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OPERATOR REPRESENTATIONS OF DEFINITIZABLE FUNCTIONS

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Abstract. Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on a separable Hilbert space $\mathscr{H}$. An $\mathscr{L}(\mathscr{H})$-valued holomorphic function F in the complement of the unit circle $\mathbf{T}$ satisfying $F(\bar{z}^{-1}) = -F(z)^*$ such that the values of F in the unit disc have nonnegative real part is called a Carathéodory operator function. A definitizable operator function is a function with the same $\mathbf{T}$-antisymmetry which can be transformed into a Carathéodory function by multiplying with a rational scalar function and by adding a rational operator function.

We prove here, relying heavily on a result of T. Ya. Azizov, that the definitizable functions are precisely those operator functions which can be represented in the form

$$F(z) = iS + \Gamma^+(U+z)(U-z)^{-1}\Gamma,$$

where S is a bounded selfadjoint operator in $\mathscr{H}$, U is a definitizable unitary operator in some Kreĭn space $\mathscr{K}$, Γ is a bounded operator of $\mathscr{H}$ in $\mathscr{K}$, and Γ^+ denotes the Kreĭn space adjoint of Γ. If the representing operator U for a given definitizable function fulfils some minimality condition, then the restrictions of U to all Pontryagin type spectral subspaces are uniquely determined up to unitary equivalence.

The main objective of this paper are the connections between sign properties of F, which were introduced in [9], and similar properties of the representing operators U. The results are carried over to a similar class of operator functions meromorphic in $\mathbf{C} \setminus \mathbf{R}$.

Let $\mathscr{H}$ be a separable complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators in $\mathscr{H}$. By $M\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ we denote the set of all $\mathscr{L}(\mathscr{H})$-valued functions F which are meromorphic outside the unit circle and at ∞ and satisfy the relation $F(\bar{z}^{-1}) = -\big(F(z)\big)^*$ at all points z of holomorphy.

A function $F \in M\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ belongs to the Carathéodory class $C\big(\mathscr{L}(\mathscr{H})\big)$, i.e., F is holomorphic in $\overline{\mathbf{C}} \setminus \mathbf{T}$, and for every z in the open unit disc $\mathbf{D}$ its real part $\frac{1}{2}\big(F(z)+F(z)^*\big)$ is a nonnegative operator if and only if it admits an integral representation of the form

$$(0.1)\qquad F(z) = iS + (4\pi)^{-1}\int_0^{2\pi} (e^{i\Theta}+z)(e^{i\Theta}-z)^{-1}\,dM(\Theta), \qquad z \in \overline{\mathbf{C}} \setminus \mathbf{T},$$

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where $S \in \mathscr{L}(\mathscr{H})$ is selfadjoint and M is a positive operator measure. Moreover, $F \in C\big(\mathscr{L}(\mathscr{H})\big)$ if and only if there exist a Hilbert space $\mathscr{K}$, a unitary operator U in $\mathscr{K}$, a mapping $\Gamma \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and a bounded selfadjoint operator S in $\mathscr{H}$ such that

$$F(z) = iS + \Gamma^*(U+z)(U-z)^{-1}\Gamma, \qquad z \in \overline{\mathbf{C}} \setminus \mathbf{T}. \tag{0.2}$$

Similarly, the classes $C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$, $\kappa = 0, 1, \dots$, of generalized Carathéodory functions introduced and studied by M. G. Kreĭn and H. Langer (see, e.g., [11]) can be described with the help of integral and operator representations. Recall that, by definition, a function $F \in M\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ belongs to $C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ if the kernel C_F,

$$C_F(z,\zeta) := (1 - z\overline{\zeta})^{-1}\big(F(z) + F(\zeta)^*\big), \tag{0.3}$$

has κ negative squares. The class $C_0\big(\mathscr{L}(\mathscr{H})\big)$ coincides with $C\big(\mathscr{L}(\mathscr{H})\big)$.

A function $F \in M\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ holomorphic at 0 belongs to one of the classes $C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ if and only if (0.2) holds with the Hilbert space $\mathscr{K}$ replaced by a Pontryagin space. For more precise results, see [11].

In the present paper, we consider a class of operator functions which was introduced in [9]; here these functions will be called *definitizable*: A function $F \in M\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ is definitizable if it can be "definitized", which here means transformed into a Carathéodory function, by multiplying with a rational scalar function and by adding a rational operator function the poles of which are points of holomorphy of F (see Definition 1.1). In [9], $D_0\big(\mathscr{L}(\mathscr{H})\big)$ is the set of all definitizable $\mathscr{L}(\mathscr{H})$-valued functions which are holomorphic at zero. The latter condition is no restriction; it can always be fulfilled with the help of a linear fractional transformation of the argument.

The functions of all the classes $C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ are definitizable ([9, Proposition 2.5]). In [9], definitizable functions were characterized by representations (0.1) with the measure M replaced by a distribution T_F satisfying some sign properties (see [9, Proposition 2.2]). In the present paper we deal with operator representations (0.2) of definitizable functions where, in general, $\mathscr{K}$ is a Kreĭn space.

In [9] we considered operator functions $z \mapsto G(U+z)(U-z)^{-1}$ as examples of definitizable functions, where G is a bounded and boundedly invertible selfadjoint operator in a Hilbert space $\big(\mathscr{K}, (\,\cdot\,,\cdot\,)\big)$ and U is a definitizable unitary operator in the Kreĭn space $\big(\mathscr{K}, (G\,\cdot\,,\cdot\,)\big)$. It is not difficult to verify (Theorem 1.7 below) that for $\Gamma \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and selfadjoint $S \in \mathscr{L}(\mathscr{H})$ the function

$$z \longmapsto iS + \Gamma^* G(U+z)(U-z)^{-1}\Gamma \tag{0.4}$$

is again definitizable.

On the other hand, by a result of T. Ya. Azizov ([1]), every function $F \in M(\mathbf{T};\mathscr{L}(\mathscr{H}))$ which is holomorphic at 0 admits an operator representation (0.4) with a unitary but not necessarily definitizable operator U.

In Section 1 of the present paper we recall some definitions from [9] and introduce the notion of a regular critical point. We prove that for every definitizable function F in its representation (0.4), the operator U is definitizable if U fulfils a minimality condition. Relations between properties of F and of a minimal representing operator U are studied.

In Section 2 we deal with the problem to which extent the representing operator U is determined by the function F. It is well known that two minimal representing operators, U_1 and U_2, for the same definitizable function are weakly isomorphic (cf. [3] or Theorem 1.8). We give an example which shows that, in general, U_1 and U_2 may not be unitarily equivalent with respect to the Kreĭn space inner products.

In Section 3 the definitions and results are carried over by linear fractional transformations to the case of operator functions which are symmetric with respect to the real axis $\mathbf{R}$. These operator functions often appear in applications. For instance, $\mathbf{R}$-symmetric definitizable functions occur in the theory of operator polynomials, in connection with Sturm–Liouville problems with floating singularity and in the extension theory of symmetric operators in Kreĭn spaces (cf. Section 3.4).

Similar results can be proved for some class of operator functions which locally behave like definitizable functions. This case will be considered in a subsequent paper.

All definitions and results of the present paper can be carried over without difficulty to the case when the $\mathbf{T}$-antisymmetry of the operator functions F is understood with respect to a Kreĭn space inner product $[\,\cdot\,,\cdot\,]$ on $\mathscr{H}$ instead of the Hilbert space scalar product $(\,\cdot\,,\cdot\,)$. Indeed, if $G' \in \mathscr{L}(\mathscr{H})$ is the boundedly invertible operator defined by $[x,y] = (G'x,y)$, $x,y \in \mathscr{H}$, then the relation $F(\bar{z}^{-1}) = -F(z)^+$ where $F(z)^+$ denotes the adjoint of $F(z)$ with respect to $[\,\cdot\,,\cdot\,]$ is equivalent to $G'F(\bar{z}^{-1}) = -\big(G'F(z)\big)^*$. The same applies for the other properties which will be studied in the following, and for $\mathbf{R}$-symmetric operator functions.

1. Representations of definitizable functions skew symmetric with respect to the unit circle

1.1. *Notation and definitions.* If $\mathscr{S}$ is a subset of the extended complex plane $\overline{\mathbf{C}}$, we set $\widehat{\mathscr{S}} := \{z \in \overline{\mathbf{C}} : \bar{z}^{-1} \in \mathscr{S}\}$. For a scalar (or $\mathscr{L}(\mathscr{H})$-valued) function f defined on a set $\mathscr{D} = \widehat{\mathscr{D}} \subset \overline{\mathbf{C}}$, we define $\hat{f}(\mu) := \overline{f(\bar{\mu}^{-1})}$ ($\hat{f}(\mu) := f(\bar{\mu}^{-1})^*$), $\mu \in \mathscr{D}$. For a Banach space X, let $\mathscr{R}_{0,\infty}(X)$ denote the set of all functions $\mathbf{C} \ni z \longmapsto \sum_{j\in\mathbf{Z}} c_j z^j$ with $c_j \in X$, where the sum is finite, $\mathscr{R}_{0,\infty} := \mathscr{R}_{0,\infty}(\mathbf{C})$. By $\mathscr{R}^s_{0,\infty}$ we denote the set of all functions $g \in \mathscr{R}_{0,\infty}$ such that $g = \hat{g}$.

The linear space of locally holomorphic functions on a compact set $K \subset \mathbf{C}$ is denoted by $H(K)$. $H(K;\mathscr{H})$ denotes the analogous space of $\mathscr{H}$-valued functions. Similar notation is used for other spaces of vector-valued functions.

We recall the definitions of the analytic functionals T_F and $T_F(\,\cdot\,,\cdot\,)$ from [9]. Let $L = \hat{L}$ be a compact subset of $\mathbf{C}$ such that $\mathbf{T} \subset L$ and $\mathbf{D} \setminus L$ is connected. Then with every $\mathscr{L}(\mathscr{H})$-valued function F which is holomorphic in $(\mathbf{D} \setminus L) \cup (\widehat{\mathbf{D}} \setminus L)$ and satisfies $F(\bar{z}^{-1}) = -\big(F(z)\big)^*$, $z \in (\mathbf{D} \setminus L) \cup (\widehat{\mathbf{D}} \setminus L)$, we connect an analytic functional $T_F \in \mathscr{L}\big(H(L), \mathscr{L}(\mathscr{H})\big)$ (cf. [9]):

$$T_F.f := -\int_{\mathscr{C}} F(z) f(z)(iz)^{-1}\,dz, \qquad f \in H(L). \tag{1.1}$$

Here $\mathscr{C}$ is the oriented boundary of a finite union Ω of smooth bounded domains containing L such that f is defined on $\overline{\Omega}$ and $0 \notin \overline{\Omega}$. Moreover, for every f as in (1.1) and $u, v \in H(L, \mathscr{H})$ we define

$$T_F(u,v).f := -\int_{\mathscr{C}} \big(F(z)u(z), v(\bar{z}^{-1})\big) f(z)(iz)^{-1}\,dz, \tag{1.2}$$

where $\mathscr{C}$ is as above and such that u and v are defined on $\overline{\Omega}$. If $f = \hat{f}$ holds, the sesquilinear form $(u,v) \longmapsto T_F(u,v).f$ is hermitian. If F is a Carathéodory function and (0.1) holds, then T_F coincides with M regarded as an operator-valued Radon measure restricted to $H(\mathbf{T})$.

The set of all $z \in \overline{\mathbf{C}}$ such that $F \in M\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ can analytically be continued in z in a unique way is denoted by $\mathrm{P}(F)$. We set $\Sigma(F) := \overline{\mathbf{C}} \setminus \mathrm{P}(F)$. Note that in [9] the notation $\sigma(F)$ has been used instead of $\Sigma(F)$.

If $F \in M\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ is holomorphic at 0 and ∞, has only a finite number of poles outside $\mathbf{T}$ and there exists an $m \in \mathbf{N}$ such that

$$\sup\big\{ \|F(z)\| \big|1 - |z|\big|^m : |z| \in (\eta, 1) \cup (1, \eta^{-1}) \big\} < \infty \tag{1.3}$$

for some $\eta \in (0,1)$, then T_F can be extended by continuity to

$$C^{m+1}(\mathbf{T}) \times H\big(\Sigma(F) \setminus \mathbf{T}\big)$$

([9, Proposition 1.1]). Under the same assumptions on F, $T_F(\,\cdot\,,\cdot\,)$ can be extended by continuity to

$$\big(C^{m+1}(\mathbf{T}, \mathscr{H}) \times H\big(\Sigma(F) \setminus \mathbf{T}, \mathscr{H}\big)\big)^2 \times \big(C^{m+1}(\mathbf{T}) \times H\big(\Sigma(F) \setminus \mathbf{T}\big)\big)$$

([9, Proposition 1.2]). We use this extended form, as in [9], to introduce sign types of open subsets of $\mathbf{T}$ with respect to F.

Let first $\mathscr{L}$ be a linear space equipped with a hermitian sesquilinear form $[\,\cdot\,,\cdot\,]$. We denote by $\kappa_+\big((\mathscr{L}, [\,\cdot\,,\cdot\,])\big)$ $\big(\kappa_-\big((\mathscr{L}, [\,\cdot\,,\cdot\,])\big)\big)$ the least upper bound

$(\leq \infty)$ of the dimensions of the subspaces of $\mathscr{L}$ which are positive (resp., negative) definite with respect to $[\,\cdot\,,\cdot\,]$.

For F as above and $f \in C^\infty(\mathbf{T}) \times H\bigl(\Sigma(F) \setminus \mathbf{T}\bigr)$ with $f = \hat{f}$, we define

$$\kappa_\pm(f;F) := \kappa_\pm\bigl(\bigl(\mathscr{R}_{0,\infty}(\mathscr{H}), T_F(\,\cdot\,,\cdot\,).f\bigr)\bigr).$$

An open subset γ of $\mathbf{T}$ is said to be of *positive type* (*negative type*, *type* π_+ and *type* π_-) with respect to F if $\kappa_-(f;F) = 0$ ($\kappa_+(f;F) = 0$, $\kappa_-(f;F) < \infty$, $\kappa_+(f;F) < \infty$) for all nonnegative functions $f \in C^\infty(\mathbf{T}) \times H\bigl(\Sigma(F) \setminus \mathbf{T}\bigr)$ with $\operatorname{supp} f \subset \gamma$. We say that γ is of *definite type* (*type* π), if it is of positive or of negative type (resp., type π_+ or type π_-). A point $\alpha \in \mathbf{T}$ is called a *critical point* (an *essential critical point*) with respect to F if α is not contained in an open subset of $\mathbf{T}$ of definite type (resp., type π) with respect to F; we write $\alpha \in \mathrm{K}(F)$ (resp., $\alpha \in \mathrm{K}_\infty(F)$).

Definition 1.1. A function $F \in M\bigl(\mathbf{T};\mathscr{L}(\mathscr{H})\bigr)$ is called *definitizable* if there exists a scalar rational function g such that gF is the sum of a Carathéodory function H, and an $\mathscr{L}(\mathscr{H})$-valued rational function h the poles of which belong to $\mathrm{P}(F)$:

$$g(z)F(z) = H(z) + h(z) \tag{1.4}$$

for all points $z \in \overline{\mathbf{C}} \setminus \mathbf{T}$ of holomorphy of gF. A rational operator function is by definition a meromorphic operator function in $\overline{\mathbf{C}}$. A function g with the property mentioned above is called a *definitizing* function for F.

Evidently, a definitizable function F has only a finite number of poles outside $\mathbf{T}$ and (1.3) holds for some $m \in \mathbf{N}$. There is no loss of generality in assuming that the definitizing function g is $\mathbf{T}$-symmetric, $g = \hat{g}$. Indeed, if (1.4) holds, then

$$\hat{g}(z)F(z) = -\bigl(g(\bar{z}^{-1})F(\bar{z}^{-1})\bigr)^* = -H(\bar{z}^{-1})^* - h(\bar{z}^{-1})^* = H(z) - h(\bar{z}^{-1})^*.$$

Hence $\frac{1}{2}(g + \hat{g}) = \frac{1}{2}(g + \hat{g})\hat{\ }$ is definitizing.

The following lemma shows, in particular, that a function $F \in M\bigl(\mathbf{T};\mathscr{L}(\mathscr{H})\bigr)$ which satisfies the conditions of the above definition except the requirement that the poles of h belong to $\mathrm{P}(F)$ is still definitizable. In this case, g is not necessarily definitizing for F in the sense of Definition 1.1. Moreover, the subsequent lemma will show that Definition 1.1 can equivalently be formulated with the help of special definitizing functions g, see the definition in [9].

Lemma 1.2. *Besides $\mathscr{H}$ let a further Hilbert space $\mathscr{H}'$ be given. Let $H \in M\bigl(\mathbf{T};\mathscr{L}(\mathscr{H}')\bigr)$ be a Carathéodory function, q a scalar rational function, $q = \hat{q}$, k an $\mathscr{L}(\mathscr{H},\mathscr{H}')$-valued rational function and h an $\mathscr{L}(\mathscr{H})$-valued rational function, $h = -\hat{h}$. Then the operator function F defined by*

$$F(z) := q(z)k(\bar{z}^{-1})^* H(z)k(z) + h(z)$$

for all $z \in \overline{\mathbf{C}}$ *where* q, H, k *and* h *are holomorphic, is definitizable. Moreover, if* $z_0 \in \overline{\mathbf{D}}$ *is a point of holomorphy of* F, *there exists a definitizing function* $g = \hat{g}$ *of* F *with the following properties*:

(i) g *has poles at most at* z_0 *and* $\bar{z}_0^{-1}$.

(ii) *There is a decomposition* $gF = H_0 + h_0$ *with the properties mentioned in Definition* 1.1 *such that, in addition,* $\mathrm{P}(F) \subset \mathrm{P}(H_0)$ *and* h_0 *has poles at most at* z_0 *and* $\bar{z}_0^{-1}$.

Proof. We assume that z_0 as in the lemma is not zero. This is no restriction. Let α_j, $j = 1, \ldots, j_0$, be the poles of the functions $q(z)^{-1}$, $k(\bar{z}^{-1})^*$, $k(z)$ and $q(z)^{-1}h(z)$ and let ν_j be the sum of the orders of α_j as a pole of these four functions, $\nu := \sum_{j=1}^{j_0} \nu_j$. We set

$$p(z) := \left(\prod_{j=1}^{j_0} (z^{-1} - \bar{\alpha}_j)^{\nu_j} (z - \alpha_j)^{\nu_j}\right)(z^{-1} - \bar{z}_0)^{-\nu}(z - z_0)^{-\nu}.$$

Then $g := pq^{-1}$ has poles at most at z_0 and $\bar{z}_0^{-1}$, and we have

$$g(z)F(z) = p(z)k(\bar{z}^{-1})^* H(z)k(z) + p(z)q(z)^{-1}h(z). \tag{1.5}$$

If $z_0 \in \mathbf{T}$, then z_0 is no accumulation point of $\Sigma(H)$, and we denote by γ a closed arc of $\mathbf{T}$ with $z_0 \notin \gamma$ and $\Sigma(H) \setminus \{z_0\} \subset \gamma$. If $z_0 \notin \mathbf{T}$, we set $\gamma = \mathbf{T}$. Outside $\gamma \cup \{z_0, \bar{z}_0^{-1}\}$, the function $p(z)k(\bar{z}^{-1})^* H(z)k(z)$, which belongs to $M\bigl(\mathbf{T}; \mathscr{L}(\mathscr{H})\bigr)$, is holomorphic. We decompose this function,

$$p(z)k(\bar{z}^{-1})^* H(z)k(z) = H_0(z) + \tilde{h}_0(z) \tag{1.6}$$

such that H_0, $\tilde{h}_0 \in M\bigl(\mathbf{T}; \mathscr{L}(\mathscr{H})\bigr)$, $\Sigma(H_0) \subset \gamma$, $\Sigma(\tilde{h}_0) \subset \{z_0, \bar{z}_0^{-1}\}$ (see, e.g., [9, Section 1.3]). By (1.5) and (1.6) and since $\tilde{h}_0$ and $pq^{-1}h$ are rational operator functions which have no poles in $\overline{\mathbf{C}} \setminus \{z_0, \bar{z}_0^{-1}\}$, it remains to verify that $H_0 \in C\bigl(\mathscr{L}(\mathscr{H})\bigr)$ and $\mathrm{P}(F) \subset \mathrm{P}(H_0)$. The latter relation follows from (1.5), (1.6) and the construction of H_0.

We choose $\varepsilon > 0$ so that $z_0 \notin \{z \in \overline{\mathbf{D}} : \operatorname{dist}(z, \gamma) \le \varepsilon\} =: \mathscr{U}_1$. Let $\mathscr{C} := \partial(\mathscr{U}_1 \cup \widehat{\mathscr{U}}_1)$. If $u \in \mathscr{R}_{0,\infty}(\mathscr{H})$, then

$$\begin{aligned} T_{H_0}(u,u).\mathbf{1} &= -\int_{\mathscr{C}} \bigl(H_0(\zeta)u(\zeta), u(\bar{\zeta}^{-1})\bigr)_{\mathscr{H}} (i\zeta)^{-1}\, d\zeta \\ &= -\int_{\mathscr{C}} p(\zeta)\bigl(H(\zeta)k(\zeta)u(\zeta), k(\bar{\zeta}^{-1})u(\bar{\zeta}^{-1})\bigr)_{\mathscr{H}'} (i\zeta)^{-1}\, d\zeta \\ &= T_H(v,v).\mathbf{1} \ge 0. \end{aligned}$$

Here v is a function coinciding with $\bigl(\prod_{j=1}^{j_0}(z-\alpha_j)^{\nu_j}\bigr)(z-z_0)^{-\nu}k(z)u(z)$ in a neighbourhood of $\mathscr{U}_1 \cup \widehat{\mathscr{U}}_1$ and equal to zero in a neighbourhood of $\{z_0, \bar{z}_0^{-1}\}$. Hence H_0 is a Carathéodory function (see [9, relation (1.6)]).

Remark 1.3. Evidently, the assertions of Lemma 1.2 remain true if $H \in M\big(\mathbf{T};\mathscr{L}(\mathscr{H}')\big)$ is a definitizable function.

The following lemma can be proved in the same way as Lemma 1.2.

Lemma 1.4. *Let $F \in M\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ be definitizable. Let $g = \hat{g}$ be a scalar rational function such that its poles α_j, $j = 1,\dots,j_0$, in $\overline{\mathbf{D}}$ belong to $\mathrm{P}(F)$. Denote by ν_j the order of the pole α_j, $\nu := \sum_{j=1}^{j_0}\nu_j$. If $\beta_k \in \overline{\mathbf{D}}\cap\mathrm{P}(F)$, $\mu_k \in \mathbf{N}$, $k = 1,\dots,k_0$, $\sum_{k=1}^{k_0}\mu_k = \nu$ and*

$$p(z) := \left(\prod_{j=1}^{j_0}(z^{-1}-\bar{\alpha}_j)^{\nu_j}(z-\alpha_j)^{\nu_j}\right)\times\left(\prod_{k=1}^{k_0}(z^{-1}-\bar{\beta}_k)^{\mu_k}(z-\beta_k)^{\mu_k}\right)^{-1},$$

then g is definitizing for F if and only if pg is definitizing for F.

In the following we mostly consider the class $D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ (in [9] denoted by $D_0\big(\mathscr{L}(\mathscr{H})\big)$) of those definitizable functions which are holomorphic at 0 and ∞. This involves no loss of generality (see Introduction). By Lemma 1.2, for $F \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$, there is always a definitizing function $g \in \mathscr{R}^s_{0,\infty}$.

Simple examples of definitizable functions are the differences $F_1 - F_2$ of two Carathéodory functions F_1, F_2 with $\Sigma(F_1)\cap\Sigma(F_2) = \emptyset$. By [9, Proposition 2.2], a definitizable function F is of this form if and only if it is holomorphic in $\overline{\mathbf{C}}\setminus\mathbf{T}$, i.e., $\Sigma(F)\subset\mathbf{T}$, and has no critical points, i.e., $\mathrm{K}(F) = \emptyset$.

Similarly, if $F_i \in C_{\kappa_i}\big(\mathscr{L}(\mathscr{H})\big)$, κ_i is a nonnegative integer, $i = 1,2$, and $\Sigma(F_1)\cap\Sigma(F_2) = \emptyset$, then $F_1 - F_2$ is definitizable. It is easy to verify that a definitizable function F is of this form if and only if all poles of F outside $\mathbf{T}$ have finite multiplicity (see [9, Lemma 1.5]) and F has no essential critical points, i.e., $\mathrm{K}_\infty(F) = \emptyset$.

1.2. *Poles and regular critical points of definitizable functions.* Let $F \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$, $\alpha \in \Sigma(F)\cap\mathbf{T}$, and assume that there is an open arc γ containing α such that $\gamma\setminus\{\alpha\}\subset\mathrm{P}(F)$. Then, by Definition 1.1, α is a pole of F. If k is the order of the pole α, then the operator function

$$\lambda \longmapsto iF\left(-\alpha\frac{\lambda-i}{\lambda+i}\right) =: G(\lambda)$$

which is holomorphic in $\{\lambda : 0 < |\lambda| < \rho\}$ for sufficiently small ρ has a pole of order k at 0. Let

$$G_{-k}\lambda^{-k} + \cdots + G_{-1}\lambda^{-1}$$

be the principal part of G at 0. Then, by the skew-symmetry of F, G_{-i}, $i = 1,\dots,k$, are bounded selfadjoint operators in $\mathscr{H}$. Let $\widetilde{G}$ denote the selfadjoint operator

$$\begin{pmatrix} G_{-1} & G_{-2} & \cdots & G_{-k} \\ G_{-2} & G_{-3} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ G_{-k} & 0 & \cdots & 0 \end{pmatrix}$$

in $\mathscr{H}^k$. In the following proposition we characterize some sign properties of the pole with the help of the operator $\widetilde{G}$.

Proposition 1.5. *Let F, α, γ, $\widetilde{G}$ be as above. Then the multiplicity of the pole α of F is equal to $\dim \mathscr{R}(\widetilde{G})$. If $f \in C^\infty(\mathbf{T}) \times H\big(\Sigma(F) \setminus \mathbf{T}\big)$, $\operatorname{supp} f \subset \gamma$, and f is identically equal to one in some neighbourhood of α, then*

$$\kappa_+(f;F) = \dim \mathscr{R}\big(E\big((-\infty,0);\widetilde{G}\big)\big), \qquad \kappa_-(f;F) = \dim \mathscr{R}\big(E\big((0,\infty);\widetilde{G}\big)\big),$$

where $E(\,\cdot\,;\widetilde{G})$ is the spectral function of $\widetilde{G}$.

Proof. The first statement is a well-known fact. If $C_{\rho/2} := \{\lambda : |\lambda| = \rho/2\}$ and $p(\lambda) = p_0 + p_1\lambda + \cdots + p_{k-1}\lambda^{k-1}$, $p_0,\dots,p_k \in \mathscr{H}$, then we have

$$\begin{aligned}
&-4\pi \left(\begin{pmatrix} G_{-1} & G_{-2} & \cdots & G_{-k} \\ G_{-2} & G_{-3} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ G_{-k} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix}, \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} \right)_{\mathscr{H}^k} \\
&\quad = -4\pi(2\pi i)^{-1} \int_{C_{\rho/2}} \lambda^{-1} \big\{ G_{-1}(p_0,p_0) + G_{-2}\big((p_0,p_1)+(p_1,p_0)\big) \\
&\qquad \cdots + G_{-k}\big((p_0,p_{k-1}) + \cdots + (p_{k-1},p_0)\big)\big\}\, d\lambda \\
&\quad = 2i \int_{C_{\rho/2}} \big(G(\lambda)p(\lambda), p(\bar\lambda)\big)\, d\lambda.
\end{aligned} \tag{1.7}$$

With the substitution

$$\lambda = i\frac{1 - z\alpha^{-1}}{1 + z\alpha^{-1}}$$

we get

$$\begin{aligned}
&2i \int_{C_{\rho/2}} \big(G(\lambda)p(\lambda), p(\bar\lambda)\big)\, d\lambda \\
&\quad = -4 \int_{\mathscr{C}} \left(F(z) \frac{1}{1 + z\alpha^{-1}} p\Big(i\frac{1 - z\alpha^{-1}}{1 + z\alpha^{-1}}\Big), \frac{1}{1 + \bar z^{-1}\alpha^{-1}} p\Big(i\frac{1 - \bar z^{-1}\alpha^{-1}}{1 + \bar z^{-1}\alpha^{-1}}\Big) \right) \frac{dz}{iz},
\end{aligned} \tag{1.8}$$

where $\mathscr{C}$ is a circle in $\mathrm{P}(F)$ such that the interior domain corresponding to $\mathscr{C}$ intersects $\Sigma(F)$ at the point α. Relations (1.7) and (1.8) imply the proposition.

If α and γ are as in Proposition 1.5, then we define the *sign type of the pole* α to be the sign type of γ with respect to F.

A critical point λ of $F \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ is called *regular*, if there exists an open arc γ, $\lambda \in \gamma$, such that T_F restricted to $\gamma \setminus \{\lambda\}$ is a bounded operator measure. The set of all regular critical points of F is denoted by $\mathrm{K}_r(F)$. A critical point which is not regular is called *singular*. We set $\mathrm{K}_s(F) := \mathrm{K}(F) \setminus \mathrm{K}_r(F)$. The following proposition will be needed below for the construction of a special operator representation.

Proposition 1.6. *Let $e^{i\Theta_0} \in \mathrm{K}(F)$ for some $\Theta_0 \in \mathbf{R}$ and let ε, $0 < \varepsilon < \pi$, be chosen so that the arcs $\{e^{i\Theta} : \Theta \in (\Theta_0 - \varepsilon, \Theta_0)\}$ and $\{e^{i\Theta} : \Theta \in (\Theta_0, \Theta_0 + \varepsilon)\}$ are of definite type with respect to F. Let $\gamma_1 := \{e^{i\Theta} : \Theta \in (\Theta_0 - \varepsilon_1, \Theta_0)\}$, $\gamma_2 := \{e^{i\Theta} : \Theta \in (\Theta_0, \Theta_0 + \varepsilon_2)\}$ for some ε_1, ε_2, $0 < \varepsilon_1, \varepsilon_2 < \varepsilon$. Then the following statements are equivalent.*

(1) *$e^{i\Theta_0} \in \mathrm{K}_r(F)$.*

(2) *There exists a decomposition of F,*

$$F = \sum_{i=0}^{3} F_i, \qquad F_i \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big),\ i = 0,1,2,3,$$

with the following properties.

(i) $\Sigma(F_0) = \{e^{i\Theta_0}\}$, $\Sigma(F_1) \subset \overline{\gamma_1}$, $\Sigma(F_2) \subset \overline{\gamma_2}$, $\Sigma(F_3) \cap (\gamma_1 \cup \{e^{i\Theta_0}\} \cup \gamma_2) = \emptyset$.

(ii) *$F_i \in C\big(\mathscr{L}(\mathscr{H})\big)$ or $-F_i \in C\big(\mathscr{L}(\mathscr{H})\big)$, $i = 1,2$. For arbitrary $x \in \mathscr{H}$, the angular limits*

$$\begin{aligned}
&\widehat{\lim}_{z \to e^{i\Theta_0}} (z - e^{i\Theta_0})(F_1(z)x, x),\\
&\widehat{\lim}_{z \to e^{i\Theta_0}} (z - e^{i\Theta_0})\big(F_2(z)x, x\big),\\
&\widehat{\lim}_{z \to e^{i(\Theta_0 - \varepsilon_1)}} (z - e^{i(\Theta_0 - \varepsilon_1)})\big(F_1(z)x, x\big),\\
&\widehat{\lim}_{z \to e^{i(\Theta_0 + \varepsilon_2)}} (z - e^{i(\Theta_0 + \varepsilon_2)})\big(F_2(z)x, x\big)
\end{aligned}$$

are zero.

If (2) holds, the functions F_i are unique up to terms of the form iS, S selfadjoint, and the set of functions $\{F_0(z) + iS\}$ does not depend on the special choice of γ_1 and γ_2.

Proof. Assume that (1) holds. Let $(\chi_{1,n})$, $(\chi_{2,n})$ be sequences of nonnegative functions in $C^\infty(\mathbf{T}) \times H\big(\Sigma(F) \setminus \mathbf{T}\big)$, $\operatorname{supp} \chi_{1,n} \subset \gamma_1$, $\operatorname{supp} \chi_{2,n} \subset \gamma_2$ with the following properties: for any compact subsets $K_1 \subset \gamma_1$, $K_2 \subset \gamma_2$, there exists an N such that for $n \geq N$, $\chi_{1,n}$ is equal to 1 on K_1 and $\chi_{2,n}$ is equal to 1 on K_2, and $(\chi_{1,n})$ and $(\chi_{2,n})$ converge monotonically increasing to 1 on γ_1 and γ_2, respectively. Then the operator functional T_i, $i = 1,2$, defined by

$$T_i.f := \lim_{n\to\infty} T_F.\chi_{i,n} f, \qquad f \in C^\infty(\mathbf{T}) \times H\big(\Sigma(F) \setminus \mathbf{T}\big),$$

is a positive or a negative operator measure with support contained in $\overline{\gamma_i}$. For arbitrary $x \in \mathscr{H}$, the measure $\big(T_i(\,\cdot\,)x, x\big)$, $i = 1, 2$, has no masses at the endpoints of γ_1 and γ_2. Let $\chi_0 \in C^\infty(\mathbf{T}) \times H\big(\Sigma(F)\setminus\mathbf{T}\big)$ with $\operatorname{supp}\chi_0 \subset \gamma_1 \cup \{e^{i\Theta_0}\} \cup \gamma_2$ such that χ_0 is identically equal to one in some neighbourhood of $e^{i\Theta_0}$ and define $T_0 := \chi_0(T_F - T_1 - T_2)$, $T_3 := T_F - T_0 - T_1 - T_2$. If we set $F_i(\lambda) := T_i.f_\lambda - (4\pi)^{-1}T_i.\mathbf{1}$, $i = 0, 1, 2, 3$, we get (cf. [9, Section 1.3])

$$F(\lambda) - i \operatorname{Im} F(0) = \sum_{i=0}^{3} F_i(\lambda).$$

This decomposition of F has the properties required in (2). It is easy to see that (2) implies (1).

We note that the second part of (ii) is equivalent to the fact that the measures $\big(T_{F_i}(\,\cdot\,)x, x\big)$, $i = 1, 2$, have no masses at the endpoints of γ_1 and γ_2. Then, if (2) holds, the uniqueness property of the functions F_i, $i = 0, 1, 2, 3$, follows from the fact that the corresponding functionals T_{F_i} are uniquely determined. This proves Proposition 1.6.

We shall say that the critical point $e^{i\Theta_0}$ of F *has a mass of positive type*, if the pole $e^{i\Theta_0}$ of F_0 is of positive type. It is the same for the other sign types.

1.3. *Definitizable functions defined by operators.* We now consider, besides $\mathscr{H}$, a further Hilbert space $\mathscr{K}$ and denote the scalar product in $\mathscr{K}$ by $(\,\cdot\,,\cdot\,)$, in the same way as in $\mathscr{H}$. Let W be a bounded selfadjoint operator in $\mathscr{K}$, let $U \in \mathscr{L}(\mathscr{K})$, $0 \in \rho(U)$ and $(WUx, Uy) = (Wx, y)$ for all $x, y \in \mathscr{K}$. If S is a selfadjoint bounded operator in $\mathscr{H}$ and $\Gamma \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and $\mathscr{U}$ is a domain in $\mathbf{C}$ such that $0 \in \mathscr{U}$ and $\mathscr{U} \cup \widehat{\mathscr{U}} \subset \rho(U) \cup \{\infty\}$, then the function F defined by

$$F(z) = iS + \Gamma^* W(U + z)(U - z)^{-1}\Gamma, \qquad z \in \mathscr{U} \cup \widehat{\mathscr{U}}, \tag{1.9}$$

satisfies the relation $F(z) = -F(\bar{z}^{-1})^*$. Indeed, from the relations $U^*W = WU^{-1}$, $(U^*)^{-1}W = WU$, by expansions of the resolvent of U at 0 and ∞ and analytic continuation, it follows that

$$(U^* - \bar{z})^{-1}W = W(U^{-1} - z)^{-1}, \qquad z \in \mathscr{U} \cup \widehat{\mathscr{U}}, \tag{1.10}$$

which implies the skew-symmetry of F.

We give some relations connecting F and U, which will be used later on. (1.9) and $U^*W = WU^{-1}$ imply the following identities:

$$\begin{aligned} 2 \operatorname{Re} F(0) &= 2\Gamma^* W \Gamma, \\ (k!)^{-1} F^{(k)}(0) &= 2\Gamma^* W U^{-k} \Gamma, \qquad k \in \mathbf{N}, \\ (k!)^{-1} \big(F^{(k)}(0)\big)^* &= 2\Gamma^* W U^{k} \Gamma, \qquad k \in \mathbf{N}. \end{aligned}$$

Then, as in relation [9, (1.21)], we verify that

$$4\pi\Gamma^* W U^l \Gamma = T_F.\dot{\lambda}^l, \qquad l \in \mathbf{Z},$$

where $\dot{\lambda}^l$ denotes the function $\lambda \longmapsto \lambda^l$, and hence,

$$4\pi\Gamma^* W g(U)\Gamma = T_F.g, \qquad g \in \mathscr{R}_{0,\infty}. \tag{1.11}$$

Therefore, if $x_{-n}, \dots, x_n \in \mathscr{H}$ and $g \in \mathscr{R}_{0,\infty}$, we find

$$\begin{aligned}(4\pi)^{-1} T_F\Big(\sum_{j=-n}^{n} x_j \dot{\lambda}^j, \sum_{k=-n}^{n} x_k \dot{\lambda}^k\Big).g &= (4\pi)^{-1} \sum_{j,k=-n}^{n} \big((T_F.g\dot{\lambda}^{j-k})x_j, x_k\big) \\ &= \sum_{j,k=-n}^{n} \big(\Gamma^* W g(U) U^{j-k}\Gamma x_j, x_k\big) \\ &= \big(Wg(U)(U^{-n}\Gamma x_{-n} + \dots + U^n\Gamma x_n\big), U^{-n}\Gamma x_{-n} + \dots + U^n \Gamma x_n).\end{aligned} \tag{1.12}$$

With the help of (1.12) and [9, relation (1.6)] one could easily verify the following theorem (cf. proof of Theorem 1.9), but we will give a short direct proof.

Theorem 1.7. *Let $\mathscr{K}$, W, U, S, Γ and $\mathscr{U}$ be as above and let F be the $\mathscr{L}(\mathscr{H})$-valued function defined by (1.9). Then the following holds.*

(i) *If for some rational function $g = \hat{g}$ the poles of which belong to $\mathscr{U} \cup \widehat{\mathscr{U}}$ we have $\big(Wg(U)x, x\big) \geq 0$ for all $x \in \mathscr{K}$, then F has an analytic continuation to a definitizable function and g is a definitizing function for F.*

(ii) *If for some rational function $g = \hat{g}$ the poles of which belong to $\mathscr{U} \cup \widehat{\mathscr{U}}$ the hermitian form $\big(Wg(U)\cdot, \cdot\big)$ on $\mathscr{K}$ has κ negative squares, then F has an analytic continuation to a definitizable function. Moreover, there exist a function $H \in C_{\kappa'}\big(\mathscr{L}(\mathscr{H})\big)$, $0 \leq \kappa' \leq \kappa$, holomorphic in $\mathscr{U} \cup \widehat{\mathscr{U}}$, and a rational operator function h holomorphic at all points where g is holomorphic such that*

$$g(z)F(z) = H(z) + h(z)$$

for all points z of holomorphy of g, F, H and h.

Proof. Assertion (i) is a consequence of (ii). In order to prove (ii), assume that $\big(Wg(U)\cdot, \cdot\big)$ has κ negative squares. We have

$$\begin{aligned} g(z)F(z) = ig(z)S &+ \Gamma^* W g(U)(U+z)(U-z)^{-1}\Gamma \\ &- \Gamma^* W\big(g(U) - g(z)\big)(U+z)(U-z)^{-1}\Gamma. \end{aligned} \tag{1.13}$$

Since the first and the third term on the right of (1.13) are rational operator functions holomorphic at all points where g is holomorphic, it remains to verify that the function

$$H(z) := \Gamma^* W g(U)(U+z)(U-z)^{-1}\Gamma$$

belongs to $C_{\kappa'}\big(\mathscr{L}(\mathscr{H})\big)$ for some κ', $0 \le \kappa' \le \kappa$. From (1.10) it follows

$$H(\zeta)^* = \Gamma^* W g(U)(U^{-1} - \bar\zeta)^{-1}(U^{-1} + \zeta)\Gamma.$$

If $x, y \in \mathscr{H}$, $z, \zeta \in \mathscr{U} \cup \widehat{\mathscr{U}}$, a simple computation gives

$$\begin{aligned}(1 - z\bar\zeta)^{-1}\big(\big(H(z) + H(\zeta)^*\big)x, y\big) &= (1 - z\bar\zeta)^{-1}\big(W g(U)\big\{(U + z)(U - z)^{-1} \\ &\qquad + (U^{-1} + \bar\zeta)(U^{-1} - \bar\zeta)^{-1}\big\}\Gamma x, \Gamma y\big) \\ &= 2\big(W g(U)(U - z)^{-1}(U^{-1} - \bar\zeta)^{-1}\Gamma x, \Gamma y\big) \\ &= 2\big(W g(U)(U - z)^{-1}\Gamma x, (U - \zeta)^{-1}\Gamma y\big).\end{aligned}$$

By assumption, this kernel has no more than κ negative squares and Theorem 1.7 is proved.

If $0 \in \rho(W)$, then $(\mathscr{K}, [\,\cdot\,,\cdot\,])$ with $[\,\cdot\,,\cdot\,] := (W\,\cdot\,,\cdot\,)$ is a Kreĭn space, U is unitary in this Kreĭn space, and in the above formulas $\Gamma^* W$ can be replaced by the Kreĭn space adjoint Γ^+.

We recall that a unitary operator U in a Kreĭn space $(\mathscr{K}, [\,\cdot\,,\cdot\,])$ is called *definitizable* if there exists a rational function $g = \hat g$ whose poles belong to $\rho(U) \cup \{\infty\}$ such that $[g(U)x, x] \ge 0$, $x \in \mathscr{K}$. A function g with this property is called a *definitizing* function for U. With this notation, Theorem 1.7(i) says that every definitizing function for U is also definitizing for $F(z) = iS + \Gamma^+(U+z)(U-z)^{-1}\Gamma$. It is well known that if for some rational function $g_0 = \hat g_0$ the form $[g_0(U)\,\cdot\,,\cdot\,]$ has a finite number of negative squares on $\mathscr{K}$, then U is definitizable.

In the following subsections we shall make use of the spectral function $E(\,\cdot\,;U)$ and the notion of the critical point of a definitizable unitary operator U (see [14], [7]). We denote by $c(U)$ the set of critical points of U.

1.4. *Operator representations of a given definitizable function.* In this section we consider operator representations of a given definitizable function. First we recall T. Azizov's result mentioned in the introduction, in a form convenient for our purpose. If T is a closed operator or a closed linear relation in a Kreĭn space $\mathscr{K}$ with $\rho(T) \ne \emptyset$, $\mathscr{R}$ is a class of functions which are locally holomorphic on the spectrum of T, and $\mathscr{M}$ is a subset of $\mathscr{K}$, then we define

$$\mathscr{K}(T, \mathscr{R}, \mathscr{M}) := \operatorname{sp}\{g(T)x : g \in \mathscr{R},\ x \in \mathscr{M}\}$$

and

$$\overline{\mathscr{K}}(T, \mathscr{R}, \mathscr{M}) := \overline{\mathscr{K}(T, \mathscr{R}, \mathscr{M})}.$$

Theorem 1.8 ([1], see also [3]). *Let F be an $\mathscr{L}(\mathscr{H})$-valued function holomorphic on $\mathbf{D}_r \cup \widehat{\mathbf{D}}_r$, $\mathbf{D}_r := \{z : |z| < r\}$, for some $r \in (0,1)$ such that*

$F(z) = -F(\bar{z}^{-1})^*$, $z \in \mathbf{D}_r \cup \widehat{\mathbf{D}}_r$. *Then there exist a Kreĭn space* $\mathscr{K}$, *a unitary operator* U *in* $\mathscr{K}$, $\mathbf{D}_r \cup \widehat{\mathbf{D}}_r \subset \rho(U) \cup \{\infty\}$, *and* $\Gamma \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ *such that*

$$F(z) = i \operatorname{Im} F(0) + \Gamma^+(U + z)(U - z)^{-1}\Gamma, \qquad z \in \mathbf{D}_r \cup \widehat{\mathbf{D}}_r, \tag{1.14}$$

and

$$\mathscr{K} = \overline{\mathscr{K}}(U, \mathscr{R}_{0,\infty}, \Gamma\mathscr{H}). \tag{1.15}$$

If F *admits two such representations with Kreĭn spaces* $(\mathscr{K}_j, [\,\cdot\,,\cdot\,]_j)$, *operators* $\Gamma_j \in \mathscr{L}(\mathscr{H}, \mathscr{K}_j)$ *and unitary operators* $U_j \in \mathscr{L}(\mathscr{K}_j)$, $j = 1, 2$, *then*

$$V\colon \sum_{i=1}^{m} f_i(U_1)\Gamma_1 x_i \longmapsto \sum_{i=1}^{m} f_i(U_2)\Gamma_2 x_i, \tag{1.16}$$

$x_i \in \mathscr{H}$, $f_i \in \mathscr{R}_{0,\infty}$, $i = 1, \ldots, m$, *defines a linear and isometric mapping of the dense linear set* $\mathscr{K}_1(U_1, \mathscr{R}_{0,\infty}, \Gamma_1\mathscr{H})$ *in* $\mathscr{K}_1$ *onto the dense linear set* $\mathscr{K}_2(U_2, \mathscr{R}_{0,\infty}, \Gamma_2\mathscr{H})$ *in* $\mathscr{K}_2$ *such that*

$$U_2 V y = V U_1 y, \qquad y \in \mathscr{K}_1(U_1, \mathscr{R}_{0,\infty}, \Gamma_1\mathscr{H}).$$

The relation (1.15) is called a *minimality* condition. The last relation can easily be verified with the help of (1.14). It is called the *weak isomorphy* of the two minimal representing operators U_1 and U_2 (cf. [3]).

Theorem 1.9. *Let* $F \in D_0\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ *and let, for some* $r \in (0, 1)$,

$$F(z) = i \operatorname{Im} F(0) + \Gamma^+(U + z)(U - z)^{-1}\Gamma, \qquad z \in \mathbf{D}_r \cup \widehat{\mathbf{D}}_r,$$

be a minimal representation of F *as in Theorem 1.8. Then the unitary operator* U *in* $\mathscr{K}$ *is definitizable and we have* $\rho(U) \cup \{\infty\} = \mathrm{P}(F)$. *Moreover, the following holds*:

(i) *Every definitizing function* $g \in \mathscr{R}^s_{0,\infty}$ *for* F *is definitizing also for* U.

(ii) *If, for some* $g \in \mathscr{R}^s_{0,\infty}$,

$$g(z)F(z) = H(z) + h(z)$$

where $H \in C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ *is holomorphic at* 0 *and* $h \in \mathscr{R}_{0,\infty}\big(\mathscr{L}(\mathscr{H})\big)$ *and equality holds for all those* $z \in \overline{\mathbf{C}} \setminus \mathbf{T}$ *where* g, F, H *and* h *are holomorphic, then* $g(U)$ *has* κ *negative squares.*

Proof. Let $g \in \mathscr{R}^s_{0,\infty}$ and H be as in assertion (ii). Since $T_H(\,\cdot\,,\cdot\,).\mathbf{1} = T_F(\,\cdot\,,\cdot\,).g$ has κ negative squares (cf. [9, relation (1.6)]), it follows with the help of (1.12) that $[g(U)\,\cdot\,,\cdot\,]$ has κ negative squares on $\mathscr{K}(U,\mathscr{R}_{0,\infty},\Gamma\mathscr{H})$. Therefore, $g(U)$ has κ negative squares. With $\kappa = 0$, we get that U is definitizable and assertion (i).

If $\mu \in \rho(U) \cup \{\infty\}$, it follows by the definition of $\mathrm{P}(F)$ that $\mu \in \mathrm{P}(F)$. It remains to verify that $\mathrm{P}(F) \subset \rho(U) \cup \{\infty\}$. By (1.12) we have

$$\begin{aligned}(1.17)\qquad &(4\pi)^{-1} \sum_{j,k=-n}^{n} \big((T_F.f\dot{\lambda}^{j-k})x_j, x_k\big) \\ &\quad = [f(U)(U^{-n}\Gamma x_{-n} + \cdots + U^n\Gamma x_n), U^{-n}\Gamma x_{-n} + \cdots + U^n\Gamma x_n]\end{aligned}$$

for all $f \in \mathscr{R}_{0,\infty}$, $x_i \in \mathscr{H}$, $i = -n,\dots,n$. Let $\mu \in \mathrm{P}(F)\setminus(\mathbf{T}\cup\{0\}\cup\{\infty\})$ and let (f_n) be a sequence in $\mathscr{R}_{0,\infty}$ converging uniformly to 1 in some neighbourhood of μ and to 0 in some neighbourhood of $\Sigma(F)\cup\big(\sigma(U)\setminus\{\mu\}\big)$. Then $f_n(U)$ converges with respect to the operator norm. By (1.17) the limit is zero, hence $\mu \in \rho(U)$.

Let $\mu \in \mathrm{P}(F)\cap\mathbf{T}$ and let γ be an open arc of $\mathbf{T}$ such that $\mu \in \gamma$, $\bar{\gamma} \subset \mathrm{P}(F)$ and the endpoints of γ do not belong to $c(U)$. Then there exists a sequence (f_n) in $\mathscr{R}_{0,\infty}$ such that (f_n) converges uniformly to 0 in a neighbourhood of $\Sigma(F)$ and $\big(f_n(U)\big)$ converges strongly to $E(\gamma;U)$. By (1.17) we have $E(\gamma;U) = 0$ and, hence, $\mu \in \rho(U)$, which proves Theorem 1.9.

Remark 1.10. Under the assumptions of Theorem 1.9, a rational function $g = \hat{g}$ the poles of which belong to $\mathrm{P}(F) = \rho(U)\cup\{\infty\}$ is definitizing for F if and only if it is definitizing for U. This follows from Theorems 1.7 and 1.9 and Lemma 1.4.

Remark 1.11. If J is a fundamental symmetry of the Kreĭn space $\mathscr{K}$ and U is a unitary operator in $\mathscr{K}$, then, as a consequence of Theorems 1.7 and 1.9, U is definitizable if and only if $z \longmapsto J(U+z)(U-z)^{-1}$ defined in a neighbourhood of $\{0,\infty\}$ can analytically be continued to a definitizable operator function. The sets of the corresponding definitizing rational functions the poles of which belong to $\rho(U)\cup\{\infty\}$ coincide.

The following theorem will show, in particular, that a definitizable function and a minimal definitizable unitary representing operator have similar sign and multiplicity properties.

First, we recall some definitions for a definitizable unitary operator U in a Kreĭn space $\mathscr{K}$. Let $E(\,\cdot\,) := E(\,\cdot\,;U)$ be the spectral function of U. A critical point α of U is called *regular* (we write $\alpha \in c_r(U)$) if there exists an open arc $\gamma_0 \ni \alpha$ with $\overline{\gamma_0}\cap(c(U)\setminus\{\alpha\}) = \emptyset$ such that the projections $E(\gamma)$, $\gamma = \bar{\gamma} \subset \gamma_0\setminus\{\alpha\}$, are uniformly bounded. The elements of $c_s(U) = c(U)\setminus c_r(U)$ are called *singular* critical points. A critical point α with the property that for every open arc γ such that $\alpha \in \gamma$ and $E(\gamma)$ is defined, the range of $E(\gamma)$ is neither a Hilbert or anti-Hilbert space nor a Pontryagin space, is called an *essential critical point.* The set of essential critical points of U is denoted by $c_\infty(U)$.

Theorem 1.12. *Let* $F \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ *and let*

$$F(z) = i\operatorname{Im} F(0) + \Gamma^+(U+z)(U-z)^{-1}\Gamma, \qquad z \in \rho(U), \tag{1.18}$$

be a representation of F *with a definitizable unitary operator* U *in a Kreĭn space* $\mathscr{K}$. *Let* E *be the spectral function of* U. *Then the following holds*:

(i) $\Sigma(F) \subset \sigma(U)$.

(ii) $\mathrm{K}(F) \subset c(U)$. *If, in addition, the representation is minimal, we have* $\mathrm{K}(F) = c(U)$.

(iii) *For every open subarc* γ *of* $\mathbf{T}$, *the endpoints of which do not belong to* $c(U)$, *and every* $g \in \mathscr{R}^s_{0,\infty}$, *we have*

$$\begin{aligned}\kappa_\pm\Big(\big(\big\{C^\infty(\mathbf{T},\mathscr{H}) \times H\big(\Sigma(F)\setminus\mathbf{T},\mathscr{H}\big)\big\}_\gamma, T_F(\cdot,\cdot).g\big)\Big)\\ \le \kappa_\pm\Big(\big(E(\gamma)\mathscr{K}, [g(U)\cdot,\cdot]\big)\Big).\end{aligned} \tag{1.19}$$

Here $\big\{C^\infty(\mathbf{T},\mathscr{H}) \times H\big(\Sigma(F)\setminus\mathbf{T},\mathscr{H}\big)\big\}_\gamma$ *denotes the linear space of all functions of* $C^\infty(\mathbf{T},\mathscr{H}) \times H(\Sigma(F)\setminus\mathbf{T},\mathscr{H})$ *with support in* γ. *In particular,*

$$\begin{aligned}&\kappa_\pm\Big(\big(\big\{C^\infty(\mathbf{T},\mathscr{H}) \times H\big(\Sigma(F)\setminus\mathbf{T},\mathscr{H}\big)\big\}_\gamma, T_F(\,\cdot\,,\cdot\,).\mathbf{1}\big)\Big)\\ &= \kappa_\pm\Big(\big(\mathscr{R}_{0,\infty}(\mathscr{H}), E_F(\,\cdot\,,\cdot\,;\gamma)\big)\Big) \le \kappa_\pm\Big(\big(E(\gamma)\mathscr{K}, [\,\cdot\,,\cdot\,]\big)\Big),\end{aligned} \tag{1.20}$$

where $E_F(\,\cdot\,,\cdot\,;\cdot\,)$ *denotes the form spectral function of* F (*see* [9, Section 2.3]). *In the case of a minimal representation we have equality in* (1.19) *and* (1.20).

(iv) *Let* $\nu \notin \mathbf{T}$ *be a pole of* F. *Then the multiplicity of the pole* ν *of* F *is less than or equal to* $\dim E(\{\nu\})\mathscr{K}$, *where* $E(\{\nu\})$ *is the Riesz–Dunford projection corresponding to* U *and* $\{\nu\}$. *In the minimal case we have equality.*

Proof. 1. Assertion (i) follows from the definition of $\mathrm{P}(F)$. In order to verify (iii), let γ be an open subarc of $\mathbf{T}$ whose endpoints do not belong to $c(F)\cup c(U)$. Let $g \in \mathscr{R}^s_{0,\infty}$. Then, by extending the relation (1.12) we get

$$\begin{aligned}&(4\pi)^{-1}T_F\bigg(\sum_{j=-n}^{n} x_j\dot\lambda^j, \sum_{k=-n}^{n} x_k\dot\lambda^k\bigg).\alpha g\\ &\quad= [\alpha(U)g(U)(U^{-n}\Gamma x_{-n} + \cdots + U^n\Gamma x_n), U^{-n}\Gamma x_{-n} + \cdots + U^n\Gamma x_n]\end{aligned} \tag{1.21}$$

for all functions $\alpha \in C^\infty(\mathbf{T}) \times H\big(\Sigma(F)\setminus\mathbf{T}\big)$ with $\operatorname{supp}\alpha \subset \gamma$. Then a density argument gives (1.19). With the help of [9, Lemma 2.8], we get (1.20). If the representation (1.18) is minimal, then the set

$$\{U^{-n}\Gamma x_n + \cdots + U^n\Gamma x_n : x_i \in \mathscr{H},\ i = -n,\ldots,n;\ n \in \mathbf{N}\}$$

is dense in $\mathscr{K}$. Therefore,

$$\{E(\gamma)(U^{-n}\Gamma x_n + \cdots + U^n \Gamma x_n) : x_i \in \mathscr{H},\ i = -n, \dots, n;\ n \in \mathbf{N}\}$$

is dense in $E(\gamma)\mathscr{K}$.

Using the fact that $E(\gamma)$ can be written as a strong limit of operators of the form $\alpha_m(U)g(U)$, $\alpha_m \in C^\infty(\mathbf{T}) \times H\big(\sigma(U) \setminus \mathbf{T}\big)$, $\operatorname{supp} \alpha_m \subset \gamma$, we easily verify, with the help of (1.21), that equality holds in (1.19) and (1.20). As a consequence, we get assertion (ii). This shows that the above additional assumption concerning the endpoints of γ is no restriction. Hence (iii) holds.

2. Let $\mu \in \sigma(U) \setminus \mathbf{T}$ and let $\varepsilon > 0$ be chosen so that

$$\mathscr{U} := \{z : 0 < |\mu - z| \le \varepsilon\} \subset \rho(U).$$

Let χ be the indicator function of $\overline{\mathscr{U}} \cup \hat{\overline{\mathscr{U}}}$. The function χ is locally holomorphic on $\sigma(U)$ and on $\Sigma(F)$. Then by the extension by continuity of the relation (1.12) we find, for $x_{-n}, \dots, x_n \in \mathscr{H}$,

$$\begin{aligned}(4\pi)^{-1} T_F\Big(\sum_{j=-n}^{n} x_j \dot{\lambda}^j, \sum_{k=-n}^{n} x_k \dot{\lambda}^k\Big).\chi \\ = [E_0(U^{-n}\Gamma x_{-n} + \cdots + U^n \Gamma x_n), U^{-n}\Gamma x_{-n} + \cdots + U^n \Gamma x_n],\end{aligned} \tag{1.22}$$

where $E_0 = E(\{\mu\}) + E(\{\bar{\mu}^{-1}\})$. By [9, Lemma 1.5], the multiplicity l of the pole μ of F coincides with the number of negative and with the number of positive squares of the symmetric form on the right hand side of (1.22). Then by (1.22)

$$l \le \kappa_\pm\big((E_0\mathscr{K}, [\,\cdot\,,\cdot\,])\big). \tag{1.23}$$

By a well-known result for unitary operators in Kreĭn space, we have

$$\kappa_\pm\big((E_0\mathscr{K}, [\,\cdot\,,\cdot\,])\big) = \dim E(\{\mu\})\mathscr{K}.$$

If the representation (1.18) is minimal, we see as above that equality holds in (1.23), and, therefore, $l = \dim E(\{\mu\})\mathscr{K}$. Hence statement (iv) holds, and Theorem 1.12 is proved.

1.5. *Representations of definitizable functions with regular critical points.* Below, in Section 2.4, we shall see that a regular critical point of a definitizable function is not necessarily a regular critical point for all minimal representing operators. But the converse implication is true.

Theorem 1.13. *If*

$$F(z) = i \operatorname{Im} F(0) + \Gamma^+(U + z)(U - z)^{-1}\Gamma, \qquad z \in \rho(U),$$

is a (not necessarily minimal) representation of $F \in D_0\big(\mathbf{T}; \mathscr{L}(\mathscr{H})\big)$ *by a definitizable unitary operator* U *in a Kreĭn space* $\mathscr{K}$, *then* $c_r(U) \subset \mathrm{K}_r(F)$.

Proof. Let $\gamma \subset \mathbf{T}$ be an open arc whose endpoints do not belong to $c_s(U)$ and assume that $E(\gamma)\mathscr{K}$ is a Hilbert space or anti-Hilbert space. Let $\mathscr{M}$ be the set of all functions $f \in C^\infty(\mathbf{T}) \times H\big(\sigma(U) \setminus \mathbf{T}\big)$ with $\operatorname{supp} f \subset \gamma$ and $\sup\{|f(s)| : s \in \gamma\} \le 1$; then the set $\{f(U) : f \in \mathscr{M}\}$ is bounded in $\mathscr{L}(\mathscr{H})$. Then, making use of the extension by continuity of the relation (1.11), we find that T_F is a bounded operator measure on γ, which proves Theorem 1.13.

The following theorem shows that for every definitizable function we can find a minimal representing operator which has the same regular critical points.

Theorem 1.14. *Let $F \in D_0\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$ and $\mathrm{K}_r(F) \neq \emptyset$. Then there exists a minimal definitizable unitary representing operator U such that $c_r(U) = \mathrm{K}_r(F)$, and for every $\alpha \in \mathrm{K}_r(F)$ we have $\kappa_\pm(\mathbf{1}, F_0) = \kappa_\pm(\mathscr{L}_\alpha)$, where F_0 is as in Proposition* 1.6 *and $\mathscr{L}_\alpha$ is the algebraic eigenspace of U corresponding to α.*

Proof. Assume first that $\mathrm{K}_r(F) = \{e^{i\Theta_0}\}$. Let $F = \sum_{i=0}^3 F_i$ be a decomposition of F as in Proposition 1.6 and let

$$F_i(z) = i \operatorname{Im} F_i(0) + \Gamma_i^+(U_i + z)(U_i - z)^{-1}\Gamma_i$$

be a minimal representation of F_i in some Kreĭn space $\mathscr{K}_i$. If we set $\mathscr{K} := \mathscr{K}_0 \times \cdots \times \mathscr{K}_3$, $U := U_0 \times \cdots \times U_3$ and $\Gamma := (\Gamma_0, \ldots, \Gamma_3)^T$, then

$$F(z) = i \operatorname{Im} F(0) + \Gamma^+(U + z)(U - z)^{-1}\Gamma. \tag{1.24}$$

Since by Theorem 1.9 the unitary operators U_i, $i = 0, 1, 2, 3$, are definitizable and the sets $\sigma(U_i) \cap \sigma(U_j)$, $i \neq j$, are empty or consist of one point, U is a definitizable unitary operator in $\mathscr{K}$. The spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ are Hilbert or anti-Hilbert spaces. Therefore, $e^{i\Theta_0} \in c_r(U)$. Making use of the definition of F_1 and F_2 in the proof of Proposition 1.6 and of (1.12) with F replaced by F_1 and F_2 one verifies without difficulty that U_1 and U_2 have no eigenvalues at the endpoints of γ_1 and γ_2. In order to verify that the representation (1.24) is minimal, let y_i, $i = 0, 1, 2, 3$, be an arbitrary element of $\mathscr{K}_i$. Every element of $\mathscr{K}_i$ can be approximated in $\mathscr{K}_i$ by elements of the form

$$\sum_{j=1}^k g_j(U_i)\Gamma_i x_j, \qquad x_j \in \mathscr{H},\ g_j \in \mathscr{R}_{0,\infty}.$$

Hence it remains to prove that every element of the form $w_{(i)} = (w_0, \ldots, w_3) \in \mathscr{K}$, where $w_j = 0$ for $j \neq i$ and $w_i = g(U_i)\Gamma_i x$, $g \in \mathscr{R}_{0,\infty}$, $x \in \mathscr{H}$, can be approximated in $\mathscr{K}$ by elements of the form $h(U)\Gamma x$, $h \in \mathscr{R}_{0,\infty}$. Since $\mathscr{R}_{0,\infty}$ is dense in $C^m(\mathbf{T}) \times H\big(\sigma(U) \setminus \mathbf{T}\big)$ for arbitrary m, it is sufficient to prove that $w_{(i)}$ is the limit of a converging sequence $\big(f_n(U)\Gamma x\big)$, $f_n \in C^\infty(\mathbf{T}) \times H\big(\sigma(U) \setminus \mathbf{T}\big)$. If $i = 1$ or $i = 2$, then the sequences $(f_n) = (\chi_{1,n} g)$ or $(f_n) = (\chi_{2,n} g)$, where $\chi_{1,n}$, $\chi_{2,n}$ are as in the proof of Lemma 1.6, have the required property. For $i = 0$

the sequence $\big(\chi_0(1-\chi_{1,n}-\chi_{2,n})g\big)$ and for $i=3$ the sequence $\big((1-\chi_0)(1-\chi_{1,n}-\chi_{2,n})g\big)$ have the required property, where χ_0 is again as in the proof of Lemma 1.6.

If F has more than one regular critical point, F has to be decomposed with respect to all of its critical points. Then a similar reasoning applies. The last assertion is a direct consequence of the minimality and Theorem 1.12(iii).

Remark 1.15. If F, U, α are as in Theorem 1.14 and, in addition, $\dim\mathscr{H}<\infty$, that is, F is a matrix function, then in view of Proposition 1.5 we have $\dim\mathscr{L}_\alpha<\infty$.

Remark 1.16. Theorem 1.14 will be supplemented in Proposition 2.7 below.

2. Relations between two minimal representing operators of the same definitizable function

2.1. We consider now the situation of the second part of Theorem 1.8. Let $(\mathscr{K}_1,[\,\cdot\,,\cdot\,]_1)$ and $(\mathscr{K}_2,[\,\cdot\,,\cdot\,]_2)$ be two Kreĭn spaces and let U_1 and U_2 be definitizable unitary operators in $\mathscr{K}_1$ and $\mathscr{K}_2$, respectively. By E_1 and E_2 we denote the spectral functions of U_1 and U_2, respectively.

Moreover, let V be an isometric linear operator from $\mathscr{K}_1$ to $\mathscr{K}_2$ with

$$\overline{\mathscr{D}(V)}=\mathscr{K}_1,\qquad \overline{\mathscr{R}(V)}=\mathscr{K}_2 \tag{2.1}$$

and

$$\begin{aligned} U_1\mathscr{D}(V)\subset\mathscr{D}(V),&\qquad U_1^{-1}\mathscr{D}(V)\subset\mathscr{D}(V),\\ U_2\mathscr{R}(V)\subset\mathscr{R}(V),&\qquad U_2^{-1}\mathscr{R}(V)\subset\mathscr{R}(V),\end{aligned}$$

such that

$$VU_1=U_2V,\qquad VU_1^{-1}=U_2^{-1}V. \tag{2.2}$$

In Sections 2.1–2.3 we do not assume that U_1 and U_2 are representing operators of the same definitizable function and that the operator V is of the form (1.16).

As a consequence of (2.2) we have

$$Vp(U_1)=p(U_2)V \tag{2.3}$$

for every $p\in\mathscr{R}^s_{0,\infty}$. Then, for every $x\in\mathscr{D}(V)$,

$$[p(U_1)x,x]_1=[Vp(U_1)x,Vx]_2=[p(U_2)Vx,Vx]_2,$$

which implies that U_1 and U_2 have the same definitizing functions in $\mathscr{R}^s_{0,\infty}$.

By (2.1), the isometric operator V is closable and $\overline{V}$ is again an isometric operator from $\mathscr{K}_1$ into $\mathscr{K}_2$. It is easy to see that all properties mentioned above for V remain valid for $\overline{V}$.

2.2. Let γ be a subarc of $\mathbf{T}$ such that there exists a definitizing polynomial for U_1 (or U_2) which does not have zeros at the endpoints of γ. Then there exists a sequence $(p_n) \subset \mathscr{R}^s_{0,\infty}$ such that

$$p_n(U_1) \longmapsto E_1(\gamma), \qquad p_n(U_2) \longmapsto E_2(\gamma)$$

in the strong sense. Then it follows from (2.3) that

$$E_1(\gamma)\mathscr{D}(\overline{V}) \subset \mathscr{D}(\overline{V}), \qquad E_2(\gamma)\mathscr{R}(\overline{V}) \subset \mathscr{R}(\overline{V})$$

and

$$\overline{V}E_1(\gamma) = E_2(\gamma)\,\overline{V}. \tag{2.4}$$

Evidently, $E_1(\gamma)\mathscr{D}(\overline{V})$ is dense in $E_1(\gamma)\mathscr{K}_1$ and $E_2(\gamma)\mathscr{R}(\overline{V})$ is dense in $E_2(\gamma)\mathscr{K}_2$. For $x \in \mathscr{D}(\overline{V})$ we get

$$[E_1(\gamma)x, x]_1 = [\overline{V}E_1(\gamma)x,\, \overline{V}x]_2 = [E_2(\gamma)\,\overline{V}x,\, \overline{V}x]_2.$$

Hence, U_1 and U_2 have the same critical points, and for every arc $\gamma \subset \mathbf{T}$ whose endpoints are not critical points, we have

$$\kappa_\pm\big((E_1(\gamma)\mathscr{K}_1, [\,\cdot\,,\cdot\,]_1)\big) = \kappa_\pm\big((E_2(\gamma)\mathscr{K}_2, [\,\cdot\,,\cdot\,]_2)\big).$$

In particular, $E_1(\gamma)\mathscr{K}_1$ is a Pontryagin space if and only if $E_2(\gamma)\mathscr{K}_2$ is a Pontryagin space.

Lemma 2.1. *If γ is a subarc of $\mathbf{T}$ such that the endpoints of γ are not critical points of U_1 and $E_1(\gamma)\mathscr{K}_1$ is a Pontryagin space, then $E_1(\gamma)\mathscr{K}_1 \subset \mathscr{D}(\overline{V})$, $\overline{V}$ is an isometric isomorphism of $E_1(\gamma)\mathscr{K}_1$ onto $E_2(\gamma)\mathscr{K}_2$ and $U_1 = \overline{V}^{-1}U_2\overline{V}$ on $E_1(\gamma)\mathscr{K}_1$.*

Proof. The first two assertions follow from the fact that a densely defined isometric operator with dense range between Pontryagin spaces is automatically continuous. The last relation follows from the relation (2.4) with $E_i(\gamma)$ replaced by $U_iE_i(\gamma)$.

Evidently, in Lemma 2.1 γ can be replaced by a finite union of pairwise disjoint arcs fulfilling the assumptions of Lemma 2.1. These arcs may also degenerate to points of $\mathbf{T}$.

In the same way, the following can be proved.

Lemma 2.2. *Let γ be a subarc of $\mathbf{T}$ such that the endpoints of γ are not critical points of U_1. Assume that $\gamma \cap c(U_1) = \gamma \cap c(U_2) = \{\lambda\}$ and that λ is a regular critical point for U_1 and U_2. Also assume that the algebraic eigenspaces corresponding to λ with respect to U_1 and U_2 are Pontryagin spaces. Then the same assertions hold as in Lemma* 2.1.

Remark 2.3. Assume that the point 1 belongs to $c(U_1)$, i.e., to $c(U_1) \cap c(U_2)$, and is not a regular critical point for at least one of the operators U_1 and U_2. Let γ be an open arc of $\mathbf{T}$ such that $1 \in \gamma$, the endpoints of γ are not critical points of U_1 and the components of $\gamma \setminus \{1\}$ are definite with respect to U_1 (and U_2). Let $q_n(\lambda) := i^n(\lambda^{-1} - \lambda)^n$, $n = 1, 2, \ldots$. We have $q_n \in \mathscr{R}^s_{0,\infty}$, and q_n has a zero of n-th order at 1. Then we see as above that

$$[q_n(U_1)E_1(\gamma)x, x]_1 = [q_n(U_2)E_2(\gamma)\overline{V}x, \overline{V}x]_2, \qquad x \in \mathscr{D}(\overline{V}).$$

Hence, for $n = 1, 2, \ldots$, we have

$$\kappa_\pm\big((E_1(\gamma)\mathscr{K}_1, [q_n(U_1)\,\cdot\,, \cdot\,]_1)\big) = \kappa_\pm\big((E_2(\gamma)\mathscr{K}_2, [q_n(U_2)\,\cdot\,, \cdot\,]_2)\big). \tag{2.5}$$

Some of these quantities may be finite even if 1 is an essential singular critical point of U_1 and U_2. The point 1 can be replaced by any other point $\lambda \in \mathbf{T}$.

2.3. The sets $\sigma(U_1) \setminus \mathbf{T}$ and $\sigma(U_2) \setminus \mathbf{T}$ are finite. Let G be a C^∞-domain such that $0 \notin \overline{G} \subset \mathbf{D}$ and $\partial G \cap \big(\sigma(U_1) \cup \sigma(U_2)\big) = \emptyset$. Let $E_1(G \cup \widehat{G})$ and $E_2(G \cup \widehat{G})$ be the Riesz–Dunford projections corresponding to U_1 and U_2 and the part of $\sigma(U_1)$ or $\sigma(U_2)$, respectively, lying in $G \cup \widehat{G}$. It is easy to see that there exists a sequence $(q_n) \subset \mathscr{R}^s_{0,\infty}$ such that

$$\lim_{n\to\infty} q_n(U_1) = E_1(G \cup \widehat{G}), \qquad \lim_{n\to\infty} q_n(U_2) = E_2(G \cup \widehat{G})$$

with respect to the norm convergence. Then we get, as above,

$$E_1(G \cup \widehat{G})\mathscr{D}(\overline{V}) \subset \mathscr{D}(\overline{V}), \qquad E_2(G \cup \widehat{G})\mathscr{R}(\overline{V}) \subset \mathscr{R}(\overline{V})$$

and

$$\overline{V}E_1(G \cup \widehat{G}) = E_2(G \cup \widehat{G})\,\overline{V}.$$

This implies, as above, that $\sigma(U_1) \setminus \mathbf{T} = \sigma(U_2) \setminus \mathbf{T}$ and that for every λ belonging to this set, the dimensions of the ranges of the Riesz–Dunford projections corresponding to λ coincide. Moreover, the following lemma holds.

Lemma 2.4. *If $\lambda \notin \mathbf{T}$ is a normal eigenvalue of U_1, then λ is also a normal eigenvalue of U_2 and $\overline{V}$ maps the algebraic eigenspace $\mathscr{L}_\lambda(U_1)$ of U_1 corresponding to λ on the algebraic eigenspace $\mathscr{L}_\lambda(U_2)$ of U_2 corresponding to λ and $U_1 = \overline{V}^{-1}U_2\overline{V}$ on $\mathscr{L}_\lambda(U_1)$.*

From Lemmas 2.1, 2.2 and 2.4 we get the following slight generalization of [2, Theorem 7.1]. The latter result was formulated for selfadjoint operators.

Theorem 2.5. *Assume that all eigenvalues of U_1 in $\mathbf{D} \cup \widehat{\mathbf{D}}$ are normal, $c_\infty(U_1) \subset c_r(U_1)$, $c_\infty(U_2) \subset c_r(U_2)$ and that all algebraic eigenspaces of U_1 and U_2 corresponding to regular critical points are Pontryagin spaces. Then U_1 and U_2 are unitarily equivalent.*

2.4. If $F \in \mathscr{D}_0(\mathbf{T};\mathscr{L}(\mathscr{H}))$ and U_1 and U_2 are two minimal representing operators for F and V is the operator introduced in Theorem 1.8, Lemmas 2.1–3 show which parts of U_1 and U_2 are unitarily equivalent. As a consequence of these local results, we get the following

Theorem 2.6. *Let $F \in \mathscr{D}_0(\mathbf{T};\mathscr{L}(\mathscr{H}))$. Assume that all poles of F in $\mathbf{D} \cup \widehat{\mathbf{D}}$ have finite multiplicities, $\mathrm{K}_\infty(F) \subset \mathrm{K}_r(F)$ and that the masses of all points of $\mathrm{K}_r(F)$ are of type π. Then all minimal representing definitizable unitary operators for F, the essential critical points of which are regular, are unitarily equivalent.*

As a consequence of Lemma 2.1 and Theorem 1.14, we get the following supplement to Theorem 1.14.

Proposition 2.7. *Let $F \in D_0(\mathbf{T};\mathscr{L}(\mathscr{H}))$ and $\alpha \in (\mathrm{K}(F) \backslash \mathrm{K}_\infty(F)) \cap \mathrm{K}_r(F)$. Then for every minimal unitary representing operator U for F we have $\alpha \in c_r(U)$.*

Now we give an example of a definitizable operator function F with $\mathrm{K}(F) = \mathrm{K}_r(F) = \{1\}$ which has two not unitarily equivalent minimal representing operators.

Example. Let $(\mathscr{G}_0, (\,\cdot\,,\cdot\,)_0)$ be a Hilbert space and let H be a positive bounded operator in $\mathscr{G}_0$ such that $0 \in \sigma(H) \setminus \sigma_p(H)$. We assume, in addition, that H is a cyclic operator. Let $e_0 \in \mathscr{D}(H^{-1/4})$ be a generating element for H. We denote by $\mathscr{G}_{-1/4}$ the completion of $\mathscr{G}_0$ with respect to the scalar product

$$(x,y)_{-1/4} := \big(H^{1/4}x, H^{1/4}y\big)_0, \qquad x, y \in \mathscr{G}_0,$$

and by $\mathscr{G}_{1/4}$ the domain of $H^{-1/4}$ provided with the graph scalar product

$$(x,y)_{1/4} := \big((H^{-1/4}+1)x, (H^{-1/4}+1)y\big)_0, \qquad x, y \in \mathscr{D}(H^{-1/4}).$$

It is easy to see that the form $(\,\cdot\,,\cdot\,)_0$ can be extended by continuity to $\mathscr{G}_{1/4} \times \mathscr{G}_{-1/4}$ and $\mathscr{G}_{-1/4} \times \mathscr{G}_{1/4}$. We provide $\mathscr{H} := \mathscr{G}_0 \times \mathscr{G}_0$ with the Kreĭn space inner product

$$\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] := (u_2, v_1)_0 + (u_1, v_2)_0, \qquad u_1, u_2, v_1, v_2 \in \mathscr{G}_0. \tag{2.6}$$

In the Kreĭn space $(\mathscr{H}, [\,\cdot\,,\cdot\,])$ we consider the positive bounded operator

$$A := \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix}.$$

It is easy to see that A is cyclic and

$$e := \begin{pmatrix} 0 \\ e_0 \end{pmatrix}$$

is a generating element.

At the same time we consider the space $\mathscr{H}' := \mathscr{G}_{-1/4} \times \mathscr{G}_{1/4}$ with an inner product $[\,\cdot\,,\cdot\,]'$ defined as in (2.6), where $u_1, v_1 \in \mathscr{G}_{-1/4}$, $u_2, v_2 \in \mathscr{G}_{1/4}$. $(\mathscr{H}', [\,\cdot\,,\cdot\,]')$ is also a Kreĭn space. The operator A' defined by

$$A' := \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix}$$

with respect to $\mathscr{H}' = \mathscr{G}_{-1/4} \times \mathscr{G}_{1/4}$ is a positive bounded operator in $\mathscr{H}'$, it is cyclic, and

$$e = \begin{pmatrix} 0 \\ e_0 \end{pmatrix}$$

is a generating element.

The operators $(A - z)^{-1}$ and $(A' - z)^{-1}$, $z \in \mathbf{C} \setminus \mathbf{R}$, can be written as

$$\begin{pmatrix} z(H - z^2)^{-1} & (H - z^2)^{-1} \\ 1 + z^2(H - z^2)^{-1} & z(H - z^2)^{-1} \end{pmatrix} \tag{2.7}$$

with respect to the above decompositions of $\mathscr{H}$ and $\mathscr{H}'$, respectively. The resolvents $(A - z)^{-1}$ and $(A' - z)^{-1}$ coincide on $\mathscr{H} \cap \mathscr{H}'$ and map this linear set into itself.

Let $\varepsilon > 0$. Making use of (2.7), we see ([16], cf. also [10]) that the spectral projections corresponding to the operators A and A' and the interval $(\varepsilon, 1)$ can be written as

$$\frac{1}{2} \begin{pmatrix} E\big((\varepsilon, 1)\big) & H^{-1/2} E\big((\varepsilon, 1)\big) \\ H^{1/2} E\big((\varepsilon, 1)\big) & E\big((\varepsilon, 1)\big) \end{pmatrix}.$$

For the spectral projections corresponding to the interval $(-1, -\varepsilon)$, we find

$$\frac{1}{2} \begin{pmatrix} E\big((\varepsilon, 1)\big) & -H^{-1/2} E\big((\varepsilon, 1)\big) \\ -H^{1/2} E\big((\varepsilon, 1)\big) & E\big((\varepsilon, 1)\big) \end{pmatrix}.$$

Here E is the spectral function of $H^{1/2}$. Since $\big\|H^{-1/2} E\big((\varepsilon, 1)\big)\big\| \longrightarrow \infty$ if $\varepsilon \longrightarrow 0$, the point 0 is a singular critical point of A. One easily verifies that 0 is a regular critical point for A'.

Now we consider the unitary operators $U = -(A - i)(A + i)^{-1}$ and $U' = -(A' - i)(A' + i)^{-1}$ in $\mathscr{H}$ and $\mathscr{H}'$, respectively. We have $\sigma(U) \subset \mathbf{T}$, $\sigma(U') \subset \mathbf{T}$, and U and U' are definitizable with the definitizing function $g(z) := i(z^{-1} - z)$, $g \in \mathscr{R}^s_{0,\infty}$. The point 1 is a singular critical point for U and a regular critical point for U'. Since A and A' are bounded, there exists a function $f \in C^1(\mathbf{T})$ such that $A = f(U)$ and $A' = f(U')$ holds. The elementary functional calculi of U and U' are continuous with respect to the topology of $C^1(\mathbf{T})$ (cf. [6, Theorem 1] for the selfadjoint case). Let (p_n), $n = 1, 2, \ldots$, be a sequence of functions from

$\mathscr{R}_{0,\infty}$ which converges to f in $C^1(\mathbf{T})$. Then $Ae = \lim_{n\to\infty} p_n(U)e$ and $A'e = \lim_{n\to\infty} p_n(U')e$. Since e is a generating element for A and A', we obtain

$$\text{(2.8)} \qquad \mathscr{H} = \operatorname{closp}\{p(U)e : p \in \mathscr{R}_{0,\infty}\}, \qquad \mathscr{H}' = \operatorname{closp}\{p(U')e : p \in \mathscr{R}_{0,\infty}\}.$$

For an arbitrary $\zeta \in \mathbf{C}\setminus\mathbf{T}$, the operator $(U+\zeta)(U-\zeta)^{-1}$ $((U'+\zeta)(U'-\zeta)^{-1})$ can be written as a linear combination of the identity and $(A-z)^{-1}$ $((A'-z)^{-1}$, respectively) with $z = i(1-\zeta)(1+\zeta)^{-1} \notin \mathbf{R}$. Hence the operators $(U+\zeta)(U-\zeta)^{-1}$ and $(U'+\zeta)(U'-\zeta)^{-1}$ coincide on $\mathscr{H}\cap\mathscr{H}'$ and map this linear set into itself.

If we define linear mappings $\Gamma\colon \mathbf{C} \longrightarrow \mathscr{H}$ and $\Gamma'\colon \mathbf{C} \longrightarrow \mathscr{H}'$ by $\mathbf{C} \ni a \longmapsto ae$, then we find

$$\begin{aligned}\Gamma^+(U+\zeta)(U-\zeta)^{-1}\Gamma &= [(U+\zeta)(U-\zeta)^{-1}e, e]\\ &= [(U'+\zeta)(U'-\zeta)^{-1}e, e] = \Gamma'^+(U'+\zeta)(U'-\zeta)^{-1}\Gamma'.\end{aligned}$$

In other words, the scalar definitizable function $F(\zeta) = [(U'+\zeta)(U'-\zeta)^{-1}e, e]$ can be represented by the operators U and U'. By (2.8) U and U' are minimal representing operators. But these operators are not unitarily equivalent since $1 \in c_s(U)\setminus c_s(U')$.

3. Definitizable functions symmetric with respect to the real axis

3.1. *Notation for the real axis case and relations to the unit circle case.* In this section we consider classes of operator functions which are closely connected with those considered in Sections 1 and 2. The real axis and the open upper half plane $\mathbf{C}^+$ take over the role the unit circle and the open unit disc were playing before. In the following, we will carry over definitions and results from [9] and Section 1 with the help of the fractional linear transformations ψ and ϕ defined by

$$\psi(\lambda) := -\frac{\lambda - i}{\lambda + i}, \qquad \phi(z) := i\frac{1-z}{1+z}.$$

We have $\phi\circ\psi = \mathrm{id}$ and $\psi(\overline{\mathbf{R}}) = \mathbf{T}$. For any subset $\mathscr{S}$ of $\overline{\mathbf{C}}$, we set $\mathscr{S}^* := \{\bar z : z \in \mathscr{S}\}$, and for any scalar (or $\mathscr{L}(\mathscr{H})$-valued) function f defined on a set $\mathscr{D} = \mathscr{D}^* \subset \overline{\mathbf{C}}$, we define $f^*(\mu) = \overline{f(\bar\mu)}$ $(f^*(\mu) = (f(\bar\mu))^*)$, $\mu \in \mathscr{D}$. Put $\mathbf{C}^- := (\mathbf{C}^+)^*$. For a Banach space X, let $\mathscr{R}_{i,-i}(X)$ denote the set of all functions of the form

$$\mathbf{C} \ni \lambda \longmapsto d_0 + \sum_{k=1}^{n} d_k(\lambda - i)^{-k} + \sum_{j=1}^{n} d_{-j}(\lambda + i)^{-j}$$

with arbitrary n, $d_l \in X$, $l = -n, \dots, n$. We put $\mathscr{R}_{i,-i} := \mathscr{R}_{i,-i}(\mathbf{C})$. $\mathscr{R}^s_{i,-i}$ denotes the set of all functions $f \in \mathscr{R}_{i,-i}$ such that $f = f^*$. Then $f \in \mathscr{R}_{i,-i}$ $(f \in \mathscr{R}^s_{i,-i})$ if and only if $f\circ\phi \in \mathscr{R}_{0,\infty}$ $(f\circ\phi \in \mathscr{R}^s_{0,\infty})$.

Let $L' = (L')^*$ be a compact subset of $\overline{\mathbf{C}}$ such that $\overline{\mathbf{R}} \subset L'$, $i \notin L'$ and $\mathbf{C}^+ \setminus L'$ is connected. Similarly to the unit circle case, we connect with every $\mathscr{L}(\mathscr{H})$-valued function G holomorphic in $(\mathbf{C}^+ \setminus L') \cup (\mathbf{C}^- \setminus L')$ with $G = G^*$ an analytic functional $S_G \in \mathscr{L}\big(H(L'), \mathscr{L}(\mathscr{H})\big)$:

$$(3.1) \qquad S_G.g = 2i \int_{\mathscr{C}'} G(\lambda) g(\lambda)(\lambda^2+1)^{-1}\, d\lambda, \qquad g \in H(L'),$$

where $\mathscr{C}'$ is the oriented boundary of a finite union Ω' of smooth domains of $\overline{\mathbf{C}}$ containing L' such that g is defined on $\overline{\Omega'}$ and $i, -i \notin \overline{\Omega'}$. For every g as in (3.1) and $u', v' \in H(L', \mathscr{H})$, we define

$$(3.2) \qquad S_G(u', v').g := 2i \int_{\mathscr{C}'} \big(G(\lambda)u'(\lambda), v'(\bar\lambda)\big) g(\lambda)(\lambda^2+1)^{-1}\, d\lambda.$$

Here $\mathscr{C}'$ is as above and such that u' and v' are defined on $\overline{\Omega'}$. If $g = g^*$, the sesquilinear form $(u', v') \longmapsto S_G(u', v').g$ is hermitian. If we put $L = \psi(L')$, $F = -i(G \circ \phi)$, $f = g \circ \phi$, $u = u' \circ \phi$, $v = v' \circ \phi$, then L, F, f, u, v fulfil the assumptions of (1.1) and (1.2) and an easy computation gives the relations

$$(3.3) \qquad T_F.(g \circ \phi) = S_G.g, \qquad T_F(u' \circ \phi, v' \circ \phi).(g \circ \phi) = S_G(u', v').g.$$

The set of all $\mathscr{L}(\mathscr{H})$-valued functions G meromorphic in $\overline{\mathbf{C}} \setminus \overline{\mathbf{R}}$ and satisfying the relation $G = G^*$ will be denoted by $M\big(\overline{\mathbf{R}}; \mathscr{L}(\mathscr{H})\big)$. For a function $G \in M\big(\overline{\mathbf{R}}; \mathscr{L}(\mathscr{H})\big)$, the set of all $\lambda \in \overline{\mathbf{C}}$ such that G can analytically be continued in λ in a unique way is denoted by $\mathrm{P}(G)$. We set $\Sigma(G) := \overline{\mathbf{C}} \setminus \mathrm{P}(G)$.

In the rest of this subsection, let $G \in M\big(\overline{\mathbf{R}}; \mathscr{L}(\mathscr{H})\big)$ be holomorphic at i and $-i$, let G have only a finite number of poles outside $\overline{\mathbf{R}}$, and assume that there exists an $m \in \mathbf{N}$ such that, for some $\eta > 0$,

$$(3.4) \qquad \sup\{\|G(\lambda)\|\, |\operatorname{Im}\lambda|^m (|\lambda|+1)^{-2m} : \lambda \in \overline{\mathbf{C}} \setminus \overline{\mathbf{R}},\ \operatorname{dist}_{\overline{\mathbf{C}}}(\lambda, \overline{\mathbf{R}}) < \eta\} < \infty,$$

where $\operatorname{dist}_{\overline{\mathbf{C}}}(\,\cdot\,,\cdot\,)$ is the distance on the Riemann complex sphere. Then $F := -iG \circ \phi$ fulfils the relation (1.3), and by (3.3) S_G and $S_G(\,\cdot\,,\cdot\,)$ can be extended by continuity to

$$C^{m+1}(\overline{\mathbf{R}}) \times H(\Sigma(G) \setminus \overline{\mathbf{R}})$$

and

$$\big(C^{m+1}(\overline{\mathbf{R}}, \mathscr{H}) \times H\big(\Sigma(G) \setminus \overline{\mathbf{R}}, \mathscr{H}\big)\big)^2 \times \big(C^{m+1}(\overline{\mathbf{R}}) \times H\big(\Sigma(G) \setminus \overline{\mathbf{R}}\big)\big),$$

respectively. Here $\overline{\mathbf{R}}$ is regarded as a real-analytic manifold in a natural way.

If $g \in C^\infty(\overline{\mathbf{R}}) \times H\big(\Sigma(G) \setminus \overline{\mathbf{R}}\big)$, $g = g^*$, we define

$$(3.5) \qquad \kappa_\pm(g; G) = \kappa_\pm\big(\big(\mathscr{R}_{i,-i}(\mathscr{H}), S_G(\,\cdot\,,\cdot\,).g\big)\big).$$

If Δ is an open subset of $\overline{\mathbf{R}}$, we say that Δ is of *positive type* (*negative type*, *type* π_+, *type* π_-) with respect to G if $\kappa_-(g;G)=0$ ($\kappa_+(g;G)=0$, $\kappa_-(g;G)<\infty$, $\kappa_+(g,G)<\infty$) for all nonnegative functions $g\in C^\infty(\overline{\mathbf{R}})\times H\big(\Sigma(G)\setminus\overline{\mathbf{R}}\big)$ with $\operatorname{supp} g\subset\Delta$. The set Δ is said to be of *definite type* (*type* π) with respect to G if Δ is of positive or negative type (of type π_+ or type π_-, respectively) with respect to G.

By (3.3) Δ has one of the properties mentioned above with respect to G if and only if $\psi(\Delta)$ has the corresponding property with respect to $-iG\circ\phi$. Then one verifies without difficulty that the criteria in [9, Lemma 1.7] for an open subset of $\mathbf{T}$ to be of positive type can be taken over to the real axis case. Here we mention only one of these criteria: An open subset Δ of $\overline{\mathbf{R}}$ is of positive type with respect to G if the following two conditions are satisfied for every $x\in\mathscr{H}$.

(i) $\liminf_{\varepsilon\downarrow 0}\big\{-i\big(\big(G(t+i\varepsilon)-G(t-i\varepsilon)\big)x,x\big)\big\}\geq 0$ for almost every $t\in\Delta\setminus\{\infty\}$.

(ii) For every bounded closed subset Δ_0 of Δ and sufficiently small $\varepsilon_0>0$,

$$\inf\big\{-i\big(\big(G(t+i\varepsilon)-G(t-i\varepsilon)\big)x,x\big):t\in\Delta_0,\ 0<\varepsilon\leq\varepsilon_0\big\}>-\infty.$$

If $\infty\in\Delta$, then, in addition, for sufficiently small $\delta_0>0$, $\varepsilon_0>0$,

$$\begin{aligned}\inf\big\{-i\big(\big(G(-(t+i\varepsilon)^{-1}\big)-G\big(-(t-i\varepsilon)^{-1}\big)\big)x,x\big):&\\ -\delta_0\leq t\leq\delta_0,\ 0<\varepsilon\leq\varepsilon_0\big\}>-\infty.&\end{aligned}$$

A point $t\in\overline{\mathbf{R}}$ is called a *critical point* of G (we write $t\in\mathrm{K}(G)$) if it is not contained in an open subset of $\overline{\mathbf{R}}$ of definite type with respect to G. By $\mathrm{K}_\infty(G)$ we denote the set of those points $t\in\overline{\mathbf{R}}$ which are not contained in an open subset of $\overline{\mathbf{R}}$ of type π with respect to G. The points of $\mathrm{K}_\infty(G)$ are called *essential* critical points. If $F:=-i(G\circ\phi)$, we have $\mathrm{K}(G)=\phi\big(\mathrm{K}(F)\big)$ and $\mathrm{K}_\infty(G)=\phi\big(\mathrm{K}_\infty(F)\big)$.

3.2. *Definitizable functions.* Let $N\big(\mathscr{L}(\mathscr{H})\big)$ denote the class of $\mathscr{L}(\mathscr{H})$-valued Nevanlinna functions, i.e., the class of all functions $G\in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ holomorphic in $\mathbf{C}^+\cup\mathbf{C}^-$, such that for every $z\in\mathbf{C}^+$ the imaginary part of G (that is, $(2i)^{-1}\big(G(z)-G(z)^*\big)$) is a nonnegative operator. By $N_\kappa\big(\mathscr{L}(\mathscr{H})\big)$, $\kappa=0,1,\dots$, we denote the classes of $\mathscr{L}(\mathscr{H})$-valued generalized Nevanlinna functions introduced and studied in [12], [13]. Recall that a function $G\in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$, by definition, belongs to $N_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ if the kernel N_G,

$$N_G(z,\zeta):=(z-\bar\zeta)^{-1}\big(G(z)-G(\zeta)^*\big),$$

has κ negative squares. The class $N_0\big(\mathscr{L}(\mathscr{H})\big)$ coincides with $N\big(\mathscr{L}(\mathscr{H})\big)$. We have $G\in N_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ if and only if $-iG\circ\phi\in C_\kappa\big(\mathscr{L}(\mathscr{H})\big)$.

Definition 3.1. A function $G \in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ is called *definitizable* if there exists a scalar rational function r such that the product rG is the sum of a Nevanlinna function N and an $\mathscr{L}(\mathscr{H})$-valued rational function n whose poles belong to $\mathrm{P}(G)$:

$$r(z)G(z) = N(z) + n(z)$$

for all points $z \in \mathbf{C}^+ \cup \mathbf{C}^-$ of holomorphy of rG. A function r with the properties mentioned above is called a *definitizing* function for G.

It follows from Definitions 1.1 and 3.1 that G is a definitizable function in $M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ if and only if $-iG\circ\phi$ is a definitizable function in $M\big(\mathbf{T};\mathscr{L}(\mathscr{H})\big)$. If r is definitizing for G, then $r\circ\phi$ is definitizing for $-iG\circ\phi$, and conversely. Therefore, by the remark following Definition 1.1, there exist always $\mathbf{R}$-symmetric definitizing functions. Lemmas 1.2 and 1.4 imply analogous results for the $\mathbf{R}$-symmetric case. In the following lemma we mention only a consequence of Lemma 1.2.

Lemma 3.2. *Let $G \in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ be definitizable and $z_0 \in \mathrm{P}(G)$. Then there exists a rational definitizing function $r_0 = r_0^*$ for G which has no poles except in z_0 and $\bar{z}_0$. In particular, if G is holomorphic at ∞, there exists a definitizing polynomial*; *if G is holomorphic at i and $-i$, there exists a definitizing function from $\mathscr{R}^s_{i,-i}$.*

In the following, we restrict ourselves to the class $D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ of those definitizable functions which are holomorphic at i and $-i$. This involves no loss of generality. The following definitizability criterion is an immediate consequence of [9, Proposition 2.2].

Proposition 3.3. *For $G \in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ the following assertions are equivalent.*

(i) $G \in D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$.

(ii) *G is holomorphic at i and $-i$, has only a finite number of poles outside $\overline{\mathbf{R}}$, and fulfils* (3.4) *for some $m \in \mathbf{N}$. Moreover, there exists a finite set $e' \subset \overline{\mathbf{R}}$ such that all components of $\overline{\mathbf{R}} \setminus e'$ are of definite type with respect to G.*

3.3. *Poles and regular critical points of definitizable functions.* The following lemma is a consequence of [9, Lemma 1.5].

Lemma 3.4. *Let $G \in M\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ and let $\mu \in \overline{\mathbf{C}} \setminus \overline{\mathbf{R}}$ be a pole of G with multiplicity l. Assume that $\chi \in H\big(\overline{\mathbf{R}} \cup \Sigma(G)\big)$ is equal to $\mathbf{1}$ in an open neighbourhood of $\{\mu,\bar{\mu}\}$ and equal to 0 in an open neighbourhood of $\big(\overline{\mathbf{R}} \cup \Sigma(G)\big) \setminus \{\mu,\bar{\mu}\}$. Then $\kappa_+(\chi;G) = \kappa_-(\chi;G) = l$.*

Let $G \in D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ and let $\beta \in \Sigma(G) \cap \overline{\mathbf{R}}$ be an isolated point of $\Sigma(G)$ and δ a connected open subset of $\overline{\mathbf{R}}$ containing β such that $\delta \setminus \{\beta\} \subset \mathrm{P}(G)$. Then β is a pole of G. Let the principal part of G at β be

$$G_{-k}(\lambda-\beta)^{-k} + \cdots + G_{-1}(\lambda-\beta)^{-1}$$

for $\beta \neq \infty$ or

$$G_{-k}\lambda^k + \cdots + G_{-1}\lambda$$

for $\beta = \infty$, and let $\widetilde{G}$ denote the selfadjoint operator

$$\begin{pmatrix} G_{-1} & G_{-2} & \cdots & G_{-k} \\ G_{-2} & G_{-3} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ G_{-k} & 0 & \cdots & 0 \end{pmatrix}$$

in $\mathscr{H}^k$. Then a straightforward calculation gives the second part of the following proposition (cf. Proposition 1.5).

Proposition 3.5. *Let G, β, δ, $\widetilde{G}$ be as above. Then the multiplicity of the pole β of G is equal to* $\dim \mathscr{R}(\widetilde{G})$. *If $g \in C^\infty(\overline{\mathbf{R}}) \times H\big(\Sigma(G) \setminus \overline{\mathbf{R}}\big)$,* $\operatorname{supp} g \subset \delta$ *and g is identically equal to one in some neighbourhood of β, then*

$$\kappa_+(g;G) = \dim \mathscr{R}\big(E\big((-\infty,0);\widetilde{G}\big)\big), \qquad \kappa_-(g;G) = \dim \mathscr{R}\big(E\big((0,\infty);\widetilde{G}\big)\big),$$

where $E(\,\cdot\,;\widetilde{G})$ is the spectral function of $\widetilde{G}$.

A critical point t of G is called *regular* (we write $t \in \mathrm{K}_r(G)$) if there exists an open connected subset δ of $\overline{\mathbf{R}}$, $t \in \delta$, such that S_G restricted to $\delta \setminus \{t\}$ is a bounded operator measure. A critical point which is not regular is called *singular.* We set $\mathrm{K}_s(G) := \mathrm{K}(G) \setminus \mathrm{K}_r(G)$.

If $F := -i(G \circ \phi)$, then the relations (3.3) imply $\mathrm{K}_r(G) = \phi\big(\mathrm{K}_r(F)\big)$ and the regularity criterion Proposition 1.6 can be carried over to the $\mathbf{R}$-symmetric case. If $\lambda_0 \in \mathrm{K}_r(G)$ and $G = \sum_{i=0}^{3} G_i$ is a decomposition of G with properties analogous to that of Proposition 1.6 and F_0 is the function connected by Proposition 1.6 with F and the critical point $e^{i\Theta_0} := \psi(\lambda_0)$, which is unique up to a constant iS, S selfadjoint, then we have

$$G_0 := i(F_0 \circ \psi) + S_0, \tag{3.6}$$

where S_0 is some selfadjoint constant. Similarly to the $\mathbf{T}$-antisymmetric case, we may define the sign types of the mass at λ_0 with the help of G_0.

3.4. *Definitizable functions defined by relations.* Let $\big(\mathscr{K}, (\,\cdot\,,\cdot\,)\big)$ be a Hilbert space and let A be a closed linear relation in $\mathscr{K}$. For linear relations and, in particular, for selfadjoint linear relations in Kreĭn spaces, we refer to [4] and [5]. We define the *extended spectrum* $\sigma_e(A)$ of A by $\sigma_e(A) = \sigma(A)$ if A is a bounded operator, and $\sigma_e(A) = \sigma(A) \cup \{\infty\}$ otherwise. Put $\rho_e(A) := \overline{\mathbf{C}} \setminus \sigma_e(A)$. If $i, -i \in \rho(A)$, then $U := \psi(A) = -1 + 2i(A+i)^{-1}$ and $U^{-1} = -1 - 2i(A-i)^{-1}$ are bounded operators in $\mathscr{K}$. We have $\sigma(U) = \psi\big(\sigma_e(A)\big)$. If $\lambda \in \rho(A)$ and $z = \psi(\lambda)$, then an easy computation gives

$$(U+z)(U-z)^{-1} = -i\big(\lambda + (\lambda^2+1)(A-\lambda)^{-1}\big).$$

Now the following analogue of Theorem 1.7 is a consequence of the above considerations.

Theorem 3.6. *Let $(\mathscr{K},(\cdot,\cdot))$ be a Hilbert space, W a bounded selfadjoint operator in $\mathscr{K}$, and $\mathscr{V}$ a domain in $\overline{\mathbf{C}}$ with $i\in\mathscr{V}$. Let A be a closed linear relation in $\mathscr{K}$, $\mathscr{V}\cup\mathscr{V}^*\subset\rho_e(A)$, such that*

$$((A+i)^{-1})^*W=W(A-i)^{-1}. \tag{3.7}$$

Let, further, S be a bounded selfadjoint operator in $\mathscr{H}$ and $\Gamma\in\mathscr{L}(\mathscr{H},\mathscr{K})$. If then, for some rational function $r=r^*$ whose poles belong to $\mathscr{V}\cup\mathscr{V}^*$, the form $(Wr(A)\cdot,\cdot)$ on $\mathscr{K}$ has κ negative squares, the function

$$G(\lambda):=S+\Gamma^*W\big(\lambda+(\lambda^2+1)(A-\lambda)^{-1}\big)\Gamma,$$

defined in $\mathscr{V}\cup\mathscr{V}^*$ has an analytic continuation to a definitizable function. Moreover, there exist a function $N\in N_{\kappa'}(\mathscr{L}(\mathscr{H}))$, $0\le\kappa'\le\kappa$, holomorphic in $\mathscr{V}\cup\mathscr{V}^*$, and a rational operator function n holomorphic at all points where r is holomorphic such that

$$r(\lambda)G(\lambda)=N(\lambda)+n(\lambda)$$

for all points λ of holomorphy of r, G, N and n.

Remark 3.7. Let, in addition to the assumptions of Theorem 3.6, A be bounded. Then (3.7) is equivalent to $A^*W=WA$. Let, moreover, $\widetilde{S}$ be a bounded selfadjoint operator in $\mathscr{H}$ and $\widetilde{\Gamma}\in\mathscr{L}(\mathscr{H},\mathscr{K})$. Then the assertions of Theorem 3.6 remain true for G replaced by $\widetilde{G}$,

$$\widetilde{G}(\lambda)=\widetilde{S}+\widetilde{\Gamma}^*W(A-\lambda)^{-1}\widetilde{\Gamma}.$$

Indeed, if we set $S=\widetilde{\Gamma}^*WA(A^2+1)^{-1}\widetilde{\Gamma}$, $\Gamma=(A-i)^{-1}\widetilde{\Gamma}$, then $\widetilde{G}(\lambda)=G(\lambda)$.

If W in Theorem 3.6 is even boundedly invertible, then A is a selfadjoint linear relation in the Kreĭn space $(\mathscr{K},[\cdot,\cdot])$, $[\cdot,\cdot]:=(W\cdot,\cdot)$, with $i,-i\in\rho(A)$. Such a linear relation is called *definitizable*, if there exists a rational function $r=r^*$ whose poles belong to $\rho_e(A)$ such that $[r(A)x,x]\ge 0$, $x\in\mathscr{K}$. It is easy to see that this definition is equivalent to the one in [5]. A is definitizable if and only if the unitary operator $U:=\psi(A)$ is definitizable. The function r is definitizing for A if and only if $r\circ\phi$ is definitizing for U. If $0\in\rho(W)$, the linear relation A in Theorem 3.6 is definitizable.

As examples for the application of Theorem 3.6 we mention two operator functions, occurring in connection with λ-nonlinear eigenvalue problems.

Examples. (1) Let L be an operator polynomial in $\mathscr{H}$,

$$L(\lambda)=\lambda^n+A_{n-1}\lambda^{n-1}+\cdots+A_1\lambda+A_0,$$

where A_i, $i=0,\dots,n-1$, are bounded selfadjoint operators in $\mathscr{H}$. Assume that L is weakly hyperbolic, i.e., for every $x\in\mathscr{H}$, $x\ne 0$, all zeros of the polynomial $(L(\lambda)x,x)$ are real. Then the companion operator A of L is a definitizable bounded selfadjoint operator in the space $\mathscr{H}^n$ provided with a certain Kreĭn space inner product ([15]). We have $\sigma(A) \subset \mathbf{R}$ and

$$L(\lambda)^{-1} = Q^+(A-\lambda)^{-1}Q, \qquad \lambda \in \rho(A),$$

where $Q \in \mathscr{L}(\mathscr{H}, \mathscr{H}^n)$ maps $\mathscr{H}$ identically onto the first factor of $\mathscr{H}^n$. Then, by Theorem 3.6 and Remark 3.7, $L(\lambda)^{-1}$ is a definitizable operator function and every definitizing polynomial for A is also definitizing for $L(\lambda)^{-1}$. The same conclusion can be drawn for L being a compact perturbation of a strongly hyperbolic operator polynomial (for this notion, see [17]).

(2) Assume that on $\mathscr{H}$, besides the Hilbert scalar product $(\cdot,\cdot)$, a Kreĭn space inner product $[\cdot,\cdot]$ is given. Let W be the Gram operator of $[\cdot,\cdot]$: $[\cdot,\cdot] = (W\cdot,\cdot)$. Let $\mathscr{H}'$ be a further Kreĭn space and let A and D be definitizable selfadjoint operators in the Kreĭn spaces $\mathscr{H}$ and $\mathscr{H}'$, respectively, such that the operator $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ is definitizable in $\mathscr{K} = \mathscr{H} \times \mathscr{H}'$ and has no finite critical points. If $B \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ is compact, then

$$M := \begin{pmatrix} A & B^+ \\ B & D \end{pmatrix}$$

is definitizable ([8]). We consider the operator function

$$T(\lambda) := A - \lambda - B^+(D-\lambda)^{-1}B, \qquad \lambda \in \rho(D). \tag{3.8}$$

Its inverse can be represented in the form

$$T(\lambda)^{-1} = I_1^+(M-\lambda)^{-1}I_1,$$

where I_1 is the natural embedding of $\mathscr{H}$ in $\mathscr{K}$. If $i \in \rho(M)$, we have

$$\begin{aligned} WI_1^+(M-\lambda)^{-1}I_1 = W\big((M-i)^{-1}I_1\big)^+ M(M-i)^{-1}I_1 \\ + W\big((M-i)^{-1}I_1\big)^+\{\lambda + (\lambda^2+1)(M-\lambda)^{-1}\}(M-i)^{-1}I_1. \end{aligned}$$

Since $W\big((M-i)^{-1}I_1\big)^+$ is the adjoint of $(M-i)^{-1}I_1$ regarded as an operator from $\big(\mathscr{H},(\cdot,\cdot)\big)$ in $\mathscr{K}$, Theorem 3.6 gives that $WT(\lambda)^{-1}$ is a definitizable function and that every definitizing function for M is definitizing for $WT(\lambda)^{-1}$.

Sturm–Liouville operators with a floating singularity are of the form (3.8), with a differential operator A and matrix multiplication operators B and D.

3.5. *Representations of a given definitizable function.* First we formulate (a slightly weaker variant of) T. Azizov's result and the minimality of the representation for the $\mathbf{R}$-symmetric case.

Theorem 3.8. *Let G be an $\mathscr{L}(\mathscr{H})$-valued function locally holomorphic in an $\mathbf{R}$-symmetric neighbourhood of $\{i,-i\}$ such that $G = G^*$. Then there exist a Kreĭn space $\mathscr{K}$, a selfadjoint linear relation A in $\mathscr{K}$ with $i,-i \in \rho(A)$, and $\Gamma \in \mathscr{L}(\mathscr{H},\mathscr{K})$ such that*

$$G(\lambda) = \operatorname{Re} G(i) + \Gamma^+\big(\lambda + (\lambda^2+1)(A-\lambda)^{-1}\big)\Gamma,$$

for all λ in some neighbourhood of $\{i,-i\}$, and

$$\mathscr{K} = \overline{\mathscr{K}}(A, \mathscr{R}_{i,-i}, \Gamma\mathscr{H}).$$

If G admits two such representations with Kreĭn spaces $(\mathscr{K}_j, [\,\cdot\,,\cdot\,]_j)$, operators $\Gamma_j \in \mathscr{L}(\mathscr{H},\mathscr{K}_j)$ and selfadjoint linear relations A_j in $\mathscr{K}_j$, $j=1,2$, then

$$V\colon \sum_{k=1}^{m} g_k(A_1)\Gamma_1 x_k \longmapsto \sum_{k=1}^{m} g_k(A_2)\Gamma_2 x_k, \tag{3.9}$$

$x_k \in \mathscr{H}$, $g_k \in \mathscr{R}_{i,-i}$, $k=1,\dots,m$, is an isometric linear mapping of the dense linear set $\mathscr{K}_1(A_1,\mathscr{R}_{i,-i},\Gamma_1\mathscr{H})$ in $\mathscr{K}_1$ onto the dense linear set $\mathscr{K}_2(A_2,\mathscr{R}_{i,-i},\Gamma_2\mathscr{H})$ in $\mathscr{K}_2$ such that

$$(A_2 \pm i)^{-1} V y = V(A_1 \pm i)^{-1} y, \qquad y \in \mathscr{K}_1(A_1,\mathscr{R}_{i,-i},\Gamma_1\mathscr{H}).$$

The following result is a direct consequence of Theorem 1.9 and Remark 1.10.

Theorem 3.9. *Let $G \in D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$, and let, for some neighbourhood $\mathscr{V}$ of i,*

$$G(\lambda) = \operatorname{Re} G(i) + \Gamma^+\big(\lambda + (\lambda^2+1)(A-\lambda)^{-1}\big)\Gamma, \qquad \lambda \in \mathscr{V} \cup \mathscr{V}^*, \tag{3.10}$$

be a minimal representation of G as in Theorem 3.8. Then the selfadjoint linear relation A in $\mathscr{K}$ is definitizable and we have $\rho_e(A) = \mathrm{P}(G)$. A rational function $r = r^*$ whose poles belong to $\rho_e(A) = \mathrm{P}(G)$ is definitizing for G if and only if it is definitizing for A.

Moreover, if for some $r \in \mathscr{R}^s_{i,-i}$

$$r(z)G(z) = N(z) + n(z),$$

where $N \in N_\kappa\big(\mathscr{L}(\mathscr{H})\big)$ is holomorphic at i and $-i$ and $n \in \mathscr{R}_{i,-i}\big(\mathscr{L}(\mathscr{H})\big)$, and equality holds for all $\lambda \in \mathbf{C}\setminus\mathbf{R}$, where r, G, N and n are holomorphic, then $r(A)$ has κ negative squares.

Remark 3.10. If $G \in D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ is holomorphic in a neighbourhood of ∞, then by Theorem 3.9 in the representation (3.10) the operator A is bounded. Then G can be represented in the form

$$G(\lambda) = \widetilde{S} + \widetilde{\Gamma}^+(A-\lambda)^{-1}\widetilde{\Gamma}$$

with $\widetilde{S} = G(\infty)$. $\widetilde{\Gamma} = (A-i)\Gamma$ (see Remark 3.7).

Remark 3.11. Let J be a fundamental symmetry of the Kreĭn space $\mathscr{K}$ and A a selfadjoint linear relation in $\mathscr{K}$ with $i,\ -i \in \rho(A)$. Then by Theorems 3.6 and 3.9, A is definitizable if and only if the function $J\{\lambda + (\lambda^2+1)(A-\lambda)^{-1}\}$ defined on a neighbourhood of $\{-i, i\}$ can be continued analytically to a definitizable operator function G. The sets of the corresponding definitizing rational functions the poles of which belong to $\rho_e(A) = \mathrm{P}(G)$ coincide.

A definitizable selfadjoint relation A possesses a spectral function $E(\,\cdot\,;A)$ with properties similar to the spectral function of a definitizable selfadjoint operator. It can be defined with the help of the Cayley transform: Let $i, -i \in \rho(A)$, $U = \psi(A)$ and let $E(\,\cdot\,;U)$ be the spectral function of U. If Δ is a connected subset of $\overline{\mathbf{R}}$ and $E(\,\cdot\,;U)$ is defined for the arc $\psi(\Delta)$, we set $E(\Delta;A) := E\big(\psi(\Delta);U\big)$. The set of critical points and its subsets are defined for A in the same way as for unitary operators; they are denoted by $c(A)$, $c_\infty(A)$, $c_r(A)$. We have $c(A) = \phi\big(c(U)\big)$, $c_\infty(A) = \phi\big(c_\infty(U)\big)$ and $c_r(A) = \phi\big(c_r(U)\big)$.

Now Theorems 1.12–1.14 can be carried over to the $\mathbf{R}$-symmetric case. The $\mathbf{R}$-symmetric versions are completely analogous to the $\mathbf{T}$-skew-symmetric ones. Here we give only the analogue of Theorem 1.14.

Theorem 3.12. *Let $G \in D_i\big(\overline{\mathbf{R}};\mathscr{L}(\mathscr{H})\big)$ and $\mathrm{K}_r(G) \neq \emptyset$. Then there exists a minimal definitizable selfadjoint representing linear relation A such that $c_r(A) = \mathrm{K}_r(G)$, and for every $\lambda_0 \in \mathrm{K}_r(G)$ we have $\kappa_\pm(\mathbf{1}, G_0) = \kappa_\pm(\mathscr{L}_{\lambda_0})$, where G_0 is as in* (3.6) *and $\mathscr{L}_{\lambda_0}$ is the algebraic eigenspace of A corresponding to λ_0.*

The formulation of the results of Section 2 for the $\mathbf{R}$-symmetric case is not difficult and is left to the reader.

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