\varepsilon\to 0} \int_X f(x)\, d\mu(x) \int_{B_X(x,\varepsilon)} \frac{1}{\varepsilon^{m+2}} \sum_{j=1}^{4} R^{(j)} \, d\mu(x'). \tag{4.11}$$

Inserting the above estimates of $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, and $R^{(4)}$ in (4.11), we obtain by application of (1.2) through (1.4) and viewing $C_c(X,[0,1])$ as an upper directed set:

$$\begin{aligned} C_3\big(E(\varphi_\kappa) + E(\varphi_{1-\kappa})\big) &\leqslant (1+4C_1)\big(E(\varphi_0) + E(\varphi_1)\big) \\ &\quad + 4C_2 \int_X \left( |\nabla \cos v|^2 + \left| \nabla \frac{\tan(\lambda v)}{\sin v} \right|^2 \right) d\mu \\ &\quad + 4 \int_X |\nabla \cos v|^2 \, d\mu + 2 \int_X |\nabla (\lambda v)|^2 \, d\mu \\ &\leqslant C_4 \left( E(\varphi_0) + E(\varphi_1) + \int_X \big( |\nabla u|^2 + |\nabla \kappa|^2 \big) \, d\mu \right) \end{aligned}$$

after an easy reduction, invoking (2.2), (4.8), and $\lambda = \kappa u/v$. This leads to (2.3) in view of (2.2).

*Step* 3. In the general case we have, by (4.4), $E(\varphi_{1/2}) \leqslant c^2\big(E(\varphi_0) + E(\varphi_1)\big)$, hence $E(\varphi_{1/4}), E(\varphi_{3/4}) \leqslant (c^2 + c^4)\big(E(\varphi_0) + E(\varphi_1)\big)$, etc. Choose an integer $n \geqslant 1$ so that $3{\cdot}2^{-n} 2R < \varrho$. For any number $\alpha \in [0,1]$ such that $2^n \alpha$ is an integer it follows that

$$E(\varphi_\alpha) \leqslant C_5\big(E(\varphi_0) + E(\varphi_1)\big), \quad C_5 := c^2 + \cdots + c^{2n}. \tag{4.12}$$

For integers $i \in [-1, 2^n + 1]$ write $2^{-n} i = \alpha_i$. For $i \in [0, 2^n - 1]$ and $x \in X$ define $\kappa_i(x) \in [0,1]$ as the number in the interval $[\alpha_{i-1}, \alpha_{i+2}]$ nearest to $\kappa(x)$; then $\kappa_i \in W^{1,2}(X)$, by [EF, Proposition 5.1(c)]. Suppose $\kappa \in \mathrm{Lip}(X)$ (cf. Step 4 below). Let $\varphi_\kappa^{(i)}$ denote the restriction of $\varphi_\kappa$ to the open set

$$X^{(i)} := \kappa^{-1}(\,]\alpha_{i-1}, \alpha_{i+2}[\,)$$

in which $\kappa = \kappa_i$ and hence $\varphi_\kappa = \varphi_{\kappa_i}$. Every connectivity component of $X^{(i)}$ is an admissible Riemannian polyhedron, like $X$; and (in case $i \in [1, 2^n - 2]$)

$$\varphi_\kappa^{(i)} = \varphi_{\kappa_i} = (1-\eta)\varphi_{\alpha_{i-1}} + \eta \varphi_{\alpha_{i+2}} \quad \text{in } X^{(i)}$$

with $\eta(x) = \frac{1}{3}2^n\big(\kappa(x) - \alpha_{i-1}\big) \in \,]0,1[$ for $x \in X^{(i)}$; furthermore, $\eta$ has finite Dirichlet integral, like $\kappa$; and

$$u^{(i)} := d_Y(\varphi_{\alpha_{i-1}}, \varphi_{\alpha_{i+2}}) = (\alpha_{i+2} - \alpha_{i-1})d_Y(\varphi_0, \varphi_1) = 3{\cdot}2^{-n}u < \varrho,$$

by choice of $n$ (since $u := d_Y(\varphi_0, \varphi_1) \leqslant 2R$). We therefore obtain from the case of (2.3) established in Step 2, but now applied with $X, \varphi_0, \varphi_1, \kappa$ replaced by $X^{(i)}, \varphi_{\alpha_{i-1}}, \varphi_{\alpha_{i+2}}, \eta$:

$$E(\varphi_\kappa^{(i)}) \leqslant C_6\bigg(E(\varphi_{\alpha_{i-1}}) + E(\varphi_{\alpha_{i+2}}) + \int_{X^{(i)}} |\nabla\eta|^2\, d\mu\bigg).$$

Here $|\nabla\eta| = \frac{1}{3}2^n|\nabla\kappa|$. In view of (4.12) applied to $\alpha = \alpha_{i-1}, \alpha_{i+2}$, this establishes (2.3) restricted to $X^{(i)}$ with $i \in [1, 2^n-2]$; the remaining cases $i = 0$ and $i = 2^n-1$ are handled similarly. Because the $X^{(i)}$, $i \in [0, 2^n - 1]$, cover $X$ we have now proved (2.3) for the whole of $X$.

*Step* 4. Suppose first that $X$ is compact, hence $\kappa \in W^{1,2}(X)$. To remove the extra hypothesis $\kappa \in \mathrm{Lip}(X)$ made in Step 3, we approximate $\kappa$ in $W^{1,2}$-norm by a sequence of functions $\kappa_j \in \mathrm{Lip}(X)$, $0 \leqslant \kappa_j \leqslant 1$; then $\varphi_{\kappa_j} \to \varphi_\kappa$ in $L^2(X, B)$ because $d_Y(\varphi_{\kappa_j}, \varphi_\kappa) = |\kappa_j - \kappa| d_Y(\varphi_0, \varphi_1)$ with $d_Y(\varphi_0, \varphi_1) \leqslant 2R$. Since (2.2) holds for each $\kappa_j$ in place of $\kappa$, and since $\int_X |\nabla\kappa_j|^2\, d\mu \to \int_X |\nabla\kappa|^2\, d\mu < \infty$, we have $\sup_j E(\varphi_{\kappa_j}) < \infty$. Consequently, (2.2) follows for $j \to \infty$ (and with the same constant $C$) by lower semicontinuity of energy [EF, Lemma 9.1].

Finally, if $X$ is noncompact, exhaust it by an increasing sequence of finite subpolyhedra $X_j$. From (2.2) applied to the restrictions of $\varphi_0$, $\varphi_1$, and $\kappa$ to each $X_j$ follows that $\varphi_\kappa \in W^{1,2}_\psi(X, B)$, and in the limit as $j \to \infty$ that (2.2) holds as it stands.

## 5. Proof of Proposition 2

In our presentation of Serbinowski's proof of his inequality (2.7) in Proposition 2 above we bring in Lemma 4 below a variant of the proof of [Se, Lemma 2.9]; and in the proof of Lemma 3 we bring some details underlying [Se, proof of Lemma 1.10], drawing for that purpose on [F2, Lemma 2]. Furthermore Proposition 1 above is indispensable.

In this section the domain of maps will be a *Riemannian domain* $(\Omega, g)$ in the sense of Korevaar and Schoen [KS, Section 1.1], i.e., a connected open subset of a Riemannian manifold $(M, g)$ such that the metric completion $\overline{\Omega}$ is a compact subset of $M$. The volume measure on $M$ is denoted by $\mu = \mu_g$. We refer to [KS, Section 1.7] for the class $\Gamma(T\Omega)$ of all *Lipschitz vector fields* $\omega$ on $\overline{\Omega}$, and the associated *directional energies* ${}^\omega E(\varphi)$ and *directional energy densities* ${}^\omega e(\varphi) = |\varphi_*(\omega)|^2$ of maps $\varphi \in W^{1,2}(\Omega, Y)$, i.e., maps $\Omega \to Y$ of finite 2-energy. For any open set $U \subset \Omega$ write $U_\varepsilon = \{x \in U : d_\Omega(x, \partial U) > \varepsilon\}$, $\varepsilon > 0$. We sometimes abbreviate $d_Y(P, Q) = PQ$ for points $P, Q \in Y$. In the following lemma, $Y$ may be any complete metric space.

**Lemma 2** ([Se, Lemmas 2.5, 2.6, 2.7]). *Let $\varphi \in W^{1,2}(\Omega, Y)$, let $\omega \in \Gamma(T\Omega)$ be a unit vector field, and denote by $\big(\bar{x}(x,t)\big)_t$ the flow generated by $\omega$. Then, for $\varepsilon > 0$,*

(a) $d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big) \leqslant \int_0^\varepsilon |\varphi_*(\omega)|\big(\bar{x}(x,t)\big)\,dt$ *for $\mu$-a.e. $x \in \Omega_{2\varepsilon}$.*

(b) *The functions $x \mapsto (1/\varepsilon) d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big)$ are locally uniformly square $\mu$-integrable for small $\varepsilon$, and*

$$\frac{1}{\varepsilon} d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big) \to |\varphi_*(\omega)| \qquad \text{in } L^2_{\mathrm{loc}}(\Omega) \text{ as } \varepsilon \to 0. \tag{5.1}$$

(c) *Defining for $\alpha > 0$*

$$A^\varepsilon_\alpha = A^\varepsilon_\alpha(\varphi) = \big\{x \in \Omega_{2\varepsilon} : d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big) > \alpha\big\} \tag{5.2}$$

*we have for any $f \in C_c(\Omega)$*

$$^{\omega}E^{\varphi}(f) := \int_\Omega |\varphi_*(\omega)|^2 f\,d\mu = \lim_{\varepsilon \to 0} \int_{\Omega \setminus A^\varepsilon_\alpha} \frac{d_Y^2\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big)}{\varepsilon^2} f(x)\,d\mu(x). \tag{5.3}$$

*Proof.* Ad (a). Let $U \subset \Omega_\varepsilon$ be a coordinate patch such that, in coordinates $x^i$ on $U$, we have $\omega = \partial/\partial x^1$. Then $d\mu(x) = \sqrt{\det g(x)}\,dx$, and for $f \in C_c^+(U_\varepsilon)$ and $n = 1, 2, \ldots$, writing $\tilde{f}(x) = f(x)\sqrt{\det g(x)}$,

$$\begin{aligned}
\int_\Omega \frac{d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big)}{\varepsilon} f(x)\,d\mu(x) &= \int_U \frac{d_Y\big(\varphi(x), \varphi(x+\varepsilon\omega)\big)}{\varepsilon} \tilde{f}(x)\,dx \\
&\leqslant \int_U \tilde{f}(x) \frac{1}{\varepsilon} \sum_{k=0}^{n-1} d_Y\left(\varphi\left(x + \frac{k}{n}\varepsilon\omega\right), \varphi\left(x + \frac{k+1}{n}\varepsilon\omega\right)\right) dx \\
&= \int_U \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}\left(x - \frac{k}{n}\varepsilon\omega\right) \frac{d_Y\big(\varphi(x), \varphi(x+\varepsilon\omega/n)\big)}{(\varepsilon/n)}\,dx.
\end{aligned}$$

By [KS, Theorem 1.8.1],

$$\frac{d_Y\big(\varphi(x), \varphi(x+\varepsilon\omega/n)\big)}{(\varepsilon/n)} \to |\varphi_*(\omega)|(x)$$

weakly as measures for $n \to \infty$. Furthermore,

$$\frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}\left(x - \frac{k}{n}\varepsilon\omega\right) \to \frac{1}{\varepsilon} \int_0^\varepsilon \tilde{f}(x - t\omega)\,dt$$

uniformly for $x \in U$ as $n \to \infty$. Thus

$$\int_\Omega d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big) f(x)\, d\mu(x) \leqslant \int_U \int_0^\varepsilon \tilde{f}(x - t\omega)|\varphi_*(\omega)|(x)\, dt\, dx$$
$$\leqslant \int_\Omega f(x) \int_0^\varepsilon |\varphi_*(\omega)|\big(\bar{x}(x,t)\big)\, dt\, d\mu(x),$$

and Part (a) of the lemma follows by varying $f$ and $U$.

Ad (b). For $\delta > 0$ define

$$V(\delta) = \sup\biggl\{\int_A |\varphi_*(\omega)|^2\, d\mu : A \subset \Omega,\ A \text{ is } \mu\text{-measurable, } \mu(A) \leqslant \delta\biggr\}. \tag{5.4}$$

Since $|\varphi_*(\omega)| \in L^2(\Omega)$ it follows that $V(\delta) \searrow 0$ as $\delta \searrow 0$.

Let $K \Subset \Omega$ be $\mu$-measurable, and choose $\varepsilon > 0$ so that $K \Subset \Omega_{2\varepsilon}$. For $|t| \leqslant \varepsilon$ define

$$K^t = \{\bar{x}(x,t) : x \in K\}\ (\Subset \Omega_\varepsilon).$$

Then we have $\mu(K^t) \leqslant c\mu(K)$ with $c$ independent of $\varepsilon$ and $t$. By Cauchy's inequality, applied to the integral in (a), we obtain after integrating over $K$

$$\int_K \frac{d_Y^2\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big)}{\varepsilon^2}\, d\mu(x) \leqslant \frac{1}{\varepsilon}\int_K \int_0^\varepsilon |\varphi_*(\omega)|^2\big(\bar{x}(x,t)\big)\, dt\, d\mu(x).$$

When combined with

$$\int_K |\varphi_*(\omega)|^2\big(\bar{x}(x,t)\big)\, d\mu(x) \leqslant C\int_{K^{-t}} |\varphi_*(\omega)|^2\, d\mu,$$

with $C$ independent of $\varepsilon$ and $t$ for $|t| \leqslant \varepsilon$, this leads to

$$\int_K \frac{d_Y^2\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big)}{\varepsilon^2}\, d\mu(x) \leqslant \frac{1}{\varepsilon}\int_0^\varepsilon C\, V\big(\mu(K^{-t})\big)\, dt \leqslant C\, V\big(c\mu(K)\big), \tag{5.5}$$

and the uniform (in small $\varepsilon$) square integrability over $K$ follows.

For brevity, write $(1/\varepsilon) d_Y\big(\varphi(x), \varphi\big(\bar{x}(x,\varepsilon)\big)\big) = u_\varepsilon(x)$, $x \in K$. According to [KS, Theorem 1.9.6], $u_\varepsilon \to |\varphi_*(\omega)|$ pointwise $\mu$-a.e. in $K$ as $\varepsilon \to 0$. By Egoroff's theorem there exists for any $\delta > 0$ a $\mu$-measurable set $E \subset K$ such that $\mu(K \setminus E) < \delta$ and that $u_\varepsilon \to |\varphi_*(\omega)|$ uniformly on $E$ as $\varepsilon \to 0$. Then

$$\int_K \big(u_\varepsilon - |\varphi_*(\omega)|\big)^2\, d\mu \leqslant \mu(K) \sup_E \big(u_\varepsilon - |\varphi_*(\omega)|\big)^2$$
$$+ 2\int_{K\setminus E} u_\varepsilon^2\, d\mu + 2\int_{K\setminus E} |\varphi_*(\omega)|^2\, d\mu,$$

and here the last two integrals are majorized by $CV(c\delta)$ and $V(\delta)$, respectively, by (5.5) (applied with $K$ replaced by $K \setminus E$, whereby we may keep $C$) and (5.4). Making first $\varepsilon \to 0$ and then $\delta \to 0$ leads to (5.1).

Ad (c). We may assume that $\varphi$ has been chosen within its equivalence class so that the map $[0,\varepsilon] \ni t \mapsto \varphi(\bar{x}(x,t))$ is (Hölder) continuous for every $x \in \Omega_\varepsilon$ off some $\mu$-nullset $N$, [KS, Lemma 1.9.2]. Define for $\alpha > 0$

$$\tilde{A}^\varepsilon_\alpha = \{x \in \Omega_{2\varepsilon} : d_Y(\varphi(x), \varphi(\bar{x}(x,t))) > \alpha \text{ for some } t \in [0,\varepsilon]\}.$$

Then $A^\varepsilon_\alpha \subset \tilde{A}^\varepsilon_\alpha$, and both sets are $\mu$-measurable, the latter because it suffices to consider rational $t$ when $x \notin N$. For $\varepsilon \to 0$ through a decreasing sequence, $\tilde{A}^\varepsilon_\alpha$ decreases to $\bigcap_\varepsilon \tilde{A}^\varepsilon_\alpha \subset N$, and hence $\mu(\widetilde{A}^\varepsilon_\alpha) \to 0$ as $\varepsilon \to 0$; in particular,

$$\mu(A^\varepsilon_\alpha) \to 0 \qquad \text{as } \varepsilon \to 0. \tag{5.6}$$

In proving (5.3) we may assume that $f \geqslant 0$. With integration over all of $\Omega$, this limit relation is contained in [KS, Theorem 1.8.1]. As it stands, (c) therefore follows from (5.5) applied to $K = K_\varepsilon := A^\varepsilon_\alpha \cap \operatorname{supp} f$, taking $\varepsilon$ small enough so that $\operatorname{supp} f \subset \Omega_{2\varepsilon}$:

$$\int_{A^\varepsilon_\alpha} \frac{d_Y^2(\varphi(x), \varphi(\bar{x}(x,\varepsilon)))}{\varepsilon^2} f(x)\, d\mu(x) \leqslant C\|f\|_{L^\infty} V(c\mu(K_\varepsilon)).$$

By (5.6), $\mu(K_\varepsilon) \to 0$, hence $V(c\mu(K_\varepsilon)) \to 0$ as $\varepsilon \to 0$. ▫

It is not known to the author whether (b) and (c) in Lemma 2 have analogues for the “full” energy functional $E^\varphi(f) = \int_\Omega e(\varphi) f\, d\mu$ in place of the directional energy functional ${}^\omega E^\varphi(f)$. Actually, rather than using (5.3), we shall in the sequel employ (5.6) combined with

$$\operatorname*{ess\,lim}_{\varepsilon \to 0} \frac{d_Y^2(\varphi(x), \varphi(\bar{x}(x,\varepsilon)))}{\varepsilon^2} = |\varphi^*(\omega)|^2(x), \tag{5.7}$$

valid for $\mu$-a.e. $x \in \Omega$ and for any $\varphi \in W^{1,2}(\Omega, Y)$, according to [KS, Theorem 1.9.6]. For a Sobolev *function* $u \in W^{1,2}(\Omega)$ this reads

$$\operatorname*{ess\,lim}_{\varepsilon \to 0} \frac{|u(x) - u(\bar{x}(x,\varepsilon))|^2}{\varepsilon^2} = |\nabla_\omega u|^2(x) \tag{5.8}$$

for $\mu$-a.e. $x \in \Omega$, as is well known.

*Henceforth we consider maps* $\varphi\colon \Omega \to B$ *with* $(Y, d_Y)$ *and* $B = B_Y(q,R)$ as in the paragraph containing (2.1); in particular $R < \frac{1}{2}\pi$.

**Lemma 3** ([Se, Lemma 2.8]). *Let* $\varphi_0, \varphi_1 \in W^{1,2}(\Omega, B)$. *For any unit vector field* $\omega \in \Gamma(T\Omega)$, *generating a flow* $(\bar{x}(x,t))_t$, *we have*

$$|(\varphi_{1/2})_*(\omega)|^2 \cos^2\left(\tfrac{1}{2}u\right) \leqslant \tfrac{1}{2}|(\varphi_0)_*(\omega)|^2 + \tfrac{1}{2}|(\varphi_1)_*(\omega)|^2 - \tfrac{1}{4}|\nabla_\omega u|^2 \tag{5.9}$$

$\mu$-a.e. *in* $\Omega$, *whereby* $u(x) = d_Y(\varphi_0(x), \varphi_1(x))$, *while* $\nabla_\omega u = \langle \omega, \nabla u\rangle$ *is the derivative of* $u \in W^{1,2}(\Omega)$ *in the direction* $\omega = \omega(x)$.

It is the factor $\cos^2\big(\frac{1}{2}u\big) \leqslant 1$ in (5.9) which causes the need for considering the map $\widehat{\varphi}_{1/2}$ instead of $\varphi_{1/2}$ in Serbinowski's inequality (2.6).

*Proof.* Proposition 1 shows that $\varphi_{1/2} \in W^{1,2}(\Omega, B)$, $B$ being convex. Let $0 < \alpha \leqslant \frac{1}{2}\pi - R$. For $\varepsilon > 0$ and $x \in \Omega_{2\varepsilon} \setminus \big(A^\varepsilon_\alpha(\varphi_0) \cup A^\varepsilon_\alpha(\varphi_1) \cup A^\varepsilon_\alpha(\varphi_{1/2})\big)$, cf. (5.2), hence

$$d_i := d_Y\big(\varphi_i(x), \varphi_i\big(\bar{x}(x,\varepsilon)\big)\big) \leqslant \alpha, \qquad i = 0, 1, \tfrac{1}{2}, \tag{5.10}$$

the ordered quadruple

$$P = \varphi_0(x), \quad Q = \varphi_0\big(\bar{x}(x,\varepsilon)\big), \quad R = \varphi_1\big(\bar{x}(x,\varepsilon)\big), \quad S = \varphi_1(x)$$

defines a quadrilateral $PQRS$ in $Y$, and $2\max\{PQ, RS\} + QR + SP < 2\pi$. According to [F2, Lemma 2(a)], $PQRS$ has a *comparison trapezoid* $\widetilde{P}\widetilde{Q}\widetilde{R}\widetilde{S}$ in the unit sphere $S^2$ in $\mathbf{R}^3$, i.e., a convex spherical quadrilateral, symmetric about a great circle $\widetilde{\gamma}$ in $S^2$, and with side lengths

$$\widetilde{P}\widetilde{S} = PS = u(x), \qquad \widetilde{Q}\widetilde{R} = QR = u\big(\bar{x}(x,\varepsilon)\big),$$

$$\widetilde{P}\widetilde{Q} = \widetilde{R}\widetilde{S} = PQ \diamond RS = d_0 \diamond d_1,$$

where the cosine mean $d_0 \diamond d_1 \in \big[0, \frac{1}{2}\pi\big]$ of numbers $d_0, d_1 \in \big[0, \frac{1}{2}\pi\big]$ is defined by $\cos(d_0 \diamond d_1) = \frac{1}{2}(\cos d_0 + \cos d_1)$, i.e., $2\sin^2\big[\frac{1}{2}(d_0 \diamond d_1)\big] = \sin^2\big(\frac{1}{2}d_0\big) + \sin^2\big(\frac{1}{2}d_1\big)$.

If $\widetilde{O} \in S^2$ denotes the pole of $\widetilde{\gamma}$ on the same side of $\widetilde{\gamma}$ as $\widetilde{R}$ and $\widetilde{S}$, the spherical cosine relation applied to $\widetilde{P}\widetilde{Q}\widetilde{O}$ yields (also if $\widetilde{P}\widetilde{Q}\widetilde{R}\widetilde{S}$ degenerates)

$$2\cos\frac{u}{2}\cos\frac{\bar{u}}{2}\sin^2\frac{\theta}{2} = \sin^2\frac{d_0}{2} + \sin^2\frac{d_1}{2} - 2\sin^2\frac{u-\bar{u}}{4}. \tag{5.11}$$

Here $u = u(x), \bar{u} = u\big(\bar{x}(x,\varepsilon)\big)$, while $\theta$ denotes the angle at $\widetilde{O}$, i.e., the spherical distance between the midpoints $\widetilde{M}, \widetilde{N} \in \widetilde{\gamma}$ of $\widetilde{P}\widetilde{S}$ and $\widetilde{Q}\widetilde{R}$, respectively. According to [F2, Lemma 2(b)], the distance $d_{1/2} = d_Y\big(\varphi_{1/2}(x), \varphi_{1/2}(\bar{x})\big) = MN$ between the midpoints $M$ and $N$ of $PS$ and $QR$, respectively, satisfies $d_{1/2} \leqslant \widetilde{M}\,\widetilde{N} = \theta$. By the triangle inequality, $|u - \bar{u}| \leqslant d_0 + d_1 \leqslant 2\alpha$, cf. (5.10).

Becausee $2\sin^2\big(\frac{1}{2}t\big) = \frac{1}{2}t^2\big(1 - O(t^2)\big)$ for $t = d_0, d_1, d_{1/2}$, and $\frac{1}{2}|u-\bar{u}|$ (all $\leqslant \alpha$), we obtain from (5.11) after division by $\frac{1}{2}\varepsilon^2\cos(\frac{1}{2}u)\cos(\frac{1}{2}\bar{u})$:

$$\frac{d_{1/2}^2}{\varepsilon^2} \leqslant \frac{1 + O(\alpha)}{\cos^2\big(\frac{1}{2}u\big)}\left(\frac{d_0^2 + d_1^2}{2\varepsilon^2} - \frac{(u-\bar{u})^2}{4\varepsilon^2}\big(1 - O(\alpha^2)\big)\right)$$

$\mu$-a.e. in $\Omega_{2\varepsilon}$. Making first $\varepsilon \to 0$ and then $\alpha \to 0$ leads in view of [KS, Theorem 1.9.6] to (5.9). For this, apply (5.6), (5.7) to $\varphi_0$, $\varphi_1$, $\varphi_{1/2}$; and (5.8) to $u \in W^{1,2}(\Omega)$, cf. (2.2), whilst noting that $1/\cos^2\big(\frac{1}{2}u\big)$ is bounded and measurable. □

**Lemma 4** ([Se, Lemma 2.9]). *Let* $\varphi \in W^{1,2}(\Omega, B)$ $(B = B_Y(q,R))$, *and write* $d_Y\big(\varphi(x), q\big) = \varrho(x)$, $x \in \Omega$. *For any function* $\eta \in W^{1,2}(\Omega)$, $0 \leqslant \eta < 1$, *the map* $\widehat{\varphi} := (1-\eta)\varphi + \eta q \in W^{1,2}(\Omega, B)$ *satisfies*

$$|\widehat{\varphi}_*(\omega)|^2 \leqslant \frac{\sin^2(\varrho - \eta\varrho)}{\sin^2 \varrho}\big(|\varphi_*(\omega)|^2 - |\nabla_\omega \varrho|^2\big) + |\nabla_\omega(\varrho - \eta\varrho)|^2 \tag{5.12}$$

$\mu$*-a.e. in* $\Omega$ *for any unit vector field* $\omega \in \Gamma(T\Omega)$ *generating a flow* $\big(\bar{x}(x,t)\big)_t$.

*Proof.* Proposition 1 shows that $\widehat{\varphi} \in W^{1,2}(\Omega, B)$, $B$ being convex; and (2.2) shows that $\varrho \in W^{1,2}(\Omega)$. Given $\varepsilon > 0$ and $0 < \alpha \leqslant \frac{1}{2}\pi - R$, let $x \in \Omega_{2\varepsilon} \setminus \big(A^\varepsilon_\alpha(\varphi) \cup A^\varepsilon_\alpha(\widehat{\varphi})\big)$, cf. (5.2). Write $\eta(x) = \eta$, $\varrho(x) = \varrho$, $\bar{x}(x,\varepsilon) = \bar{x}$, and

$$d_Y\big(\varphi(x), \varphi(\bar{x})\big) = d, \quad d_Y\big(\widehat{\varphi}(x), \widehat{\varphi}(\bar{x})\big) = \hat{d}, \quad \eta(\bar{x}) = \bar{\eta}, \quad \varrho(\bar{x}) = \bar{\varrho}.$$

Then $d, \hat{d} \leqslant \alpha$. At points $x$ where $\hat{d} \geqslant d$ we find that

$$\frac{1}{2!}(\hat{d}^{\,2} - d^2) - \frac{1}{4!}(\hat{d}^{\,4} - d^4) \leqslant \cos d - \cos \hat{d} \tag{5.13}$$

because $d \leqslant \hat{d} \leqslant \pi < \sqrt{15}$. Indeed, in the series (with terms of alternating signs)

$$\sum_{n=2}^{\infty} (-1)^n \frac{\hat{d}^{\,2n} - d^{2n}}{(2n)!} = \cos \hat{d} - \cos d + \frac{\hat{d}^{\,2} - d^2}{2!}$$

the terms do not increase in absolute value because

$$\hat{d}^{\,2n+2} - d^{2n+2} \leqslant (\hat{d}^{\,2n} - d^{2n})(\hat{d}^{\,2} + d^2)$$

and $\hat{d}^{\,2} + d^2 \leqslant 2\pi^2 < 30 \leqslant (2n+1)(2n+2)$ for $n \geqslant 2$. The sum of the series is therefore no bigger than its first term $\frac{1}{4!}(\hat{d}^{\,4} - d^4)$, as claimed in (5.13).

By triangle comparison for the triangle $\varphi(x)\varphi(\bar{x})q$ we obtain $\hat{d} \leqslant d + \frac{1}{2}\pi|\eta - \bar{\eta}|$, cf. [F1, equation (8.1)], and hence $\hat{d}^{\,4} - d^4 \leqslant (d^2 + |\bar{\eta} - \eta|^2)O(|\eta - \bar{\eta}|)$. Inserting this in (5.13) yields

$$\tfrac{1}{2}\hat{d}^{\,2} \leqslant \tfrac{1}{2}d^2 + (\cos d - \cos \hat{d}) + (d^2 + |\bar{\eta} - \eta|^2)O(|\eta - \bar{\eta}|). \tag{5.14}$$

The same holds for $\hat{d} \leqslant d$, even without the remainder term, as shown by applying the meanvalue theorem to the function $-\cos\sqrt{t}$ with derivative $\leqslant \frac{1}{2}$ for $t > 0$.[3]

[3] Arguments and estimates corresponding to those of the paragraphs containing (5.13) and (5.14) should have been used in the remainder term in [F1, equation (8.3)] and in the former inequality [EF, equation (10.17)], both of which therefore should be replaced (without further consequences) by

$$\tfrac{1}{2}(d_\lambda^2 - d^2) \leqslant (\cos d - \cos d_\lambda) + (d^2 + |\lambda - \lambda'|^2)\|\lambda\|_{\mathrm{Lip}} O(\varepsilon).$$

Now combine (5.14) with the following analogue of (4.9) (cf. [EF, equation (10.17)] or [F1, equation (8.2)]):

$$\cos d - \cos \hat{d} \leqslant R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)}, \tag{5.15}$$

where the $R^{(j)}$ are defined by the respective former expression in (4.10), etc. (and likewise defined in [EF, p. 193]), now with $v, \lambda$ replaced by $\varrho, \eta$. Note that $2\sin^2\big(\frac{1}{2}d\big) = \frac{1}{2}d^2\big(1-O(\alpha)\big)$, and that the function $[\sin(\varrho-\eta\varrho)]/\sin\varrho$ of $\varrho \in \,]0,R[$ and $\eta \in [0,1[$, extends smoothly to $[0,R]\times[0,1]$, and therefore is of class $W^{1,2}(\Omega)$ as a function of $x \in \Omega$, along with $\varrho$ and $\eta$, by the chain rule [EF, Lemma 5.2]. Inserting these expressions for the $R^{(j)}$ in (5.15) combined with (5.14), dividing by $\frac{1}{2}\varepsilon^2$, and making first $\varepsilon \to 0$ whilst using (5.6), (5.7), (5.8), and then $\alpha \to 0$, we obtain $\mu$-a.e. in $\Omega$ (cf. end of proof of the preceding lemma):

$$\begin{aligned} |\widehat{\varphi}_*(\omega)|^2 \leqslant \frac{\sin^2(\varrho-\eta\varrho)}{\sin^2\varrho}|\varphi_*(\omega)|^2 + 2\cos^2(\eta\varrho)\Big\langle \nabla_\omega \cos\varrho, \nabla_\omega \frac{\tan(\eta\varrho)}{\sin\varrho}\Big\rangle \\ + 2\frac{\sin^2(\eta\varrho)}{\sin^2\varrho}|\nabla_\omega\cos\varrho|^2 + |\nabla_\omega(\eta\varrho)|^2 - \frac{\sin^2(\eta\varrho)}{\sin^2\varrho}|\nabla_\omega\varrho|^2. \end{aligned}$$

After further applications of the chain rule this reduces to (5.12) because

$$\sin^2(\varrho-\eta\varrho) + 2\sin\varrho\sin(\eta\varrho)\cos(\varrho-\eta\varrho) \equiv \sin^2\varrho + \sin^2(\eta\varrho). \ \square$$

*Proof of Proposition* 2 (cf. [Se]). In proving (2.6) and hence (2.7), it suffices to consider each $m$-simplex of $X$ separately, and so we may assume that $X$ is an open simplex and hence a Riemannian domain $\Omega$, cf. footnote 2 to (2.7). In view of Lemma 4 (to be applied to $\varphi_{1/2} \in W^{1,2}(\Omega)$, cf. Proposition 1) we define the function $\eta$ on $\Omega$, $0 \leqslant \eta < 1$, by (2.5):

$$\frac{\sin[(1-\eta)\varrho]}{\sin\varrho} = \cos\frac{u}{2}, \tag{5.16}$$

understood so that $\eta(x) = 1 - \cos\big(\frac{1}{2}u(x)\big)$ when $\varrho(x) = 0$. Then

$$1-\eta = \frac{1}{\varrho}\operatorname{Arcsin}\big(\cos\big(\tfrac{1}{2}u\big)\sin\varrho\big)$$

extended to a smooth function of $(\varrho,u) \in [0,R]\times[0,2R]$. Since $\varrho, u \in W^{1,2}(\Omega)$, cf. (2.2), it follows by the chain rule [EF, Lemma 5.2] that $\eta \in W^{1,2}(\Omega)$. Moreover, $(1-\eta)\varrho \in W^{1,2}(\Omega)$, and

$$|\nabla_\omega[(1-\eta)\varrho]|^2 = \frac{\big|\nabla_\omega\big(\cos\big(\frac{1}{2}u\big)\sin\varrho\big)\big|^2}{1-\cos^2\big(\frac{1}{2}u\big)\sin^2\varrho}.$$

By Lemma 4, applied to $\varphi_{1/2}$, and Lemma 3 we obtain, after inserting (5.16),

$$
\begin{aligned}
|(\widehat{\varphi}_{1/2})_*(\omega)|^2 \leqslant \tfrac{1}{2}|(\varphi_0)_*(\omega)|^2 + \tfrac{1}{2}|(\varphi_1)_*(\omega)|^2 - \tfrac{1}{4}|\nabla_\omega u|^2 \\
- \cos^2\frac{u}{2}|\nabla_\omega \varrho|^2 + \frac{\big|\nabla_\omega\big(\cos\big(\tfrac{1}{2}u\big)\sin\varrho\big)\big|^2}{1-\cos^2\big(\tfrac{1}{2}u\big)\sin^2\varrho}.
\end{aligned} \tag{5.17}
$$

Simple computations show that

$$
\begin{aligned}
\frac{1}{4}|\nabla_\omega u|^2 + \cos^2\frac{u}{2}|\nabla_\omega\varrho|^2 - \frac{\big|\nabla_\omega\big(\cos\big(\tfrac{1}{2}u\big)\sin\varrho\big)\big|^2}{1-\cos^2\big(\tfrac{1}{2}u\big)\sin^2\varrho} \\
= \frac{\cos^4\varrho\cos^4\big(\tfrac{1}{2}u\big)}{1-\sin^2\varrho\cos^2\big(\tfrac{1}{2}u\big)}\left|\nabla_\omega\frac{\tan\big(\tfrac{1}{2}u\big)}{\cos\varrho}\right|^2 \\
\geqslant \cos^8 R\left|\nabla_\omega\frac{\tan\big(\tfrac{1}{2}u\big)}{\cos\varrho}\right|^2.
\end{aligned}
$$

Inserting this in (5.17) leads to

$$
|(\widehat{\varphi}_{1/2})_*(\omega)|^2 \leqslant \frac{1}{2}|(\varphi_0)_*(\omega)|^2 + \frac{1}{2}|(\varphi_1)_*(\omega)|^2 - \cos^8 R\left|\nabla_\omega\frac{\tan\big(\tfrac{1}{2}u\big)}{\cos\varrho}\right|^2
$$

$\mu$-a.e. in $\Omega$. And integrating that with respect to $\omega$ over the unit sphere $S^{m-1}$ in $\mathbf{R}^m$ establishes (2.6), by [KS, Theorem 1.8.1]:

$$
e(\widehat{\varphi}_{1/2}) \leqslant \frac{1}{2}e(\varphi_0) + \frac{1}{2}e(\varphi_1) - c_m\cos^8 R\left|\nabla\frac{\tan\big(\tfrac{1}{2}u\big)}{\cos\varrho}\right|^2 \tag{5.18}
$$

$\mu$-a.e. in $\Omega$. Writing $\nabla\big[\big(\tan(\tfrac{1}{2}u)\big)/\cos\varrho\big] = \xi \in T(\Omega)$, we have in fact

$$
\int_{S^{m-1}} \langle\omega,\xi\rangle^2\,d\sigma(\omega) = |\xi|^2\int_{S^{m-1}}\omega_1^2\,d\sigma(\omega) = \frac{|\xi|^2}{m}|S^{m-1}| = |\xi|^2\omega_m,
$$

where $\omega_m$ denotes the volume of the unit ball in $\mathbf{R}^m$. This leads to (5.18) with $c_m = \omega_m$; and that is the same as (2.6). [4]

[4] Actually, we have defined $c_m = \omega_m/(m+2)$ in [EF] because it was chosen, in the definition of the approximate energy density $e_\varepsilon(\varphi)$ in [EF, (9.2)], (cf. (1.1) above) to suppress the factor $m+2$ (occurring as $n+p$ with $p=2$ in [KS, (1.2vii)]). In [EF, Corollary 9.2], $c_m$ is inadvertently written as $\omega_m/(m+1)$, while the correct expression $c_m = \omega_m/(m+2)$ is written elsewhere in [EF], in particular on [EF, pp. 168, 171].

Finally suppose that $u \in W_0^{1,2}(X)$, and let us establish that $\sigma := \eta\varrho = d_Y(\varphi_{1/2}, \widehat{\varphi}_{1/2}) \in W_0^{1,2}(X)$. From $\sin(\varrho - \sigma) = \sin\varrho\cos\bigl(\frac{1}{2}u\bigr)$, by (5.16), we easily obtain

$$\sin\bigl(\tfrac{1}{2}\sigma\bigr)\cos\bigl(\varrho - \tfrac{1}{2}\sigma\bigr) = \sin\varrho\sin^2\bigl(\tfrac{1}{4}u\bigr), \tag{5.19}$$

where $\varrho - \frac{1}{2}\sigma = (1 - \frac{1}{2}\eta)\varrho < R < \frac{1}{2}\pi$. It follows that $1/\cos\bigl(\varrho - \frac{1}{2}\sigma\bigr) \in W^{1,2}(X)$ and hence $\sin\bigl(\frac{1}{2}\sigma\bigr) \in W_0^{1,2}(X)$, so that indeed $\sigma \in W_0^{1,2}(X)$. Finally, by (5.19),

$$\eta = \frac{\sigma}{\varrho} = \frac{\sigma}{\sin\bigl(\frac{1}{2}\sigma\bigr)}\frac{\sin\varrho}{\varrho}\frac{\sin^2\bigl(\frac{1}{4}u\bigr)}{\cos\bigl(\varrho - \frac{1}{2}\sigma\bigr)} \in W_0^{1,2}(X)$$

after removing the singularities of $\sigma/\sin\bigl(\frac{1}{2}\sigma\bigr)$ and $(\sin\varrho)/\varrho$.

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