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MEROMORPHIC FUNCTIONS WITH SHARED LIMIT VALUES

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Abstract. We introduce the notion of limit value sharing for meromorphic functions in the plane. This is closely connected to Ahlfors's theory of covering surfaces.

1. Introduction

The central notion in this paper is the *sharing of limit values*. With this we mean that two (transcendental) meromorphic functions in the plane converge to certain given values $a \in \widehat{\mathbf{C}}$ on the same sequences $z_n \to \infty$.

Shared limit values fit very well to Ahlfors's theory of covering surfaces [2]. One could say that shared limit values in Ahlfors's theory are an analogue of shared values in Nevanlinna's theory of meromorphic functions [25]. This shows in the fact that shared limit values lead to shared *islands* in a suitable sense. Despite of the formal similarities between shared values and shared limit values, problems connected to the latter are more topological in nature. A consequence of this is that the proof of the five point theorem does not carry over. Already simple examples like f and $f + 1/z$ show that meromorphic functions can share *all* limit values without being identical. Nonetheless we will show that there is a remarkable difference between four and five shared limit values.

In Section 2 we summarize some facts from value distribution theory for later reference. (For a complete treatment we refer to the monographs [13], [19], [26], [27] and [37].) In the third section we give growth estimations in terms of the characteristic functions of two mappings that share limit values. The next two sections are devoted to the construction of non-trivial examples of functions sharing limit values. The reader who is interested in more theoretical results may skip this part. The following sections treat the case of five shared limit values, extension properties of limit value sharing, a generalization of limit value sharing and filling disks. In these sections the theory of normal families plays a central role. We refer to [34].

I would like to thank my teacher F. Pittnauer. The idea of shared limit values is his. Further I want to thank J.K. Langley. The examples in Section 5 were constructed by him.

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2. Value distribution theory

We set $\mathbf{D}_r(a) := \{z \in \mathbf{C} \mid |z-a| < r\}$ (with obvious modification for $a = \infty$), $\mathbf{D}_r := \mathbf{D}_r(0)$ and $\mathbf{D} := \mathbf{D}_1$. The closure of $\mathbf{D}_r$ is $\overline{\mathbf{D}}_r := \{z \in \mathbf{C} \mid |z| \le r\}$. We denote by $\widehat{\mathbf{C}}$ the Riemann sphere and by χ the chordal metric. The complement of a set A will be denoted A^c. It will always be clear from the context whether the complement is taken with respect to $\mathbf{C}$ or $\widehat{\mathbf{C}}$.

For meromorphic $f\colon U \to \widehat{\mathbf{C}}$ with $U \subset \mathbf{C}$ the spherical derivative of f is

$$f^{\#}(z) := \lim_{w\to z} \frac{\chi\big(f(z), f(w)\big)}{|z-w|} = \frac{|f'(z)|}{1+|f(z)|^2}$$

with $z \in U$. We have $f^{\#} = (1/f)^{\#}$, hence $f^{\#}$ is defined in poles of f. Further $f^{\#}\colon U \to \mathbf{R}^+$ is continuous.

Let $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be meromorphic and $D \subset \widehat{\mathbf{C}}$ be a domain. The bounded components of $f^{-1}(D)$ are called *islands* (over D). Let $\overline{n}(r, f, D)$ be the number of islands of f over D contained in $\overline{\mathbf{D}}_r$. Further

$$A(r, f) := \frac{1}{\pi} \int_{\mathbf{D}_r} \big(f^{\#}(z)\big)^2\, dz = \frac{1}{\pi} \int_0^{2\pi} \int_0^r \frac{|f'(\varrho e^{i\varphi})|^2 \varrho}{\big(1+|f(\varrho e^{i\varphi})|^2\big)^2}\, d\varrho\, d\varphi.$$

The function $\pi \cdot A(r, f)$ is the spherical area of the image of $\mathbf{D}_r$ under f (with regard to multiplicities).

The Ahlfors–Shimizu characteristic is defined (see [13, p. 10])

$$T(r, f) := \int_0^r \frac{A(t, f)}{t}\, dt. \tag{1}$$

The function $T(r, f)$ is, up to a bounded term, Nevanlinna's characteristic. Similarly to (1) let $\overline{N}(r, f, D)$ be the logarithmic integral of $\overline{n}(r, f, D)$. As usual we denote by $S(r, f)$ functions such that $S(r, f) = o\big(T(r, f)\big)$ (outside a possible exceptional set of finite linear measure).

With this notation Ahlfors's second fundamental theorem takes the form:

Theorem 2.1 (Ahlfors). *Let $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be meromorphic and $D_1, \dots, D_q$ be Jordan domains with disjoint closures. Then*

$$(q-2)T(r, f) \le \sum_{k=1}^{q} \overline{N}(r, f, D_k) + S(r, f).$$

The estimation of the integrated error term is due to Miles [21].

Let now $\{a_1, \dots, a_q\} \subset \widehat{\mathbf{C}}$ and $D_1, \dots, D_q$ disks with disjoint closures and $a_k \in D_k$ for $k = 1, \dots, q$, and $\overline{n}(r, f, a_k)$ be the number of preimages in $f^{-1}(\{a_k\})$ contained in $\overline{\mathbf{D}}_r$. The trivial inequality $\overline{n}(r, f, D_k) \le \overline{n}(r, f, a_k)$ shows with Theorem 2.1 the second fundamental theorem of Nevanlinna theory.

Theorem 2.2 (Nevanlinna). *Let* $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be meromorphic and* $\{a_1, \ldots, a_q\} \subset \widehat{\mathbf{C}}$. *Then*

$$(q-2)T(r,f) \le \sum_{k=1}^{q} \overline{N}(r,f,a_k) + S(r,f).$$

Of course $\overline{N}(r,f,a_k) = \int_0^r \overline{n}(t,f,a_k)/t\,dt$. (If $f(0) = a_k$ a simple modification is necessary.)

Let I be an island of f over D. If $f\colon I \to D$ is bijective, then I is called a *simple* island.

Theorem 2.3 (five islands theorem). *Let* $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be a transcendental meromorphic function. Then over at least one of five given Jordan domains with disjoint closures* f *possesses infinitely many simple islands.*

For a proof see [2] or the recent paper [4].

We note that f and g are said to *share* the value $a \in \widehat{\mathbf{C}}$, if for all $z \in \mathbf{C}$

$$f(z) = a \quad \Longleftrightarrow \quad g(z) = a,$$

i.e. if f and g have the same preimages for a. (A survey on shared value problems can be found in [22].) The basic result on shared values is Nevanlinna's five point theorem [25].

Theorem 2.4 (five point theorem). *Let* $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be meromorphic and not both constant. If* f *and* g *share five values then* $f = g$.

To illustrate the strength of this theorem, let us note how Picard's theorem follows from the five point theorem: Suppose there exists a non-constant meromorphic function f in the plane that omits the third roots of unity. Let γ be a non-trivial third root of unity, then f and γf are distinct, non-constant and share five values, namely 0, ∞ and the third roots of unity, contradicting the five point theorem.

From the second fundamental theorem it follows easily:

Proposition 2.5. *Let* $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be meromorphic and non-constant.*

(i) *If* f *and* g *share three values then* $T(r,f) \le 3 \cdot T(r,g) + S(r,f)$.

(ii) *If* f *and* g *share four values then* $T(r,f) = T(r,g) + S(r,f)$.

3. Shared limit values

Picard's theorem shows that if f is transcendental meromorphic then $\widehat{\mathbf{C}} \setminus \{a,b\} \subset f(\mathbf{D}_r^c)$ for all $r > 0$ with suitable $a, b \in \widehat{\mathbf{C}}$. In particular we have the Casorati–Weierstraß theorem: For each $a \in \widehat{\mathbf{C}}$ there is a sequence $z_n \in \mathbf{C}$ with $z_n \to \infty$ and $f(z_n) \to a$.

We will consider transcendental meromorphic functions f and g on $\mathbf{C}$ that converge on the same sequences $z_n \to \infty$ to given $a \in \widehat{\mathbf{C}}$.

Definition 3.1. Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions and $a \in \widehat{\mathbf{C}}$. We say that f and g *share* the *limit value* a, if for all sequences $z_n \to \infty$:

$$f(z_n) \to a \quad \Longleftrightarrow \quad g(z_n) \to a.$$

The formal similarity of value sharing and limit value sharing lets one hope for uniqueness theorems like the five point theorem. Simple examples show, that such statements are not true (without further restrictions). Let e.g. f be arbitrary and $f_1 := (z+1)f$ and $f_2 := zf$. Since $f_1/f_2 = (z+1)/z$ it follows that f_1 and f_2 share all limit values but their difference is $f_1 - f_2 = f$. This already shows that the proof of the five point theorem cannot be mimicked. The given example is trivial since the quotient is rational. We will construct in Section 5 examples where there are no such simple relations between f and g, yet all limit values are shared.

The following lemma makes it possible to use Ahlfors's theory. For $M, N \subset \mathbf{C}$ we write $N \mathrel{\tilde{\subset}} M$ if $N \setminus M$ is bounded.

Lemma 3.2. *Meromorphic functions $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ share the limit value $a \in \widehat{\mathbf{C}}$ if and only if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$g^{-1}\bigl(\mathbf{D}_\delta(a)\bigr) \mathrel{\tilde{\subset}} f^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr) \qquad \textit{and} \qquad f^{-1}\bigl(\mathbf{D}_\delta(a)\bigr) \mathrel{\tilde{\subset}} g^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr).$$

Proof. We can assume that a is finite.

"$\Rightarrow$" Suppose there is $\varepsilon > 0$ such that the sets $M_n := g^{-1}\bigl(\mathbf{D}_{1/n}(a)\bigr) \setminus f^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr)$ with $n \in \mathbf{N}$ are unbounded. Choose $z_n \in M_n$ with $|z_n| \ge n$. Then $z_n \to \infty$ and since $|g(z_n) - a| < 1/n$ we have $g(z_n) \to a$. Now $z_n \notin f^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr)$ implies $|f(z_n) - a| \ge \varepsilon$ hence $f(z_n) \not\to a$, a contradiction. Symmetry shows necessity.

"$\Leftarrow$" Suppose there is $z_n \to \infty$ with $f(z_n) \to a$ but $g(z_n) \not\to a$. Passing to a subsequence we can assume $|g(z_n) - a| > \varepsilon$ for some $\varepsilon > 0$ and for all $n \in \mathbf{N}$. Since $\{z_n\} \mathrel{\tilde{\subset}} f^{-1}\bigl(\mathbf{D}_\delta(a)\bigr)$ for all $\delta > 0$ and $\{z_n\} \cap g^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr) = \emptyset$ we deduce that $f^{-1}\bigl(\mathbf{D}_\delta(a)\bigr) \setminus g^{-1}\bigl(\mathbf{D}_\varepsilon(a)\bigr)$ is unbounded for all $\delta > 0$. This is a contradiction. Symmetry shows the rest. □

We introduce a further notion.

Definition 3.3. Let f and g be meromorphic functions and $a \in \widehat{\mathbf{C}}$. We call a a *completely unshared limit value* of f and g if there is no sequence $z_n \to \infty$ with $f(z_n) \to a$ and $g(z_n) \to a$.

A simple "diagonal argument" shows:

Proposition 3.4. *Let f and g be meromorphic functions. Then the set of completely unshared limit values is open.*

Proposition 3.5 *Let f and g be transcendental meromorphic functions. Then the set V of all completely unshared limit values is a proper subset of $\widehat{\mathbf{C}}$.*

Proof. Suppose $V = \widehat{\mathbf{C}}$. Then $f - g$ and f/g are not transcendental since otherwise there are sequences z_n and w_n with $(f-g)(z_n) \to 0$ and $(f/g)(w_n) \to 1$. It easily follows that f and g are rational. $\square$

Let f be a transcendental entire function and $g := e^f + f$. It is easy to show that f and g share no limit value and that $V = \mathbf{C}$ is maximal. Another example is $f(z) := e^z + e^{-z}$ and $g(z) := e^z - e^{-z}$. Again $V = \mathbf{C}$ but f and g share the limit value ∞.

4. Growth estimations

If f and g share values then the charcteristic functions are of comparable growth (cf. Proposition 2.5). The same is true for shared limit values.

Theorem 4.1. *Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions that share $q \geq 3$ limit values. Then*

$$T(r,f) \leq \frac{q}{q-2} T(r,g) + S(r,f).$$

Proof. Let $a_1, \ldots, a_q$ be the shared limit values. Choose $\varepsilon > 0$ such that the disks $\mathbf{D}_\varepsilon(a_1), \ldots, \mathbf{D}_\varepsilon(a_q)$ have disjoint closures. Lemma 3.2 shows the existence of $\delta > 0$ with $g^{-1}\big(\mathbf{D}_\delta(a_k)\big) \mathrel{\tilde{\subset}} f^{-1}\big(\mathbf{D}_\varepsilon(a_k)\big)$ for $k = 1, \ldots, q$. We show that, with at most finitely many exceptions, every island of f over $\mathbf{D}_\varepsilon(a_k)$ contains an island of g over $\mathbf{D}_\delta(a_k)$. Suppose it exists a sequence of islands I_n of f over $\mathbf{D}_\varepsilon(a_k)$ which do not contain islands of g over $\mathbf{D}_\delta(a_k)$. Since $g^{-1}\big(\mathbf{D}_\delta(a_k)\big) \mathrel{\tilde{\subset}} f^{-1}\big(\mathbf{D}_\varepsilon(a_k)\big)$ it follows $I_n \cap g^{-1}\big(\mathbf{D}_\delta(a_k)\big) = \emptyset$ for $n \geq n_0$. Each I_n contains z_n with $f(z_n) = a_k$, hence $f(z_n) \to a_k$. Then $|g(z_n) - a_k| \geq \delta$ gives a contradiction. It follows that $\overline{n}\big(r, f, \mathbf{D}_\varepsilon(a_k)\big) \leq \overline{n}\big(r, g, \mathbf{D}_\delta(a_k)\big) + O(1) \leq \overline{n}(r, g, a_k) + O(1)$. Logarithmic integration and Theorem 2.1 shows

$$(q-2)T(r,f) \leq \sum_{k=1}^{q} \overline{N}\,(r, g, a_k) + S(r,f) \leq q \cdot T(r,g) + S(r,f). \ \square$$

Corollary 4.2. *Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions that share infinitely many limit values. Then there exists for every $\varepsilon > 0$ a set $E_\varepsilon \subset \mathbf{R}_+$ of finite linear measure such that*

$$T(r,f) \leq (1+\varepsilon) T(r,g)$$

outside E_ε.

In view of Proposition 2.5(ii) the question arises if the conclusion of Corollary 4.2 is already true for $q \geq 4$ shared limit values.

The above proof shows that limit value sharing implies a kind of *island sharing.* Uniqueness theorems do not follow (without further restrictions) since one cannot conclude that $f - g$ has zeros in these islands.

We now prove a uniqueness theorem under strong additional assumptions. It is only a partial result and it would be interesting to know whether the assumptions can be weakened (especially concerning the growth of f).

For the proof we need the following corollary from Wiman's theorem ([6, p. 224]):

Theorem 4.3 (Wiman). *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a non-constant entire function with order $\varrho(f) < \frac{1}{2}$ and $M \subset \mathbf{C}$ be unbounded and connected. Then $f(M)$ is unbounded.*

Theorem 4.4. *Let $f, g\colon \mathbf{C} \to \mathbf{C}$ be zero-free entire functions that share all limit values of a curve surrounding 0. If*

$$\varrho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} < \frac{1}{2},$$

then $f = g$.

Proof. Since $\varrho_1(f) < \frac{1}{2}$ we have $f = \exp(\varphi)$ with an entire function φ with order $\varrho(\varphi) < \frac{1}{2}$. From our growth estimates it follows $g = \exp(\psi)$ with $\varrho(\psi) < \frac{1}{2}$. Let C be the curve consisting of shared limit values and γ be a component of $f^{-1}(C)$. Then γ is unbounded and connected. Consider $f/g = \exp(\varphi - \psi)$. Suppose $\varphi - \psi$ is not constant. Since $\varrho(\varphi - \psi) < \frac{1}{2}$ it follows from Wiman's theorem that $\Gamma := (\varphi - \psi)(\gamma)$ is unbounded (and of course connected). We have $f/g(z) \to 1$ for $z \to \infty$ in $f^{-1}(C)$ since C is compact, hence in particular $f/g \to 1$ on γ. For all $\delta > 0$ it follows $\Gamma \mathrel{\tilde{\subset}} \exp^{-1}\big(\mathbf{D}_\delta(1)\big)$. This implies a contradiction since for suitable $\delta > 0$ the preimage $\exp^{-1}\big(\mathbf{D}_\delta(1)\big)$ consists only of islands. We conclude that $\varphi - \psi$ is constant and it follows that $f = g$. □

5. Examples of functions that share all limit values

In this section we construct entire functions f and g that share all limit values in $\widehat{\mathbf{C}}$ and are not related by simple transformations.

The construction is due to J.K. Langley [16].

The idea is to use two Weierstraß products with close zeros. We need some technical preparation. Let $a_n \to \infty$ be a complex sequence with $|a_n| \le |a_{n+1}|$ for all $n \in \mathbf{N}$. We define

$$\operatorname{dist}(r) := \inf\big\{|a_j - a_k| \mid j \ne k,\ |a_j| \ge r,\ |a_k| \ge r\big\}.$$

For $z \in \mathbf{C}$ we denote by $k(z)$ the index such that $|z - a_{k(z)}| \le |z - a_j|$ for all $j \in \mathbf{N}$. If several a_j have minimal distance from z then we choose $k(z)$ maximal.

Lemma 5.1. *Let $a_n \to \infty$ be a complex sequence with $\operatorname{dist}(r) \to \infty$ as $r \to \infty$. Then there exists a function $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $\varphi(r) \to \infty$ for $r \to \infty$ such that for all $z \in \mathbf{C}$ and $j \ne k(z)$:*

$$|z - a_j| \ge \varphi(|z|).$$

Proof. Let $|z| = r$ and $j \in \mathbf{N}$ such that a_j has second minimal distance from z. (Note that $|a_{k(z)} - z| = |a_j - z|$ is possible.) First we assume $|z - a_j| < \frac{1}{2}r$. Then $|a_j|, |a_{k(z)}| \geq \frac{1}{2}r$ and thus $|z - a_j| \geq \frac{1}{2}\mathrm{dist}\left(\frac{1}{2}r\right)$. Otherwise $|a_j - a_{k(z)}| \leq |z - a_j| + |z - a_{k(z)}| < \frac{1}{2}\mathrm{dist}\left(\frac{1}{2}r\right) + \frac{1}{2}\mathrm{dist}\left(\frac{1}{2}r\right) = \mathrm{dist}\left(\frac{1}{2}r\right)$ gives a contradiction. It follows $|z - a_j| \geq \min\left\{\frac{1}{2}r, \frac{1}{2}\mathrm{dist}\left(\frac{1}{2}r\right)\right\} =: \varphi(r)$. □

Theorem 5.2. *For every $\delta \in [0,1)$ there exist entire functions f and g with $\delta = \varrho(f) = \varrho(g) = \varrho(f/g)$ which share all limit values.*

Proof. We set

$$f(z) := \prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right)$$

with $a_k := 2^k$ if $\delta = 0$ and $a_k := k^{1/\delta}$ for $\delta \in (0,1)$. These functions are well known and it follows from the basic results on the exponent of convergence of canonical Weierstraß products that $\varrho(f) = \delta$ for $\delta \in [0,1)$ (see [27]). We choose a second sequence b_k with $|b_k| = a_k$ and $|b_k - a_k| = \varepsilon_k \to 0$ with a positive sequence ε_k which we prescribe later. Put

$$h(z) := \prod_{k=1}^{\infty}\left(1 - \frac{z}{b_k}\right).$$

Clearly $\varrho(h) = \delta$. For large z we have

$$|f(z)| + |h(z)| \leq \exp(|z|).$$

On the circles $|z - a_k| = \frac{1}{2}(a_k - a_{k-1})$ with large z and k:

$$\frac{|f(z)|}{|z - a_k|} \leq \frac{\exp(2a_k)}{\frac{1}{2}(a_k - a_{k-1})} \leq \exp(2a_k),$$

since $\frac{1}{2}(a_k - a_{k-1}) \to \infty$. Now $f(z)/(z - a_k)$ is holomorphic on $|z - a_k| \leq \frac{1}{2}(a_k - a_{k-1})$. The maximum principle shows that on $|z - a_k| \leq \exp(-3a_k)$ it holds $|f(z)| \leq \exp(-a_k)$. The same inequality is true for h on $|z - b_k| \leq \exp(-3a_k)$. Set $\varepsilon_k := \exp(-4a_k)$. Then

$$|f(z)| \leq \exp(-a_k) \qquad \text{and} \qquad |h(z)| \leq \exp(-a_k) \tag{2}$$

simultanuously on $|z - c_k| < \frac{1}{2}\exp(-3a_k)$ where $c_k := \frac{1}{2}(a_k + b_k)$. The union of these disks will be denoted by E, i.e.

$$E := \left\{z \in \mathbf{C} \mid |z - c_k| < \tfrac{1}{2}\exp(-3a_k)\right\}.$$

We estimate h/f outside E. Again we denote by $k(z)$ the index so that $|z - a_{k(z)}|$ is minimal. It is easy to see that a_k fulfills the assumptions of Lemma 5.1. Thus there is function φ with $\varphi(r) \to \infty$ and $|z - a_j| \geq \varphi(|z|)$ for $j \neq k(z)$. For the function

$$u_k(z) := \frac{z - b_k}{z - a_k}$$

we get

$$|u_k(z) - 1| = \frac{\varepsilon_k}{|z - a_k|}.$$

We show $|u_{k(z)}(z) - 1| \to 0$ for $z \to \infty$ outside E. Let $z_n \to \infty$ be a sequence outside E and consider $k(z_n)$. If $k(z_n)$ is bounded so is $a_{k(z_n)}$ and it follows $|u_{k(z_n)}(z_n) - 1| \to 0$. If $k(z_n)$ is unbounded we can assume $k(z_n) \to \infty$. We obtain

$$\begin{aligned}|u_{k(z_n)}(z_n) - 1| = \frac{\varepsilon_{k(z_n)}}{|z_n - a_{k(z_n)}|} &\leq 2\frac{\exp(-4a_{k(z_n)})}{\exp(-3a_{k(z_n)}) - \exp(-4a_{k(z_n)})} \\ &\leq 12\exp(-a_{k(z_n)}) \to 0,\end{aligned}$$

since $a_{k(z_n)} \to \infty$. Thus for $z \notin E$

$$\sum_{k=1}^{\infty} |u_k(z) - 1| \leq \sum_{\substack{k=1 \\ k\neq k(z)}}^{\infty} |u_k(z) - 1| + o(1) \leq \frac{1}{\varphi(|z|)} \sum_{k=1}^{\infty} |\varepsilon_k| + o(1) = o(1).$$

It follows that $v(z) := \prod_{k=1}^{\infty} u_k(z)$ converges compactly in $\mathbf{C} \setminus E$ and $v(z) \to 1$ for $z \to \infty$ in $\mathbf{C} \setminus E$. It is easy to show

$$\alpha := \prod_{k=1}^{\infty} \frac{b_k}{a_k} \neq 0.$$

Hence for $z \to \infty$ in $\mathbf{C} \setminus E$

$$\frac{h(z)}{f(z)} = \frac{v(z)}{\alpha} \to \frac{1}{\alpha}.$$

We set $g := \alpha h$ and claim that f and g share all limit values. Let $z_n \to \infty$ be a sequence with $f(z_n) \to a \in \widehat{\mathbf{C}} \setminus \{0\}$. From (2) it follows $z_n \notin E$ for $n \geq n_0$. The behaviour of v shows $g(z_n) \to a$. Similarly we get from $g(z_n) \to a \in \widehat{\mathbf{C}} \setminus \{0\}$ that $f(z_n) \to a$. Hence f and g share all limit values in $\widehat{\mathbf{C}} \setminus \{0\}$ and it is easy to conclude that all limit values are shared.

It is clear that $\varrho(f/g) \leq \delta$. Since the zeros of f are the zeros of f/g it follows from the exponent of convergence that $\varrho(f/g) = \delta$. □

Considering $f(z^n)$ and $g(z^n)$ with the above constructed f and g one gets examples for every finite order. If $F\colon \mathbf{C} \to \widehat{\mathbf{C}}$ is meromorphic and bounded on E then $F \cdot f$ and $F \cdot g$ share all limit values. Such F can easily be constructed with results from complex approximation.

6. Examples of functions that share finitely many limit values

It is known from the theory of shared values that examples with finitely many shared values can be constructed by $f = p \circ \varphi$ and $g = q \circ \varphi$ with $p, q\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ rational and $\varphi\colon \mathbf{C} \to \widehat{\mathbf{C}}$ meromorphic. A well-known example of Gundersen [11] for functions sharing four values is of the form $f(z) = p \circ e^z$ and $g(z) = q \circ e^z$. Since $\exp(\mathbf{C}) = \mathbf{C} \setminus \{0\}$ the preimages of p and q for the shared values must be equal in $\mathbf{C} \setminus \{0\}$.

To get examples in the case of limit value sharing, p and q have to share the values on the whole sphere (already because of the Casorati–Weierstraß theorem). It follows that examples with four shared limit values cannot be constructed with the above method. It was noted in [1] that rational functions that share four values on the sphere are identical.

Let $p\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be rational and non-constant (this will be assumed throughout this section), $N := \{a_1, \dots, a_n\} \subset \widehat{\mathbf{C}}$ and $M := p^{-1}(N)$. It can be proved elementarily that

$$(n-2)\deg p \le |M| - 2 \tag{3}$$

where $\deg p$ is the degree of p and $|\cdot|$ is the cardinality.

The similarity to the formulas in Theorem 2.1 and 2.2 is obvious. With (3) an adaption of the proof of the five point theorem shows:

Theorem 6.1. *If rational $p, q\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ share four values then $p = q$.*

As noted in Proposition 2.5 we have $T(r, f) \le 3 \cdot T(r, g) + S(r, f)$ if two transcendental meromorphic functions $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ share three values. See also [12] where it is shown that the constant 3 is sharp. For rational functions (3) gives:

Proposition 6.2. *If rational $p, q\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ share three values then*

$$\deg p \le 3 \deg q - 2.$$

An example of two rational functions with equal degree that share three values on the sphere was given in [30]. We now give two examples that are extremal for the inequality in Proposition 6.2. This answers a question in [30, Question III].

Examples 6.3. First note that if $\deg q = 1$ then Proposition 6.2 shows $\deg p = 1$. It follows $p = q$. For the case $\deg p = 4$ and $\deg q = 2$ we have:

$$p(z) := \frac{(z+1)^3(z-3)}{(z-1)^3(z+3)}, \qquad q(z) := \frac{(z+1)(z-3)}{(z-1)(z+3)}; \tag{4}$$

p and q share the values $0, 1, \infty$.

The next extremal case is $\deg p = 7$ and $\deg q = 3$. We cannot give such an example. For $\deg p = 10$, $\deg q = 4$ we have

$$
(5) \qquad
\begin{aligned}
p(z) &:= \frac{(z-1)^7(z+3)\big(z+2-i\sqrt{3}\,\big)\big(z+2+i\sqrt{3}\,\big)}{(z+1)^7(z-3)\big(z-2+i\sqrt{3}\,\big)\big(z-2-i\sqrt{3}\,\big)}\\
&= \frac{z^{10}-9z^8+42z^6-210z^4+384z^3-315z^2+128z-21}{z^{10}-9z^8+42z^6-210z^4-384z^3-315z^2-128z-21},\\
q(z) &:= \frac{(z-1)(z+3)\big(z+2-i\sqrt{3}\,\big)\big(z+2+i\sqrt{3}\,\big)}{(z+1)(z-3)\big(z-2+i\sqrt{3}\,\big)\big(z-2-i\sqrt{3}\,\big)}\\
&= \frac{z^4+6z^3+12z^2+2z-21}{z^4-6z^3+12z^2-2z-21};
\end{aligned}
$$

p and q again share $0, 1, \infty$. We were not able to construct examples of higher degree.

Let now $\varphi\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be a transcendental meromorphic function. Then with the rational functions p, q from (4) and (5) it follows that

$$f := p \circ \varphi \qquad \text{and} \qquad g := q \circ \varphi$$

share the values and the limit values 0, 1 and ∞. Then (see e.g. [15, Theorem 2.2.5]) $T(r,f) = 2\cdot T(r,g) + S(r,f)$ and $T(r,f) = 2.5\cdot T(r,g) + S(r,f)$. Theorem 4.1 shows that if three limit values are shared then $T(r,f) \le 3\cdot T(r,g) + S(r,f)$. We believe that examples of the above type with arbitrary large degree exist so that the constant 3 should be (at least asymptotically) sharp.

Functions that share four limit values gives an example of Reinders [32]. He constructs two elliptic functions (on the same torus) that share four values. It is easy to see that value sharing and limit value sharing are identical for such functions. Consider $p, q\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ rational that share $0, 1, \infty$ and take the square root of p and q. Then uniformize the algebraic functions by an elliptic function. This method works for

$$
(6) \qquad p(z) := \frac{(z+1)^3(z-3)}{(z-1)^3(z+3)}, \qquad q(z) := \frac{(z+1)(z-3)^3}{(z-1)(z+3)^3}
$$

and an elliptic function $\varphi\colon \mathbf{C} \to \widehat{\mathbf{C}}$ which solves

$$(\varphi')^2 = (\varphi+1)(\varphi-1)(\varphi+3)(\varphi-3).$$

This leads to

$$f = \sqrt{p} \circ \varphi = \varphi' \cdot \frac{\varphi+1}{(\varphi-1)^2(\varphi+3)}, \qquad g = \sqrt{q} \circ \varphi = \varphi' \cdot \frac{\varphi-3}{(\varphi-1)(\varphi+3)^2};$$

f and g share the values and the limit values $0, 1, -1, \infty$. Our constants are different from Reinders's original example. The uniqueness theorem in [32] shows that both examples are equal up to a Möbius transformation. We have $p(z) = q(z)$ with p, q from (6) for $z \in \{0, 1, -1, 3, -3, \infty, i\sqrt{3}, -i\sqrt{3}\}$. In 0, 1, -1, 3, -3, ∞ the shared limit values are taken. For $\varphi \to \pm i\sqrt{3}$ it holds

$$\frac{f}{g} \to \frac{(\pm i\sqrt{3}+1)(\pm i\sqrt{3}+3)}{(\pm i\sqrt{3}-1)(\pm i\sqrt{3}-3)} = -1.$$

Hence the set of all completely unshared limit values is $\widehat{\mathbf{C}} \setminus \{0, 1, -1, \infty\}$. In the next section we show that if five limit values are shared then this set is empty.

All constructed examples in this section have in common that the shared limit values are also shared values. This is not necessary. Let p_1, p_2, q_1, q_2 be rational functions with $p_1(\infty) = q_1(\infty) = 1$ and $p_2(\infty) = q_2(\infty) = 0$ then with the above examples we get $\tilde{f} := p_1 \cdot f + p_2$, $\tilde{g} := q_1 \cdot g + q_2$ which share the limit values but not necessarily the values.

7. A five limit value theorem

First we note a statement which is known as the *Zalcman lemma* [38]. The idea essentially comes from a paper of Lohwater and Pommerenke [20].

Lemma 7.1 *Let $\mathscr{F}$ be a family of meromorphic functions in $\mathbf{D}_r$. Then the following is equivalent*:

(i) *$\mathscr{F}$ is not normal in $\mathbf{D}_r$.*

(ii) *There is a sequence $f_j \in \mathscr{F}$, a sequence of linear transformations M_j with $M_j \to c \in \mathbf{D}_r$ compactly in $\mathbf{C}$ such that $f_j \circ M_j \to F$ compactly in $\mathbf{C}$ with a non-constant meromorphic function F.*

We need the following result of Lehto [17].

Theorem 7.2. (i) *Let f be a transcendental entire function. Then there exists a sequence $w_j \to \infty$ with*

$$|w_j| f^{\#}(w_j) \to \infty.$$

(ii) *Let f be a transcendental meromorphic function. Then there exists a sequence $w_j \to \infty$ with*

$$\limsup_{j\to\infty} |w_j| f^{\#}(w_j) \geq \tfrac{1}{2}.$$

In this section we will only use (ii) which, except for the sharp constant $\frac{1}{2}$, can already be found in [18]. Statement (i) will be used later in connection with Julia directions. We note that (i) was improved by Clunie and Hayman [5]. See also Pommerenke [31].

We now prove a theorem for five shared limit values. An important argument is Nevanlinna's five point theorem.

Theorem 7.3. *Let f and g be transcendental meromorphic functions that share five limit values. Then for each $a \in \widehat{\mathbf{C}}$ there is a sequence $z_j \to \infty$ such that $f(z_j) \to a$ and $g(z_j) \to a$, i.e. the set of all completely unshared limit values is empty.*

Proof. According to Theorem 7.2(ii) there is a sequence $w_j \to \infty$ with $f^{\#}(w_j) \geq (3|w_j|)^{-1}$. We set for $z \in \mathbf{D}$:

$$\Phi_j(z) := w_j |w_j|^{z/(1-z)}.$$

Then $\Phi_j(0) = w_j$. Further $z/(1-z)$ maps the unit disk onto the half plane $\bigl\{\operatorname{Re} z \geq -\frac{1}{2}\bigr\}$. Hence

$$|\Phi_j(z)| \geq |\Phi_j(-1)| = \sqrt{|w_j|} \to \infty.$$

Put $f_j := f \circ \Phi_j \colon \mathbf{D} \to \widehat{\mathbf{C}}$. It follows

$$f_j^{\#}(0) = |\Phi_j'(0)| f^{\#}\bigl(\Phi_j(0)\bigr) = |w_j| \log|w_j| f^{\#}(w_j) \geq \tfrac{1}{3} \cdot \log|w_j| \to \infty.$$

Hence f_j has no convergent subsequence since $f_j \to F$ implies $f_j^{\#} \to F^{\#}$.

The Zalcman lemma shows the existence of a sequence of linear transformations $M_j \to c \in \mathbf{D}$ such that a subsequence $f_j \circ M_j \to F$ compactly on $\mathbf{C}$ with F meromorphic and non-constant. We claim that $g_j \circ M_j$ with $g_j := g \circ \Phi_j$ is normal in $\mathbf{C}$. Suppose this is not the case and $g_j \circ M_j$ is not normal in $\mathbf{D}_r$ for some $r > 0$. The Zalcman lemma gives a sequence of linear transformations $T_j \to d \in \mathbf{D}_r$ such that a subsequence $g_j \circ M_j \circ T_j$ converges compactly on $\mathbf{C}$ to a non-constant meromorphic function H. Since $f_j \circ M_j \to F$ compactly on $\mathbf{C}$ it follows $f_j \circ M_j \circ T_j \to F(d)$ compactly on $\mathbf{C}$. Let $a_1, \dots, a_5$ be the five shared limit values. Since H is non-constant H takes one of the shared limit values, say $H(z_0) = a_1$. Then $z_j := \Phi_j \circ M_j \circ T_j(z_0) \to \infty$ and $g(z_j) \to a_1$. Since a_1 is a shared limit value we get $f(z_j) \to a_1$ and therefore $F(d) = a_1$. Hence $f \circ \Phi_j \circ M_j \circ T_j(z) \to a_1$ for all $z \in \mathbf{C}$ and thus since $\Phi_j \circ M_j \circ T_j(z) \to \infty$ also $g \circ \Phi_j \circ M_j \circ T_j(z) \to a_1$. This contradicts $H \neq$ const. We conclude that a subsequence of $g_j \circ M_j$ converges compactly on $\mathbf{C}$ to a meromorphic function G.

Let a_i be a shared limit value and $F(z_0) = a_i$. Then $z_j := \Phi_j \circ M_j(z_0) \to \infty$ and $f(z_j) \to a_i$. It follows $g(z_j) \to a_i$ which shows $G(z_0) = a_i$. Symmetry shows that the five limit values are shared values of F and G. Since F is non-constant the five point theorem shows $F = G$. The Picard theorem gives for every $a \in \widehat{\mathbf{C}}$, with at most two exceptions, $z_a \in \mathbf{C}$ with $F(z_a) = a$ and hence a sequence $z_j^a := \Phi_j \circ M_j(z_a) \to \infty$ with $f(z_j^a) \to a$. Since $F = G$ it follows $g(z_j^a) \to a$. Proposition 3.4 shows that the set of all completely unshared limit values is open. Hence there are similar sequences for the two possible exceptional values. □

An important tool in our proof is the mapping Φ_j. It is the universal covering of $\mathbf{D}$ onto $\mathbf{C} \setminus \bigl\{|z| \leq \sqrt{|w_j|}\,\bigr\}$.

If f and g are entire so are F and G. It follows:

Theorem 7.4. *Let f and g be transcendental entire functions that share four finite limit values. Then for every $a \in \widehat{\mathbf{C}}$ there is a sequence $z_j \to \infty$ with $f(z_j) \to a$ and $g(z_j) \to a$.*

We state the following conjecture:

Conjecture 7.5. *Let f and g be transcendental meromorphic functions that share five limit values. Then f and g share all limit values.*

It seems to us that this is the correct analogue of the five point theorem.

We feel it would be interesting to investigate whether the conclusion holds if the five point set is replaced by a non-discrete set. In the next section we will show that it holds if the set of shared values has non-empty interior.

The above method of proof proposes to each uniqueness theorem concerning at least three shared values without multiplicities a corresponding statement for limit value sharing. In [23] it was proved:

Theorem 7.6. *Let $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be meromorphic and non-constant. If f and f' share three finite values then $f = f'$.*

A modification of the above reasoning shows:

Theorem 7.7. *Let $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental and meromorphic. If f and f' share three finite limit values then for every $a \in \widehat{\mathbf{C}}$ there is a sequence $z_j \to \infty$ with $f(z_j) \to a$ and $f'(z_j) \to a$.*

8. Extension properties of limit value sharing

Now we assume that f and g share all limit values from an open set. We will show that then f and g share *all* limit values. We describe the idea of the proof in a special case.

Let $M \subset \widehat{\mathbf{C}}$ be the open set of shared limit values. We can assume $\mathbf{D}_r^c \subset M$ for some $r > 0$. It remains to prove that f and g share all limit values in $\mathbf{D}_r$. If $f^{-1}(\mathbf{D}_r)$ and $g^{-1}(\mathbf{D}_r)$ consist only of islands (this is our special assumption), it follows from the maximum principle that $(f-g)(z) \to 0$ for $z \to \infty$ in these islands. Clearly $\partial f^{-1}(\mathbf{D}_r) \subset f^{-1}(\partial \mathbf{D}_r)$ and $\partial \mathbf{D}_r \subset M$ is compact. It follows $f - g \to 0$ on $\partial f^{-1}(\mathbf{D}_r)$. Symmetry shows that f and g share all limit values in $\mathbf{D}_r$.

In the general case we need statements that play the role of the maximum principle for unbounded domains. *Universal Phragmén–Lindelöf theorems* (see [8]) are theorems of this type. The next theorem was conjectured by Newman [28] and proved by Fuchs [9].

Theorem 8.1. *Let G be an unbounded domain and f be holomorphic on G such that for every finite $w \in \partial G$*

$$\limsup_{z \to w,\, z \in G} |f(z)| \le 1.$$

Set

$$M_G(r,f) := \sup_{z\in\partial\mathbf{D}_r\cap G} |f(z)|.$$

If $\liminf_{r\to\infty} M_G(r,f)/r = 0$ *then* $|f(z)| \le 1$ *for all* $z\in G$.

Note that no geometrical or topological assumptions are made for the domain.

We will further need a general version of a theorem of Lindelöf (see e.g. [6, p. 226]) which states that e.g. in every sector each bounded holomorphic function which converges to 0 for $z\to\infty$ on the boundary converges to 0 in the interior. Such a statement follows from the following result of Sakai [33] (see also [8]).

Theorem 8.2 (Sakai). *Let* G *be an unbounded domain with unbounded boundary and* f *be holomorphic on* G. *If for each* $w\in\partial G$ *with* $w\notin\overline{\mathbf{D}}$ *when approximated from* $G\setminus\overline{\mathbf{D}}$:

$$\limsup_{z\to w} |f(z)| \le 1$$

and if

$$|f(z)| \le a|z|^b$$

in $G\setminus\mathbf{D}$ *for some* $a,b>0$ *then*

$$\limsup_{z\to\infty,\, z\in G} |f(z)| \le 1.$$

Corollary 8.3. *Let* G *be an unbounded domain with unbounded boundary and* f *be continuous and bounded on* $\overline{G}$ *and holomorphic in* G. *If* f *converges to* 0 *on the boundary, i.e. if*

$$\lim_{\substack{z\to\infty\\ z\in\partial G}} f(z) = 0,$$

then f *converges to* 0 *in* G:

$$\lim_{\substack{z\to\infty\\ z\in G}} f(z) = 0.$$

Proof. For all $\varepsilon>0$ there exists $r>0$ such that Sakai's theorem can be applied to $f(rz)/\varepsilon$ on G/r. Hence $\limsup_{z\to\infty,\, z\in G}|f(z)| \le \varepsilon$. With $\varepsilon\to 0$ the claim follows. □

Now we can prove our extension theorem.

Theorem 8.4. *Let* $f,g\colon \mathbf{C}\to\widehat{\mathbf{C}}$ *be transcendental meromorphic functions that share all limit values of an open set. Then* f *and* g *share all limit values in* $\widehat{\mathbf{C}}$.

Proof. Let $M\subset\widehat{\mathbf{C}}$ be an open set of shared limit values and $\mathbf{D}_r^c\subset M$. Consider $f^{-1}(\mathbf{D}_r)$. According to the remarks at the beginning of this section it is sufficient to consider the unbounded components of $f^{-1}(\mathbf{D}_r)$. Suppose there exists a sequence $z_n\to\infty$ with $f(z_n)\to a\in\mathbf{D}_r$ but $g(z_n)\not\to a$. Then there must be a subsequence z_n which lies in the unbounded components of $f^{-1}(\mathbf{D}_r)$. Let Z_n be the component that contains z_n. If the Z_n tend to ∞ then

$$\lim_{n\to\infty} \max_{z\in\partial Z_n} |(f-g)(z)| = 0.$$

Further $f-g$ is asymptotically bounded in $f^{-1}(\mathbf{D}_r)$. Otherwise there is a sequence $w_n \to \infty$ in $f^{-1}(\mathbf{D}_r)$ with $g(w_n) \to \infty$ and $f(w_n)$ bounded. This contradicts the assumption that f and g share the limit value ∞. Theorem 8.1 shows

$$\lim_{n\to\infty} \max_{z\in Z_n} |(f-g)(z)| = 0,$$

in contradiction to $(f-g)(z_n) \not\to 0$. Hence almost all z_n lie in a fixed Z. Since

$$\lim_{\substack{z\to\infty \\ z\in\partial Z}} (f-g)(z) = 0$$

and $f-g$ is asymptotically bounded on Z we get from Corollary 8.3

$$\lim_{\substack{z\to\infty \\ z\in Z}} (f-g)(z) = 0,$$

again a contradiction. Symmetry proves the rest. □

Instead of Theorem 8.1 one can also use a generalized maximum principle as can be found e.g. in [6].

9. A generalization of limit value sharing

In this section we study the following situation: Suppose there is an open set $M \subset \widehat{\mathbf{C}}$ and a function $\varphi\colon M \to \widehat{\mathbf{C}}$ with

$$f(z_n) \to a \in M \quad \Longleftrightarrow \quad g(z_n) \to \varphi(a) \in \varphi(M).$$

We will show that φ must be conformal. In fact, we show that φ is the restriction of a Möbius transformation and that the relation between the limit values of f and g extends to all of $\widehat{\mathbf{C}}$.

Lemma 9.1. *Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions, $M \subset \widehat{\mathbf{C}}$ and $\varphi\colon M \to \widehat{\mathbf{C}}$ such that for all sequences $z_n \to \infty$:*

$$f(z_n) \to a \in M \quad \Rightarrow \quad g(z_n) \to \varphi(a).$$

Then φ is continuous.

Proof. Let $a \in M$ and a_k be a sequence in M with $a_k \to a$. We can assume $a, \varphi(a), a_k, \varphi(a_k) \in \mathbf{C}$. Choose a sequence $z_n^{(k)} \to \infty$ with $f(z_n^{(k)}) \to a_k$ for $n \to \infty$ and hence $g(z_n^{(k)}) \to \varphi(a_k)$. Pick n_k such that $|f(z_{n_k}^{(k)}) - a_k| < 1/k$, $|g(z_{n_k}^{(k)}) - \varphi(a_k)| < 1/k$ and $|z_{n_k}^{(k)}| \geq k$. Then $z_{n_k}^{(k)} \to \infty$ with $k \to \infty$ and $f(z_{n_k}^{(k)}) \to a$. Then $g(z_{n_k}^{(k)}) \to \varphi(a)$ and it follows $\varphi(a_k) \to \varphi(a)$. ◻

Note that we did not even use continuity.

It is less obvious that φ is meromorphic.

Lemma 9.2. *With the assumptions of Lemma* 9.1 *with* $M \subset \widehat{\mathbf{C}}$ *open* $\varphi\colon M \to \widehat{\mathbf{C}}$ *is meromorphic.*

Proof. Let $a \in M$. We may assume $a, \varphi(a) \in \mathbf{C}$ and $\varphi \neq \infty$ on $\overline{\mathbf{D}}_\varepsilon(a)$ with suitable $\varepsilon > 0$. The five islands theorem shows that, with at most four exceptions, f possesses infinitely many simple islands over $\mathbf{D}_\varepsilon(a)$ for each $a \in M$ (maybe one has to decrease $\varepsilon > 0$). Let I_n be a sequence of simple islands of f over $\mathbf{D}_\varepsilon(a)$ and $f_n^{-1}\colon \mathbf{D}_\varepsilon(a) \to I_n$ be the corresponding branches of the inverse function of f. We consider $F_n\colon \mathbf{D}_\varepsilon(a) \to \widehat{\mathbf{C}}$ with $F_n := g \circ f_n^{-1}$. Now g is asymptotically bounded on $I := \cup I_n$. Otherwise there exists a sequence $z_n \in I$ with $z_n \to \infty$ and $g(z_n) \to \infty$. Passing to a subsequence we may assume $f(z_n) \to b \in \overline{\mathbf{D}}_\varepsilon(a)$. It follows $\varphi(b) = \lim_{n\to\infty} g(z_n) = \infty$ in contradiction to the choice of ε. Hence F_n is a normal sequence and converges pointwise to the continuous function $\varphi|_{\mathbf{D}_\varepsilon(a)}$. According to Vitali's theorem F_n converges compactly. Thus φ is holomorphic on $\mathbf{D}_\varepsilon(a)$. The continuity of φ shows the removability of the four possible exceptional points. ◻

Theorem 9.3. *Let* $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ *be transcendental meromorphic functions,* $M \subset \widehat{\mathbf{C}}$ *be open and* $\varphi\colon M \to \widehat{\mathbf{C}}$ *such that for every sequence* $z_n \to \infty$:

$$f(z_n) \to a \in M \quad \Longleftrightarrow \quad g(z_n) \to \varphi(a) \in \varphi(M).$$

Then φ *can be extended to a Möbius transformation and*

$$f(z_n) \to a \in \widehat{\mathbf{C}} \quad \Longleftrightarrow \quad g(z_n) \to \varphi(a) \in \widehat{\mathbf{C}},$$

i.e. $\varphi \circ f$ *and* g *share all limit values.*

Proof. We may assume $\mathbf{D}^c \subset M$ and $\varphi(\infty) = \infty$, i.e. f and g share the limit value ∞. We show that φ can be extended to a holomorphic function in $\mathbf{C}$. Suppose the power series expansion of φ around ∞ has as its circle of convergence $\partial \mathbf{D}_r$ with $0 < r < 1$. Let $a \in \partial \mathbf{D}_r$. We choose ε with $0 < \varepsilon < r$ and consider

$$U_\varepsilon(a) := \{z \in \mathbf{C} \mid r - \varepsilon < |z| < 2,\ |\arg z - \arg a| < \varepsilon\}.$$

According to the five islands theorem there are at most four points $a \in \partial \mathbf{D}_r$ such that f has only finitely many simple islands over $U_\varepsilon(a)$ for all $\varepsilon > 0$. Suppose

a is not such an exceptional point. Then there is a sequence I_n of simple islands over $U_\varepsilon(a)$ with suitable $\varepsilon > 0$. Let $f_n^{-1}\colon U_\varepsilon(a) \to I_n$ be the corresponding inverse functions. Since f and g share the limit value ∞ it follows that g is asymptotically bounded on $f^{-1}\big(U_\varepsilon(a)\big)$. Hence $F_n := g \circ f_n^{-1}$ is normal on $U_\varepsilon(a)$ and converges on the part of $U_\varepsilon(a)$ which lies in $\mathbf{D}^c$ pointwise to φ. Vitali's theorem shows that F_n converges on $U_\varepsilon(a)$ compactly to the extension of φ.

Let now $a \in \partial \mathbf{D}_r$ be such that f possesses only finitely many simple islands over $U_\varepsilon(a)$ for every $\varepsilon > 0$. We denote by $V_{\varepsilon,\delta}(a)$ the sets $V_{\varepsilon,\delta}(a) := U_\varepsilon(a) \setminus \overline{U_\delta(a)}$ with $\delta < \varepsilon$. We claim the existence of $\varepsilon > 0$ such that f has infinitely many simple islands over each $V_{\varepsilon,\delta}(a)$ with $\delta \in (0,\varepsilon)$. Suppose ε does not exist. For $\varepsilon_1 > 0$ choose $\delta_1 > 0$ such that f has only finitely many simple islands over $V_{\varepsilon_1,\delta_1}(a)$. For $\varepsilon_2 > 0$ with $\varepsilon_2 < \delta_1$ choose δ_2 such that f possesses only finitely many islands over $V_{\varepsilon_2,\delta_2}(a)$ and continue inductively. Since the $V_{\varepsilon_k,\delta_k}(a)$ are Jordan domains with disjoint closures the five islands theorem shows that this process must stop after three repetitions and we obtain a contradiction. Hence there is $\varepsilon > 0$ such that f has for all $\varepsilon > \delta > 0$ infinitely many simple islands over $V_{\varepsilon,\delta}(a)$. As above one shows that φ can be extended to $\overline{\mathbf{D}}_r^c \cup V_{\varepsilon,\delta}(a)$ for all δ with $\varepsilon > \delta > 0$. Now $\delta \to 0$ shows that a is an isolated singularity of φ. Since ∞ is a shared limit value g is asymptotically bounded on $\bigcup_{0<\delta<\varepsilon} f^{-1}\big(V_{\varepsilon,\delta}(a)\big)$. It follows that φ is bounded in a neighbourhood of a and therefore a is removable. Hence the radius of convergence of φ around ∞ is not finite, i.e. φ is holomorphic in $\widehat{\mathbf{C}} \setminus \{0\}$. The origin is again a removable singularity. Thus $\varphi\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is rational with a single pole at ∞. This pole is simple since φ is bijective in a neighbourhood of ∞. Hence φ is a polynomial of first degree.

Consider $\tilde{f} := \varphi \circ f$ and g. Then $\tilde{f}$ and g share all limit values in M. According to Theorem 8.4, $\tilde{f}$ and g share all limit values in $\widehat{\mathbf{C}}$. The theorem is proved. □

We generalize the foregoing theorem. For this we need the following lemma.

Lemma 9.4. *Let f and g be meromorphic functions that share the limit value $a \in \widehat{\mathbf{C}}$. If there exists a neighbourhood $\mathbf{D}_\varepsilon(a)$ such that for all $b \in \mathbf{D}_\varepsilon(a)$ and all sequences $z_n \to \infty$:*

$$f(z_n) \to b \quad \Rightarrow \quad g(z_n) \to b,$$

then f and g share all limit values in $\widehat{\mathbf{C}}$.

Proof. We may assume that f and g share the limit value $a = 0$. According to Lemma 3.2 there exists δ with $0 < \delta < \varepsilon$ such that $g^{-1}(\mathbf{D}_\delta) \mathrel{\tilde{\subset}} f^{-1}(\mathbf{D}_{\varepsilon/2})$. Suppose there is a sequence $z_n \to \infty$ with $g(z_n) \to b \in \mathbf{D}_\delta$ but $f(z_n) \not\to b$. Then there exists a subsequence $z_n \to \infty$ with $f(z_n) \to c \neq b$. Since $\{z_n\} \mathrel{\tilde{\subset}} f^{-1}(\mathbf{D}_{\varepsilon/2})$ it follows $c \in \mathbf{D}_\varepsilon$ and therefore $g(z_n) \to c$, a contradiction. Hence f and g share all limit values in $\mathbf{D}_\delta$ and Theorem 8.4 shows that all limit values are shared. □

Theorem 9.5. *Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions which share the limit value ∞. Further let M be a neighbourhood of ∞ and $\varphi\colon M \to \widehat{\mathbf{C}}$ such that for all $z_n \to \infty$:*

$$f(z_n) \to a \in M \quad \Rightarrow \quad g(z_n) \to \varphi(a).$$

Then φ can be extended to a polynomial and $\varphi \circ f$ and g share all limit values.

Proof. That φ is a polynomial follows exactly as in the proof of Theorem 9.3. We set $K := \sup_{z \in M^c} |\varphi(z)| < \infty$ and choose $R > K$. Then $\varphi^{-1}(\overline{\mathbf{D}}_R^c) \subset M$ and $\varphi \circ f$ and g share the limit value ∞ since $\varphi^{-1}(\{\infty\}) = \{\infty\}$. Let $a \in \overline{\mathbf{D}}_R^c$ and $z_n \to \infty$ such that $\varphi \circ f(z_n) \to a$. Then all accumulation points of $f(z_n)$ are contained in $\varphi^{-1}(\{a\}) \in M$. It follows $g(z_n) \to \varphi\big(\varphi^{-1}(\{a\})\big) = a$. With $a = \infty$ the assumptions of Lemma 9.4 are fulfilled. Hence $\varphi \circ f$ and g share all limit values. □

It is important that ∞ is a shared limit value. With an approximation theorem of Arakelian (see [10]) it is possible to construct entire functions g with

$$\exp(z_n) \to a \in \mathbf{D} \quad \Rightarrow \quad g(z_n) \to \varphi(a)$$

where $\varphi\colon \mathbf{D} \to \mathbf{C}$ is an arbitrary holomorphic and continuous function in $\overline{\mathbf{D}}$.

If φ is defined globally we obtain:

Theorem 9.6. *Let $f, g\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be transcendental meromorphic functions and $\varphi\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that for all $z_n \to \infty$:*

$$f(z_n) \to a \in \widehat{\mathbf{C}} \quad \Rightarrow \quad g(z_n) \to \varphi(a).$$

Then φ is rational and $\varphi \circ f$ and g share all limit values.

Proof. Lemma 9.2 shows that φ is rational. Let $a \in \widehat{\mathbf{C}}$ and $z_n \to \infty$ such that $\varphi \circ f(z_n) \to a$. The accumulation points of $f(z_n)$ are again contained in $\varphi^{-1}(\{a\})$. Then $g(z_n) \to \varphi\big(\varphi^{-1}(\{a\})\big) = a$. It follows that $\varphi \circ f$ and g share all limit values. □

10. Filling disks and Julia directions

A sequence of disks

$$\mathbf{D}_j := \{z \in \mathbf{C} \mid |z - z_j| < \varepsilon_j |z_j|\} \tag{7}$$

with $z_j \to \infty$ and $\varepsilon_j \to 0$ is called a sequence of *filling disks* for meromorphic $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ if in every infinite union $\cup \mathbf{D}_{j_k}$ f takes all values of $\widehat{\mathbf{C}}$ with at most two exceptions.

Every transcendental entire function possesses a sequence of filling disks. The z_j are chosen such that $f^{\#}(z_j)$ is large. Theorem 7.2 shows the existence of a sequence z_j with $M_j := |z_j| f^{\#}(z_j) \to \infty$. Set e.g. $\varepsilon_j := 1/\sqrt{M_j}$ and consider for $z \in \mathbf{D}$ the functions

$$f_j(z) := f(\varepsilon_j |z_j| z + z_j).$$

Then $f_j^{\#}(0) = \varepsilon_j |z_j| f^{\#}(z_j) = \sqrt{M_j} \to \infty$. Hence f_j has no convergent subsequence. Montel's theorem shows that each subsequence of f_j takes all values in $\widehat{\mathbf{C}}$, with at most two exceptions, infinitely often. This shows that the $\mathbf{D}_j$ are filling disks.

The functions f_j are of the form $f_j = f \circ \Phi_j$, with $\Phi_j(\mathbf{D}) \to \infty$.

Lemma 10.1. *Let f and g be transcendental meromorphic functions that share three limit values and $\Phi_j \colon \mathbf{D} \to \mathbf{C}$ holomorphic such that $\Phi_j(\mathbf{D}) \to \infty$. Set $f_j := f \circ \Phi_j$ and $g_j := g \circ \Phi_j$. Then f_j is normal in $\mathbf{D}$ if and only if g_j is normal in $\mathbf{D}$.*

Proof. Suppose f_j is not normal in $\mathbf{D}$ but g_j is normal. According to the Zalcman lemma there exists a sequence of linear transformations $M_j \to c \in \mathbf{D}$ such that after passing to a subsequence $f_j \circ M_j \to F$ compactly on $\mathbf{C}$ with a non-constant meromorphic function F.

Since g_j is normal, a subsequence converges compactly $g_j \to G$ with a meromorphic function G in $\mathbf{D}$ and thus $g_j \circ M_j \to G(c)$ compactly on $\mathbf{C}$. Since F is non-constant, F takes one of the shared limit values a_1, a_2, a_3. We assume $F(z_0) = a_1$. Hence $f_j \circ M_j(z_0) = f \circ \Phi_j \circ M_j(z_0) \to a_1$. From the properties of M_j and Φ_j it follows $z_j := \Phi_j \circ M_j(z_0) \to \infty$. Since a_1 is a shared limit value of f and g we obtain $g(z_j) = g_j \circ M_j(z_0) \to a_1$ and thus $G(c) = a_1$. It follows $g_j \circ M_j(z) \to a_1$ for all $z \in \mathbf{C}$ and since $\Phi_j \circ M_j(z) \to \infty$ also $f_j \circ M_j(z) \to a_1$. This implies $F \equiv a_1$, contradicting $F \neq$ const. □

If f is entire, so are the functions resulting from the Zalcman lemma.

Lemma 10.2. *Let f and g be transcendental entire functions that share two finite limit values. Let Φ_j, f_j and g_j be as above. Then f_j is normal in $\mathbf{D}$ if and only if g_j is normal in $\mathbf{D}$.*

Suppose f possesses filling disks with centres z_j. If α is an accumulation point of $\arg z_j$ then in each sector

$$J_\varepsilon := \{z \in \mathbf{C} \mid |\arg z - \alpha| < \varepsilon\}$$

around the ray $J := \{z \in \mathbf{C} \mid \arg z = \alpha\}$ lie infinitely many filling disks. It follows that f takes in every J_ε each value, with at most two exceptions, infinitely often, i.e. J is a *Julia direction* (see [14]). Hence every transcendental entire function possesses a Julia direction.

For meromorphic functions the situation is more complicated. In fact there are meromorphic functions without Julia directions. From the above argumentation for entire functions we see that in this case necessarily:

$$f^{\#}(z) = O\left(\frac{1}{|z|}\right) \tag{8}$$

for all $z \in \mathbf{C}$. Functions with (8) are called *Julia exceptional functions* [29] (see also [18, Theorem 3]).

Inequality (8) and (1) immediately imply $T(r,f) = O\big((\log r)^2\big)$.

Hence if

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = \infty, \tag{9}$$

then there exists $z_j \to \infty$ with $|z_j| f^{\#}(z_j) \to \infty$. It follows again that f has a sequence of filling disks and hence a Julia direction.

We note that Julia directions need not be generated by filling disks. An example of a meromorphic function with (8) and a Julia direction was given in [39]. An example of an entire function with a Julia direction where the sectors J_ε do not contain filling disks can be found in [3]. We will call a Julia direction which is obtained from filling disks a *Milloux* direction.

We call $z_j \to \infty$ a *singular sequence* for f if for all ε with $0 < \varepsilon < 1$ and each subsequence $\mathbf{D}^\varepsilon_{j_k}$ of the disks

$$\mathbf{D}^\varepsilon_j := \{z \in \mathbf{C} \mid |z - z_j| < \varepsilon |z_j|\}$$

f takes in $\cup \mathbf{D}^\varepsilon_{j_k}$ all values, with at most two exceptions, infinitely often. A similar notion was introduced in [7].

Proposition 10.3. *Let $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ be meromorphic and $z_j \to \infty$. Then the following statements are equivalent:*

(i) *z_j is a singular sequence of f.*

(ii) *$f_j\colon \mathbf{D} \to \widehat{\mathbf{C}}$ with $f_j(z) := f(\varepsilon|z_j|z + z_j)$ has no convergent subsequence for all $\varepsilon \in (0,1)$.*

(iii) *There exists a sequence u_j with $|z_j - u_j| = o(|z_j|)$ and $|u_j| \cdot f^{\#}(u_j) \to \infty$.*

(iv) *There exists a sequence $\varepsilon_j \to 0$ such that the disks (7) are filling disks for f.*

We omit the simple proof.

Theorem 10.4. *Let f and g be trancendental meromorphic functions that share three limit values. Then f and g have the same singular sequences.*

Proof. Let z_j be a singular sequence of f. For $\varepsilon \in (0,1)$ set $\Phi_j(z) := \varepsilon \cdot |z_j| \cdot z + z_j$. Since $\varepsilon < 1$ it follows $\Phi_j(\mathbf{D}) \to \infty$. Proposition 10.3 shows that $f_j = f \circ \Phi_j$ has no convergent subsequence. Suppose $g_j = g \circ \Phi_j$ has a convergent subsequence g_j. According to Lemma 10.1 the corresponding subsequence f_j is normal and possessses hence a convergent subsequence in contradiction to Proposition 10.3. It follows that g_j has no convergent subsequence and Proposition 10.3 shows that z_j is a singular sequence for g. Symmetry proves the rest. □

Corollary 10.5. *Let f and g be transcendental meromorphic functions that share three limit values. Then*:

(i) *f and g have the same Milloux directions.*
(ii) *Either f and g are both Julia exceptional functions or f and g have a common Milloux direction.*
(iii) *If* (9) *holds for f then* (9) *is true for g and f and g have a common Milloux direction.*

For (iii) apply Theorem 4.1.

Corollary 10.6. *Let f and g be transcendental entire functions that share two finite limit values. Then*:

(i) *f and g have the same Milloux directions.*
(ii) *f and g have a common Milloux direction.*

We note that in [35] (see also [36]) it was shown that functions with a Valiron deficient value are not Julia exceptional. This gives obvious versions of the above statements for such functions.

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