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ON THE GROWTH AND FACTORIZATION OF ENTIRE SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we discuss the growth and factorization of entire solutions of some classes of first-order algebraic differential equations and prove that any entire solution of a first-order algebraic differential equation must be pseudo-prime.

1. Introduction

In the study of the solutions of complex differential equations, the growth of a solution is a very important property. For linear differential equations of the form

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = a(z), \tag{1.1}$$

where $a(z)$, $a_0(z), \dots, a_{(n-1)}(z)$ are polynomials, it is known that any entire solution must be of finite and positive order; see Laine [13, pp. 52–73, pp. 144–164], Gundersen, et al. [6]. This can be proved by mainly using the Wiman–Valiron theory. However, there are only a few results concerning the growth of the solutions of a nonlinear algebraic differential equation

$$P(z, f, f', \dots, f^{(n)}) = 0, \tag{1.2}$$

where P is a polynomial in all its arguments. Equation (1.2) can be rewritten in the form

$$\sum_{\lambda\in I} a_\lambda(z) f^{i_0} \cdots \bigl(f^{(n)}\bigr)^{i_n} = 0, \tag{1.3}$$

where I is a finite set of multi-indices $(i_0, \dots, i_n) = \lambda$. We define a differential monomial in f as

$$M[f] = a_\lambda(z) f^{i_0} \cdots \bigl(f^{(n)}\bigr)^{i_n}.$$

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The degree γ_M and the weight Γ_M of M are defined by

$$\gamma_M = i_0 + i_1 + \cdots + i_n, \quad \Gamma_M = i_0 + 2i_1 + \cdots + (n+1)i_n.$$

Then the left-hand side of equation (1.3) can be expressed as a finite sum of differential monomials. From now on we shall call this a differential poynomial in f, i.e.

$$P[f] = P(z, f, f', \ldots, f^n) = \sum_{\lambda \in I} M_\lambda[f].$$

The degree γ_P and the weight Γ_P of P are defined by $\gamma_P = \max_{\lambda\in I} \gamma_{M_\lambda}$, $\Gamma_P = \max_{\lambda \in I} \Gamma_{M_\lambda}$. Some results have been obtained on the growth estimates for solutions of algebraic differential equations; see e.g. [13]. However, in general, a complete growth estimate for nonlinear algebraic differential equations remains to be resolved. The first important result on the growth estimate was due to A. Gol'dberg [4] (or see e.g. [13]). Some variations of Gol'dberg's result were obtained by Bank–Kaufman [1], W. Bergweiler [2], and others.

Theorem A (Gold'berg). *Let $P(u_1, u_2, u_3)$ be a polynomial in all of its arguments u_1, u_2 and u_3 and consider the first-order algebraic differential equation*

$$P(z, f, f') = 0. \tag{1.4}$$

Then all meromorphic solutions of (1.4) *are of finite order of growth.*

Note that the order can be zero (see e.g. [13]).

For nonlinear differential equations of second-order Steinmetz [18] proved the following theorem (see also e.g. [13]):

Theorem B. *Suppose that in equation* (1.2) *P is homogeneous in f, f' and f''. Then all meromorphic solutions of* (1.2) *satisfy*

$$T(r, f) = O(\exp r^b)$$

as $r \to \infty$ for some $b > 1$ depending only on the degrees of the polynomial coefficients of (1.3).

Recently W.K. Hayman [8] studied the growth of solutions of (1.2) and posed the following

Conjecture. *If $f(z)$ is an entire solution of* (1.2), *then*

$$T(r, f) < a \exp_{n-1}(br^c), \qquad 0 \le r < \infty$$

where a, b and c are positive constants and $\exp_l(x)$ is the exponential iterated l times.

Hayman also showed that the conjecture is true for some special class of equations.

Set

$$\Lambda = \{\lambda = (i_0, i_1, \dots, i_n) \mid \gamma_{M_\lambda} = \gamma_P,\ \Gamma_{M_\lambda} = \Gamma_P\}.$$

Theorem C (Hayman [8]). *Suppose that equation* (1.3) *holds with* Λ *defined as above. Let* d *be the maximum degree among all the polynomials* $a_\lambda(z)$ *in* (1.3) *and suppose that*

$$\sum_{\lambda\in\Lambda} a_\lambda(z) \not\equiv 0. \tag{1.5}$$

Then any entire solution of (1.3) *has finite order* ρ, *with* $\rho \le \max\{2d, 1+d\}$.

On the other hand, Sh. Strelitz [20] proved

Theorem D. *Every entire transcendental solution of a first-order algebraic differential equation with rational coefficients has an order no less than* $\frac{1}{2}$.

Definition (Gross [5]). A meromorphic function f is called pseudo-prime if whenever $f = g(h)$ with g, h entire or meromorphic, implies that either g is a rational function or h is a polynomial.

As regards the factorization of the solutions of differential equations, the following two results are well known.

Theorem E (Steinmetz [17]). *Any meromorphic solution of* (1.1) *is pseudo-prime.*

Theorem F (Mues [14]). *Let* f *be a meromorphic solution of the Riccati differential equation*

$$w' = a(z) + b(z)w + c(z)w^2,$$

where $a(z)$, $b(z)$, $c(z)$ *are polynomials. Then* f *is pseudo-prime.*

For some other related results on the factorization of solutions of some first-order algebraic differential equations, see e.g. He–Laine [9] and He–Yang [10]. In this paper we shall mainly discuss the growth and factorization of the transcendental entire solutions of the first-order algebraic differential equation in its most general form. Subsequently, we always assume that f denotes an entire function and write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We denote the maximum term of f by $\mu(r,f)$, the central index by $\nu(r,f)$, and the maximum modulus by $M(r,f)$, i.e.

$$\mu(r,f) = \max_{|z|=r} |a_n z^n|,\ \ \nu(r,f) = \sup\{n \mid |a_n| r^n = \mu(r,f)\},\ \ M(r,f) = \max_{|z|=r} |f(z)|.$$

As usual, we use $T(r,f)$ to denote the Nevanlinna characteristic function of f and $\rho(f)$ to denote the order of f.

2. Lemmas

In order to obtain our theorems, we need some basic results of the Wiman–Valiron theory .

Lemma 1 (Laine [13]). *If f is an entire function of order ρ, then*

$$\rho = \limsup_{r\to\infty} \frac{\log^+ \nu(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log^+ \mu(r,f)}{\log r}. \tag{2.1}$$

Lemma 2 (Laine [13]). *Let f be a transcendental entire function, and let $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r,f)\nu(r,f)^{-(1/4)+\delta} \tag{2.2}$$

holds. Then there exists a set $F \subset R^+$ of finite logarithmic measure, i.e., $\int dt/t < +\infty$ such that

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m \big(1 + o(1)\big) f(z) \tag{2.3}$$

holds for all $m \geq 0$ and all $r \notin F$.

By Lemma 1, we can easily derive

Lemma 3. *Let f be a transcendental entire function of order $\rho < 1$. Then*

$$\lim_{r\to\infty} \frac{\nu(r,f)}{r} = 0. \tag{2.4}$$

The following lemma due to Polya [15] plays a very important role in our discussions.

Lemma 4. *If f and g are entire functions, the composite function $f \circ g$ is of infinite order unless* (a) *f is of finite order and g is a polynomial or* (b) *f is of zero order and g is of finite order.*

Lemma 5 (Steinmetz [16]). *Let $f_1, \dots, f_n$ and g be entire functions and let $h_1, \dots, h_n$ be meromorphic functions such that the inequality*

$$\sum_{j=1}^{n} T(r, h_j) \leq KT(r, g)$$

holds, with K a constant. Suppose that f_j and h_j $(j = 1, 2, \dots, n)$ satisfy

$$f_1\big(g(z)\big)h_1(z) + \cdots + f_n\big(g(z)\big)h_n(z) \equiv 0.$$

Then there exist two sets of polynomials $\{P_j\}$ and $\{Q_j\}$ $(j = 1, 2, \dots, n)$ not all identically zero in either of the two sets such that

$$P_1\big(g(z)\big)h_1(z) + \cdots + P_n\big(g(z)\big)h_n(z) \equiv 0 \tag{2.5}$$

and

$$f_1(z)Q_1(z) + \cdots + f_n(z)Q_n(z) \equiv 0. \tag{2.6}$$

Lemma 6 (Zimogljad [21]). *Every entire transcendental solution of a second-order algebraic differential equation with rational coefficients has a positive order.*

3. Growth of solutions of certain types of first-order algebraic differential equations

In this section, we shall provide a more precise estimation of the growth of entire solutions of algebraic differentional equations of the form

$$C(z,w)(w')^2 + B(z,w)w' + A(z,w) = 0, \tag{3.1}$$

where $C(z,w) \not\equiv 0, B(z,w)$, and $A(z,w)$ are polynomials in z and w. Here we refer the reader to Ishizaki's works [11], [12] for the cases where the cofficients of the powers of w in $A(z,w)$, $B(z,w)$ and $C(z,w)$ are transcendental functions. Steinmetz [19] showed that if (3.1) has a transcendental meromorphic solution, then

$$\deg_w C(z,w) = 0, \qquad \deg_w B(z,w) \le 2, \qquad \deg_w A(z,w) \le 4.$$

Thus we can assume that $C(z,w) \equiv a(z)$ is a polynomial in z only and that (3.1) can be rewritten in the following form:

$$\begin{aligned} a(z){w'}^2 + \big(b_2(z)w^2 + b_1(z)w + b_0(z)\big)w' = d_4(z)w^4 + d_3(z)w^3 \\ + d_2(z)w^2 + d_1(z)w + d_0(z), \end{aligned} \tag{3.2}$$

where $a(z)$, $b_i(z)$ $(i = 0,1,2)$ and $d_j(z)$ $(j = 0,\dots,4)$ are polynomials. If (3.2) has a transcendental entire solution, one can derive $d_4(z) \equiv 0$ by comparing the characteristic functions on both sides of (3.2). Finally equation (3.2) can be reduced to the following form:

$$a(z){f'}^2 + \big(b_2(z)f^2 + b_1(z)f + b_0(z)\big)f' = d_3(z)f^3 + d_2(z)f^2 + d_1(z)f + d_0(z). \tag{3.3}$$

Theorem 1. *If* $\deg d_2(z) \ne \deg a(z) - 1$ *in* (3.3) *and* $f(z)$ *is a transcendental entire solution of equation* (3.2), *then* $\rho(f) \ge 1$.

Proof. We assume that f has an order $\rho(f) < 1$ and satisfies equation (3.3). Now we can rewrite (3.3) as follows:

$$\begin{aligned} \left(d_3(z) - b_2(z)\frac{f'(z)}{f(z)}\right) = \Bigg[a(z)\frac{\big(f'(z)\big)^2}{\big(f(z)\big)^2} + b_1(z)\frac{f'(z)}{f(z)} + b_0(z)\frac{f'(z)}{\big(f(z)\big)^2} \\ + d_2(z) + \frac{d_1(z)}{f(z)} + \frac{d_0(z)}{\big(f(z)\big)^2}\Bigg]\frac{1}{f(z)}. \end{aligned} \tag{3.4}$$

We choose $r_n \notin F$ and z_n such that $r_n \to \infty$, $n \to \infty$, $|z_n| = r_n$, $|f(z_n)| = M(r_n,f)$. From Lemmas 2 and 3 we have

$$\frac{f'(z_n)}{f(z_n)} = \frac{\nu(r_n,f)}{z_n}\big(1 + o(1)\big) \to 0. \tag{3.5}$$

Thus

$$\left|a(z)\frac{\big(f'(z)\big)^2}{\big(f(z)\big)^2}+b_1(z)\frac{f'(z)}{f(z)}+b_0(z)\frac{f'(z)}{\big(f(z)\big)^2}+d_2(z)+\frac{d_1(z)}{f(z)}+\frac{d_0(z)}{\big(f(z)\big)^2}\right|<cr^m, \tag{3.6}$$

where c is a constant and m is the degree of $d_2(z)$. Hence

$$\lim_{n\to\infty}\left[d_3(z_n)-b_2(z_n)\frac{\nu(r_n,f)}{r_n}\big(1+o(1)\big)\right]=0. \tag{3.7}$$

If $b_2(z)\not\equiv 0$, then

$$\lim_{n\to\infty}\frac{d_3(z_n)-b_2(z_n)\big(\nu(r_n,f)/r_n\big)\big(1+o(1)\big)}{b_2(z_n)}=0. \tag{3.8}$$

Equation (3.7) and Lemma 3 yield

$$\lim_{n\to\infty}\frac{d_3(z_n)}{b_2(z_n)}=0. \tag{3.9}$$

It follows that $\deg d_3(z)<\deg b_2(z)$, and hence

$$\lim_{n\to\infty}\frac{z_n d_3(z_n)}{b_2(z_n)}=t, \tag{3.10}$$

where t is a finite constant. If $d_3(z)\not\equiv 0$, then $\deg b_2(z)\ge 1$. Then, again from (3.7), and noting that $|z/b_2(z)|$ is bounded for sufficiently large $r=|z|$ and f is a transcendental function, we have

$$\lim_{n\to\infty}\frac{z_n d_3(z_n)}{b_2(z_n)}=\lim_{n\to\infty}\nu(r_n,f)\big(1+o(1)\big)=\infty. \tag{3.11}$$

This contradicts (3.10). Hence $d_3(z)\equiv 0$. It follows that equation (3.3) becomes

$$a(z){f'}^2+\big(b_2(z)f^2+b_1(z)f+b_0(z)\big)f'=d_2(z)f^2+d_1(z)f+d_0(z). \tag{3.12}$$

Assume that $a(z)\not\equiv 0$. If $b_2(z)\not\equiv 0$, then

$$\begin{aligned} f'(z)=&\frac{d_2(z)}{b_2(z)}+\frac{d_1(z)}{b_2(z)}\frac{1}{f(z)}+\frac{d_0(z)}{b_2(z)}\frac{1}{\big(f(z)\big)^2}\\ &-\frac{b_1(z)}{b_2(z)}\frac{f'(z)}{f(z)}-\frac{b_0(z)}{b_2(z)}\frac{f'(z)}{\big(f(z)\big)^2}-\frac{a(z)}{b_2(z)}\frac{\big(f'(z)\big)^2}{\big(f(z)\big)^2}.\end{aligned} \tag{3.13}$$

Applying Lemmas 2 and 3 to the above equation, the following result holds for a sequence of $r_n \to \infty$:

$$(1+o(1))\nu(r_n,f)M(r_n,f) \le A{r_n}^B, \tag{3.14}$$

where A and B are constants. This is impossible. Thus $b_2(z) \equiv 0$. Similarly, if $a(z) \equiv 0$, we can also conclude $b_2(z) \equiv 0$. Therefore (3.12) reduces to

$$a(z){f'}^2 + \big(b_1(z)f + b_0(z)\big)f' = d_2(z)f^2 + d_1(z)f + d_0(z). \tag{3.15}$$

Now if $a(z) \not\equiv 0$, by (3.15) and Lemma 2 we have for a sequence of $r_n \to \infty$ (as in (3.5))

$$\begin{aligned} &a(z_n)\left(\frac{\nu(r_n,f)}{r_n}\right)^2\big(1+o(1)\big) + b_1(z_n)\frac{\nu(r_n,f)}{r_n}\big(1+o(1)\big) - d_2(z_n) \\ &\quad = \left[b_0(z_n)\frac{\nu(r_n,f)}{r_n}\big(1+o(1)\big) + d_1(z_n) + \frac{d_0(z_n)}{f(z_n)}\right]\frac{1}{f(z_n)}. \end{aligned} \tag{3.16}$$

By Lemma 3, (3.16) and noting $\lim_{r\to\infty}\big(r^k/M(r,f)\big) = 0$ for any k, we have

$$\lim_{n\to\infty}\left\{a(z_n)\left(\frac{\nu(r_n,f)}{r_n}\right)^2\big(1+o(1)\big) + b_1(z_n)\frac{\nu(r_n,f)}{r_n} - d_2(z_n)\right\} = 0. \tag{3.17}$$

Now we will discuss three cases separately.

Case 1: $\deg b_1(z) > \deg a(z)$. From (3.17) we have

$$\lim_{n\to\infty}\frac{d_2(z_n)}{b_1(z_n)} = 0. \tag{3.18}$$

Thus $\deg d_2(z) < \deg b_1(z)$ and hence

$$\lim_{n\to\infty}\frac{z_n d_2(z_n)}{b_1(z_n)} = c, \tag{3.19}$$

where c is a finite constant. But, on the other hand, from (3.17) we have

$$\lim_{n\to\infty}\left[\frac{b_1(z_n)}{a(z_n)}\frac{\nu(r_n,f)}{r_n} - \frac{d_2(z_n)}{a(z_n)}\right] = 0. \tag{3.20}$$

By (20), and noting that $|za(z)/b_1(z)|$ is bounded for sufficiently large $r = |z|$ and f is a transcendental function, we can get a conclusion which contradicts (3.19) by using the same argument as in the derivation of (3.11).

Case 2: $\deg b_1(z) = \deg a(z)$. Then

$$\lim_{n\to\infty} \frac{d_2(z_n)}{a(z_n)} = c \neq 0. \tag{3.21}$$

From (3.17) and Lemma 3 we have

$$\lim_{n\to\infty} \frac{d_2(z_n)}{a(z_n)} = 0. \tag{3.22}$$

It follows that from this, (3.16), and (3.21) we have

$$\left| \big(c + o(1)\big) \frac{\nu(r_n, f)}{r_n} \right| M(r_n, f) \leq A r_n^k, \tag{3.23}$$

where A, k are constants. This is impossible.

Case 3: $\deg b_1(z) < \deg a(z)$. In this case, we also have (3.22). This means $\deg a(z) > \deg d_2(z)$. However, by the assumption of the theorem, we have $\deg a(z) > \deg d_2(z) + 1$. If $d_2(z) \not\equiv 0$, then $\deg a(z) > 1$. Thus

$$\lim_{n\to\infty} \frac{r_n b_1(z_n)}{a(z_n)} = c_1, \qquad \lim_{n\to\infty} \frac{r_n^2 d_2(z_n)}{a(z_n)} = c_2. \tag{3.24}$$

From (3.17) we get

$$\lim_{n\to\infty} \left\{ \big(\nu(r_n, f)\big)^2 + \frac{r_n b_1(z_n)}{a(z_n)} \nu(r_n, f) - \frac{r_n^2 d_2(z_n)}{a(z_n)} \right\} = 0. \tag{3.25}$$

This is also impossible. Now we assume that $d_2(z) \equiv 0$. Then by (3.16) and noting that $|z b_1(z)/a(z)|$ is bound for sufficiently large $|z|$, we have

$$\left| \big(1 + o(1)\big) a(z_n) \big(\nu(r_n, f)\big)^2 \right| M(r_n, f) < A r_n^k, \tag{3.26}$$

where A, k are constants. This is again impossible. From the above discussions we can conclude that $a(z) \equiv 0$ if f satisfies (3.15), i.e., f must satisfy

$$\big(b_1(z) f + b_0(z)\big) f' = d_2(z) f^2 + d_1(z) f + d_0(z). \tag{3.27}$$

By Lemmas 2 and 3, and noting $\lim_{r\to\infty} \big(r^k / M(r, f)\big) = 0$ for any k, we have

$$\lim_{n\to\infty} \left\{ b_1(z_n) \frac{\nu(r_n, f)}{r_n} \big(1 + o(1)\big) - d_2(z_n) \right\} = 0. \tag{3.28}$$

By the same argument as in the case of (3.7), we have $b_1(z) \equiv 0$ and $d_2(z) \equiv 0$. Thus (3.15) becomes

$$b_0(z)f'(z) = d_1(z)f(z) + d_0(z). \tag{3.29}$$

From this and Lemma 2, we have for a sequence of z_n, ($|z_n| = r_n \to \infty$),

$$b_0(z_n)\frac{\nu(r_n, f)}{r_n}\big(1 + o(1)\big) = d_1(z_n) + \frac{d_0(z_n)}{f(z_n)}, \tag{3.30}$$

where $f(z_n) = M(r_n, f)$. From this it is easily seen that either $\lim_{n\to\infty} \nu(r_n, f) = c$, where c is a constant, or $\nu(r_n, f) \geq Ar_n^k + o(1)$, where A is a constant, and k is a positive integer. However, both cases are impossible. The proof is thus completed.

Remark 1. The condition $\deg d_2(z) \neq \deg a(z) - 1$ in Theorem 1 cannot be omitted. For example, $f(z) = \cos\sqrt{z}$, which has an order $\rho(f) = \frac{1}{2}$. However, it satisfies the following first-order algebraic differential equation:

$$4z(w')^2 + w^2 - 1 = 0.$$

Remark 2. The conclusion is sharp in Theorem 1. There exists the function $f(z) = z\sin z$, which is of order one and satisfies the following first-order algebraic differential equation:

$$z^3(1 - z^2)(w')^2 - 2z^2ww' + (z^2 + z)w^2 - z^5 = 0.$$

4. Factorization of the solutions of first-order algebraic differential equations

In general, a solution of a higher-order algebraic differential equation may not be pseudo-prime. For example $f_2(z) = e^{e^z}$ satisfies the homogeneous second-order algebraic differential equation

$$ww'' - {w'}^2 - ww' = 0.$$

Furthermore, according to Hayman [8], $f_n(z) = \exp_n(z)$ satisfies a homogeneous nth-order algebraic differential equation, where $\exp_n(z)$ is the nth iterate of exponential function. In this section, we consider the factorization of the entire solutions of the most general first-order algebraic differential equation,

$$\sum_{\lambda\in I} a_\lambda(z)f^{i_0}(f')^{i_1} = 0, \tag{4.1}$$

where $a_\lambda(z)$ denotes a rational function. First we prove

Theorem 2. *All transcendental entire solutions of* (4.1) *are pseudo-prime.*

Proof. If f is a transcendental entire solution of (4.1), then $\rho(f) < \infty$ by Theorem A. Now we assume that f is not pseudo-prime. i.e.,

$$f = g(h), \tag{4.2}$$

where g is a transcendental meromorphic function and h is a transcendental entire function. First we assume that g is entire. From Lemma 4 we have $\rho(g) = 0$. By substituting (4.2) into (4.1), we have

$$\sum_{\lambda\in I} a_\lambda(z)\big[g\big(h(z)\big)\big]^{i_0}\big[g'\big(h(z)\big)\big]^{i_1}[h'(z)]^{i_1} = 0. \tag{4.3}$$

We denote $F_\lambda(w) = \big(g(w)\big)^{i_0}\big(g'(w)\big)^{i_1}$. Then we can rewrite (4.3) as

$$\sum_{\lambda\in I} a_\lambda(z)F_\lambda\big(h(z)\big)[h'(z)]^{i_1} = 0. \tag{4.4}$$

Noting that $T(r,h') = m(r,h') \le m(r,h) + m(r,h'/h) = T(r,h) + S(r,h)$ and using Lemma 5, we see that there exist some polynomials Q_λ which are not all identically zero such that

$$\sum_{\lambda\in I} Q_\lambda F_\lambda \equiv 0,$$

i.e.

$$\sum_{\lambda\in I} Q_\lambda(z)\big(g(z)\big)^{i_0}\big(g'(z)\big)^{i_1} \equiv 0. \tag{4.5}$$

From this and Theorem D we have $\rho(g) > 0$. This contradicts the fact $\rho(g) = 0$. Now if g is not entire, it is easily shown that g has one and only one pole w_1. Thus $g(w) = g_1(w)/(w-w_1)^n$, where $g_1(z)$ is a transcendental entire function. Then

$$f(z) = \frac{g_1\big(h(z)\big)}{h(z) - w_1}. \tag{4.6}$$

By substituting (4.6) into (4.1), we get

$$\sum_{\lambda\in J} b_\lambda(z)Q_\lambda\big(h(z),h'(z)\big)\big[g_1\big(h(z)\big)\big]^{i_0}\big[{g'}_1\big(h(z)\big)\big]^{i_1} = 0, \tag{4.7}$$

where $b_\lambda(z)$ denotes a rational function, $Q_\lambda(\eta,\zeta)$ being a rational function of η and ζ. From this and (2.6) it follows that g_1 satisfies a first-order algebraic differential equation. This will lead to $\rho(g_1) > 0$, which contradicts the fact that $\rho(g_1) = \rho(g) = 0$. Thus f must be pseudo-prime, which also completes the proof of the theorem.

By using Lemmas 5 and 6, and the argument similar to that used in the proof of Theorem 2, we can prove

Theorem 3. *Every finite-order transcendental entire solution of a second-order algebraic differential equation with rational coefficients must be pseudo-prime.*

Remark 3. The proof of Theorem 2 cannot be used to show that every transcendental meromorphic solution of (4.1) must be pseudo-prime, since it is known that there exists a transcendental meromorphic (non-entire) function f which satisfies a first-order algebraic differential equation and is of zero order. For example, there exists a meromorphic function H (see [13]), which satisfies the differential equation

$$(z^2-4)H'(z)^2 = 4\big(H(z)-e_1\big)\big(H(z)-e_2\big)\big(H(z)-e_3\big) \tag{4.8}$$

with the growth condition

$$T(r,H) = O(\log r)^2, r\to\infty. \tag{4.9}$$

Question. *Is every transcendental meromorphic solution of a first-order algebraic differential equation pseudo-prime*?

The answer to the question is negative. As indicated by W. Bergweiler, the meromorphic function $f(z) = H\big(g(z)\big)$ satisfies the differential equation

$$f'(z)^2 = 4(f(z)-e_1)\big(f(z)-e_2\big)\big(f(z)-e_3\big),$$

where H satisfies (4.8) and g satisfies $(g^2-4) = {g'}^2$. Apparently f is not pseudo-prime.

Remark 4. For some higher-order algebraic differential equations with rational coefficients, periodic entire solutions of finite order have been presented which are not pseudo-prime (see e.g. [3, pp. 164, Theorem 4.13]). Also all known non-pseudo-prime entire solutions of some higher-order algebraic differential equations are periodic functions. However, by using the same method as in [3], one can easily construct non-periodic entire functions which are not pseudo-prime and satisfy some higher-order algebraic differential equations.

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References

[1] Bank, S., and R. Kaufman: On meromorphic solutions of first-order differential equations. - Comment. Math. Helv. 51, 1976, 289–299.

[2] Bergweiler, W.: On a theorem of Gol'dberg concerning meromorphic solutions of algebraic differential equations. - Complex Variables Theory Appl. (to appear).

[3] Chuang, C.T., C.C. Yang, Y. He and G.C. Wen: Some Topics in Function Theory of One Complex Variable. - Basic Series of Modern Mathematics, Science Press, Beijing, China, 1995 (in Chinese).

[4] Gol'dberg, A.A.: On single-valued solutions of first-order differential equations. - Ukraïn. Mat. Zh. 8, 1956, 254–261.

[5] Gross, F.: On factorization of meromorphic functions. - Trans. Amer. Math. Soc. 131, 1968 , 215–222.

[6] Gross, F.: On factorization of meromorphic functions. - Trans. Amer. Math. Soc. 131, 1968, 215–222.

[7] Gundersen, G., M. Steinbart and S. Wang: The possible orders of linear differential equations with polynomial coefficients. - Trans. Amer. Math. Soc. 350, 1998, 1225–1247.

[8] Hayman, W.K.: The growth of solutions of algebraic differential equations. - Rend. Mat. Accad. Lincei 7, 1996, 67–73.

[9] He, Y., and I. Laine: Factorization of meromorphic solutions to the differential equation $(f')^n = R(z,f)$. - Rev. Roumaine Math. Pures Appl. 39, 1994, 675–689.

[10] He, Y., and C.C. Yang: On pseudo-primality of the product of some pseudo-prime meromorphic functions. - In: Analysis of One Complex Variable, Proceedings of the American Mathematical Society's 821st Wyoming 1985 meeting, 113–124.

[11] Ishizaki, I.: Meromorphic solutions of complex differential equations. - Ph.D. Dissertation, Chiba, 1993.

[12] Ishizaki, I.: A result for a certain algebraic differential equation. - BHKMS, 1, 1997, 301–308.

[13] Laine, I.: Nevanlinna Theory and Complex Differential Equations. - Walter de Gruyter, Berlin–New York, 1993.

[14] Mues, E.: Über faktorisierbare Lösungen von Riccatischen Differentialgleichungen. - Math. Z. 121, 1971, 145–156.

[15] Polya, G.: Zur Untersuchung der Grössenordnung ganzer Funktionen, die einer Differentialgleichung genügen. - Acta Math. 42, 1920, 309–316.

[16] Steinmetz, N.: Eigenschaften eindeutiger Lösungen gewöhnlicher Differentialgleichungen in Komplexen. - Karlsruhe Dissertations, 1978.

[17] Steinmetz, N.: Über die faktorisierbaren Lösungen gewöhnlicher Differentialgleichungen. - Math. Z. 170, 1980, 168–180.

[18] Steinmetz, N.: Über das Anwachsen der Lösungen homogener algebraischer Differentialgleichungen zweiter Ordnung. - Manuscripta Math. 32, 1980, 303–308.

[19] Steinmetz, N.: Ein Malmquistscher Satz für algebraische Differentialgleichungen erster Ordnung. - J. Reine Angew. Math. 316, 1980, 44–53.

[20] Strelitz, Sh.: Three theorems on the growth of entire transcendental solutions of algebraic differential equations. - Canad. J. Math. 35, 1983, 1110–1128.

[21] Zimogljad, V.V.: On the growth of entire transcendental solutions of second-order algebraic differential equations. - Mat. Sb. 85(127), 2(6), 1971, 283–382.

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