Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 26, 2001, 483–488

SOLUTIONS OF $f'' + P(z)f = 0$ THAT HAVE ALMOST ALL REAL ZEROS

Gary G. Gundersen

University of New Orleans, Department of Mathematics
New Orleans, Louisiana 70148, U.S.A.; ggunders@math.uno.edu

Abstract. We show that for any given real constants $a > 0$ and $b \geq 0$, there exist certain real constants λ such that the equation $f'' + (az^4 + bz^2 - \lambda)f = 0$ possesses a solution f that has an infinite number of real zeros and at most a finite number of nonreal zeros. When $b > 0$, these equations are new examples of equations of the form $f'' + P(z)f = 0$, where $P(z)$ is a polynomial, that possess exceptional solutions of this kind. When $b = 0$, these equations are earlier examples of Titchmarsh.

1. Introduction

Consider a second order linear differential equation of the form

$$f'' + P_n(z)f = 0, \tag{1}$$

where $P_n(z)$ is a nonconstant polynomial of degree $n \geq 1$. It is well known that every solution f of equation (1) is an entire function.

It is an interesting question to ask the following: For which nonconstant polynomials $P_n(z)$ will equation (1) admit a solution that has an infinite number of real zeros and at most a finite number of nonreal zeros? (See [3], [4], and [2, Problem 2.71].) To the author's knowledge, it appears that up until now, the only examples of this kind in the literature are Examples 1 and 2 below. The purpose of this paper is to give new examples of this kind.

Example 1. It is well known that the classical Airy differential equation

$$f'' - zf = 0 \tag{2}$$

possesses a special contour integral solution $f(z) = \mathrm{Ai}(z)$ called the Airy integral, where all the zeros of $\mathrm{Ai}(z)$ are real and negative, and where $\mathrm{Ai}(z)$ has an infinite number of zeros (see [8, pp. 413–415]). Thus if $a \neq 0$ and b are any real constants, then from a suitable linear change of the independent variable in (2), we obtain that the equation

$$f'' + (az + b)f = 0 \tag{3}$$

possesses a solution with only real zeros and infinitely many.

2000 Mathematics Subject Classification: Primary 34M10; Secondary 30D35.

Example 2. This example comes from a simple transformation of an equation of Titchmarsh (see [10, pp. 172–173] and [3, p. 289] for the details). There exists an infinite sequence of positive constants λ_k satisfying $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, where $\lambda_k \to \infty$ as $k \to \infty$, such that for each λ_k, the equation

$$f'' + (z^4 - \lambda_k)f = 0 \tag{4}$$

possesses a solution $f = f_k(z)$ that has an infinite number of real zeros and at most a finite number of nonreal zeros. Any nonreal zero of f_k must lie on the imaginary axis, and f_k possesses exactly k zeros that lie on the imaginary axis. When k is even, f_k is an even function, and when k is odd, f_k is an odd function. It follows that f_0 and f_1 each have only real zeros.

From equation (4) we see that if $a \neq 0$ and b are any real constants, then the equation

$$f'' + [a^2(az+b)^4 - a^2\lambda_k]f = 0 \tag{5}$$

possesses the solution $f = f_k(az+b)$ which has an infinite number of real zeros and at most a finite number of nonreal zeros. Thus when $k = 0$ or $k = 1$, equation (5) possesses a solution that has only real zeros and infinitely many.

On the other hand, we now list several results which show that equations of the form (1) which admit the kind of solutions in Examples 1 and 2 are exceptional. In [5] it was shown that equation (1) cannot possess two linearly independent solutions that each have only real zeros. More generally, we have the following result.

Theorem A [3]. *Let f_1 and f_2 be any two linearly independent solutions of equation* (1). *Then at least one of f_1, f_2 has the property that its sequence of nonreal zeros has exponent of convergence equal to $\frac{1}{2}(n+2)$.*

The next four results show that there are many equations of the form (1) which do not admit the kind of exceptional solutions in Examples 1 and 2. The following result shows that when the degree of the polynomial $P_n(z)$ is one of the numbers $2, 6, 10, \ldots$, then equation (1) cannot possess a solution that has an infinite number of real zeros and at most a finite number of nonreal zeros.

Theorem B [3], [4]. *Let $P_n(z)$ be a polynomial of degree $n = 2 + 4k$ for some integer $k \geq 0$, and suppose that $f \not\equiv 0$ is a solution of equation* (1). *Then either f has only a finite number of zeros, or the exponent of convergence of the sequence of nonreal zeros of f is equal to $\frac{1}{2}(n+2)$.*

The next result is an immediate corollary of Theorem 3 and Lemma 5 in [3]. Here we recall that an entire function is said to be *real* if it is real on the real axis.

Theorem C [3]. *If equation* (1) *possesses a solution f that has an infinite number of real zeros, then $P_n(z)$ is a real polynomial and f is a constant multiple of a real solution of* (1).

The next theorem and corollary show that when an equation of the form (1) possesses a solution that has only real zeros and infinitely many, then this puts a restriction on the number of real zeros that $P_n(z)$ can have.

Theorem D [9]. *Suppose that equation* (1) *possesses a solution f that has only real zeros and infinitely many. Then the number of real zeros of $P_n(z)$ counting multiplicities is less than $\frac{1}{2}(n+2)$.*

Corollary [9]. *If equation* (1) *possesses a solution f with only real zeros and infinitely many, and if $P_n(z)$ has only real zeros, then equation* (1) *is an equation of the form* (3) *and f is a constant multiple of $\mathrm{Ai}(\alpha z + \beta)$, where $\mathrm{Ai}(z)$ is the Airy integral and α and β are real constants.*

2. The new examples

We now state the new examples. For any given real constants $a > 0$ and $b \geq 0$, we will show that there exists an infinite sequence of real constants λ_k satisfying $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, where $\lambda_k \to \infty$ as $k \to \infty$, such that for each λ_k, the equation

$$f'' + (az^4 + bz^2 - \lambda_k)f = 0 \tag{6}$$

possesses a solution $f = f_k(z)$ that has an infinite number of real zeros and at most a finite number of nonreal zeros. By making a linear change of the independent variable in equation (6), we can obtain more equations that possess solutions that have an infinite number of real zeros and at most a finite number of nonreal zeros.

Remark. When $b > 0$, equation (6) does not have the form of equation (5). When $b = 0$, we see that equation (6) is Example 2. Hence these examples in equation (6) extend Example 2.

3. Proof of the new examples

We will prove the existence of these examples in equation (6) by combining the classical theory of eigenvalues and eigenfunctions with the asymptotic integration theory of equation (6).

Hille used his theory of asymptotic integration together with the Liouville transformation to obtain many basic properties of the solutions of equations of the form (1); see Chapter 7.4 in [6]. The following theorem contains some of these properties.

Theorem E [6, Chapter 7.4]. *In equation* (1) *we set*

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \ldots, a_n$ *are constants with* $a_n \neq 0$.

The critical rays $\theta_0, \theta_1, \ldots, \theta_{n+1}$ *of equation* (1) *are defined as follows*:

$$\theta_j = \frac{2\pi j - \arg a_n}{n+2} \qquad \text{for } 0 \leq j \leq n+1.$$

For convenience with notation, we set $\theta_{n+2} = \theta_0$ *and* $\theta_{n+3} = \theta_1$.

Let $f \not\equiv 0$ *be a solution of equation* (1), *and let* $\varepsilon > 0$ *be a small fixed constant. Then the following statements hold*:

(i) *For each* $j = 0, 1, 2, \ldots, n+1$, *we have either*

$$f(z) \to \infty \quad \text{as } z \to \infty \quad \text{in } \theta_j + \varepsilon \leq \arg z \leq \theta_{j+1} - \varepsilon,$$

or

$$f(z) \to 0 \quad \text{as } z \to \infty \quad \text{in } \theta_j + \varepsilon \leq \arg z \leq \theta_{j+1} - \varepsilon.$$

(ii) *For each* $j = 0, 1, 2, \ldots, n+1$, $f(z)$ *has at most a finite number of zeros in* $\theta_j + \varepsilon \leq \arg z \leq \theta_{j+1} - \varepsilon$.

(iii) *If for some particular* $j \in \{1, 2, \ldots, n+1, n+2\}$, *we have either*

$$f(z) \to 0 \quad \text{as } z \to \infty \quad \text{in } \theta_{j-1} + \varepsilon \leq \arg z \leq \theta_j - \varepsilon,$$

or

$$f(z) \to 0 \quad \text{as } z \to \infty \quad \text{in } \theta_j + \varepsilon \leq \arg z \leq \theta_{j+1} - \varepsilon,$$

then f *has at most a finite number of zeros in* $\theta_j - \varepsilon \leq \arg z \leq \theta_j + \varepsilon$.

Although [6, Chapter 7.4] contains several more properties of the solutions of (1), the properties in Theorem E are enough for our purposes.

We now prove the existence of the examples in equation (6). To this end, we consider the differential equation

$$\psi'' + (\lambda - az^4 + bz^2)\psi = 0,$$

where λ, a, and b are real constants such that $a > 0$ and $b \geq 0$. From the classical theory of eigenvalues and eigenfunctions, it is well known that for any given real constants $a > 0$ and $b \geq 0$, there exists an infinite sequence of real constants λ_k (the eigenvalues) satisfying $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, where $\lambda_k \to \infty$ as $k \to \infty$, such that for each λ_k, the equation

$$\psi'' + (\lambda_k - az^4 + bz^2)\psi = 0 \tag{7}$$

possesses a solution $\psi = \psi_k(z)$ (an eigenfunction) that is real on the real axis and has exactly k real zeros, such that $\psi_k(z)$ is an even function when k is even and is an odd function when k is odd, and where the set of eigenfunctions $\psi_0, \psi_1, \psi_2, \ldots$ form a complete orthonormal set for $L^2(\mathbf{R})$. Proofs of these statements are contained in Chapters 2 and 5 in [10].

The critical rays of equation (7) are $\pm\pi/6$, $\pm\pi/2$, and $\pm 5\pi/6$. We now fix a nonnegative integer k and consider the solution ψ_k of (7). Let $\varepsilon > 0$ be a small fixed constant. Since ψ_k is in $L^2(\mathbf{R})$, we can deduce from Theorem E(i) that $\psi_k(z) \to 0$ as $z \to \infty$ in $-\pi/6+\varepsilon \le \arg z \le \pi/6-\varepsilon$. Thus from Theorem E(iii), we obtain that $\psi_k(z)$ has at most a finite number of zeros in each of the following two angles: (A) $-\pi/6-\varepsilon \le \arg z \le -\pi/6+\varepsilon$, and (B) $\pi/6-\varepsilon \le \arg z \le \pi/6+\varepsilon$. From Theorem E(ii), $\psi_k(z)$ has at most a finite number of zeros in each of the following three angles: (A) $-\pi/2+\varepsilon \le \arg z \le -\pi/6-\varepsilon$, (B) $-\pi/6+\varepsilon \le \arg z \le \pi/6-\varepsilon$, and (C) $\pi/6+\varepsilon \le \arg z \le \pi/2-\varepsilon$. Therefore, $\psi_k(z)$ has at most a finite number of zeros in $-\pi/2+\varepsilon \le \arg z \le \pi/2-\varepsilon$.

Now suppose that $\zeta = Re^{i\beta}$ is a zero of $\psi_k(z)$, where $\pi/2-\varepsilon < \beta < \pi/2$. Arguing as on pp. 172–173 of [10], we start with the Green's transform of equation (7) (see [7, p. 509]):

$$(8)\quad \int_0^\zeta (\lambda_k - az^4 + bz^2)|\psi_k(z)|^2\,dz = \psi_k'(0)\overline{\psi_k(0)} - \psi_k'(\zeta)\overline{\psi_k(\zeta)} + \int_0^\zeta |\psi_k'(z)|^2\,\overline{dz}.$$

If k is even, then $\psi_k(z)$ is an even function, and so $\psi_k'(0) = 0$, while if k is odd, then $\psi_k(z)$ is an odd function, and so $\psi_k(0) = 0$. Since we also have $\psi_k(\zeta) = 0$, we obtain from (8) that

$$\int_0^\zeta (\lambda_k - az^4 + bz^2)|\psi_k(z)|^2\,dz = \int_0^\zeta |\psi_k'(z)|^2\,\overline{dz}.$$

By integrating along the ray $\arg z = \beta$, we obtain

$$\int_0^R (\lambda_k - ar^4e^{i4\beta} + br^2e^{i2\beta})|\psi_k(re^{i\beta})|^2\,dr = e^{-i2\beta}\int_0^R |\psi_k'(re^{i\beta})|^2\,dr.$$

Taking imaginary parts gives

$$(9)\qquad \int_0^R (-ar^4\sin 4\beta + br^2\sin 2\beta)|\psi_k(re^{i\beta})|^2\,dr = -\sin 2\beta\int_0^R |\psi_k'(re^{i\beta})|^2\,dr.$$

Since $\pi/2-\varepsilon < \beta < \pi/2$, the left-hand side of (9) is positive and the right-hand side of (9) is negative. Hence we have a contradiction. Therefore, $\psi_k(z)$ cannot have any zeros in $\pi/2-\varepsilon < \arg z < \pi/2$. Since we have already shown that $\psi_k(z)$ has at most a finite number of zeros in $-\pi/2+\varepsilon \le \arg z \le \pi/2-\varepsilon$, we obtain that $\psi_k(z)$ has at most a finite number of zeros in $-\pi/2+\varepsilon \le \arg z < \pi/2$. Since $\psi_k(z)$ is real on the real axis, it follows that $\psi_k(z)$ has at most a finite number of zeros in $-\pi/2 < \arg z < \pi/2$. Furthermore, since $\psi_k(z)$ is either an even function or an odd function, it follows that $\psi_k(z)$ has at most a finite number of zeros that do not lie on the imaginary axis.

Since $\psi = \psi_k(z)$ is a solution of equation (7), it follows that $\psi_k(z)$ has order 3 (see [1]). If k is even, then $\psi_k(z)$ is an even function, and therefore $\psi_k(\sqrt{z})$ is an entire function of order $\frac{3}{2}$, which implies that $\psi_k(z)$ has an infinite number of zeros. If k is odd, then $\psi_k(z)$ is an odd function, and therefore $\sqrt{z}\,\psi_k(\sqrt{z})$ is an entire function of order $\frac{3}{2}$, which also implies that $\psi_k(z)$ has an infinite number of zeros. It follows that $\psi_k(z)$ has an infinite number of zeros that lie on the imaginary axis and at most a finite number of zeros that do not lie on the imaginary axis.

Now set $f_k(z) = \psi_k(iz)$. Since $\psi = \psi_k(z)$ is a solution of equation (7), we obtain that $f = f_k(z)$ is a solution of equation (6), and $f_k(z)$ has an infinite number of real zeros and at most a finite number of nonreal zeros. This completes the proof of the existence of the examples in equation (6).

References

[1] Bank, S., and I. Laine: On the oscillation theory of $f'' + Af = 0$ where A is entire. - Trans. Amer. Math. Soc. 273, 1982, 351–363.

[2] Brannan, D.A., and W.K. Hayman: Research problems in complex analysis. - Bull. London Math. Soc. 21, 1989, 1–35.

[3] Gundersen, G.: On the real zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire. - Ann. Acad. Sci. Fenn. Math. 11, 1986, 275–294.

[4] Hellerstein, S., and J. Rossi: Zeros of meromorphic solutions of second order linear differential equations. - Math. Z. 192, 1986, 603–612.

[5] Hellerstein, S., L.-C. Shen, and J. Williamson: Real zeros of derivatives of meromorphic functions and solutions of second order differential equations. - Trans. Amer. Math. Soc. 285, 1984, 759–776.

[6] Hille, E.: Lectures on Ordinary Differential Equations. - Addison-Wesley, Reading, Mass., 1969.

[7] Ince, E.L.: Ordinary Differential Equations. - Dover Publications, New York, 1956.

[8] Olver, F.: Asymptotics and Special Functions. - Academic Press, New York, 1974.

[9] Rossi, J., and S. Wang: The radial oscillation of solutions to ODE's in the complex domain. - Proc. Edinburgh Math. Soc. 39, 1996, 473–483.

[10] Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-order Differential Equations, Part I, second edition. - Oxford University Press, London, 1962.

Received 28 March 2000