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ENTIRE FUNCTIONS AND LOGARITHMIC SUMS OVER NONSYMMETRIC SETS OF THE REAL LINE

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Abstract. We give uniform estimates of entire functions of exponential type less than a numerical constant having sufficiently small logarithmic sums over certain nonsymmetric discrete subsets of the real line. We thereby generalize earlier results about logarithmic sums over symmetric sets, in particular the set of integers.

1. Introduction

The set of polynomials p satisfying

$$\sum_{n\in\mathbf{Z}} \frac{\log^+ |p(n)|}{n^2+1} \leq \eta$$

is, for η small enough, a normal family in the whole complex plane. This result was published by Paul Koosis in 1966 (for even polynomials) and later for general polynomials. See [2] and [3, Chapter VIII, B]. Recently, other methods of proof, based on the investigation of so-called least superharmonic majorants, have been found, see [10], [7] and [6].

In this paper we shall deal with the question of how much the structure of the set of integers is involved in these results. In [11], an investigation of the situation where the integers are replaced by so-called h-dense subsets of the real line was begun. There, the main results in [10] were generalized to *symmetric* h-dense subsets. Symmetry of the set played a crucial role. This was mainly because a deep result, concerning the existence of least superharmonic majorants, is only available for even functions. However, in [7] and [6], use of that result was completely dispensed with. That makes it possible for us to generalize the results of [11] to nonsymmetric h-dense sets. We recall that a discrete subset Λ of the real line is called h-dense if, outside a bounded subset of the real line, any closed interval of length h contains at least one element of Λ.

Before stating the main theorem we shall recall the definition of a certain numerical constant T_* (from [10]):

$$T_* = \pi/M_*,$$

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where

$$M_* = \inf_{s>0}\left\{\frac{1}{s}\int_0^{\pi/2} \exp\big(s(1+\sin\theta)\big)\,d\theta\right\}.$$

The theorem is as follows.

Theorem 1.1. *Suppose that Λ is an h-dense subset of the real line. For any $A_0 < T_*/h$ and $\varepsilon > 0$ there is $\eta_0 > 0$ such that for any $\eta \le \eta_0$ there is a constant $C_\eta > 0$ with the property that*

$$|f(z)| \le C_\eta \exp(A|y| + \varepsilon|z|)$$

for all complex z and all entire functions f of exponential type $\le A \le A_0$ satisfying

$$\sum_{\lambda\in\Lambda} \frac{\log^+|f(\lambda)|}{\lambda^2+1} \le \eta.$$

The constant T_* is approximately equal to 0.44. In the situation where Λ is the set of integers, Theorem 1.1 is true with T_*/h replaced by π. See [7] and [6].

Let us briefly indicate the main steps in the proof of the theorem above and in particular, the visible differences in comparison to the arguments in [11].

The main work is carried out for entire functions f of exponential type $\le A \le A_0 < T_*/h$, without zeros in the upper half-plane, satisfying $f(0) = 1$ and

$$1 \le |f(x)| \le \mathrm{Const}\,(|x|+1)$$

for real x. For such functions the logarithmic integral

$$\mathscr{I}(f) = \int_{-\infty}^{\infty} \frac{\log|f(t)|}{t^2}\,dt \tag{1}$$

exists; see for example Problem 27 in [3], and we shall compare it with the logarithmic sum

$$\sum_{\lambda\in\Lambda} \frac{\log|f(\lambda)|}{\lambda^2}$$

of f over Λ. For $b > 0$ we define

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|\log|f(t)|}{|z-t|^2}\,dt - b|y|$$

and we consider the least superharmonic majorant $\mathscr{M}F$ of F and its Riesz measure ϱ on the real line. The logarithmic sum can, in a weak form, be bounded from below by an integral involving $\mathscr{M}F$ and ϱ, namely

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2}\,d\varrho(x); \tag{2}$$

see Theorem 6.2. The main part of this paper consists of finding a good lower bound for this integral, depending only on the parameter b and the logarithmic integral of f; see Theorem 5.1. The main result of this paper follows from these two theorems; see the remarks following Theorem 6.2.

In Section 2 we give some fundamental properties of the least superharmonic majorant. We give its Riesz representation (Proposition 2.1) and we show that the Riesz measure ϱ has bounded Radon–Nikodym derivative (Proposition 2.2).

Section 3 deals with the asymptotic behaviour of ϱ. We prove for example that the distribution function $\varrho(t)$ is differentiable at the origin. This is based on a version of Kolmogorov's theorem on the harmonic conjugate, suitable for functions u satisfying

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{t^2}\,dt < \infty;$$

see Theorem 3.2 and Proposition 3.3.

In Sections 4 and 5 the lower bound of the integral (2) is found. The considerations involve energy integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\frac{1}{|x-t|}\,d\tau(t)\,d\tau(x)$$

associated with certain real measures τ satisfying $\tau(\mathbf{R}) = 0$. In this setup it is possible to equip the linear space of potentials

$$u_\tau(x) = \int_{-\infty}^{\infty} \log\frac{1}{|x-t|}\,d\tau(t)$$

with an inner product structure. We shall use the Hilbert space obtained by completing the inner product space. A weak compactness argument (see Theorem 4.5) will give us

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x) \ge \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,\frac{\varrho(x)}{x}\,dx.$$

Section 5 is devoted to estimating the right-hand side of this relation. This is done by using the second mean value theorem and an integration by parts method.

The integral (2) plays the same role in this paper as the integral

$$\int_{0}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)$$

does in the previous papers [10], [11], [7] and [6]. In those papers, $|f|$ is assumed to be even on the real axis, so that $\mathscr{M}F$ is also even on the real line. Its Riesz measure is thus symmetric and can be represented as a measure ϱ on the half-line $[0,\infty)$. The general method used to estimate (2) is the same as the method used to estimate the integral in the even case. However, as we shall see, many difficulties appear in the present more general situation.

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2. The least superharmonic majorant

We fix in this section an entire function f of exponential type less than or equal to A, with $f(0)=1$ and $|f(x)|\geq 1$ for all real x. Furthermore, we suppose that

$$|f(x)| \leq \mathrm{Const}\,(1+|x|) \tag{3}$$

for real x and that f has no zeros in the upper half-plane.

For $b>0$ we construct

$$F(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|y|\log|f(t)|}{|z-t|^2}\,dt-b|y|. \tag{4}$$

This function, defined by an integral over the entire real line, has a finite superharmonic majorant in the whole complex plane. Indeed, for a suitable choice of $K>0$, the entire function of exponential type b, $\phi(z)=K(\sin bz)/z$ satisfies $|\phi(x)f(x)|\leq 1$ on the real line. This means that the superharmonic function $-\log|\phi(z)|$, not identically equal to infinity, is a majorant of F in the whole complex plane. The least superharmonic majorant of F, $\mathscr{M}F$, is thus at our disposal.

We remark that (3) assures the existence of a non-zero entire function ϕ of exponential type $\leq b$ making $f\phi$ bounded on the real axis. Such an entire function is called a multiplier of type $\leq b$ associated with f. Such multipliers exist (for any $b>0$) for any entire function f of exponential type satisfying the much weaker condition

$$\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{t^2+1}\,dt<\infty.$$

This result was published by Beurling and Malliavin in 1962 (see [1]). All results in this section and in the following two sections are still valid if we drop the assumption (3) (see again [3, Problem 27]). However, one of the surprising aspects of the whole approach is that Beurling and Malliavin's theorem is not needed and indeed can be obtained as a corollary, see [6] and [7].

We refer to [3, p. 363] for a general introduction to least superharmonic majorants. We recall some general properties of $\mathscr{M}F$:

It is a continuous function in the whole plane.

It is harmonic where F is harmonic.

It is harmonic where it is $>F$.

We conclude that F is harmonic in the upper and lower half-planes and therefore $\mathscr{M}F$ is harmonic in the whole plane except the closed set E, defined as

$$E=\{x\in\mathbf{R}\mid \log|f(x)|=\mathscr{M}F(x)\}. \tag{5}$$

Since $\mathscr{M}F$ is a majorant of F we must have $\mathscr{M}F(x)\geq\log|f(x)|$ for all real x. In particular $\mathscr{M}F(x)\geq 0$ for all $x\in\mathbf{R}$.

We turn to representations of $\mathscr{M}F$. First of all,

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(t)}{t^2+1}\,dt < \infty \tag{6}$$

and we have the Poisson representation

$$\mathscr{M}F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|\mathscr{M}F(t)}{|z-t|^2}\,dt - b|y|; \tag{7}$$

(see for example [4, p. 374]). We also note the following global Riesz representation of $\mathscr{M}F$.

Proposition 2.1. *We have*

$$\mathscr{M}F(z) = \mathscr{M}F(0) - \gamma x - \int_{|t|\le 1} \log\left|1-\frac{z}{t}\right| d\varrho(t) - \int_{|t|\ge 1}\left(\log\left|1-\frac{z}{t}\right| + \frac{x}{t}\right) d\varrho(t),$$

where γ is a real constant and ϱ is a positive measure on the real line satisfying

$$\int_{|t|\le 1} \bigl|\log|t|\bigr|\,d\varrho(t) < \infty$$

and $\varrho(t)/t \le \mathrm{Const}$ for $|t|\ge 1$. Here the distribution function $\varrho(t)$ is normalized so that $\varrho(0)=0$. Furthermore, ϱ is concentrated on the set E given in (5).

If $\mathscr{M}F(0)>0$ we must have $0\notin E$ and ϱ must vanish on a small neighbourhood of the origin. Therefore one may write the representation of $\mathscr{M}F$ as

$$\mathscr{M}F(z) = \mathscr{M}F(0) - \tilde{\gamma}x - \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{x}{t}\right) d\varrho(t),$$

with some other constant $\tilde{\gamma}$. A proof of the representation in this situation can be found in [4, p. 376]. However, when $\mathscr{M}F(0)=0$, the origin is in E and it is necessary to split the integral into two parts as in Proposition 2.1. The proof in this situation is similar to the one given in [4, p. 376] and we shall not give it here; see also Problem 57 in [4].

We start our investigation of ϱ by giving the following fundamental result.

Proposition 2.2. *The Riesz measure ϱ is absolutely continuous and*

$$d\varrho(t) \le \frac{A+b}{\pi}\,dt.$$

In the proof of this result we shall make use of the Stieltjes representation of $\log|f|$.

Lemma 2.3. *Let $\{z_k\}$ denote the zeros of f, counting multiplicities, ordered so that $|z_1| \le |z_2| \le \cdots$. For $y \ge 0$ we have*

$$\log|f(x+iy)| = a_1 x - a_2 y + \int_{|t|\ge 1} \left(\log\left|1-\frac{z}{t}\right| + \frac{x}{t}\right) d\nu(t)$$
$$+ \lim_{N\to\infty} \sum_1^N \frac{1}{\pi}\left\{\int_{|t|\le 1} \log\left|1-\frac{z}{t}\right| \frac{-y_k}{|z_k-t|^2}\,dt - \int_{|t|\ge 1} \frac{x}{t}\frac{-y_k}{|z_k-t|^2}\,dt\right\}.$$

Here a_1, a_2 are real constants and

$$d\nu(t) = \frac{1}{\pi}\left(\sum_1^\infty \frac{-y_k}{|z_k-t|^2}\right) dt$$

satisfies $\int_{-\infty}^{\infty} \bigl(1/(t^2+1)\bigr)\,d\nu(t) < \infty$.

Proof. The proof is based on a version of Levinson's theorem, see for example [5]. By that theorem we may write f as

$$f(z) = e^{az} \lim_N \prod_1^N \left(1 - \frac{z}{z_k}\right). \tag{8}$$

The product is not in general absolutely convergent (see [5, p. 39]). Here $a = a_1 + ia_2$ is a complex number. We write $z_k = x_k + iy_k$ with $y_k < 0$. From the elementary Poisson formula

$$\log\left|1-\frac{z}{z_k}\right| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|1-\frac{z}{t}\right| \frac{-y_k}{|z_k-t|^2}\,dt$$

we obtain

$$\begin{aligned}\log|f(x+iy)| &= a_1 x - a_2 y + \lim_N \sum_1^N \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|1-\frac{z}{t}\right| \frac{-y_k}{|z_k-t|^2}\,dt \\ &= a_1 x - a_2 y + \int_{|t|\ge 1} \left(\log\left|1-\frac{z}{t}\right| + \frac{x}{t}\right) d\nu(t) \\ &\quad + \lim_{N\to\infty} \sum_1^N \frac{1}{\pi}\left\{\int_{|t|\le 1} \log\left|1-\frac{z}{t}\right| \frac{-y_k}{|z_k-t|^2}\,dt - \int_{|t|\ge 1} \frac{x}{t}\frac{-y_k}{|z_k-t|^2}\,dt\right\}.\end{aligned}$$

The lemma is proved.

Proof of Proposition 2.2. In the product representation (8) of f we may assume that a is real; replacing it by its real part cannot increase the type of f and at the same time it leaves $|f(t)|$ unchanged for real t. We may thus take $a_2 = 0$ in Lemma 2.3. We have, furthermore, for $y \geq 0$,

$$\log|f(z)| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\log|f(t)|}{|z-t|^2}\,dt + Ay.$$

This implies

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\log|f(t)|}{|z-t|^2}\,dt - by \geq \log|f(z)| - (A+b)y.$$

When $x \in E$ we therefore have

$$\begin{aligned}\mathscr{M}F(x) - \log|f(x)| = 0 &\leq \mathscr{M}F(x+iy) - F(x+iy)\\ &\leq \mathscr{M}F(x+iy) - \log|f(x+iy)| + (A+b)y.\end{aligned}$$

This is, by Proposition 2.1 and Lemma 2.3, the same as

$$\begin{aligned}\int_{|t|\geq 1} \log\left|\frac{1-z/t}{1-x/t}\right| d(\varrho+\nu)(t) + \int_{|t|\leq 1} \log\left|\frac{1-z/t}{1-x/t}\right| d\varrho(t)&\\ + \lim_{N\to\infty}\sum_{1}^{N}\frac{1}{\pi}\int_{|t|\leq 1}\log\left|\frac{1-z/t}{1-x/t}\right|\frac{-y_k}{|z_k-t|^2}\,dt &\leq (A+b)y.\end{aligned}$$

Since

$$\log\left|\frac{1-z/t}{1-x/t}\right| = \frac{1}{2}\log\left(1+\left(\frac{y^2}{(t-x)^2}\right)\right)$$

we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}\log\left(1+\left(\frac{y^2}{(t-x)^2}\right)\right)d(\varrho+\nu)(t) \leq (A+b)y,$$

for $x \in E$. From this relation one finds that ϱ is absolutely continuous and

$$d\varrho(t) \leq \frac{A+b}{\pi}\,dt;$$

see [4, p. 406].

Corollary 2.4. *We have*

$$0 \leq \mathscr{M}F(x) - \mathscr{M}F(x+iy) \leq (A+b)y.$$

The first inequality is evident from the Riesz representation of $\mathscr{M}F$ and the second follows from Proposition 2.2, since

$$\mathscr{M}F(x) - \mathscr{M}F(x+iy) = \frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \left(\frac{y^2}{(t-x)^2}\right)\right) d\varrho(t).$$

Lemma 2.5. *In the situation where* $\mathscr{M}F(0) = 0$ *we have*

$$0 \le \int_{-\infty}^{\infty} \frac{\mathscr{M}F(t)}{t^2}\, dt \le \pi b.$$

Proof. From the Poisson and Riesz representation of $\mathscr{M}F$ we find, for $y > 0$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{M}F(t)}{t^2+y^2}\, dt - b = \frac{\mathscr{M}F(iy)}{y} = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(1+y^2/t^2)}{y}\, d\varrho(t) \le 0.$$

As y decreases to zero we get by monotone convergence that the integral

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(t)}{t^2}\, dt$$

converges and that it is less than or equal to πb.

3. Asymptotic behaviour of the Riesz measure

The function $\mathscr{M}F$ is in the upper half-plane the real part of the analytic function

$$\Psi(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{t^2+1}\right) \mathscr{M}F(t)\, dt + ibz.$$

We see that, by Proposition 2.1, $\mathscr{M}F$ is also the real part of

$$\begin{aligned}\Phi(z) = &-\int_{|t|\ge 1} \left(\log\left(1 - \frac{z}{t}\right) + \frac{z}{t}\right) d\varrho(t)\\ &- \int_{|t|\le 1} \log\left(1 - \frac{z}{t}\right) d\varrho(t) - \gamma z + \mathscr{M}F(0).\end{aligned}$$

The imaginary parts of these two functions must therefore agree up to an additive constant:

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1}\right) \mathscr{M}F(t)\, dt + bx\\ &= -\int_{|t|\ge 1} \left(\arg\left(1 - \frac{z}{t}\right) + \frac{y}{t}\right) d\varrho(t) - \int_{|t|\le 1} \arg\left(1 - \frac{z}{t}\right) d\varrho(t) - \gamma y + C.\end{aligned} \tag{9}$$

We shall draw several conclusions from (9). For $z = i$ it implies a relation between C, γ and ϱ. We shall, however, need another relation in the case of $\mathscr{M}F(0) = 0$:

Lemma 3.1. *If* $\mathscr{M}F(0)=0$, *the constant* C *in* (9) *is equal to*

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t(t^2+1)}\,dt.$$

Proof. We put $z=iy$ in (9) and obtain, after integration by parts,

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t(y^2-1)}{(t^2+y^2)(t^2+1)}\mathscr{M}F(t)\,dt\\ &=-\gamma y+C+\big(\varrho(1)+\varrho(-1)\big)y-\int_{|t|\geq 1}\frac{\varrho(t)y^3}{t^2(t^2+y^2)}\,dt+\int_{|t|\leq 1}\frac{\varrho(t)y}{t^2+y^2}\,dt.\end{aligned}\tag{10}$$

Then we let y tend to zero. By Proposition 2.2 we have $|\varrho(t)|\leq \mathrm{Const}\,|t|$ so that

$$\left|\frac{\varrho(t)y}{t^2+y^2}\right|\leq\frac{\mathrm{Const}\,|t|y}{t^2+y^2}\leq \mathrm{Const}.$$

By the dominated convergence theorem the integrals on the right-hand side of (10) tend to zero. The left-hand side tends to

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t(t^2+1)}\,dt,$$

also by dominated convergence. Indeed,

$$\left|\frac{t(y^2-1)}{(t^2+1)(t^2+y^2)}\right|\leq\frac{1}{(t^2+1)|t|},$$

for $0\leq y\leq 1$ and $\int_{-\infty}^{\infty}(\mathscr{M}F(t)/t^2)\,dt<\infty$ by Lemma 2.5. The lemma follows.

The next result describes the asymptotic behaviour of $\varrho(t)/t$ and it plays an important role in what follows.

Theorem 3.2. *In the situation where* $\mathscr{M}F(0)>0$,

$$\frac{\varrho(t)}{t}\to\frac{b}{\pi}\qquad\text{as } t\to\pm\infty,$$

and ϱ *is zero close to the origin. In the situation where* $\mathscr{M}F(0)=0$ *we have*

$$\begin{aligned}&\frac{\varrho(t)}{t}\to\frac{b}{\pi}&&\text{as } t\to\pm\infty,\\ &\frac{\varrho(t)}{t}\to\frac{b}{\pi}-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt&&\text{as } t\to 0.\end{aligned}$$

The proof of Theorem 3.2 is based on Lemma 3.5 below and suitable versions of Kolmogorov's theorem on the harmonic conjugate.

For a real-valued function u satisfying $\int_{-\infty}^{\infty} |u(t)|/(t^2+1)\,dt < \infty$, the harmonic conjugate is defined a.e. on the real axis by

$$\tilde{u}(x) = \lim_{y\to 0_+} \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1}\right)u(t)\,dt.$$

Kolmogorov's theorem on the harmonic conjugate states that, for $\lambda > 0$,

$$\int_{\{x||\tilde{u}(x)|>\lambda\}} \frac{dx}{x^2+1} \le \frac{4}{\lambda}\int_{-\infty}^{\infty} \frac{|u(t)|}{t^2+1}\,dt.$$

This result, or rather a corollary to it, asserting that the integral on the left-hand side is $o(1/\lambda)$ as λ tends to infinity, can be used to find the asymptotic behaviour at $\pm\infty$ of $\varrho(t)/t$. We need another version of Kolmogorov's theorem in order to describe the behaviour near the origin.

We suppose that u is a real-valued function satisfying

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{t^2}\,dt < \infty.$$

In this situation we define

$$H(u)(x) = \lim_{y\to 0_+} \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2} + \frac{1}{t}\right)u(t)\,dt.$$

This function exists a.e. on the real line and we have

$$H(u)(x) = \tilde{u}(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t)}{t(t^2+1)}\,dt. \tag{11}$$

In Appendix A we give a proof of the following version of Kolmogorov's theorem.

Proposition 3.3. *We have*

$$\int_{\{x||H(u)(x)|>\lambda\}} \frac{dx}{x^2} \le \frac{4}{\lambda}\int_{-\infty}^{\infty} \frac{|u(t)|}{t^2}\,dt.$$

We shall also need to know the asymptotic behaviour of $H(u)$ near zero if u is known to vanish near the origin:

Lemma 3.4. *If $u(t)=0$ when $|t|\le\delta$ then*

$$\left|H(u)(x) + \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{u(t)}{t^2}\,dt\right| \le \frac{2x^2}{\pi\delta}\int_{-\infty}^{\infty}\frac{|u(t)|}{t^2}\,dt$$

for $|x| \le \frac{1}{2}\delta$.

Proof. Assume $|x| \le \frac{1}{2}\delta$. We have

$$\begin{aligned}\pi H(u)(x) &= \lim_{y\to 0+} \int_{|t-x|\ge\delta/2} \left(\frac{x-t}{(x-t)^2+y^2} + \frac{1}{t}\right) u(t)\,dt \\ &= \int_{|x-t|\ge\delta/2} \left(\frac{1}{x-t} + \frac{1}{t}\right) u(t)\,dt \\ &= \int_{|x-t|\ge\delta/2} \left(\frac{1}{x-t} + \frac{1}{t} + \frac{x}{t^2}\right) u(t)\,dt - x\int_{|x-t|\ge\delta/2} \frac{u(t)}{t^2}\,dt \\ &= x^2 \int_{|x-t|\ge\delta/2} \frac{u(t)}{t^2(x-t)}\,dt - x\int_{|x-t|\ge\delta/2} \frac{u(t)}{t^2}\,dt.\end{aligned}$$

The lemma follows.

The relation between the harmonic conjugate of $\mathscr{M}F$ and the Riesz measure ϱ is given in the next lemma.

Lemma 3.5. *We have, for real* x,

$$\pi\varrho(x) = \widetilde{\mathscr{M}F}(x) + bx - C,$$

where C *is the constant in the relation* (9).

The lemma follows from (9) by letting y tend to zero. We shall not give the proof.

Proof of Theorem 3.2. A proof in the case where $\mathscr{M}F(0) > 0$ can be found in [4, p. 376]. There, a version of Levinson's theorem is used; one may also follow the arguments below, based directly on Kolmogorov's theorem.

We thus only consider the case where $\mathscr{M}F(0) = 0$. From Lemma 3.5 we have

$$\pi\frac{\varrho(x)}{x} = b + \frac{\widetilde{\mathscr{M}F}(x)}{x} - \frac{C}{x}.$$

We find the asserted asymptotic behaviour of $\varrho(x)/x$ as $x \to \pm\infty$ since $\widetilde{\mathscr{M}F}(x)/x$ tends to zero as x tends to $\pm\infty$. That this is the case follows from the elementary arguments in [3, p. 68]. One should take a number $\lambda > 1$, very close to 1. Then one should use the corollary to Kolmogorov's theorem on the harmonic conjugate already mentioned to obtain, for given ε, $x_n \in [\lambda^n, \lambda^{n+1}]$ such that $|\widetilde{\mathscr{M}F}(x_n)| \le \varepsilon\lambda^n$ for all large n and finally use the monotoneity of ϱ; see [3, p. 68].

Proposition 3.3 allows us to argue similarly when x is very close to zero. Lemma 3.5, Lemma 3.1 and (11) give us

$$\pi\frac{\varrho(x)}{x} = \frac{H(\mathscr{M}F)(x)}{x} + b. \tag{12}$$

We show that

$$\lim_{x\to 0}\frac{\varrho(x)}{x}=\frac{b}{\pi}-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt$$

by obtaining

$$\frac{H(\mathscr{M}F)(x)}{x}\to-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt$$

as $x\to 0$. We shall only consider the case where $x>0$; the case where $x<0$ may be treated similarly. To ease notation in this proof we put

$$I(h)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(t)}{t^2}\,dt$$

for functions h making this integral absolutely convergent.

Let $\varepsilon>0$ be given. Choose $\lambda\in(0,1)$ so close to 1 that

$$b\frac{1-\lambda^2}{\lambda^2}\le\frac{\varepsilon}{2}. \tag{13}$$

Then choose $\varepsilon_1>0$ such that

$$\varepsilon_1<\frac{1-\lambda}{8\lambda} \tag{14}$$

and $\delta>0$ such that

$$\int_{|t|\le\delta}\frac{\mathscr{M}F(t)}{t^2}\,dt\le\varepsilon_1^2.$$

(This is possible by Lemma 2.5.) We put $\varphi(t)=\mathscr{M}F(t)\chi(t)$, where χ is the characteristic function of the set $\{|t|>\delta\}$, and we consider the two sets

$$A_n=\{x\in[\lambda^{n+1},\lambda^n]\mid |H(\mathscr{M}F)(x)+I(\varphi)x|<\varepsilon_1\lambda^n\}$$

and

$$B_n=\{x\in[\lambda^{n+1},\lambda^n]\mid |H(\mathscr{M}F)(x)-H(\varphi)(x)|<\varepsilon_1\lambda^n/2\}.$$

If $x\in B_n$ then

$$\begin{aligned}|H(\mathscr{M}F)(x)+I(\varphi)x|&<\frac{\varepsilon_1\lambda^n}{2}+|H(\varphi)(x)+I(\varphi)x|\\&\le\frac{\varepsilon_1\lambda^n}{2}+\frac{2}{\pi\delta}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t^2}\,dt\,x^2\\&\le\frac{\varepsilon_1\lambda^n}{2}+\frac{2}{\pi\delta}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t^2}\,dt\,\lambda^{2n}\\&\le\varepsilon_1\lambda^n,\end{aligned}$$

for all sufficiently large n. We can use Lemma 3.4 on φ since, by construction, $\varphi \equiv 0$ on $[-\delta, \delta]$. Thus $B_n \subseteq A_n$ for all sufficiently large n.

We claim that A_n is non-empty for n sufficiently large. Indeed, if it were empty then

$$\begin{aligned}\frac{1-\lambda}{\lambda^{n+1}} &= \int_{\lambda^{n+1}}^{\lambda^n} \frac{dx}{x^2} = \int_{[\lambda^{n+1},\lambda^n]\setminus A_n} \frac{dx}{x^2} \\ &\leq \int_{[\lambda^{n+1},\lambda^n]\setminus B_n} \frac{dx}{x^2} \leq \int_{\{x \mid |H(\mathscr{M}F-\varphi)(x)| \geq \varepsilon_1 \lambda^n/2\}} \frac{dx}{x^2}.\end{aligned}$$

Thus, by Proposition 3.3,

$$\frac{1-\lambda}{\lambda^{n+1}} \leq \frac{8}{\varepsilon_1 \lambda^n} \int_{|t|\leq\delta} \frac{\mathscr{M}F(t)}{t^2}\, dt \leq \frac{8\varepsilon_1}{\lambda^n}.$$

This contradicts (14). Therefore, when n is sufficiently large, there is $x_n \in [\lambda^{n+1}, \lambda^n]$ such that

$$|H(\mathscr{M}F)(x_n) + I(\varphi)x_n| < \varepsilon_1 \lambda^n.$$

From (12) we see that $H(\mathscr{M}F)(x) + bx$ is increasing. The idea is now to estimate $H(\mathscr{M}F)(x)/x$ when $\lambda^{n+1} \leq x \leq \lambda^n$, using the points x_{n+1} and x_{n-1}. When $\lambda^{n+1} \leq x \leq \lambda^n$,

$$H(\mathscr{M}F)(x_{n+1}) + bx_{n+1} \leq H(\mathscr{M}F)(x) + bx \leq H(\mathscr{M}F)(x_{n-1}) + bx_{n-1}.$$

Therefore

$$\begin{aligned}H(\mathscr{M}F)(x_{n+1}) + I(\varphi)x_{n+1} - b(x - x_{n+1}) &\leq H(\mathscr{M}F)(x) + I(\varphi)x \\ &\leq H(\mathscr{M}F)(x_{n-1}) + I(\varphi)x_{n-1} + b(x_{n-1} - x).\end{aligned}$$

We thus get

$$H(\mathscr{M}F)(x) + I(\varphi)x \leq \varepsilon_1 \lambda^{n-1} + b(\lambda^{n-1} - \lambda^{n+1}),$$

so that

$$\frac{H(\mathscr{M}F)(x)}{x} + I(\varphi) \leq \lambda^{-2}\big(\varepsilon_1 + b(1-\lambda^2)\big).$$

Similarly we find that

$$\frac{H(\mathscr{M}F)(x)}{x} + I(\varphi) \geq -\big(\varepsilon_1 + b(1-\lambda^2)/\lambda\big).$$

Therefore, and by (13) and since we may assume $\varepsilon_1/\lambda^2 \le \varepsilon/2$,

$$\left|\frac{H(\mathscr{M}F)(x)}{x} + I(\varphi)\right| \le \varepsilon_1 \frac{1}{\lambda^2} + b\frac{1-\lambda^2}{\lambda^2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finally implies that, since we can also assume $\varepsilon_1^2/\pi \le \varepsilon$,

$$\begin{aligned}\left|\frac{H(\mathscr{M}F)(x)}{x} + I(\mathscr{M}F)\right| &\le \left|\frac{H(\mathscr{M}F)(x)}{x} + I(\varphi)\right| + |I(\mathscr{M}F) - I(\varphi)|\\ &\le \varepsilon + \varepsilon_1^2/\pi \le 2\varepsilon,\end{aligned}$$

for all $x \in [\lambda^{n+1}, \lambda^n]$ and for all large n. The claim follows and the theorem is proved.

We shall need some relations involving the measure ϱ and the real constant γ in the Riesz representation of $\mathscr{M}F$. After division by y, (10) reads

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t(y^2-1)}{y(t^2+y^2)(t^2+1)}\mathscr{M}F(t)\,dt = -\gamma + \frac{C}{y} + \bigl(\varrho(1)+\varrho(-1)\bigr)\\ - \int_{|t|\ge 1} \frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt + \int_{|t|\le 1} \frac{\varrho(t)}{t^2+y^2}\,dt.\end{aligned}$$

We then let y tend to infinity. The integral on the left-hand side tends, by dominated convergence, to zero and so does the last term on the right-hand side. We therefore obtain

$$\int_{|t|\ge 1} \frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt \to -\gamma + \bigl(\varrho(1)+\varrho(-1)\bigr) \qquad \text{as } y\to\infty.$$

From this relation it follows that

$$\int_{1\le|t|\le y} \frac{\varrho(t)}{t^2}\,dt \to -\gamma + \bigl(\varrho(1)+\varrho(-1)\bigr) \qquad \text{as } y\to\infty. \tag{15}$$

Indeed,

$$\begin{aligned}\int_{1\le|t|\le y} \frac{\varrho(t)}{t^2}\,dt - \int_{|t|\ge 1}\frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt &= \int_{1\le|t|\le y}\frac{\varrho(t)}{t^2+y^2}\,dt + \int_{|t|\ge y}\frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt\\ &= \int_{1/y\le|s|\le 1}\frac{\varrho(sy)}{sy}\frac{s}{s^2+1}\,ds\\ &\quad + \int_{|s|\ge 1}\frac{\varrho(sy)}{sy}\frac{1}{s(s^2+1)}\,ds.\end{aligned}$$

By Theorem 3.2 and the dominated convergence theorem (recalling that $|\varrho(t)| \le \text{Const}\,|t|$), these two integrals tend to

$$\int_{|s|\le 1} \frac{b}{\pi}\frac{s}{s^2+1}\,ds + \int_{|s|\ge 1} \frac{b}{\pi}\frac{1}{s(s^2+1)}\,ds = 0.$$

Lemma 3.6. *In the situation where* $\mathscr{M}F(0) = 0$,

$$\int_{y\le|t|\le 1} \frac{\varrho(t)}{t^2}\,dt \to \gamma - \big(\varrho(1)+\varrho(-1)\big) \qquad \text{as } y\to 0.$$

Proof. We substitute the value of C, given by Lemma 3.1, into relation (10) and add it to the left-hand side. We divide by y and obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{M}F(t)}{t^2}\frac{yt}{t^2+y^2}\,dt = -\gamma + \big(\varrho(1)+\varrho(-1)\big) + \int_{|t|\le 1}\frac{\varrho(t)}{t^2+y^2}\,dt$$
$$- \int_{|t|\ge 1}\frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt.$$

The integral on the left-hand side of this relation tends to zero as y tends to zero. This is true by dominated convergence since the function $\mathscr{M}F(t)/t^2$ can be used as an integrable majorant in view of Lemma 2.5. The last integral on the right-hand side of the relation tends to zero as well; here one may simply use $|\varrho(t)/t^4|$, $|t|\ge 1$, as an integrable majorant. We thus obtain

$$\int_{|t|\le 1}\frac{\varrho(t)}{t^2+y^2}\,dt \to \gamma - \big(\varrho(1)+\varrho(-1)\big) \qquad \text{as } y\to 0.$$

Furthermore,

$$\int_{y\le|t|\le 1}\frac{\varrho(t)}{t^2}\,dt - \int_{|t|\le 1}\frac{\varrho(t)}{t^2+y^2}\,dt = \int_{y\le|t|\le 1}\frac{\varrho(t)y^2}{t^2(t^2+y^2)}\,dt - \int_{|t|\le y}\frac{\varrho(t)}{t^2+y^2}\,dt$$
$$= \int_{1\le|s|\le 1/y}\frac{\varrho(sy)}{sy}\frac{1}{s(s^2+1)}\,ds - \int_{|s|\le 1}\frac{\varrho(sy)}{sy}\frac{s}{s^2+1}\,ds.$$

By Theorem 3.2 and dominated convergence we see that the above expression tends to

$$\left(\lim_{t\to 0}\frac{\varrho(t)}{t}\right)\left(\int_{|s|\ge 1}\frac{1}{s(s^2+1)}\,ds - \int_{|s|\le 1}\frac{s}{s^2+1}\,ds\right) = 0.$$

Therefore,

$$\int_{y\le|t|\le 1}\frac{\varrho(t)}{t^2}\,dt \to \gamma - \big(\varrho(1)+\varrho(-1)\big) \qquad \text{as } y\to 0$$

and the lemma is proved.

Corollary 3.7. *In the situation where* $\mathscr{M}F(0)=0$,

$$\int_r^R \frac{\varrho(t)+\varrho(-t)}{t^2}\,dt \to 0$$

as $r\to 0$ *and* $R\to\infty$.

The corollary follows by combining the lemma above with relation (15). In the situation where $\mathscr{M}F(0)\neq 0$, and thus ϱ is zero in a neighbourhood of the origin, the integral in the corollary converges to

$$\big(\varrho(1)+\varrho(-1)\big)-\gamma+\int_{|t|\le 1}\frac{\varrho(t)}{t^2}\,dt.$$

The linear term γx in the Riesz representation of $\mathscr{M}F$ makes $\mathscr{M}F(x)/x^2$ integrable on the real line. It is there to compensate for the inbalance of ϱ. The corollary expresses that, when $\mathscr{M}F(0)=0$, the overall inbalance of ϱ, measured by the integral in the corollary, is zero and thus does not involve γ.

4. Energy

In this section we shall find a good lower bound on the integral

$$\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)-\mathscr{M}F(0)}{t^2}\,d\varrho(t).$$

We recall that this integral is convergent: the measure ϱ is concentrated on the set E where $\mathscr{M}F(t)=\log|f(t)|$ (see (5)) so the integral is the same as

$$\int_{-\infty}^{\infty}\frac{\log|f(t)|-\mathscr{M}F(0)}{t^2}\,d\varrho(t).$$

This integral is convergent in view of Proposition 2.2, relation (1) and the observation that ϱ is zero in a neighbourhood of the origin if $\mathscr{M}F(0)\neq 0$.

The lower bound is found by following a procedure involving a certain energy integral. We begin by giving a short introduction to these integrals. We define L as the set of real measures σ on the real line satisfying $\sigma(\mathbf{R})=0$ and making the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\frac{1}{|x-t|}\,d\sigma(t)\,d\sigma(x)$$

absolutely convergent. This double integral is called the energy associated with the measure σ. For such a measure we put

$$u_\sigma(x)=\int_{-\infty}^{\infty}\log\frac{1}{|x-t|}\,d\sigma(t). \tag{16}$$

In the recent paper [9] it is proved that one can define an inner product on the space of u_σ's by putting

$$\langle u_{\sigma_1}, u_{\sigma_2}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\frac{1}{|x-t|}\, d\sigma_1(t)\, d\sigma_2(x);$$

see Corollary 2.5 and Example 3.3 of [9]. The norm of u_σ (considered as an element of this inner product space) is thus the square root of the energy associated with σ. We shall denote the Hilbert space, obtained by completion of this inner product space, by $\mathscr{H}$.

When σ has compact support, this inner product structure is mentioned in some books on potential theory; see for example [8], but the assumption of compact support is too restrictive for us.

We remark that an inner product space result, general enough for our purpose, may be obtained in quite an elementary fashion. One could consider the space of real-valued measurable functions ψ on the real line satisfying

$$|\psi(x)| \le \frac{\text{Const}}{x^2+1} \qquad \text{a.e. on } \mathbf{R},$$

and $\int_{-\infty}^{\infty}\psi(t)\,dt = 0$. One may now show that $\langle u_{\psi_1\,dt}, u_{\psi_2\,dt}\rangle$ defines a semi inner product on the $u_{\psi\,dt}$'s. This can be done by bringing in the Riesz kernels (see for example [8, p. 80])

$$k_\varepsilon(x) = |x|^{-\varepsilon}, \qquad \varepsilon > 0,$$

known to be positive definite. Thus

$$0 \le \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|x-t|^{\varepsilon}}\psi(t)\psi(x)\,dt\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|x-t|^{-\varepsilon}-1}{\varepsilon}\psi(t)\psi(x)\,dt\,dx.$$

We split this integral into one over the set where $|x-t|\le 1$ and one over the set where $|x-t|\ge 1$. In the first of these integrals we use, for $\varepsilon<\frac{1}{2}$,

$$(|x-t|^{-\varepsilon}-1)/\varepsilon \le (-\log|x-t|)|x-t|^{-1/2}$$

and in the second we use

$$\bigl|(|x-t|^{-\varepsilon}-1)/\varepsilon\bigr| \le \log|x-t|.$$

By letting $\varepsilon\to 0$, we obtain, by the dominated convergence theorem,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\frac{1}{|x-t|}\psi(t)\psi(x)\,dt\,dx \ge 0.$$

The measures to which we shall apply these Hilbert space results are of the form $d(\varrho_n(t)/t)$, where ϱ_n is the restriction of ϱ to $1/n\le|t|\le n$. Such restrictions have all the properties mentioned above.

For a positive measure μ on the real line satisfying $d\mu(t) \le \operatorname{Const} dt$ and normalized so that $\mu(0) = 0$ we have

$$
\begin{aligned}
x\int_{|t|\le r} \log\left|1-\frac{t}{x}\right| d\left(\frac{\mu(t)}{t}\right) = {}& \int_{|t|\le r} \log\left|1-\frac{x}{t}\right| d\mu(t) \\
&+ \frac{\mu(r)}{r} x\log\left|1-\frac{r}{x}\right| + \frac{\mu(-r)}{r} x\log\left|1+\frac{r}{x}\right| \\
&- \mu(r)\log\left|1-\frac{x}{r}\right| + \mu(-r)\log\left|1+\frac{x}{r}\right|
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
x\int_{|t|\ge r} \log\left|1-\frac{x}{t}\right| d\left(\frac{\mu(t)}{t}\right) = {}& \int_{|t|\ge r} \left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right) d\mu(t) \\
&+ \mu(r)x\left(\frac{1}{x}-\frac{1}{r}\right)\log\left|1-\frac{x}{r}\right| \\
&- \mu(-r)x\left(\frac{1}{x}+\frac{1}{r}\right)\log\left|1+\frac{x}{r}\right| \\
&+ \frac{\mu(r)+\mu(-r)}{r}x.
\end{aligned} \tag{18}
$$

These relations are found by partial integration. As an example

$$
\begin{aligned}
x\int_{|t|\le r} \log\left|1-\frac{t}{x}\right| d\left(\frac{\mu(t)}{t}\right) = x\biggl\{ & \int_{|t|\le r} \frac{1}{t}\log\left|1-\frac{t}{x}\right| d\mu(t) \\
& - \int_{|t|\le r} \frac{\mu(t)}{t^2}\log\left|1-\frac{t}{x}\right| dt \biggr\},
\end{aligned}
$$

and here one should perform integration by parts on the second term, using

$$\int \frac{1}{t^2}\log\left|1-\frac{t}{x}\right| dt = \frac{1}{x}\log\left|1-\frac{x}{t}\right| - \frac{1}{t}\log\left|1-\frac{t}{x}\right|.$$

Relation (17) follows. To obtain (18) one should proceed in the same way, using

$$\int \frac{1}{t^2}\log\left|1-\frac{x}{t}\right| dt = \left(\frac{1}{x}-\frac{1}{t}\right)\log\left|1-\frac{x}{t}\right| + \frac{1}{t}.$$

We shall now return to the investigation of the least superharmonic majorant. It is convenient to define

$$\Delta(x) = \frac{\varrho(x)+\varrho(-x)}{x} \tag{19}$$

and

$$\Gamma(r,R)=\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\,dt. \tag{20}$$

The Riesz representation of $\mathscr{M}F$ can be written as

$$\begin{aligned}\mathscr{M}F(x)=\mathscr{M}F(0)&-\int_{|t|\le r}\log\Bigl|1-\frac{x}{t}\Bigr|\,d\varrho(t)\\&-\int_{|t|\ge r}\Bigl(\log\Bigl|1-\frac{x}{t}\Bigr|+\frac{x}{t}\Bigr)\,d\varrho(t)+x\Bigl(\int_{r\le|t|\le 1}\frac{1}{t}\,d\varrho(t)-\gamma\Bigr),\end{aligned} \tag{21}$$

and, since

$$\int_{r\le|t|\le R}\frac{1}{t}\,d\varrho(t)=\Gamma(r,R)+\Delta(R)-\Delta(r), \tag{22}$$

we obtain from (17) and (18) the following corollary.

Corollary 4.1. *For $x\in\mathbf{R}$ and $r<1$ we have*

$$\begin{aligned}\mathscr{M}F(x)=\mathscr{M}F(0)&-x\int_{|t|\le r}\log\Bigl|1-\frac{t}{x}\Bigr|\,d\Bigl(\frac{\varrho(t)}{t}\Bigr)-x\int_{|t|\ge r}\log\Bigl|1-\frac{x}{t}\Bigr|\,d\Bigl(\frac{\varrho(t)}{t}\Bigr)\\&+x\Bigl(\Delta(r)\log\frac{r}{|x|}+\Delta(1)+\Gamma(r,1)-\gamma\Bigr).\end{aligned}$$

We put

$$\varrho_n(t)=\begin{cases}\varrho(-n)-\varrho(-1/n), & t\le -n,\\ \varrho(t)-\varrho(-1/n), & -n\le t\le -1/n,\\ 0, & -1/n\le t\le 1/n,\\ \varrho(t)-\varrho(1/n), & 1/n\le t\le n,\\ \varrho(n)-\varrho(1/n), & n\le t,\end{cases}$$

and

$$u_n(x)=\int_{-\infty}^{\infty}\log\frac{1}{|x-t|}\,d\Bigl(\frac{\varrho_n(t)}{t}\Bigr).$$

It is not hard to see that u_n belongs to the Hilbert space $\mathscr{H}$. Our aim is to show (and then use) the property that some subsequence of the u_n's converges weakly in $\mathscr{H}$. The key to this is the following lemma. We denote by E_n the energy associated with the measure $d(\varrho_n(t)/t)$,

$$E_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\frac{1}{|x-t|}\,d\Bigl(\frac{\varrho_n(t)}{t}\Bigr)\,d\Bigl(\frac{\varrho_n(x)}{x}\Bigr). \tag{23}$$

The norm of u_n in $\mathscr{H}$ is the square root of this quantity E_n.

Lemma 4.2. *The energy E_n remains bounded as n tends to infinity.*

Proof. First we rewrite u_n. We have, by definition and (18), with $r = 1/n$,

$$
\begin{aligned}
u_n(x) &= -\int_{-\infty}^{\infty} \log\Big|1 - \frac{x}{t}\Big|\, d\Big(\frac{\varrho_n(t)}{t}\Big) - \int_{-\infty}^{\infty} \log|t|\, d\Big(\frac{\varrho_n(t)}{t}\Big) \\
&= -\frac{1}{x}\int_{-\infty}^{\infty} \Big(\log\Big|1 - \frac{x}{t}\Big| + \frac{x}{t}\Big)\, d\varrho_n(t) - \int_{-\infty}^{\infty} \log|t|\, d\Big(\frac{\varrho_n(t)}{t}\Big).
\end{aligned} \tag{24}
$$

According to (21) this implies

$$
\begin{aligned}
u_n(x) = -\frac{1}{x}\Big\{ &- \int_{|t|\le 1/n} \log\Big|1 - \frac{x}{t}\Big|\, d\varrho(t) + x\int_{1/n\le|t|\le 1} \frac{1}{t}\, d\varrho(t) \\
&- \int_{|t|\ge n} \Big(\log\Big|1 - \frac{x}{t}\Big| + \frac{x}{t}\Big)\, d\varrho(t) - \big(\mathscr{M}F(x) - \mathscr{M}F(0)\big) - \gamma x\Big\} \\
&- \int_{-\infty}^{\infty} \log|t|\, d\Big(\frac{\varrho_n(t)}{t}\Big).
\end{aligned}
$$

Therefore, by relation (22),

$$
\begin{aligned}
u_n(x) = &\frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x} + \frac{1}{x}\int_{|t|\le 1/n} \log\Big|1 - \frac{x}{t}\Big|\, d\varrho(t) \\
&+ \frac{1}{x}\int_{|t|\ge n} \Big(\log\Big|1 - \frac{x}{t}\Big| + \frac{x}{t}\Big)\, d\varrho(t) + \Gamma(1,n) + \Delta(n) - \Delta(1) + \gamma.
\end{aligned} \tag{25}
$$

We have also used

$$
\int_{-\infty}^{\infty} \log|t|\, d\Big(\frac{\varrho_n(t)}{t}\Big) = -\Gamma(1/n, n) + \Delta(1/n) - \Delta(n). \tag{26}
$$

From (25) we first of all see that

$$
u_n(x) \to \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x} \tag{27}
$$

as n tends to infinity. This is because $\Delta(n) \to 0$ and $\Gamma(1,n) \to \Delta(1) - \gamma$ as n tends to infinity (see Theorem 3.2 and (15)). Furthermore, since $a \mapsto a + \log|1-a|$ is non-positive for $|a| \le 1$, we obtain

$$
\begin{aligned}
\frac{u_n(x)}{x} \le &\frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2} + \frac{1}{x^2}\int_{|t|\le 1/n} \log\Big|1 - \frac{x}{t}\Big|\, d\varrho(t) \\
&+ \frac{1}{x}\big(\Gamma(1,n) + \Delta(n) - \Delta(1) + \gamma\big),
\end{aligned}
$$

for $|x| \le n$.

We now rewrite the energy E_n (in (23)) as

$$E_n = \int_{-\infty}^{\infty} \frac{u_n(x)}{x}\, d\varrho_n(x) - \int_{-\infty}^{\infty} u_n(x)\frac{\varrho_n(x)}{x^2}\, dx. \tag{28}$$

The first term is equal to

$$\int_{1/n \le |x| \le n} \frac{u_n(x)}{x}\, d\varrho(x)$$

and is therefore bounded from above by

$$\begin{aligned}
&\int_{1/n \le |x| \le n} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2}\, d\varrho(x) \\
&\quad + \int_{1/n \le |x| \le n} \frac{1}{x^2} \int_{|t| \le 1/n} \log\left|1 - \frac{x}{t}\right| d\varrho(t)\, d\varrho(x) \\
&\quad + \big(\Gamma(1,n) + \Delta(n) - \Delta(1) + \gamma\big) \int_{1/n \le |x| \le n} \frac{1}{x}\, d\varrho(x).
\end{aligned} \tag{29}$$

The last term remains bounded as n tends to infinity; in fact it tends, by (15), (22), Theorem 3.2 and the remarks following Corollary 3.7, to zero. Since $\log^+|1 - x/t| \le \log(1 + |x/t|)$, we find by Proposition 2.2 that the double integral in (29) is less than or equal to $4\big((A+b)/\pi\big)^2$ times

$$\int_{1/n}^{n} \frac{1}{x^2} \int_0^{1/n} \log\left(1 + \frac{x}{t}\right) dt\, dx.$$

This integral can be estimated:

$$\begin{aligned}
\int_{1/n}^{n} \frac{1}{x^2} \int_0^{1/n} \log\left(1 + \frac{x}{t}\right) dt\, dx &= \int_{1/n}^{n} \int_0^{1/nx} \log\left(1 + \frac{1}{s}\right) ds\, \frac{dx}{x} \\
&= \int_{1/n^2}^{1} \int_0^{y} \log\left(1 + \frac{1}{s}\right) ds\, \frac{dy}{y} \\
&\le \int_0^1 \int_0^y \log\left(1 + \frac{1}{s}\right) ds\, \frac{dy}{y} \\
&= \int_0^1 \int_s^1 \frac{dy}{y} \log\left(1 + \frac{1}{s}\right) ds \\
&= \int_0^1 (-\log s) \log\left(1 + \frac{1}{s}\right) ds,
\end{aligned}$$

a finite quantity.

The first term in (29),

$$\int_{1/n\le|x|\le n} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)$$

is bounded as n tends to infinity: indeed, since ϱ is concentrated on the set E, where $\mathscr{M}F$ is equal to $\log|f|$, it is equal to

$$\int_{1/n\le|x|\le n} \frac{\log|f(x)|}{x^2}\,d\varrho(x) - \int_{1/n\le|x|\le n} \frac{\mathscr{M}F(0)}{x^2}\,d\varrho(x).$$

By Proposition 2.2 the first term is bounded by

$$\frac{A+b}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(x)|}{x^2}\,dx < \infty.$$

The second term is non-positive since $\mathscr{M}F(0)\ge 0$ (it is bounded anyway). We thus see that the first term in (28) remains bounded from above as n tends to infinity.

We consider the second term in (28):

$$\begin{aligned}
-\int_{-\infty}^{\infty} u_n(x)\frac{\varrho_n(x)}{x^2}\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log|x-t|\,d\left(\frac{\varrho_n(t)}{t}\right)\frac{\varrho_n(x)}{x^2}\,dx\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\left|1-\frac{x}{t}\right|\,d\left(\frac{\varrho_n(t)}{t}\right)\frac{\varrho_n(x)}{x^2}\,dx\\
&\quad+\int_{-\infty}^{\infty} \log|t|\,d\left(\frac{\varrho_n(t)}{t}\right)\int_{-\infty}^{\infty}\frac{\varrho_n(x)}{x^2}\,dx.
\end{aligned}$$

The first term in the last member of this relation can be computed using the corollary in Appendix B. Since, as already used in (26),

$$\int_{-\infty}^{\infty} \frac{\varrho_n(t)}{t^2}\,dt = \Gamma(1/n,n)+\Delta(n)-\Delta(1/n),$$

we therefore get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\left|1-\frac{x}{t}\right|\,d\left(\frac{\varrho_n(t)}{t}\right)\frac{\varrho_n(x)}{x^2}\,dx = \tfrac{1}{2}\bigl(\Gamma(1/n,n)+\Delta(n)-\Delta(1/n)\bigr)^2.$$

Therefore

$$\begin{aligned}
-\int_{-\infty}^{\infty} u_n(x)\frac{\varrho_n(x)}{x^2}\,dx &= \tfrac{1}{2}\bigl(\Gamma(1/n,n)+\Delta(n)-\Delta(1/n)\bigr)^2\\
&\qquad-\bigl(\Gamma(1/n,n)+\Delta(n)-\Delta(1/n)\bigr)^2\\
&= -\tfrac{1}{2}\bigl(\Gamma(1/n,n)+\Delta(n)-\Delta(1/n)\bigr)^2,
\end{aligned}$$

a non-positive quantity (it is even bounded as n tends to infinity). We conclude that the energy E_n remains bounded as n tends to infinity. The lemma follows.

The lemma is used in the theorem below. There we shall also need an estimate of the size of u_n. This is furnished by the following two lemmas.

Lemma 4.3. *We have*

$$\int_{|t|\geq r}\left|\log\left|1-\frac{x}{t}\right|+\frac{x}{t}\right|dt \leq \left(\mathrm{Const}+\left(1+\frac{r}{|x|}\right)\log\left(1+\frac{|x|}{r}\right)+\left(1-\frac{r}{|x|}\right)\log\left|1-\frac{|x|}{r}\right|\right)|x|,$$

where the constant is independent of r and x.

Proof. After a substitution $s = t/x$, the integral we wish to estimate is $|x|$ times

$$\int_{|s|\geq r/|x|}\left|\log\left|1-\frac{1}{s}\right|+\frac{1}{s}\right|ds. \tag{30}$$

The function $s \mapsto 1/s + \log|1-1/s|$ is negative for $s<0$ and decreases there from 0 to $-\infty$. On $(0,1)$ it is also decreasing from ∞ to $-\infty$ and there is a unique $s_0 \in (0,1)$ such that $1/s_0 + \log|1-1/s_0| = 0$. For $s>1$, the function is increasing and increases from $-\infty$ to 0. Furthermore

$$\left((1-s)\log\left|1-\frac{1}{s}\right|\right)' = -\left(\log\left|1-\frac{1}{s}\right|+\frac{1}{s}\right).$$

A routine calculation of (30) yields, when $|x| \geq r/s_0$,

$$\begin{aligned}\int_{|s|\geq r/|x|}\left|\log\left|1-\frac{1}{s}\right|+\frac{1}{s}\right|ds &= -2\log\left|1-\frac{1}{s_0}\right|(1-s_0)\\ &\quad+\left(1+\frac{r}{|x|}\right)\log\left(1+\frac{|x|}{r}\right)+\left(1-\frac{r}{|x|}\right)\log\left|1-\frac{|x|}{r}\right|.\end{aligned}$$

When $r \leq |x| < r/s_0$ or when $|x| < r$ we perform similar computations and the lemma follows with a suitable choice of the constant.

Lemma 4.4.

$$\int_{|t|\leq 1}\left|\log\left|1-\frac{x}{t}\right|\right|dt \leq \mathrm{Const} + 2\log(1+|x|).$$

Proof. We may assume that x is positive. The integral in question is equal to

$$\int_0^1\left|\log\left|1-\frac{x}{t}\right|\right|dt + \int_0^1 \log\left|1+\frac{x}{t}\right|dt.$$

The second integral in this expression equals $(x+1)\log(x+1)-x\log x$. To calculate the first integral one should consider three different cases, namely $x>2$, $1<x<2$ and $0<x<1$. If $x>2$, the first integral equals $x\log x-(x-1)\log(x-1)$. For $1<x<2$ it has the value $(x-1)\log(x-1)-x\log x+2x\log 2$, and for $0<x<1$ the value $-(1-x)\log(1-x)-x\log x+2x\log 2$. The lemma follows.

Theorem 4.5. *The following estimate holds*

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x) \geq \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,\frac{\varrho(x)}{x}\,dx.$$

Proof. We resort to a weak compactness argument in the Hilbert space $\mathscr{H}$. By Lemma 4.2, $\|u_n\|$ remains bounded as n tends to infinity and hence a subsequence $\{u_{n_k}\}$ converges weakly to some element u of $\mathscr{H}$. Therefore

$$\|u\|^2 = \lim_k \langle u, u_{n_k}\rangle,$$

and also, for each k,

$$\langle u, u_{n_k}\rangle = \lim_l \langle u_{n_l}, u_{n_k}\rangle = \lim_l \int_{-\infty}^{\infty} u_{n_l}(x)\,d\Big(\frac{\varrho_{n_k}(x)}{x}\Big).$$

We now claim that

$$\lim_l \langle u_{n_l}, u_{n_k}\rangle = \lim_l \int_{-\infty}^{\infty} u_{n_l}(x)\,d\Big(\frac{\varrho_{n_k}(x)}{x}\Big) = \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x}\,d\Big(\frac{\varrho_{n_k}(x)}{x}\Big).$$

This is true by dominated convergence. Indeed, as noted in (27), $u_n(x)$ tends pointwise a.e. to $\big(\mathscr{M}F(x)-\mathscr{M}F(0)\big)/x$. By (24), (22), (26) and Lemmas 4.3 and 4.4 we see that

$$\begin{aligned}
|u_n(x)| &= \bigg|\frac{1}{x}\int_{1\le|t|\le n}\Big(\log\Big|1-\frac{x}{t}\Big|+\frac{x}{t}\Big)\,d\varrho(t) + \frac{1}{x}\int_{1/n\le|t|\le 1}\log\Big|1-\frac{x}{t}\Big|\,d\varrho(t) \\
&\qquad + \int_{1/n\le|t|\le 1}\frac{1}{t}\,d\varrho(t) + \int_{-\infty}^{\infty}\log|t|\,d\Big(\frac{\varrho_n(t)}{t}\Big)\bigg| \\
&\le \frac{1}{|x|}\int_{1\le|t|\le n}\Big|\log\Big|1-\frac{x}{t}\Big|+\frac{x}{t}\Big|\,d\varrho(t) + \frac{1}{|x|}\int_{1/n\le|t|\le 1}\Big|\log\Big|1-\frac{x}{t}\Big|\Big|\,d\varrho(t) \\
&\quad + |\Gamma(1,n)+\Delta(n)-\Delta(1)| \\
&\le \frac{A+b}{\pi}\bigg\{\text{Const} + \Big(1+\frac{1}{|x|}\Big)\log(1+|x|) \\
&\qquad + \Big(1-\frac{1}{|x|}\Big)\log\big|1-|x|\big| + \frac{\text{Const}}{|x|} + \frac{2\log(1+|x|)}{|x|}\bigg\} + \text{Const},
\end{aligned}$$

for all n, with constants independent of n. This may be used as a majorant of $|u_{n_l}(x)|$, integrable with respect to the measure $|d(\varrho_{n_k}(x)/x)|$. The claim follows. We have, furthermore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho_{n_k}(x) &= \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x}\,d\Big(\frac{\varrho_{n_k}(x)}{x}\Big) \\
&\quad + \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,\frac{\varrho_{n_k}(x)}{x}\,dx.
\end{aligned}$$

As k tends to infinity, the integral on the left-hand side tends to

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2}\, d\varrho(x)$$

by dominated convergence. The first integral on the right-hand side tends to $\|u\|^2$, which of course is non-negative, and the second to

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2} \frac{\varrho(x)}{x}\, dx,$$

again by dominated convergence. The theorem is proved.

5. Computation of a certain integral

We wish to estimate the (convergent) integral

$$\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2} \frac{\varrho(x)}{x}\, dx \tag{31}$$

from below. We recall that it is convergent by Lemma 2.5, Proposition 2.2 and the fact that ϱ is zero close to the origin if $\mathscr{M}F(0) \neq 0$. In this section we shall compute (31). We use a procedure based on the second mean value theorem (see for example [13, Section 12.3]) and the integration by parts method of Appendix B.

From Corollary 4.1 we have

$$\begin{aligned}\frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x} = &-\int_{|t|\le r} \log\left|1 - \frac{t}{x}\right| d\left(\frac{\varrho(t)}{t}\right) - \int_{|t|\ge r} \log\left|1 - \frac{x}{t}\right| d\left(\frac{\varrho(t)}{t}\right)\\ &+ \Delta(r)\log\left(\frac{r}{|x|}\right) + \Delta(1) + \Gamma(r,1) - \gamma.\end{aligned}$$

The integral (31) is thus equal to

$$\begin{aligned}-\int_{r\le|x|\le R} \frac{\varrho(x)}{x^2} \int_{|t|\le r} \log\left|1 - \frac{t}{x}\right| d\left(\frac{\varrho(t)}{t}\right) dx&\\ - \int_{r\le|x|\le R} \frac{\varrho(x)}{x^2} \int_{r\le|t|\le R} \log\left|1 - \frac{x}{t}\right| d\left(\frac{\varrho(t)}{t}\right) dx&\\ - \int_{r\le|x|\le R} \frac{\varrho(x)}{x^2} \int_{|t|\ge R} \log\left|1 - \frac{x}{t}\right| d\left(\frac{\varrho(t)}{t}\right) dx&\\ + \Delta(r) \int_{r\le|x|\le R} \frac{\varrho(x)}{x^2} \log\left(\frac{r}{|x|}\right) dx&\\ + \Gamma(r,R)\big(\Gamma(r,1) + \Delta(1) - \gamma\big) + \varepsilon(r,R),&\end{aligned} \tag{32}$$

where $\varepsilon(r,R)\to 0$ as $r\to 0$, $R\to\infty$. The first and the third term above tend to zero in absolute value as $r\to 0$, $R\to\infty$. This is seen by using the second mean value theorem: for $\delta<r$ we have a $\xi\in[\delta,r]$, depending on δ, r and x ($|x|\geq r$), such that

$$\int_\delta^r \log\Big|1-\frac{t}{x}\Big|\, d\Big(\frac{\varrho(t)}{t}\Big) = \log\Big|1-\frac{\delta}{x}\Big|\Big(\frac{\varrho(\xi)}{\xi}-\frac{\varrho(\delta)}{\delta}\Big)+\log\Big|1-\frac{r}{x}\Big|\Big(\frac{\varrho(r)}{r}-\frac{\varrho(\xi)}{\xi}\Big).$$

Letting δ tend to zero, we get

$$\Big|\int_0^r \log\Big|1-\frac{t}{x}\Big|\, d\Big(\frac{\varrho(t)}{t}\Big)\Big| \leq \Big|\log\Big|1-\frac{r}{x}\Big|\Big| D(r),$$

where

$$D(r)=\sup_{|s|,|t|\leq r}\Big|\frac{\varrho(s)}{s}-\frac{\varrho(t)}{t}\Big|.$$

Since ϱ is differentiable at 0 (by Theorem 3.2), $D(r)\to 0$ as $r\to 0$. Similarly

$$\Big|\int_{-r}^0 \log\Big|1-\frac{t}{x}\Big|\, d\Big(\frac{\varrho(t)}{t}\Big)\Big| \leq \Big|\log\Big|1+\frac{r}{x}\Big|\Big| D(r),$$

and therefore

$$\begin{aligned}
&\Big|\int_{r\leq|x|\leq R}\frac{\varrho(x)}{x^2}\int_{|t|\leq r}\log\Big|1-\frac{t}{x}\Big|\, d\Big(\frac{\varrho(t)}{t}\Big)\, dx\Big| \\
&\leq \Big(\int_{r\leq|x|\leq R}\Big|\frac{\varrho(x)}{x^2}\Big|\Big|\log\Big|1-\frac{r}{x}\Big|\Big|\, dx+\int_{r\leq|x|\leq R}\Big|\frac{\varrho(x)}{x^2}\Big|\Big|\log\Big|1+\frac{r}{x}\Big|\Big|\, dx\Big)D(r) \\
&= \int_{r\leq|x|\leq R}\frac{\varrho(x)}{x^2}\log\Big|\frac{1+r/x}{1-r/x}\Big|\, dx\, D(r),
\end{aligned}$$

taking into account the sign of $\varrho(x)/x^2$ and of $\log|1\pm r/x|$. We have, furthermore, $\varrho(x)/x\leq(A+b)/\pi$ so

$$\int_{r\leq|x|\leq R}\frac{\varrho(x)}{x^2}\log\Big|\frac{1+r/x}{1-r/x}\Big|\, dx \leq \frac{A+b}{\pi}\int_{|s|\geq 1}\log\Big|\frac{1+s}{1-s}\Big|\frac{ds}{s}=\frac{A+b}{\pi}\frac{\pi^2}{2},$$

where we have used the value

$$\int_0^1\log\Big|\frac{1+s}{1-s}\Big|\frac{ds}{s}=\frac{\pi^2}{4}, \tag{33}$$

and therefore the first term in (32) tends to zero. A similar argument shows that the third term tends to zero as well. The second term is, by Proposition B.2 in Appendix B, equal to

$$-\Big[\frac{\varrho(x)}{x}\int_{r\leq|t|\leq R}\frac{\varrho(t)}{t^2}\log\Big|1-\frac{t}{x}\Big|\, dt\Big]_{r\leq|x|\leq R}-\frac{1}{2}\Gamma(r,R)^2.$$

(Here $[G(x)]_{r\le|x|\le R}$ is short for $\big(G(R)-G(r)\big)+\big(G(-r)-G(-R)\big)$). We therefore find

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\frac{\varrho(x)}{x}\,dx\\
&= -\frac{\varrho(R)}{R}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1-\frac{t}{R}\Big|\,dt-\frac{\varrho(-R)}{R}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1+\frac{t}{R}\Big|\,dt\\
&\quad+\frac{\varrho(r)}{r}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1-\frac{t}{r}\Big|\,dt+\frac{\varrho(-r)}{r}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1+\frac{t}{r}\Big|\,dt\\
&\quad+\Delta(r)\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big(\frac{r}{|t|}\Big)\,dt\\
&\quad+\Gamma(r,R)\big(\Gamma(r,1)+\Delta(1)-\gamma\big)-\tfrac12\Gamma(r,R)^2+\varepsilon(r,R)\\
&= -\frac{\varrho(R)}{R}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1-\frac{t}{R}\Big|\,dt-\frac{\varrho(-R)}{R}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1+\frac{t}{R}\Big|\,dt\\
&\quad+\frac{\varrho(r)}{r}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1-\frac{r}{t}\Big|\,dt+\frac{\varrho(-r)}{r}\int_{r\le|t|\le R}\frac{\varrho(t)}{t^2}\log\Big|1+\frac{r}{t}\Big|\,dt\\
&\quad+\Gamma(r,R)\big(\tfrac12\Gamma(r,1)+\Delta(1)-\gamma-\tfrac12\Gamma(1,R)\big)+\varepsilon(r,R).
\end{aligned}$$

As $r\to 0$, $R\to\infty$ we find, by using substitutions of the form $s=t/R$, $s=t/r$ and by the dominated convergence theorem, that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\frac{\varrho(x)}{x}\,dx\\
&\qquad=\Big(\lim_{t\to\pm\infty}\frac{\varrho(t)}{t}\Big)^2\Big\{\int_{|s|\le1}\log|1+s|\,\frac{ds}{s}-\int_{|s|\le1}\log|1-s|\,\frac{ds}{s}\Big\}\\
&\qquad\quad-\Big(\lim_{t\to0}\frac{\varrho(t)}{t}\Big)^2\Big\{\int_{|s|\ge1}\log\Big|1+\frac1s\Big|\,\frac{ds}{s}-\int_{|s|\ge1}\log\Big|1-\frac1s\Big|\,\frac{ds}{s}\Big\}\\
&\qquad\quad+\lim_{r,R}\Gamma(r,R)\big\{\tfrac12\Gamma(r,1)+\Delta(1)-\gamma-\tfrac12\Gamma(1,R)\big\}.
\end{aligned}$$

If $\mathscr{M}F(0)=0$ then $\Gamma(r,R)\to0$ as $r\to0, R\to\infty$ (Corollary 3.7) and also

$$\tfrac12\Gamma(r,1)+\Delta(1)-\gamma-\tfrac12\Gamma(1,R)\to0$$

((15) and Lemma 3.6). If $\mathscr{M}F(0)>0$, the limit in the last line in the equation above is equal to

$$\Big(\int_{|t|\le1}\frac{\varrho(t)}{t^2}\,dt+\Delta(1)-\gamma\Big)\Big(\frac12\int_{|t|\le1}\frac{\varrho(t)}{t^2}\,dt+\frac12\Delta(1)-\frac12\gamma\Big),$$

a non-negative quantity (see again (15)). Therefore

$$\int_{-\infty}^{\infty}\frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\frac{\varrho(x)}{x}\,dx\ge\frac{\pi^2}{2}\Big\{\Big(\lim_{t\to\pm\infty}\frac{\varrho(t)}{t}\Big)^2-\Big(\lim_{t\to0}\frac{\varrho(t)}{t}\Big)^2\Big\},$$

where we have again used (33), leading to the following theorem.

Theorem 5.1. *If $\mathscr{M}F(0)>0$ then*

$$\int_{-\infty}^{\infty}\frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)\ge\frac{b^2}{2}.$$

If $\mathscr{M}F(0)=0$ then

$$\int_{-\infty}^{\infty}\frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)\ge\frac{b}{2\pi}\int_{-\infty}^{\infty}\frac{\log|f(t)|}{t^2}\,dt.$$

Proof. This follows from Theorem 4.5 and, in the case of $\mathscr{M}F(0)=0$,

$$\begin{aligned}\left(\lim_{t\to\pm\infty}\frac{\varrho(t)}{t}\right)^2-\left(\lim_{t\to 0}\frac{\varrho(t)}{t}\right)^2&=\left(\frac{b}{\pi}\right)^2-\left(\frac{b}{\pi}-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt\right)^2\\&=\frac{1}{\pi^3}\left(2b-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt\right)\int_{-\infty}^{\infty}\frac{\mathscr{M}F(t)}{t^2}\,dt.\end{aligned}$$

The quantity inside the parentheses is, by Lemma 2.5, $\ge b$, and the integral involving $\mathscr{M}F$ is, since $\mathscr{M}F$ is a majorant of $\log|f|$, greater than or equal to the logarithmic integral of f. The theorem is proved.

6. Obtaining the main result

We shall describe how to obtain the main results of this paper, based on the properties of the least superharmonic majorant in the previous sections. In this situation, the technical differences when comparing to [11] and to the two papers [7] and [6] are relatively small and therefore we shall not give complete proofs.

We let Λ denote an h-dense subset of the real line, having no finite accumulation point. We shall furthermore suppose that $0\notin\Lambda$ and that $\Lambda=\{\lambda_n\}$ (in increasing order) is separated in the sense that

$$\lambda_{n+2}-\lambda_n\ge h \tag{34}$$

for all n. For an entire function f of exponential type $<\pi$, the assumption

$$\sum_{\lambda\in\Lambda}\frac{\log^+|f(\lambda)|}{\lambda^2}<\infty$$

implies that f is of zero exponential growth on the real line; see [11].

Knowing this, we may reduce the problem to entire functions that are bounded on the real axis, but of slightly larger exponential type, see [10, Section 7]. This is based on a result about weighted approximation by sums of imaginary exponentials, going back to de Branges; see for example [3, p. 215].

A further reduction can be made, based on the following lemma.

Lemma 6.1. *To any given α there is $M > 0$ such that for any $g\colon \Lambda \to \mathbf{C}$ satisfying*

$$\sum_{\lambda\in\Lambda} \frac{\log^+ |g(\lambda)|}{\lambda^2} \le \alpha$$

we have

$$\sum_{\lambda\in\Lambda} \frac{\log\bigl(1+\lambda^2|g(\lambda)|^2/M\bigr)}{\lambda^2} \le 6\alpha.$$

Taking now any entire function g, with logarithmic sum less than or equal to some α and bounded on the real axis, we choose an entire function f of the same type as g, with no zeros in the upper half-plane, and satisfying

$$f(z)\overline{f(\overline{z})} = 1 + z^2 g(z)\overline{g(\overline{z})}/M.$$

Here M is the number from the lemma above. We note that (after multiplication by a suitable complex number) f has the properties

$$f(0) = 1, \qquad 1 \le |f(x)| \le \mathrm{Const}\,(|x|+1),$$

mentioned and used in the previous sections.

For these functions f one can give a lower bound on the logarithmic sum in terms of an integral involving the least superharmonic majorant of

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|\log|f(t)|}{|z-t|^2}\,dt - b|y|.$$

As in [11] one has:

Theorem 6.2. *Let $A < T_*/h$ and f of exponential type $\le A$ satisfy the conditions above. If the parameter b is sufficiently small we have*

$$\sum_{\lambda\in\Lambda,\,|\lambda|\ge m} \frac{\log|f(\lambda)|}{\lambda^2} \ge -\mathrm{Const} \sum_{\lambda\in\Lambda,\,|\lambda|\ge m} \frac{1}{\lambda^2} + \mathrm{Const}\int_{|x|\ge m} \frac{\mathscr{M}F(x)}{x^2}\,d\varrho(x).$$

Here the constants are positive and depend only on A, b and h.

The ideas of the proof are basically the same as in [11] and we shall not give the proof here.

The integral in the above theorem can be estimated from below:

$$\begin{aligned}
&\int_{|x|\ge m} \frac{\mathscr{M}F(x)}{x^2}\,d\varrho(x) \ge \int_{|x|\ge m} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)\\
&= \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x) - \int_{|x|\le m} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x)\\
&\ge \int_{-\infty}^{\infty} \frac{\mathscr{M}F(x)-\mathscr{M}F(0)}{x^2}\,d\varrho(x) - \frac{A+b}{\pi}\int_{|x|\le m} \frac{\log|f(x)|}{x^2}\,dx.
\end{aligned}$$

The fact that $\mathscr{M}F(0) \geq 0$ is used in both the first and the last inequality. The last inequality then follows since $\mathscr{M}F$ is equal to $\log|f|$ on the support of ϱ and since, by Proposition 2.2, $d\varrho(t) \leq \big((A+b)/\pi\big)\,dt$. We thus see that for $A < T_*/h$ and b small enough we have, for certain constants,

$$\begin{aligned}\sum_{\lambda\in\Lambda} \frac{\log|f(\lambda)|}{\lambda^2} \geq -\mathrm{Const} &\sum_{\lambda\in\Lambda,\,|\lambda|\geq m} \frac{1}{\lambda^2}\\ &+ \mathrm{Const}\int_{-\infty}^{\infty} \frac{\mathscr{M}F(x) - \mathscr{M}F(0)}{x^2}\,d\varrho(x)\\ &- \mathrm{Const}\,\frac{A+b}{\pi}\int_{|x|\leq m} \frac{\log|f(x)|}{x^2}\,dx.\end{aligned}$$

Here one should invoke Theorem 5.1 and one may now obtain the following comparison result: if $\{f_k\}$ is any sequence of functions of exponential type $\leq A_0 < T_*/h$ satisfying the conditions above for which

$$\sum_{\lambda\in\Lambda} \frac{\log|f_k(\lambda)|}{\lambda^2} \to 0$$

then

$$\int_{-\infty}^{\infty} \frac{\log|f_k(x)|}{x^2}\,dx \to 0.$$

This is seen as in [6] and [7] (see also 11]). Once we have that, the main result, Theorem 1.1, follows; see [10]. We may also prove:

Theorem 6.3. *Any entire function f of exponential type less than T_*/h, having finite logarithmic sum over Λ, belongs to the Cartwright class.*

7. Applications

In this section we describe a few applications of Theorem 1.1. They are connected with weighted approximation theory and with the classical moment problem on the real line.

We let Λ denote an h-dense sequence of the real line and suppose that we are presented with a function $W\colon \Lambda \to [1,\infty)$, called a weight. We consider the Banach space

$$\mathscr{C}_W = \{\varphi : \Lambda \to \mathbf{C} \mid |\varphi(\lambda)|/W(\lambda) \to 0 \text{ as } |\lambda| \to \infty\}$$

with norm $\|\varphi\| = \sup_\lambda |\varphi(\lambda)|/W(\lambda)$.

We denote by $\mathscr{E}_A$ the set of entire functions of exponential type $\leq A$ and bounded on the real axis. If $W(\lambda) \to \infty$ as $|\lambda| \to \infty$ then $\mathscr{E}_A \subseteq \mathscr{C}_W$ for any $A > 0$ and we put

$$W_A(z) = \sup\{|h(z)| \mid h \in \mathscr{E}_A \text{ and } \|h\| \leq 1\}.$$

We may think of W_A as a lower regularization of W in terms of entire functions. We have the following theorem.

Theorem 7.1. *Suppose that Λ is relatively h-dense and separated in the sense of (34). For $A < T_*/h$, the space $\mathscr{E}_A$ is dense in $\mathscr{C}_W$ if and only if*

$$\sum_{\lambda\in\Lambda} \frac{\log W_A(\lambda)}{\lambda^2+1} = \infty. \tag{35}$$

The proof of this theorem follows the same lines as the proof of [10, Theorem 9.1]. For the readers' convenience we sketch the proof. That (35) is sufficient for denseness of $\mathscr{E}_A$ in $\mathscr{C}_W$ is seen as in [3, p. 523]: if $\mathscr{E}_A$ is not dense in $\mathscr{C}_W$ then, by the second theorem in [3, p. 174],

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{x^2+1}\,dx < \infty$$

(continuity of W plays no role in that part of the theorem). Therefore, as in [10, p. 522],

$$\log W_A(\lambda) \le 2A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2}{(\lambda-t)^2+4}\log W_A(t)\,dt$$

so that

$$\sum_{\lambda\in\Lambda} \frac{\log W_A(\lambda)}{\lambda^2+1} \le \sum_{\lambda\in\Lambda}\frac{2A}{\lambda^2+1} + \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{\lambda\in\Lambda}\left(\frac{2}{(\lambda-t)^2+4}\frac{1}{\lambda^2+1}\right)\log W_A(t)\,dt.$$

We have, furthermore,

$$\sum_{\lambda\in\Lambda}\frac{1}{(\lambda-t)^2+1}\frac{1}{\lambda^2+1} \le \frac{\text{Const}}{t^2+1}.$$

This relation follows by replacing the sum by a suitable integral as indicated in the proof of [10, Proposition 1.4] (one could estimate the sum over Λ by the sum over all integers by choosing, for each λ, an integer n_λ such that $|\lambda - n_\lambda| \le$ some c, where c does only depend on Λ and not on the particular λ. Separation of Λ is needed in order to avoid too many repetitions among the integers n_λ). We thus obtain that the sum in (35) converges. Theorem 1.1 is needed for the necessity of (35). We suppose that the sum in (35) converges. We pick $\lambda_0 \in \Lambda$ set out to show that the function δ_{λ_0}, 1 at λ_0 and 0 elsewhere, cannot be approximated in $\mathscr{C}_W$ by functions from $\mathscr{E}_A$. Indeed, if $f_n \in \mathscr{E}_A$ and $f_n \to \delta_{\lambda_0}$ in $\mathscr{C}_W$ then by dominated convergence

$$\sum_{\lambda\in\Lambda}\frac{\log^+|f_n(\lambda)|}{\lambda^2+1} \to \sum_{\lambda\in\Lambda}\frac{\log^+|\delta_{\lambda_0}(\lambda)|}{\lambda^2+1} = 0.$$

(This is where convergence of the sum in (35) is used.) By Theorem 1.1, $\{f_n\}$ forms a normal family and furthermore, to any given $\varepsilon > 0$, a certain subsequence $\{f_{n_k}\}$ converges u.c.c. to some entire function f of exponential type $\le A + \varepsilon < \pi/h$. Since

$$f(\lambda) = \lim_k f_{n_k}(\lambda) = \delta_{\lambda_0}(\lambda), \qquad \lambda \in \Lambda,$$

we see that f is zero at all points of Λ except at λ_0 where it takes the value 1. This is a contradiction by [11, Lemma 4.2]. The theorem follows.

We give a similar application in the theory of the classical moment problem on the real line. A positive Borel measure σ on the real line is said to have moments of all orders if any polynomial is integrable with respect to σ. The set of such measures we denote by $\mathscr{M}$. The moments of $\sigma \in \mathscr{M}$ is the sequence $\{s_n\}$ given by

$$s_n = \int_{-\infty}^{\infty} x^n \, d\sigma(x), \qquad n \ge 0.$$

A measure σ is called indeterminate if there is another measure in $\mathscr{M}$, different from σ, having the same moments as σ. The following theorem is a discrete analogue of a classical result of Krein.

Theorem 7.2. *Suppose that Λ is relatively h-dense, for some $h > 0$, and that $\sigma = \sum_\Lambda b_\lambda \varepsilon_\lambda$ belongs to $\mathscr{M}$. If*

$$\sum_{\lambda \in \Lambda} \frac{\log b_\lambda}{\lambda^2 + 1} > -\infty$$

then σ is indeterminate.

We may, almost exactly as above, prove that δ_{λ_0} is not in the closure of the polynomials in any $L^p(\sigma)$ space. The polynomials are therefore not dense in $L^1(\sigma)$ and hence σ must be indeterminate; in fact it is not even an extreme point in the convex set of positive measures having the same moments as σ. The theorem generalizes earlier results on symmetric h-dense sequences, see [12].

Appendix A. A version of Kolmogorov's theorem on the harmonic conjugate

We suppose that u is a real-valued function satisfying

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{t^2} \, dt < \infty.$$

We put

$$H(u)(x) = \lim_{y \to 0_+} \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{x-t}{(x-t)^2 + y^2} + \frac{1}{t} \right) u(t) \, dt$$

and we set out to verify the inequality

$$\int_{\{|H(u)(x)|>\lambda\}} \frac{dx}{x^2} \le \frac{4}{\lambda} \int_{-\infty}^{\infty} \frac{|u(t)|}{t^2}\, dt \tag{36}$$

for $\lambda > 0$.

Let us first assume that $u \ge 0$ on $\mathbf{R}$. We consider the analytic function

$$F(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{1}{t} \right) u(t)\, dt$$

in the upper half-plane. We find

$$\operatorname{Re} F(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} u(t)\, dt, \tag{37}$$

and

$$\operatorname{Im} F(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{x-t}{(x-t)^2 + y^2} + \frac{1}{t} \right) u(t)\, dt. \tag{38}$$

First of all, $\operatorname{Re} F$ is non-negative. This is the main reason for assuming $u \ge 0$. For fixed $\lambda > 0$ we consider the auxiliary function

$$f(z) = 1 + \frac{F(z) - \lambda}{F(z) + \lambda}.$$

This function is again analytic in the upper half-plane and it is bounded by 2 there. Therefore we have the Poisson representation

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(t)}{|z-t|^2}\, dt,$$

involving the boundary values of f (see for example [3, p. 59]). In particular,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t^2 + y^2}\, dt = \frac{f(iy)}{y} = \frac{2F(iy)}{y} \frac{1}{F(iy) + \lambda}. \tag{39}$$

In view of the relations (37) and (38) we find

$$\frac{\operatorname{Re} F(iy)}{y} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t^2 + y^2}\, dt \to \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t^2}\, dt$$

as $y \to 0_+$, by dominated (or, for that matter, monotone) convergence, and

$$\frac{\operatorname{Im} F(iy)}{y} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t) y}{t(t^2 + y^2)}\, dt \to 0$$

as $y \to 0_+$. Here we have again used dominated convergence, since the integrand is bounded by $2|u(t)|/t^2$. From relation (39) we thus find

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2+y^2}\,dt \to \frac{2}{\pi\lambda}\int_{-\infty}^{\infty} \frac{u(t)}{t^2}\,dt, \tag{40}$$

as $y \to 0_+$.

The linear fractional transformation

$$T\colon w \mapsto 1 + \frac{w-\lambda}{w+\lambda}$$

maps the right half-plane onto the disk of radius 1 centered at 1. It takes the points of the closed right half-plane of absolute value $\geq \lambda$ onto those points of the closed disk that have real part ≥ 1. For almost all $t \in \mathbf{R}$,

$$\operatorname{Re} F(t+iy) \to u(t)$$

and

$$\operatorname{Im} F(t+iy) \to H(u)(t)$$

as $y \to 0_+$. So, since $f(z) = T\big(F(z)\big)$, we must have $\operatorname{Re} f(t) \geq 0$ for almost all real t, and furthermore

$$\{t \mid |H(u)(t)| \geq \lambda\} \subseteq \{t \mid |u(t) + iH(u)(t)| \geq \lambda\} \subseteq \{t \mid \operatorname{Re} f(t) \geq 1\}. \tag{41}$$

By monotone convergence,

$$\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2+y^2}\,dt \to \int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2}\,dt,$$

as $y \to 0$, so that by (40),

$$\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2}\,dt = \frac{2}{\lambda}\int_{-\infty}^{\infty} \frac{u(t)}{t^2}\,dt.$$

Finally, by (41) we see that

$$\int_{\{|H(u)(t)|\geq\lambda\}} \frac{dt}{t^2} \leq \int_{\{\operatorname{Re} f(t)\geq 1\}} \frac{dt}{t^2} \leq \int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2}\,dt$$

and we have thus verified the inequality (36), with 2 in place of 4, for non-negative functions u.

When u is real-valued, we split it into positive and negative parts, $u = u_+ - u_-$, and observe that $H(u) = H(u_+) - H(u_-)$. Therefore

$$\{t \mid |H(u)(t)| \geq \lambda\} \subseteq \{t \mid |H(u_+)(t)| \geq \tfrac{1}{2}\lambda\} \cup \{t \mid |H(u_-)(t)| \geq \tfrac{1}{2}\lambda\}$$

and we now use the inequality that we obtained for non-negative functions. The inequality (36) is now verified since $|u| = u_+ + u_-$.

Appendix B. An integration by parts formula

We are given a positive measure μ on the real line satisfying

$$d\mu(t) \le C\,dt \tag{42}$$

everywhere. The distribution function $\mu(t)$ is normalized so as to have $\mu(0) = 0$. For given positive numbers r and R ($r < R$) we define

$$H(t) = \int_{r\le|x|\le R} \frac{\mu(x)}{x^2}\log|t-x|\,dx,$$
$$h(t) = \frac{1}{t}\int_{r\le|x|\le R} \log\left|1-\frac{t}{x}\right| d\left(\frac{\mu(x)}{x}\right).$$

Lemma B.1. *The function h is locally integrable on the real line and*

$$\int_0^t h(s)\,ds = H(t) - H(0) + \psi(t),$$

where

$$\psi(t) = \left[\frac{\mu(x)}{x}\int_0^t \frac{1}{s}\log\left|1-\frac{s}{x}\right| ds\right]_{r\le|x|\le R}.$$

The square brackets mean that we evaluate the differences of the function inside at the endpoints of each of the indicated intervals and then add these differences. The proof of this lemma is a simple application of Fubini's theorem followed by partial integration and we shall not give it here.

Proposition B.2. *We have*

$$\int_{r\le|x|\le R}\int_{r\le|t|\le R} \log\left|1-\frac{x}{t}\right| d\left(\frac{\mu(t)}{t}\right)\frac{\mu(x)}{x^2}\,dx$$
$$= \left[\frac{\mu(x)}{x}\int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\log\left|1-\frac{t}{x}\right| dt\right]_{r\le|x|\le R} + \frac{1}{2}\left(\int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\,dt\right)^2.$$

Proof. By Fubini's theorem followed by Lemma B.1,

$$\int_{r\le|x|\le R}\int_{r\le|t|\le R} \log\left|1-\frac{x}{t}\right| d\left(\frac{\mu(t)}{t}\right)\frac{\mu(x)}{x^2}\,dx$$
$$= \int_{r\le|t|\le R} H(t)\,d\left(\frac{\mu(t)}{t}\right) - \int_{r\le|t|\le R}\log|t|\,d\left(\frac{\mu(t)}{t}\right)\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx$$
$$= \int_{r\le|t|\le R}\int_0^t h(s)\,ds\,d\left(\frac{\mu(t)}{t}\right) + H(0)\int_{r\le|t|\le R} d\left(\frac{\mu(t)}{t}\right)$$
$$- \int_{r\le|t|\le R}\psi(t)\,d\left(\frac{\mu(t)}{t}\right) - \int_{r\le|t|\le R}\log|t|\,d\left(\frac{\mu(t)}{t}\right)\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx.$$

Integration by parts gives us

$$\int_{r\le|t|\le R}\int_0^t h(s)\,ds\,d\Big(\frac{\mu(t)}{t}\Big) = \Big[\frac{\mu(t)}{t}\big(H(t)-H(0)+\psi(t)\big)\Big]_{r\le|t|\le R} - \int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\int_{r\le|x|\le R}\log\Big|1-\frac{t}{x}\Big|\,d\Big(\frac{\mu(x)}{x}\Big)\,dt.$$

The last term on the right-hand side of this equation is the same integral as the one we started to compute. We therefore find that

$$\begin{aligned}
2&\int_{r\le|x|\le R}\int_{r\le|t|\le R}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}\,dx\\
&=\Big[\frac{\mu(t)}{t}\big(H(t)-H(0)+\psi(t)\big)\Big]_{r\le|t|\le R} + H(0)\int_{r\le|t|\le R} d\Big(\frac{\mu(t)}{t}\Big)\\
&\quad -\int_{r\le|t|\le R}\psi(t)\,d\Big(\frac{\mu(t)}{t}\Big) - \int_{r\le|t|\le R}\log|t|\,d\Big(\frac{\mu(t)}{t}\Big)\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx\\
&=\Big[\frac{\mu(t)}{t}H(t)\Big]_{r\le|t|\le R} + \int_{r\le|t|\le R}\psi'(t)\frac{\mu(t)}{t}\,dt\\
&\quad -\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx\Big\{\Big[(\log|t|)\frac{\mu(t)}{t}\Big]_{r\le|t|\le R} - \int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\,dt\Big\}\\
&=\Big[\frac{\mu(x)}{x}\Big(\int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1-\frac{t}{x}\Big|\,dt + H(x)\Big)\Big]_{r\le|x|\le R}\\
&\quad -\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx\Big\{\Big[(\log|t|)\frac{\mu(t)}{t}\Big]_{r\le|t|\le R} - \int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\,dt\Big\}\\
&=2\Big[\frac{\mu(x)}{x}\int_{r\le|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1-\frac{t}{x}\Big|\,dt\Big]_{r\le|x|\le R} + \Big(\int_{r\le|x|\le R}\frac{\mu(x)}{x^2}\,dx\Big)^2.
\end{aligned}$$

The result follows.

This proposition is used in both Section 4 and 5. We give a corollary suitable for the use in Section 4.

Corollary B.3. *If μ satisfies the condition (42) and if, furthermore, μ is zero close to the origin and $\mu(t)$ is constant for $|t|$ large enough then*

$$x\mapsto\int_{-\infty}^{\infty}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}$$

is integrable on the real line and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}\,dx = \frac{1}{2}\Big(\int_{-\infty}^{\infty}\frac{\mu(x)}{x^2}\,dx\Big)^2.$$

Proof. The assumptions on μ assure integrability on the real line and thus that the double integral in question equals

$$\begin{aligned}\varepsilon(R)&+\int_{|x|\le R}\int_{|t|\le R}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}\,dx\\&+\int_{|x|\le R}\int_{|t|\ge R}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}\,dx,\end{aligned}\tag{43}$$

where $\varepsilon(R)\to 0$ as $R\to\infty$. To estimate the second integral in this relation, one should use the second mean value theorem (see for example [13, Section 12.3]). By that theorem one obtains, for $|x|\le R$,

$$\left|\int_R^\infty \log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\right|\le\left|\log\Big|1-\frac{x}{R}\Big|\right|\delta(R),$$

where $\delta(R)$ is the supremum of $|\mu(t)/t-\mu(s)/s|$ when $|s|,|t|\ge R$. Similarly,

$$\left|\int_{-\infty}^{-R} \log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\right|\le\left|\log\Big|1+\frac{x}{R}\Big|\right|\delta(R),$$

so that

$$\left|\int_{|x|\le R}\int_{|t|\ge R}\log\Big|1-\frac{x}{t}\Big|\,d\Big(\frac{\mu(t)}{t}\Big)\frac{\mu(x)}{x^2}\,dx\right|\le\delta(R)\int_{|x|\le R}\log\left|\frac{1+x/R}{1-x/R}\right|\frac{\mu(x)}{x^2}\,dx.$$

The integral on the right-hand side remains bounded as $R\to\infty$ because of (42). Since $\mu(t)$ is constant for $|t|$ large, $\delta(R)$ tends to zero. The second double integral in (43) tends to zero. The first double integral in (43) is, by Proposition B.2, equal to

$$\left[\frac{\mu(x)}{x}\int_{|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1-\frac{t}{x}\Big|\,dt\right]_{|x|\le R}+\frac12\left(\int_{|t|\le R}\frac{\mu(t)}{t^2}\,dt\right)^2.$$

(One may take $r=0$ there since μ is supposed to vanish close to the origin.) As R tends to infinity, the second term tends to

$$\frac12\left(\int_{-\infty}^\infty\frac{\mu(t)}{t^2}\,dt\right)^2,$$

again by the assumptions on μ. The first term is equal to

$$\frac{\mu(R)}{R}\int_{|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1-\frac{t}{R}\Big|\,dt+\frac{\mu(-R)}{R}\int_{|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1+\frac{t}{R}\Big|\,dt,$$

and this tends to zero as well. Indeed, $\mu(\pm R)/R\to 0$ and

$$\int_{|t|\le R}\frac{\mu(t)}{t^2}\log\Big|1\pm\frac{t}{R}\Big|\,dt=\int_{|s|\le 1}\frac{\mu(sR)}{sR}\log|1\pm s|\,\frac{ds}{s}\to 0$$

by (42) and dominated convergence. The corollary follows.

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