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A SHORT PROOF OF NON-GROMOV HYPERBOLICITY OF TEICHMÜLLER SPACES

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Abstract. The purpose of this note is to present a short new proof of a theorem of H. Masur and M. Wolf [MaW] to the effect that the Teichmüller space is not hyperbolic in Gromov's sense. This proof demonstrates that an old construction of Masur [Ma] used by him to prove that the Teichmüller space is not hyperbolic in the much stronger sense of Busemann is sufficient to prove non-Gromov hyperbolicity, if combined with a simple and fundamental property of Gromov-hyperbolic spaces, namely, with the exponential divergence of geodesics.

In [Ma], H. Masur proved that Teichmüller spaces are not hyperbolic in the sense of H. Busemann [B], despite an earlier claim by S. Kravetz [Kr]. Later on, a weaker and much more flexible notion of hyperbolicity had emerged, namely, the one of M. Gromov [G]. In [MaW], H. Masur and M. Wolf proved that Teichmüller spaces are not hyperbolic in Gromov's sense either. The purpose of this note is to suggest a short new proof of this result. This proof demonstrates that the original construction of Masur [Ma] is sufficient to prove non-Gromov hyperbolicity, if combined with a simple and fundamental property of Gromov-hyperbolic spaces, namely, with the *exponential divergence of geodesics.* In fact, this property can be taken as the defining property of Gromov-hyperbolic spaces; see, for example, [A]. Our proof was inspired by a recent proof of Masur–Wolf theorem due to J. McCarthy and A. Papadopoulos [McP]. In contrast with [McP], we do not rely on the advanced theory of Gromov-hyperbolic spaces (namely, on the theory of boundaries) and restrict our tools by the intuitively appealing property of the exponential divergence of geodesics. The prerequisites are also lesser on the side of the theory of Teichmüller spaces: everything we need can be found in Kerckhoff's paper [Ke].

To begin with, let us recall the notion of divergence of geodesics at the rate e, where $e\colon \mathbf{R}_{\geq 0} \to \mathbf{R}$ is a function (we denote by $\mathbf{R}_{\geq 0}$ the set of non-negative real numbers). Namely, two geodesic rays r, r' starting at the same point x of a metric space X are said to be divergent at the rate e, if for every $T, t \in \mathbf{R}_{\geq 0}$ the following property holds (where we consider geodesic rays as isometric embeddings $\mathbf{R}_{\geq 0} \to X$).

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Suppose that the distance $d\big(r(T), r'(T)\big) > e(0)$. *If a path* α *connects* $r(T+t)$ *with* $r'(T+t)$ *in the closure of the complement of* $B_{T+t}(x)$ *(the ball of radius* $T+t$ *with the center* x*), then the length of* α *is* $\geq e(t)$.

By Theorem 2.19 from [A], for any Gromov-hyperbolic space X there is an exponentially growing function e, depending only on X, such that any two geodesic rays in X starting at the same point are divergent at least at the rate e. Although we will not need this fact, this property of the exponential divergence of geodesics can be taken as the definition of the Gromov hyperbolicity (as follows from [A, Theorem 2.19 and Proposition 2.20], for example). Let us now state our key lemma.

Lemma. *Suppose that* X *is a metric space containing a family of geodesic rays* $r_\mu \colon \mathbf{R}_{\geq 0} \to \mathbf{R}$ *starting at the same point* x *with the following properties:*

(i) *for any two rays* r_μ, $r_{\mu'}$ *the distance* $d\big(r_\mu(t), r_{\mu'}(t)\big)$ *is bounded independently of* t;

(ii) *for any* $C > 0$ *there is a pair of rays* r_μ, $r_{\mu'}$ *and* $T > 0$ *such that the distance* $d\big(r_\mu(T), r_{\mu'}(T)\big) > C$.

Then X *is not Gromov hyperbolic.*

Proof. If X is Gromov hyperbolic, then there is an exponentially growing function e such that every two geodesic rays in X starting at the same point diverge at the rate e. Using the property (ii), let us choose two rays $r = r_\mu$, $r' = r_{\mu'}$ starting at x and a real number $T > 0$ such that $d\big(r(t), r'(t)\big) > e(0)$. By the property (i), there is an upper bound $B > 0$ for the function $t \mapsto d\big(r(t), r'(t)\big)$. For any $t \in \mathbf{R}$ consider the path α consisting of the following three pieces: the segment of r from $r(T+t)$ to $r(T+t+B)$; the geodesic segment from $r(T+t+B)$ to $r'(T+t+B)$; the segment of r' from $r'(T+t+B)$ to $r'(T+t)$. Since the length of the second piece is $\leq B$, the path α is contained in the closure of the complement of $B_{T+t}(x)$. Clearly, the length of α is $\leq 3B$. In particular, this length is bounded independently of t. On the other hand, this length should be $\geq e(t)$. The contradiction shows that X is not Gromov hyperbolic and completes the proof of the lemma.

Now we are ready to prove that Teichmüller spaces are not Gromov hyperbolic. Let S be a compact Riemann surface of genus g representing a point x in the Teichmüller space T_g. Let $\sigma = (\sigma_1, \ldots, \sigma_p)$ be a collection of pairwise non-isotopic non-trivial circles on S, and let $m = [m_1 : \ldots : m_p]$ be a p-tuple of positive real numbers considered up to a common factor; $1 \leq p \leq 3g-3$. To these data one can associate a quadratic differential $\theta_{\sigma,m}$ on S, namely, the unique Jenkins–Strebel differential with the core circles of cylinders isotopic to $\sigma_1, \ldots, \sigma_p$ and with the moduli of these cylinders proportional to $m_1, \ldots, m_p$. Cf. [Ke, Section 2]. In its turn, $\theta_{\sigma,m}$ defines a geodesic ray $r_{\sigma,m}$ in T_g starting at x. As above we consider $r_{\sigma,m}$ as an isometric embedding $\mathbf{R}_{\geq 0} \to T_g$.

Masur showed in [Ma] that any two rays $r_{\sigma,m}$, $r_{\sigma,m'}$ stay on a bounded distance. In other words, the Teichmüller distance $d\big(r_{\sigma,m}(t), r_{\sigma,m'}(t)\big)$ is a bounded function of $t \geq 0$. If $m \neq m'$, then these two rays are obviously different. These properties contradict Busemann hyperbolicity, but, by themselves, do not contradict Gromov hyperbolicity. On the other hand, it follows immediately from an argument of Kerckhoff (see [Ke, Section 4, the second paragraph of the proof of Proposition 2]) that for any given $C > 0$ one can find m, m' such that $d\big(r_{\sigma,m}(T), r_{\sigma,m'}(T)\big) > C$ for all sufficiently big T. In particular, the above distance is not bounded uniformly in m, m'. Implicitly, the latter result is contained already in [Ma]; see [Ma, Section 7]. In view of our lemma, this result contradicts the Gromov hyperbolicity. This contradiction shows that T_g is not Gromov hyperbolic.

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