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ON THE BERS FIBER SPACES

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Abstract. It is well known that a Bers fiber space $F(\Gamma)$ for a torsion free finitely generated Fuchsian group Γ of the first kind can be identified with a Teichmüller space. If Γ has torsion, a theorem of Earle–Kra [9] asserts that in almost all cases, $F(\Gamma)$ is not isomorphic to any Teichmüller space. However, there are 39 cases which remain unsettled. Our results remove 27 from the 39 previously unknown cases.

1. Introduction

This paper is an investigation on relationships between Teichmüller spaces and Bers fiber spaces. All Fuchsian groups considered in this paper are finitely generated Fuchsian groups of the first kind which act on the upper half plane U. Let Γ be a Fuchsian group which has signature $(g,n;\nu_1,\dots,\nu_n)$, where g is the genus of the orbifold U/Γ, n is the number of distinguished points (denoted by $x_1,\dots,x_n$) and $\nu_1,\dots,\nu_n \in \{2,3,\dots\}\cup\{\infty\}$ are the ramification numbers of $x_1,\dots,x_n$, respectively. $\nu_1,\dots,\nu_n$ are arranged so that $2\le\nu_1\le\cdots\le\nu_n\le\infty$. The pair (g,n) is called the type of Γ. There is a Fuchsian group Γ of signature $(g,n;\nu_1,\dots,\nu_n)$ if and only if

$$2g-2+\sum_{j=1}^{n}\left(1-\frac{1}{\nu_j}\right)>0.$$

The Teichmüller space $T(\Gamma)$, with $g\ge 0$, $n\ge 0$, and $3g-3+n>0$, admits a representation as a simply connected domain in $\mathbf{C}^{3g-3+n}$. Associated to each point $x\in T(\Gamma)$ there is a certain Jordan domain D_x depending holomorphically on x. The Bers fiber space $F(\Gamma)$ over $T(\Gamma)$ is the set of points (x,z) with $x\in T(\Gamma)$ and $z\in D_x$. The Bers fiber space $F(\Gamma)$ is a simply connected domain in $\mathbf{C}^{3g-2+n}$.

A theorem of Bers–Greenberg [7] implies that $T(\Gamma)$ depends only on the type of Γ. For this reason, we denote by $T(g,n)$ the Teichmüller space $T(\Gamma)$ for Γ of type (g,n). On the other hand, some examples (which we will see later in this paper) show that $F(\Gamma)$ depends on the signature of Γ. We write $F(\Gamma)=F(g,n;\nu_1,\dots,\nu_n)$ to emphasize this dependence.

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In this paper, we consider the problem of finding all biholomorphic maps (we use the term "isomorphisms" throughout this paper) between Teichmüller spaces and Bers fiber spaces. The motivation for tackling this question originally stems from [19], in which Royden proved that all automorphisms of $T(g,0)$ with $g \geq 3$ are induced by self-maps of a surface of genus g. Later, Earle and Kra [9] generalized this result to all cases of analytically finite Riemann surfaces. On the other hand, in [18], Patterson gave a complete solution to the problem of finding all isomorphisms among finite dimensional Teichmüller spaces. Since then, important progress concerning the isomorphisms between Bers fiber spaces and Teichmüller spaces has been made in [5], in which Bers proved that

$$F(g,n;\infty,\ldots,\infty) \cong T(g,n+1). \tag{1.1}$$

Then Bers asked whether $F(\Gamma)$ is isomorphic to a Teichmüller space if Γ has torsion. The study of this question was initiated by Earle and Kra [9]. They proved that in most cases the answer to the Bers question is "no". The statement of their result is:

Theorem 1. *Suppose that Γ contains elliptic elements and $F(\Gamma)$ is isomorphic to $T(\Gamma')$ for some group Γ'. If the types of Γ and Γ' are (g,n) and (g',n'), respectively, then the pair $\big((g,n),(g',n')\big)$ is among the entries of the table*:

$((0,3),(0,4))$	$((0,3),(1,1))$	$((0,4),(0,5))$	$((0,4),(1,2))$
$((1,1),(1,2))$	$((1,1),(0,5))$	$((0,5),(1,3))$	$((0,5),(0,6))$
$((0,5),(2,0))$	$((1,2),(1,3))$	$((1,2),(0,6))$	$((1,2),(2,0))$
$((0,6),(1,4))$	$((0,6),(2,1))$	$((0,7),(2,2))$	$((0,8),(3,0))$

Table A

Moreover, every elliptic element of Γ has order 2, unless Γ is of type $(0,3)$.

The remaining question is: what happens if the pair $\big((g,n),(g',n')\big)$ lies in Table A? There are, of course, some obvious isomorphisms between $F(g,n;\nu_1,\ldots,\nu_n)$ and $T(g',n')$ if the pair $\big((g,n),(g',n')\big)$ lies in Table A. To enumerate all well-known isomorphisms, we note that if Γ is of type $(0,3)$, then $T(\Gamma)$ is a single point. So $F(0,3;\nu_1,\nu_2,\nu_3)$ is a disk for ν_1, ν_2, $\nu_3 \in \{2,3,\ldots\}\cup\{\infty\}$ with

$$\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}<1.$$

This leads to the isomorphism:

$$F(0,3;\nu_1,\nu_2,\nu_3) \cong T(0,4) \cong T(1,1). \tag{1.2}$$

We also observe that a Riemann surface S of type $(2,0)$, $(1,2)$ or $(1,1)$ always admits a hyperelliptic involution. Let j denote the hyperelliptic involution in each case, then $S/\langle j\rangle$ is an orbifold of signature $(0,6;2,\dots,2)$ if S is of type $(2,0)$; an orbifold of signature $(0,5;2,\dots,2,\infty)$ if S is of signature $(1,2;\infty,\infty)$; and an orbifold of signature $(0,4;2,2,2,\infty)$ if S is of signature $(1,1;\infty)$. In addition, any Riemann surface S of signature $(0,4;\infty,\dots,\infty)$ admits three conformal involutions. To explain this fact, we take an arbitrary Riemann surface S of signature $(0,4;\infty,\dots,\infty)$, and let $x_1,\dots,x_4$ denote the punctures. By a result of [9] there are three elliptic Möbius transformations (of order 2) j_1, j_2 and j_3 defined on S, where j_1 maps x_1 to x_3, x_2 to x_4, j_2 maps x_1 to x_4, x_2 to x_3, and j_3 maps x_1 to x_2, x_3 to x_4. Note that the three corresponding quotient spaces (orbifolds) $S/\langle j_1\rangle$, $S/\langle j_2\rangle$ and $S/\langle j_3\rangle$ are of signature $(0,4;2,2,\infty,\infty)$. The above observation leads to the following equivalences:

$$
\begin{aligned}
F(0,6;2,\dots,2) &\cong F(2,0;\_\_),\\
F(0,5;2,\dots,2,\infty) &\cong F(1,2;\infty,\infty),\\
F(0,4;2,2,2,\infty) &\cong F(0,4;2,2,\infty,\infty) \cong F(0,4;\infty,\dots,\infty).
\end{aligned}
$$

On the other hand, (1.1) tells us that

$$
\begin{gathered}
F(2,0;\_\_) \cong T(2,1),\\
F(0,4;\infty,\dots,\infty) \cong T(0,5) \cong T(1,2)
\end{gathered}
$$

and

$$F(1,2;\infty,\infty) \cong T(1,3).$$

Thus, we obtain some other isomorphisms:

$$F(0,6;2,\dots,2) \cong T(2,1), \tag{1.3}$$

$$F(0,5;2,\dots,2,\infty) \cong T(1,3), \tag{1.4}$$

$$F(0,4;2,2,2,\infty) \cong T(1,2) \cong T(0,5) \cong F(0,4;2,2,\infty,\infty). \tag{1.5}$$

Except for (1.1)–(1.5), it is not known whether or not there are any other isomorphisms between Bers fiber spaces and Teichmüller spaces. The following conjecture was posed in 1974:

Conjecture (Earle–Kra [9]). *If* Γ *contains elliptic elements, then* (1.2)–(1.5) *exhaust all possible isomorphisms between Bers fiber spaces and Teichmüller spaces.*

As we see, Theorem 1 is a significant step towards this conjecture. What is left unanswered is a finite number of cases. As a matter of fact, we can immediately find that there remain 39 unknown cases, which are exhibited in Table B, where row 5 and row 6 are distinct since $T(0,6)$ or $T(2,0)$ are not isomorphic to $T(1,3)$ by Patterson's result [18].

signature $(g,n;\nu_1,\ldots,\nu_n)$	type (g',n')	# of cases
$(0,8;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{8-m})\,,\ 0<m\le 8\,,$	$(3,0)$	8
$(0,7;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{7-m})\,,\ 0<m\le 7\,,$	$(2,2)$	7
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})\,,\ 0<m<6\,,$	$(2,1)$	5
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})\,,\ 0<m<6\,,$	$(1,4)$	5
$(0,5;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{5-m})\,,\ 0<m\le 5\,,\ m\ne 4\,,$	$(0,6)$ or $(2,0)$	4
$(0,5;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{5-m})\,,\ 0<m\le 5\,,\ m\ne 4\,,$	$(1,3)$	4
$(1,2;2,m)\,,\ m=2$ or $m=\infty$	$(1,3)$	2
$(1,2;2,m)\,,\ m=2$ or $m=\infty$	$(0,6)$ or $(2,0)$	2
$(0,4;2,\infty,\infty,\infty)$	$(0,5)$ or $(1,2)$	1
$(1,1;2)$	$(0,5)$ or $(1,2)$	1

Table B

This paper is a contribution to the Earle–Kra conjecture. What we attempt to do is to eliminate most entries of Table B. More precisely, we prove

Theorem 2. *Suppose that* Γ *has torsion and* $F(\Gamma)$ *is isomorphic to* $T(\Gamma')$ *for some group* Γ'. *If* Γ *has signature* $(g,n;\nu_1,\ldots,\nu_n)$ *and* Γ' *has type* (g',n'), *then the pair* $\big((g,n;\nu_1,\ldots,\nu_n),(g',n')\big)$ *is among the entries of Table C*:

signature $(g,n;\nu_1,\ldots,\nu_n)$ of Γ	type (g',n') of Γ'
$(0,8;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{8-m})\,,\quad m=3,6\,,$	$(3,0)$
$(0,7;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{7-m})\,,\quad m=2,4,6\,,$	$(2,2)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})\,,\quad m=1,2,3,4,6\,,$	$(2,1)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})\,,\quad m=3,4\,,$	$(1,4)$
$(0,5;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{5-m})\,,\quad m=2,4\,,$	$(1,3)$
$(0,4;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{4-m})\,,\quad m=2,3\,,$	$(1,2)$ or $(0,5)$
$(0,3;\nu_1,\nu_2,\nu_3)\,,\ \nu_1,\nu_2,\nu_3\in\{2,3,\cdots\}\cup\{\infty\}$ with $1/\nu_1+1/\nu_2+1/\nu_3<1$	$(1,1)$ or $(0,4)$

Table C

An interesting consequence is:

Theorem 3. *Let* Γ *be a finitely generated Fuchsian group of the first kind of type* (g, n). *Then the Bers fiber space* $F(\Gamma)$ *is isomorphic to the Teichmüller space* $T(g, n+1)$ *if and only if one of the following conditions is satisfied*:

(i) Γ *is of type* $(0, 3)$.

(ii) *The signature of* Γ *is either* $(0, 4; 2, 2, 2, \infty)$, *or* $(0, 4; 2, 2, \infty, \infty)$.

(iii) Γ *is torsion free.*

Idea of proof of Theorem 2. To describe what we intend to do in this paper, we need to review the method in [9] that was used to prove Theorem 1. The group Γ is considered a group of holomorphic automorphisms of $F(\Gamma)$. Assume that the pair $\big((g, n), (g', n')\big)$ does not lie in Table A, where (g, n) and (g', n') are the types of Γ and Γ', respectively. Then a cyclic subgroup $\mathscr{G}$ of Γ (with prime order) can be chosen so that $\mathscr{G}$ acts on $F(\Gamma)$ as a group of holomorphic automorphisms, but $\mathscr{G}$ cannot act on $T(\Gamma')$ as a group of holomorphic automorphisms. In addition, $\mathscr{G}$ acts trivially on the image of the holomorphic section of $\pi\colon F(\Gamma) \to T(\Gamma)$ corresponding to the elliptic generator of $\mathscr{G}$.

Our method is different from the above. We construct a cyclic group $\mathscr{G}'$ (with prime order, too) of holomorphic automorphisms of $F(\Gamma)$ satisfying the condition that $\mathscr{G}'$ is not a subgroup of Γ, but it is still fiber-preserving and leaves invariant the image of a special holomorphic section of $\pi\colon F(\Gamma) \to T(\Gamma)$. This construction depends essentially on the signature of Γ. We will check that if the signature $(g, n; \nu_1, \dots, \nu_n)$ is in Table B but not in Table C and not $(0, 6; 2, 2, 2, 2, 2, \infty)$, then $\mathscr{G}'$ cannot act as holomorphic automorphisms on the corresponding Teichmüller space.

The above method must fail in handling the case that $(g, n; \nu_1, \dots, \nu_n) = (0, 6; 2, 2, 2, 2, 2, \infty)$ and $(g', n') = (2, 1)$. To see this, we note that every periodic authomorphism on $F(0, 6; 2, 2, 2, 2, 2, \infty)$ that we can construct, also acts on $F(0, 6; 2, 2, 2, 2, 2, 2)$, and all periodic automorphisms on $F(0, 6; 2, 2, 2, 2, 2, 2)$ act on $T(2, 1)$ as well, by (1.3). We need to develop a new method. By using a construction of periodic self-maps of $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ (see Magnus [15]), we prove that any two components of the hyperelliptic locus in $T(2, 1)$ are modular equivalent. Then we observe that the Teichmüller modular group Mod $(2, 1)$ can be identified with the modular group mod $(2, 0)$. Note also that mod $(2, 0)$ acts fiber-preservingly on $F(2, 0; \_\_) \cong F(0, 6; 2, 2, 2, 2, 2, 2)$, which leads to the argument that there is a parabolic automorphism which acts on $F(0, 6; 2, 2, 2, 2, 2, \infty)$ but does not act on $T(2, 1)$ as a parabolic transformation. This contradicts a theorem of Royden [19] which states that the Techmüller metric coincides with the Kobayashi metric in any finite dimensional Teichmüller space.

This paper is organized as follows. Section 2 is a background of Teichmüller space theory. Some known results are reviewed. In Section 3, we reprove Theorem 1. In Section 4, we construct certain useful automorphisms on special orbifolds. These automorphisms are used in Section 5 to construct holomorphic automorphisms of various Bers fiber spaces which have some nice properties. In Section 6, we study certain elliptic modular transformations of Teichmüller spaces of low dimensions. Section 7 is devoted to the proofs of Theorem 3 and most cases of Theorem 2. In Section 8, we introduce a new method to investigate the Bers fiber space $F(\Gamma)$ for Γ of type $(0,6)$, and complete the proof of Theorem 2. The appendix is devoted to the proof of a topological fact that any two components of hyperelliptic locus in the Teichmüller space $T(2,1)$ are modular equivalent. The method is similar to [16].

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2. The Teichmüller and Bers fiber spaces

In this section we review some basic definitions and fundamental results in Teichmüller theory. We are mainly concerned with the Bers fiber space for a group with torsion. For more background on this material, see Bers [5], Earle–Kra [9], [10] and Kra [12].

Let Γ be a finitely generated Fuchsian group of the first kind acting on the upper half plane U. Assume that the orbifold U/Γ is of the type (g,n). Let $M(\Gamma)$ denote the space of Beltrami coefficients for Γ; that is, $M(\Gamma)$ consists of all measurable functions μ on U satisfying

(i) $(\mu\circ\gamma)\cdot\overline{\gamma'}/\gamma'=\mu$, for all $\gamma\in\Gamma$, and

(ii) $\|\mu\|_\infty=\text{ess sup }\{|\mu(z)|:z\in U\}<1$.

Two elements μ, $\mu'\in M(\Gamma)$ are *equivalent* (write $\mu\sim\mu'$) if $w^\mu=w^{\mu'}$ on $\widehat{\mathbf{R}}=\partial U$, where w^μ is the unique quasiconformal self-map of $\widehat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ which fixes 0, 1, ∞, is conformal on the lower half plane L, and satisfies the Beltrami equation $w_{\bar z}=\mu w_z$ on U (see Ahlfors–Bers [3]). Note that the theorem of Ahlfors–Bers also implies that for a fixed $z\in\mathbf{C}$, $\mu\mapsto w^\mu(z)$ is a holomorphic function on $M(\Gamma)$. The *Teichmüller space* $T(\Gamma)$ is the space of equivalence classes $[\mu]$ for $\mu\in M(\Gamma)$. An important theorem of Ahlfors [2] (see also Bers [4]) asserts that $T(g,n)$ is a complex manifold of dimension $3g-3+n$.

Let $Q(\Gamma)$ denote the group of quasiconformal self-maps w of U with $w\Gamma w^{-1}$ a Fuchsian group. The *Teichmüller distance* between two points $[\mu]$ and $[\mu']\in T(\Gamma)$ is defined by

$$\langle[\mu],[\mu']\rangle=\tfrac{1}{2}\log\inf K(w),$$

where w runs over those quasiconformal self-maps w' in $Q(\Gamma)$ for which w' agrees with $w_\mu\circ(w_{\mu'})^{-1}$ on $\mathbf{R}$ and $K(w)$ is the maximal dilatation of w. It is well known that $\big(T(\Gamma),\langle\ ,\ \rangle\big)$ is a complete metric space.

Since $T(\Gamma)$ is a complex manifold, the *Kobayashi pseudo-metric* on $T(\Gamma)$ can be defined as the largest pseudo-metric d so that

$$d\big(f(z_1), f(z_2)\big) \le \varrho_U(z_1, z_2),$$

for all holomorphic maps f of U into $T(\Gamma)$ and for all z_1, $z_2 \in U$, where ϱ_U is the Poincaré metric (with constant negative curvature -1) on U. A theorem of Royden [19] asserts that the Kobayashi metric on $T(\Gamma)$ coincides with the Teichmüller metric.

It is well known that as a complex manifold $T(\Gamma)$ depends only on the type of Γ (see Bers–Greenberg [7]). We usually denote by $T(g,n)$ the Teichmüller space $T(\Gamma)$ for Γ of type (g,n).

For a moment, let Γ be of signature $(2,0;\_\_)$. Then U/Γ admits a hyperelliptic involution j which leaves precisely 6 points (which are called *Weierstrass points*) fixed. Lifting j to U, we obtain a $\tilde{j} \in \mathrm{PSL}\,(2,\mathbf{R})$ such that $\tilde{j}$ and Γ generate a Fuchsian group Γ_0. It is clear that Γ is the normal subgroup of Γ_0 with index 2 and the signature of Γ_0 is $(0,6;2,\dots,2)$. Note also that $\dim T(\Gamma_0) = \dim T(\Gamma) = 3$. This implies that $T(\Gamma_0) \cong T(\Gamma)$. Similar phenomena occur when Γ is of signature $(1,2;\infty,\infty)$ or $(1,1;\infty)$. We thus obtain

$$T(0,6) \cong T(2,0), \qquad T(0,5) \cong T(1,2), \qquad T(0,4) \cong T(1,1). \tag{2.1}$$

A natural question arises as to whether or not there are any other isomorphisms between Teichmüller spaces. This question is answered by a theorem of Patterson [18] which states that (2.1) exhausts all isomorphisms between Teichmüller spaces with distinct types.

An automorphism θ of Γ is called *geometric* if there is a $w \in Q(\Gamma)$ such that $\theta(\gamma) = w \circ \gamma \circ (w)^{-1}$, for all $\gamma \in \Gamma$. Let $w' \in Q(\Gamma)$. Then $w' \circ \gamma \circ (w')^{-1} = w \circ \gamma \circ (w)^{-1}$, for all $\gamma \in \Gamma$ if and only if $w|_{\mathbf{R}} = w'|_{\mathbf{R}}$. The *Teichmüller modular group* $\mathrm{Mod}\,\Gamma$ is defined as the group of geometric automorphisms (denoted by $\mathrm{mod}\,\Gamma$) modulo the normal subgroup of inner automorphisms. The action of $\chi \in \mathrm{Mod}\,\Gamma$ on $T(\Gamma)$ is defined as follows: let χ be the image of θ under the quotient map $q\colon \mathrm{mod}\,\Gamma \to \mathrm{Mod}\,\Gamma$, and let $w \in Q(\Gamma)$ be chosen so that $\theta(\gamma) = w \circ \gamma \circ (w)^{-1}$, for all $\gamma \in \Gamma$. For every $\mu \in M(\Gamma)$, the Beltrami coefficient μ' of the map $w_\mu \circ w^{-1}$ is given by the formula:

$$\mu'(z) = \frac{\nu(z) + (\mu \circ w^{-1})(z) \cdot \overline{(\partial w^{-1}/\partial z)}/(\partial w^{-1}/\partial z)}{1 + \bar{\nu}(z) \cdot (\mu \circ w^{-1})(z) \cdot \overline{(\partial w^{-1}/\partial z)}/(\partial w^{-1}/\partial z)}, \tag{2.2}$$

where ν is the Beltrami coefficient of w^{-1}. It is easy to check that μ' belongs to $M(\Gamma)$ and $[\mu'] \in T(\Gamma)$ depends on χ but not on θ and w. Hence, (2.2) induces an action of χ on $T(\Gamma)$ as a holomorphic automorphism. We see that the group $\mathrm{Mod}\,\Gamma$ acts on $T(\Gamma)$ as a group of holomorphic automorphisms.

Let Mod (g,n) denote the Teichmüller modular group Mod Γ for a torsion free Fuchsian group Γ of type (g,n), and let Mod $(g,n;\nu_1\cdots\nu_n)$ denote Mod Γ for Γ of signature $(g,n;\nu_1\cdots\nu_n)$. Royden's theorem [19] and its generalization, due to Earle–Kra [9], assert that the full group of holomorphic automorphisms of $T(g,n)$ is isomorphic to Mod (g,n) except when $(g,n)=(0,3),(0,4),(1,1),(1,2)$ or $(2,0)$; for any one of these special cases, the action of Mod (g,n) on $T(g,n)$ is not faithful, by which we mean that distinct elements of Mod (g,n) do not necessarily induce distinct holomorphic automorphisms of $T(g,n)$. Let Aut $T(g,n)$ denote the group of holomorphic automorphisms of $T(g,n)$. Precisely, we have

$$\begin{aligned}
&\operatorname{Aut} T(2,0) \cong \operatorname{Mod}(2,0)/\mathbf{Z}_2 \cong \operatorname{Mod}(0,6;2,2,2,2,2,2) \cong \operatorname{Mod}(0,6),\\
&\operatorname{Aut} T(1,2) \cong \operatorname{Mod}(1,2)/\mathbf{Z}_2 \cong \operatorname{Mod}(0,5;2,2,2,2,\infty) \subsetneqq \operatorname{Mod}(0,5),\\
&\operatorname{Aut} T(1,1) \cong \operatorname{Aut}\ T(0,4) \cong \operatorname{PSL}(2,\mathbf{R}),\\
&\operatorname{Aut} T(0,3) = \{\mathrm{id}\},
\end{aligned} \tag{2.3}$$

where $\mathbf{Z}_2$ stands for the subgroup of Mod $(2,0)$ (respectively Mod $(1,2)$) determined by the hyperelliptic involution on a surface of type $(2,0)$ (respectively a surface of type $(1,2)$).

In [6] Bers introduced a classification for elements χ of the Teichmüller modular group Mod Γ for a torsion free group Γ of type (g,n), $3g-3+n>0$. Let

$$a(\chi)=\inf_{\tau\in T(\Gamma)}\langle\tau,\chi(\tau)\rangle,$$

where $\langle\ ,\ \rangle$ is the Teichmüller metric. χ is called *elliptic* if it has a fixed point in $T(\Gamma)$; *parabolic* if there is no fixed point and $a(\chi)=0$; *hyperbolic* if $a(\chi)>0$ and $a(\chi)$ is assumed; and *pseudo-hyperbolic* if $a(\chi)>0$ and $a(\chi)$ is not assumed.

The element χ is induced by a self-map f of a surface S of type (g,n). The isotopy class of a self-map f of S can be topologically classified as follows (see Thurston [20]). A (non-empty) finite set of simple curves $c=\{c_1,\ldots,c_r\}$, $r\geq 1$, is called *admissible* if c_i is not homotopic to a point, a puncture, or some c_j, for $j\neq i$. f is called a *reduced map* if it keeps c invariant. f is *reducible* if it is isotopic to a reduced map. f is *irreducible* if it is not reducible. A theorem of Bers [6] states that an element $\chi\in\operatorname{Mod}\Gamma$ is elliptic if and only if f is isotopic to a periodic map; if f is not isotopic to a periodic map, then $\chi\in\operatorname{Mod}\Gamma$ is hyperbolic if and only if f is an irreducible map. A reducible non-periodic self-map f corresponds to either parabolic or pseudo-hyperbolic element χ. More precisely, let f be a reduced non-periodic self-map which induces a parabolic element χ, then there exists an $n\in\mathbf{Z}^+$ such that f^n restricts to id on all parts $S-\{N(c)\}$, where $N(c)$ is an arbitrarily small neighborhood of c.

The *Bers fiber space* over $T(\Gamma)$, denoted by $F(\Gamma)$, is a subset of $T(\Gamma)\times\mathbf{C}$ consisting of pairs $([\mu],z)$, where $\mu\in M(\Gamma)$, and $z\in w^\mu(U)$. Let $\pi\colon F(\Gamma)\to T(\Gamma)$

denote the natural (holomorphic) projection onto the first factor. By definition, the fiber of π over a point $[\mu] \in T(\Gamma)$ is the quasidisk $w^{\mu}(U)$ which depends only on the equivalence class of μ. $F(\Gamma)$ is an open connected and simply connected subset of $\mathbf{C}^{3g-2+n}$. As mentioned in the introduction, $F(\Gamma)$ depends on the signature of Γ. Let $F(g,n;\nu_1,\dots,\nu_n)$ denote the Bers fiber space $F(\Gamma)$ for Γ of signature $(g,n;\nu_1,\dots,\nu_n)$.

Examples (Dependence of Bers fiber spaces on the signatures of their groups). If the signature $(g,n;\nu_1,\dots,\nu_n)$ of Γ has the property that $2 < \nu_i < \infty$ for some i, $i \in \{1,\dots,n\}$, then Theorem 1 asserts that $F(g,n;\nu_1,\dots,\nu_n)$ cannot be isomorphic to any Teichmüller space. On the other hand, (1.1) states that $F(g,n;\infty,\dots,\infty)$ is isomorphic to $T(g,n+1)$. Hence, $F(g,n;\nu_1,\dots,\nu_n)$ is not isomorphic to $F(g,n;\infty,\dots,\infty)$ if $2 < \nu_i < \infty$ for some $i \in \{1,2,\dots,n\}$.

Another example is given implicitly by Theorems 1 and 2. From (1.5) we have $F(0,4;2,2,\infty,\infty) \cong F(0,4;2,2,2,\infty) \cong F(0,4;\infty,\infty,\infty,\infty) \cong T(0,5) \cong T(1,2)$. Theorem 2 asserts that $F(0,4;2,\infty,\infty,\infty)$ is not isomorphic to $T(0,5) \cong T(1,2)$. By Theorem 1, we know that when Γ is of type $(0,4)$ or $(1,1)$, $T(0,5)$ and $T(1,2)$ are the only two possible Teichmüller spaces to which $F(\Gamma)$ could be isomorphic. We conclude that $F(0,4;2,\infty,\infty,\infty)$ is isomorphic to neither $F(0,4;2,2,\infty,\infty)$ nor $F(0,4;2,2,2,\infty)$, while $F(0,4;2,2,\infty,\infty)$ and $F(0,4;2,2,2,\infty)$ are isomorphic to each other. This example shows that two Bers fiber spaces $F(\Gamma)$ and $F(\Gamma')$ may or may not be isomorphic to each other even if Γ and Γ' have the same type, both contain only elliptic elements of the same order, but their signatures are distinct.

The modular group $\operatorname{mod}\Gamma$ is defined as the group of geometric automorphisms of Γ. An element $\theta \in \operatorname{mod}\Gamma$ acts biholomorphically on $F(\Gamma)$ as follows: let θ be represented by $w \in Q(\Gamma)$, then

$$\theta([\mu], z) = ([\nu], \hat{z}), \tag{2.4}$$

where ν is the Beltrami coefficient of the map $w^{\mu} \circ w^{-1}$, and $\hat{z} = w^{\nu} \circ w \circ (w^{\mu})^{-1}(z)$. By Theorem 6 of Bers [5], the action of $\operatorname{mod}\Gamma$ on $F(\Gamma)$ is *effective*, by which we mean that there is an $x \in F(\Gamma)$ with $\theta(x) \neq x$ whenever $\theta \in \operatorname{mod}\Gamma$ is non-trivial. We usually identify the group $\operatorname{mod}\Gamma$ with its action on $F(\Gamma)$.

Since Γ is centerless, Γ is isomorphic to the group of inner automorphisms of Γ by associating to each $\delta \in \Gamma$ the automorphism $\gamma \mapsto \delta \circ \gamma \circ \delta^{-1}$; namely, Γ acts by conjugation as automorphisms of Γ. Thus Γ is isomorphic to a normal subgroup of $\operatorname{mod}\Gamma$ which we denote by Γ also. Γ, as a subgroup of $\operatorname{mod}\Gamma$, now also acts on $F(\Gamma)$ as a group of holomorphic automorphisms. The action of $\gamma \in \Gamma$ on $F(\Gamma)$ is given by

$$\gamma([\mu], z) = \big([\mu], \gamma^{\mu}(z)\big) = \big([\mu], w^{\mu} \circ \gamma \circ (w^{\mu})^{-1}(z)\big) \qquad \text{for all } z \in w^{\mu}(U). \tag{2.5}$$

By definition, the Teichmüller modular group $\operatorname{Mod}\Gamma$ is the factor group $\operatorname{mod}\Gamma/\Gamma$. If $\theta \in \operatorname{mod}\Gamma$, and χ is the image of θ in $\operatorname{Mod}\Gamma$ via the natural quotient homomorphism q: $\operatorname{mod}\Gamma \to \operatorname{Mod}\Gamma$, then the following diagram is commutative:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\theta} & F(\Gamma) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T(\Gamma) & \xrightarrow{\chi} & T(\Gamma). \end{array}$$

Note that the action of $\theta \in \operatorname{mod}\Gamma$ on $F(\Gamma)$ is biholomorphic, fiber-preserving, and effective. Moreover, every element of Γ can be viewed as a holomorphic automorphism of $F(\Gamma)$ which leaves invariant each fiber of π: $F(\Gamma) \to T(\Gamma)$.

Let $e \in \Gamma$ be an elliptic element, and let z_0 denote the fixed point of e in U. For any $[\mu] \in T(\Gamma)$, there is only one fixed point $z_\mu = w^\mu(z_0)$ of $e^\mu = w^\mu \circ e \circ (w^\mu)^{-1}$ in the quasidisk $w^\mu(U)$. This implies that the map s: $T(\Gamma) \to F(\Gamma)$ defined by sending $[\mu] \in T(\Gamma)$ to $([\mu], z_\mu) \in F(\Gamma)$ is a section of π: $F(\Gamma) \to T(\Gamma)$. Since w^μ depends holomorphically on $\mu \in M(\Gamma)$, the section s is a holomorphic section. The image $s\big(T(\Gamma)\big)$ under the map s is a complex manifold which is embedded in $F(\Gamma)$ and is isomorphic to $T(\Gamma)$. The section defined by an elliptic element of Γ is called a *canonical section* of π (see Earle–Kra [9]).

3. Proof of Theorem 1

The objective of this section is to give another proof of Theorem 1. The information obtained from our argument is also useful in proving Theorem 2.

A Riemann surface S of type (g,n) with $2g-2+n>0$, $n \geq 1$, is called *hyperelliptic* if S admits a hyperelliptic involution. Here by a hyperelliptic involution on S we mean a conformal involution on S (hence on the compactification $\overline{S}$) which has $2g+2$ fixed points on $\overline{S}$, interchanges pairwise the n punctures if n is even, and fixes one puncture and interchanges the other $n-1$ punctures pairwise if n is odd. The subset of a Teichmüller space consisting of hyperelliptic Riemann surfaces is called the *hyperelliptic locus*. In general, the hyperelliptic locus is not connected, but each component has the same dimension. The dimension of the hyperelliptic locus is defined by the dimension of one of its components. We need

Proposition 3.1. *Let Γ be a torsion free finitely generated Fuchsian group of type (g,n) with $2g-2+n>0$, and let $\chi \in \operatorname{Mod}\Gamma$ be an elliptic modular transformation of prime order, with the property that the restriction of χ to a non-empty component l of the hyperelliptic locus is* id. *Then χ is either* id *or a hyperelliptic involution.*

Proof. Suppose that $\chi \in \operatorname{Mod}\Gamma$ is non-trivial. Let $T(g,n)^\chi = \{x \in T(g,n) : \chi(x) = x\}$. By hypothesis, $l \subset T(g,n)^\chi$. It is well known that $T(g,n)^\chi$ is again a Teichmüller space of type (g^*,n^*) (see Kravetz [13], or Earle–Kra [10]), where g^* and n^* are defined as follows. Let χ be induced by a conformal self-map h on a Riemann surface S of type (g,n) (Nielsen's theorem [17] asserts that such an S exists), g^* is the genus of the surface $S/\langle h\rangle$, and n^* is the number of the distinguished points (including punctures) on $S/\langle h\rangle$. Since we assume that l is a non-empty component of the hyperelliptic locus, Lemma 1 of Patterson [18] implies that

$$\dim l = 2g - 1 + \left[\tfrac{1}{2}n\right], \tag{3.1}$$

where $[x]$ denote the largest integer less than or equal to x. Let k denote the number of fixed points of h on the compactification $\overline{S}$ of S, and let m be the number of the punctures fixed by h. Since h defines a branched covering $\overline{S} \to \overline{S}/\langle h\rangle$, the Riemann–Hurwitz formula (see for example, Theorem I.2.7 of Farkas–Kra [11]) shows that

$$2g - 2 = \mathrm{ord}\,(h) \cdot (2g^* - 2) + B, \tag{3.2}$$

where B is the total branch number. The number of fixed points of a conformal automorphism on $\overline{S}$ is at most $2g+2$. By definition, there are $n-m$ punctures on S which are not fixed by h. Since the order of h is ≥ 2, the number of orbits of these $n-m$ points under h is at most $\frac{1}{2}(n-m)$. Note that any one of these orbits projects to a distinguished point on $\overline{S}/\langle h\rangle$. We thus obtain

$$n^* \leq k + \tfrac{1}{2}(n-m). \tag{3.3}$$

Our claim is $g^* = 0$. Suppose that $g^* \geq 1$. There are three possibilities to consider.

Case I. $g = 0$. The left-hand side of (3.2) is negative, while the right-hand side is positive (since $g^* \geq 1$, by hypothesis). We see that this case cannot occur.

Case II. $g \geq 2$. Since ord $(h) \geq 2$, and $k \leq B$, from (3.2), we obtain

$$2g - 2 = \mathrm{ord}\,(h) \cdot (2g^* - 2) + B \geq 2(2g^* - 2) + k,$$

or

$$g^* \leq \tfrac{1}{2}g + \tfrac{1}{2} - \tfrac{1}{4}k. \tag{3.4}$$

Since h is not a hyperelliptic involution (otherwise $g^* = 0$), by Corollary 2 to V.1.5 and Proposition III.7.11 of Farkas–Kra [11], we see that $k \leq 4$ if $\overline{S}$ is hyperelliptic, and $k \leq 2g-1$ otherwise. Since $g \geq 2$, we have

$$k \leq 2g. \tag{3.5}$$

Now from (3.1), (3.3), (3.4), and (3.5), we obtain

$$\begin{aligned} 2g - 1 + \left[\tfrac{1}{2}n\right] &= \dim l \leq \dim T(g^*, n^*) = 3g^* - 3 + n^* \\ &\leq 3\left(\tfrac{1}{2}g + \tfrac{1}{2} - \tfrac{1}{4}k\right) - 3 + k + \tfrac{1}{2}(n-m) \leq 2g - \tfrac{3}{2} + \tfrac{1}{2}n \\ &= \begin{cases} 2g - \frac{3}{2} + \left[\frac{1}{2}n\right], & \text{if } n \text{ is even;} \\ 2g - \frac{3}{2} + \left[\frac{1}{2}n\right] + \frac{1}{2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.6}$$

So (3.6) is a contradiction if n is even. If n is odd, we claim that either ord (h) is not 2, or $m > 0$. Otherwise, these n points produce $\frac{1}{2}n$ orbits, contradicting that n is odd. We see that (3.6) is also impossible when n is odd. It follows that (3.6) cannot hold in any case.

Case III. $g = 1$. In this case, from (3.2) and the hypothesis that $g^* \geq 1$, we see that $g^* = 1$, $B = k = m = 0$, and $n^* \leq \frac{1}{2}n$. A similar computation as (3.6) shows that

$$1 + \left[\tfrac{1}{2}n\right] \leq 3g^* - 3 + n^* \leq \tfrac{1}{2}n.$$

But this is impossible. In summary, we conclude that the case of $g^* \geq 1$ cannot occur.

Next, we consider the case of $g^* = 0$. From (3.3), we have

$$\begin{aligned} 2g - 1 + \left[\tfrac{1}{2}n\right] &= \dim l \leq -3 + k + \tfrac{1}{2}(n-m) \\ &\leq -3 + 2g + 2 + \tfrac{1}{2}(n-m) = 2g - 1 + \tfrac{1}{2}(n-m). \end{aligned} \tag{3.7}$$

If n is even, then $\left[\frac{1}{2}n\right] = \frac{1}{2}n$. (3.7) cannot hold unless $m = 0$ and all equalities in (3.7) hold. It follows that $k = 2g + 2$ and that h is a hyperelliptic involution. If n is odd, then $m \leq 1$. We claim that m is not zero. Suppose for the contrary that $m = 0$. Then n cannot be one, and we must have $n \geq 3$. From (3.7) once again, we have ord $(h) = 2$. Observe that

$$n = m_1 + 2m_2,$$

where m_i is the number of orbits of the punctures with period i. Note also that $m_1 = m$. We conclude that $m = 1$ if n is odd, and zero if n is even. It follows from the definition that h is a hyperelliptic involution. This completes the proof. □

As an easy consequence, we obtain

Corollary 3.2. *A modular transformation* χ *of* $T(\Gamma)$ *is* id *if its restriction to a subspace with dimension greater than the dimension of the hyperelliptic locus is* id.

Proof. χ must be an elliptic modular transformation. Let $a = bp$ be the order of χ, where p is a prime number. By the same computation as in Proposition 3.1, we conclude that χ^b is id, which occurs only if χ is id, as claimed. □

Now we are able to prove Theorem 1. Let Γ, Γ' be finitely generated Fuchsian groups of the first kind. Assume that Γ contains at least one elliptic element and is of type (g, n), and that Γ' is torsion free with type (g', n'). We need two simple lemmas.

Lemma 3.3. *Assume that* (g', n') *is not* $(0,5)$, $(1,3)$, $(0,6)$, $(1,4)$, $(2,1)$, $(2,2)$, *or* $(3,0)$. *Then the codimension of the hyperelliptic locus of* $T(\Gamma')$ *is not one, and the codimension of the hyperelliptic locus of* $T(\Gamma')$ *is zero if and only if* (g', n') *is* $(0,3)$, $(0,4)$, $(1,1)$, $(1,2)$, *or* $(2,0)$.

Proof. Suppose that the codimension of the hyperelliptic locus is one. The dimension of the hyperelliptic locus can be computed; it is $2g'-1+\left[\frac{1}{2}n'\right]$. By assumption, we have

$$2g'-1+\left[\tfrac{1}{2}n'\right]+1=3g'-3+n',$$

which says that $(g',n')=(0,5),(1,3),(0,6),(1,4),(2,1),(2,2),$ or $(3,0)$, contradicting our hypothesis. The second statement is true because there is a non-trivial modular transformation with order 2 which acts trivially on the Teichmüller space $T(g',n')$ when $(g',n')=(0,3),(0,4),(1,1),(1,2)$ or $(2,0)$. See (2.3). □

Lemma 3.4. *Assume that* $(g',n')=$ $(0,5)$, $(1,3)$, $(0,6)$, $(1,4)$, $(2,1)$, $(2,2)$ *or* $(3,0)$, *and that there is an isomorphism* φ: $F(\Gamma)\to T(\Gamma')$. *Then the pair of types* $\big((g,n),(g',n')\big)$ *is among the entries of the following table*:

$((0,4),(0,5))$	$((1,1),(0,5))$	$((0,5),(1,3))$	$((0,5),(0,6))$
$((1,2),(1,3))$	$((1,2),(0,6))$	$((0,6),(1,4))$	$((0,6),(2,1))$
$((0,7),(2,2))$	$((0,8),(3,0))$		

Table D

Proof. The Teichmüller space $T(\Gamma)$ is biholomorphically equivalent to the image of a canonical section which has codimension one in the fiber space $F(\Gamma)$. We thus have

$$\dim T(\Gamma)=\dim F(\Gamma)-1=\dim T(\Gamma')-1.$$

That is,

$$3g-3+n=3g'-4+n'.$$

The assertion then follows by solving this equation. □

Remark. Table D constitutes a core part of Table A in the introduction. Table A can be easily obtained by adding relations (2.1) into Table D.

Proof of Theorem 1. Suppose that there is an isomorphism φ: $F(\Gamma)\to T(\Gamma')$, and that the pair $\big((g,n),(g',n')\big)$ does not belong to the entries of Table A in the introduction. In particular, $\big((g,n),(g',n')\big)$ does not belong to the entries of Table D. By Lemma 3.4, (g',n') is not of $(0,5),(1,3),(0,6),(1,4),(2,1),(2,2),$ or $(3,0)$. Since Γ has torsion, we can choose an elliptic element $\gamma\in\Gamma$ with prime order. The image of holomorphic section s of π: $F(\Gamma)\to T(\Gamma)$, which is determined by the fixed point of γ in U, is equivalent to $T(\Gamma)$. It is obvious that $\varphi\circ s\big(T(\Gamma)\big)$ has codimension one in $T(\Gamma')$. On the other hand, Lemma 3.3 says that the codimension of a component of the hyperelliptic locus of $T(\Gamma')$ is at least of codimension two. We conclude that

$$\dim\varphi\circ s\big(T(\Gamma)\big)>\dim\{\text{hyperelliptic locus of } T(\Gamma')\}.$$

Now $\gamma \in \operatorname{mod}\Gamma$ fixes $s\big(T(\Gamma)\big)$ pointwise, which implies that $\gamma' = \varphi \circ \gamma \circ \varphi^{-1} \in \operatorname{Mod}\Gamma'$ fixes $\varphi \circ s\big(T(\Gamma)\big)$ pointwise as well. It follows from Corollary 3.2 that γ' is id. This leads to a contradiction.

Also, from the proof of Proposition 3.1, we can deduce that every elliptic element of Γ must be of order 2^n for some positive integer n if an isomorphism of $F(\Gamma)$ onto $T(\Gamma')$ exists. Now the second part of Theorem 1 follows from the fact that any non-trivial modular transformation of $T(\Gamma')$ which acts trivially on another Teichmüller space $T(\Gamma'')$ ($\cong$ a component of the hyperelliptic locus in $T(\Gamma')$) must be of order 2. Details are omitted. □

We proceed to investigate the cases when the pair of types $\big((g,n),(g',n')\big)$ lie in Table A. Obviously, the argument of this section fails to derive a contradiction. As a matter of fact, if we denote by $s\colon T(\Gamma) \to F(\Gamma)$ the canonical section which is determined by the fixed point of an elliptic element $e \in \Gamma$, we have

Lemma 3.5. *Assume that Γ is not of type $(0,3)$ and that there is an isomorphism $\varphi\colon F(\Gamma) \to T(\Gamma')$ for a torsion free Fuchsian group Γ'. Assume also that $\big((g,n),(g',n')\big)$ lies in Table D. Then $\varphi \circ s\big(T(\Gamma)\big)$ is a component of the hyperelliptic locus in $T(\Gamma')$; that is, any marked Riemann surface $S' \in \varphi \circ s\big(T(\Gamma)\big)$ admits a hyperelliptic involution j' determined by $J' = \varphi \circ e \circ \varphi^{-1}$.*

Proof. We only prove the case that Γ is of type $(0,6)$. Other cases can be handled similarly. By Royden's theorem [19], [9], $J' = \varphi \circ e \circ \varphi^{-1}$ is an elliptic modular transformation of order 2. Next, by examining Table D, we see that Γ' must be of type $(2,1)$ or $(1,4)$.

Case I. Γ' is of type $(2,1)$. In this case, by Lemma 1 of Patterson [18], we conclude that

$$\dim \varphi \circ s\big(T(\Gamma)\big) \le 2g' - 1 + \left[\tfrac{1}{2}n'\right] = 3. \tag{3.8}$$

Since $s\big(T(\Gamma)\big) \subset F(\Gamma)$ is equivalent to $T(\Gamma)$, from (3.8), we obtain

$$3 = \dim T(\Gamma) = \dim \varphi \circ s\big(T(\Gamma)\big) \le 3.$$

We thus have equality in the above inequality. In particular, we have

$$\dim \varphi \circ s\big(T(\Gamma)\big) = 2g' - 1 + \left[\tfrac{1}{2}n'\right] = 3. \tag{3.9}$$

From (3.9) and the second part of Lemma 1 of Patterson [18], we conclude that $J' = \varphi \circ e \circ \varphi^{-1} \in \operatorname{Mod}(2,1)$ is induced by a hyperelliptic involution j' of a hyperelliptic Riemann surface of type $(2,1)$, which in turn implies that $\varphi \circ s\big(T(\Gamma)\big)$ is a component of the hyperelliptic locus.

Case II. Γ' is of type $(1,4)$. We use the same argument as above. Note that (3.8) and (3.9) also hold in this case. □

In order to extend Theorem 1 we need to introduce more delicate methods which will allow us to construct new periodic holomorphic automorphisms of $F(\Gamma)$.

4. Periodic automorphisms of special orbifolds

The purpose of this section is to construct certain useful periodic automorphisms of some special orbifolds. In what follows, we use the term "self-map in the sense of orbifolds" to denote a quasiconformal self-map of an orbifold which carries regular points to regular points, punctures to punctures, and branch points to branch points of the same order.

Let Γ be a finitely generated Fuchsian group of the first kind with signature $\mathrm{sig} = (g, n; \nu_1, \ldots, \nu_n)$. Assume that Γ contains at least one elliptic element.

Lemma 4.1. *In each of the following cases, we may choose a self-map f (in the sense of orbifolds) of U/Γ so that f fixes at least one branch point of U/Γ and there is an integer α with f^α isotopic to* id *on $U/\Gamma - \{$all branch points$\}$, where*

(1) $\alpha = 3$ *when* $\mathrm{sig} = (0, 8; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{8-m})$, $m = 1, 2, 4, 5, 7, 8$,

(2) $\alpha = 2$ *when* $\mathrm{sig} = (0, 7; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{7-m})$, $m = 1, 3, 5, 7$,

(3) $\alpha = 4$ *when* $\mathrm{sig} = (0, 6; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{6-m})$, $m = 1, 2, 5$,

(4) *either* $\alpha = 4$ *or* $\alpha = 3$ *when* $\mathrm{sig} = (0, 5; 2, \infty, \ldots, \infty)$ *or* $(0, 5; 2, \ldots, 2)$,

(5) $\alpha = 3$ *when* $\mathrm{sig} = (0, 5; 2, 2, \infty, \infty, \infty)$,

(6) $\alpha = 2$ *when* $\mathrm{sig} = (0, 5; 2, 2, 2, \infty, \infty)$, *and*

(7) $\alpha = 3$ *when* $\mathrm{sig} = (0, 4; 2, \infty, \infty, \infty)$.

Proof. Take the standard unit sphere

$$\mathbf{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

in $\mathbf{R}^3$. Let A be the rotation around z-axis with rotation angle β. Then A leaves invariant the circle $\Sigma = \{(x, y, 0) \mid x^2 + y^2 = 1\}$.

Let x_i, $i = 1, \ldots, n$, be distinguished points on $\mathbf{S}^2$ (see Figure 1). Each x_i is either a puncture or a branch point of order 2. Let x_1 be the point $(0, 0, 1)$. All distinguished points x_i which lie in Σ must divide Σ into equal pieces. The following table provides various special orbifolds as well as certain periodic automorphisms of the orbifolds. Set $\nu_i = 2$ if the distinguished point x_i is a branch point; and $\nu_i = \infty$ if x_i is a puncture. The special orbifolds so obtained are denoted by S.

m	The orbifold S of type $(0,8)$: Figure 1(a)	β	f	α
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,\dots,8$.	$2\pi/6$	A^2	3
2	$\nu_i=2$, $i=1,2$; $\nu_j=\infty$, $j=3,\dots,8$.	$2\pi/6$	A^2	3
4	$\nu_i=2$, $i=1,3,5,7$; $\nu_j=\infty$, $j=2,4,6,8$.	$2\pi/6$	A^2	3
5	$\nu_i=2$, $i=1,2,3,5,7$; $\nu_j=\infty$, $j=4,6,8$.	$2\pi/6$	A^2	3
7	$\nu_i=2$, $i=1,3,4,5,6,7,8$; $\nu_2=\infty$.	$2\pi/6$	A^2	3
8	$\nu_i=2$, $i=1,\dots,8$.	$2\pi/6$	A^2	3
m	The orbifold S of type $(0,7)$: Figure 1(b)	β	f	α
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,\dots,7$.	$2\pi/6$	A^3	2
3	$\nu_i=2$, $i=1,2,5$; $\nu_j=\infty$, $j=3,4,6,7$.	$2\pi/6$	A^3	2
5	$\nu_i=2$, $i=1,2,3,5,6$; $\nu_j=\infty$, $j=4,7$.	$2\pi/6$	A^3	2
7	$\nu_i=2$, $i=1,\dots,7$.	$2\pi/6$	A^3	2
m	The orbifold S of type $(0,6)$: Figure 1(c)	β	f	α
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,\dots,6$.	$2\pi/4$	A	4
2	$\nu_i=2$, $i=1,2$; $\nu_j=\infty$, $j=3,4,5,6$.	$2\pi/4$	A	4
5	$\nu_i=2$, $i=1,3,4,5,6$; $\nu_2=\infty$.	$2\pi/4$	A	4
m	The orbifold S of type $(0,5)$: Figure 1(d) or Figure 1(e)	β	f	α
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,\dots,5$. (Figure 1(d))	$2\pi/4$	A	4
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,\dots,5$. (Figure 1(e))	$2\pi/3$	A	3
2	$\nu_i=2$, $i=1,2$; $\nu_j=\infty$, $j=3,4,5$. (Figure 1(e))	$2\pi/3$	A	3
3	$\nu_i=2$, $i=1,3,5$; $\nu_j=\infty$, $j=2,4$. (Figure 1(d))	$2\pi/4$	A^2	2
5	$\nu_i=2$, $i=1,\dots,5$. (Figure 1(d))	$2\pi/4$	A	4
5	$\nu_i=2$, $i=1,\dots,5$. (Figure 1(e))	$2\pi/3$	A	3
m	The orbifold S of type $(0,4)$: Figure 1(f)	β	f	α
1	$\nu_1=2$; $\nu_j=\infty$, $j=2,3,4$.	$2\pi/3$	A	3

Notice that in each case f is a conformal self-map (Möbius transformation) of S in the sense of orbifolds. Suppose that Γ is a Fuchsian group whose signature is in (1)–(7) of the lemma. Then there is an orbifold S, chosen from the above table, so that a quasiconformal map h (in the sense of orbifolds) of S onto U/Γ is defined. It is obvious that the map $h\circ f\circ h^{-1}$ is the required quasiconformal self-map of U/Γ. ▫

Consider now an orbifold S of signature $(1,1;2)$. By removing the branch point, one obtains a punctured torus S_0 which can then be represented as $\mathbf{C}/G_\tau - \{0\}$, where $G_\tau=\langle B_1,B_\tau\rangle$ is the group generated by translations $B_1\colon z\mapsto z+1$ and $B_\tau\colon z\mapsto z+\tau$, for some $\tau\in U$. A fundamental region for G_τ is shown in Figure 2(A), where the origin projects to the puncture. Consider the involution $j\colon z\mapsto -z$. A computation shows that

$$j\circ C(z)=C^{-1}\circ j(z)\qquad \text{for any } C\in G_\tau. \tag{4.1}$$

This means that j can be projected to a hyperelliptic involution of S_0. Hence, a hyperelliptic involution (call it j also) on S is defined. Note that $S/\langle j\rangle$ is an orbifold of signature $(0,4;2,2,2,4)$.

Lemma 4.2. *A similar assertion as in Lemma* 4.1 *is true if* Γ *is of signature* $(1,1;2)$*; that is, there exists a self-map* f *of* U/Γ *so that* f^3 *is isotopic to* id *on the corresponding punctured torus.*

Proof. A Fuchsian group Γ_0 of signature $(0,4;2,2,2,4)$ can be chosen so that U/Γ_0 is the orbifold drawn in Figure 1(f), where x_1 is a branch point of order 4, and x_2, x_3, x_4 are branch points of order 2. There is a subgroup $\Gamma \subset \Gamma_0$ of index 2. Γ has signature $(1,1;2)$ and the two-sheeted branched covering $U/\Gamma \to U/\Gamma_0$ is holomorphic. The self-map A described in Lemma 4.1(7) is in the sense of orbifolds, and hence it can be lifted to a self-map f of U/Γ which fixes the branch point (see Birman–Hilden [8]). Since $A^3 = \mathrm{id}$, f^3 is either id or the hyperelliptic involution j. If $f^3 = \mathrm{id}$, we are done; otherwise we take $f_0 = j \circ f$. ◻

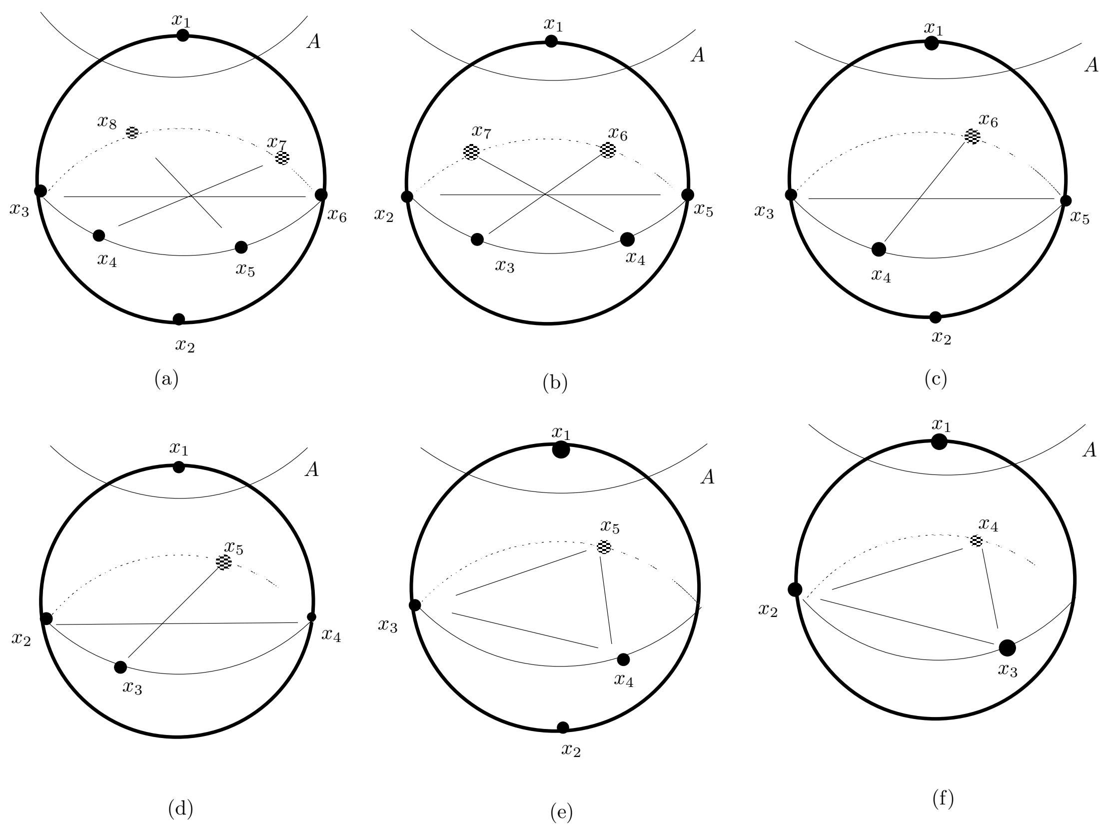

Figure 1.

We need to introduce another method to discuss the case that Γ is of type $(1,2)$. The method will also be used in Section 8.

Let $\Delta = \{z : |z| < 2\}$ be parametrized by the polar coordinates (r, α). Define $\hat{\sigma}$: $\Delta \to \Delta$ as $\hat{\sigma}(r,\alpha) = (r, \alpha - r\pi)$. Let x_i, $i = 1, 2, \ldots, n$, $n \geq 3$, be n distinct points on $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, and let τ_i, $i = 1, \ldots, n-1$, be embeddings of Δ into $\widehat{\mathbf{C}}$ with the properties that $\tau_i(\Delta)$ contains x_i and x_{i+1}, but is disjoint from all x_j for $j \neq i, i+1$. Suppose that $\tau_i(1,0) = x_i$, and $\tau_i(1,\pi) = x_{i+1}$. We obtain self-maps σ_i on $\widehat{\mathbf{C}}$ defined by

$$\sigma_i(x) = \begin{cases} x & \text{if } x \in \widehat{\mathbf{C}} - \tau_i(\Delta), \\ \tau_i \circ \hat{\sigma} \circ \tau_i^{-1}(x) & \text{if } x \in \tau_i(\Delta). \end{cases} \tag{4.2}$$

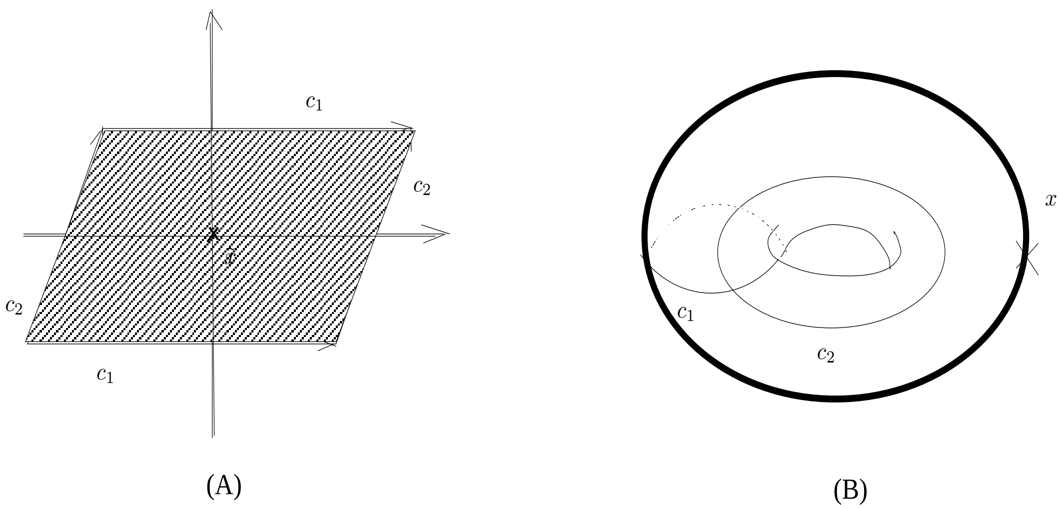

Figure 2.

Consider the following self-map of $\widehat{\mathbf{C}}$:

$$\sigma = \sigma_{n-2} \circ \cdots \circ \sigma_1. \tag{4.3}$$

It is easily seen that σ fixes x_n and realizes a permutation of the set $\{x_1, \ldots, x_n\}$. A theorem of Magnus [15] shows that, as a self-map of $\widehat{\mathbf{C}} - \{x_1, \ldots, x_n\}$, σ is periodic and its order is $n-1$ (up to isotopy). The isotopy $I\colon \widehat{\mathbf{C}} \times [0,1] \to \widehat{\mathbf{C}}$ between σ^{n-1} and id may be chosen so that $I(x_i, t) = x_i$, for $0 \leq t \leq 1$ and $i = 1, \ldots, n$.

Let S be a hyperelliptic Riemann surface, and let j be the corresponding hyperelliptic involution. Let $\zeta\colon S \to S/\langle j \rangle$ denote the natural projection. j has $2g+2$ fixed points on $\overline{S}$, which are Weierstrass points of $\overline{S}$. We see that $\overline{S}/\langle j \rangle$ is an orbifold with signature $(0, 2g+2; 2, \ldots, 2)$. In what follows, $\overline{S}$ is always taken as a symmetrically embedded surface (about x-axis) in $\mathbf{R}^3$. In this setting, j is a $180°$ rotation around x-axis. Choose a simple closed curve c on S that is symmetric about the x-axis. It is obvious that

(i) $j(c) = c$; and

(ii) j reverses the orientation of c.

The curve which has properties (i) and (ii) is called a *canonical curve*. A *Dehn twist* h_d about a simple closed curve d on a surface S is defined as follows. Let

$N(d)$ be an annular neighborhood of d parametrized by the polar coordinates (r,θ), $-1 \le r \le 1$, and $0 \le \theta \le 2\pi$ such that d is defined by $r = 0$. Then, $h_d = \mathrm{id}$ outside of $N(d)$. For $x = (r,\theta) \in N(d)$, we have

$$h_d(r,\theta) = \begin{cases} (r, \theta + 2\pi r), & 0 \le r \le 1, \\ (r,\theta), & -1 \le r \le 0. \end{cases}$$

Obviously, the homotopy class of the map h_d depends only on the free homotopy class of the curve d. Thus, the modular transformation induced by h_d depends only on the free homotopy class of d. Another property of h_d is that if f is a self-map of S, then

$$f \circ h_d \circ f^{-1} = h_{f(d)}. \tag{4.4}$$

Let c_i be a canonical curve on S which passes through the Weierstrass points x_i and x_{i+1}, and let h_{c_i} be the Dehn twist about c_i. Recall that σ_i is defined by (4.2).

Lemma 4.3. *The Dehn twist h_{c_i} of S is isotopic to a self-map h'_{c_i} which can be projected to a self-map σ'_i of $S/\langle j\rangle$ in the sense of orbifolds. Furthermore, σ'_i is isotopic to σ_i defined by* (4.2).

Proof. See Birman–Hilden [8]. ▫

Lemma 4.4. *The same assertion as in Lemma* 4.1 *remains true if Γ is of signature $(1,2;2,2)$ or $(1,2;2,\infty)$; namely, there is a self-map f of U/Γ so that it fixes a branch point and f^4 is isotopic to* id *on the corresponding surface of signature $(1,2;\infty,\infty)$.*

Proof. We only deal with the case that Γ is of signature $(1,2;2,\infty)$. The other case can be handled similarly. Let S be an orbifold of signature $(1,2;2,\infty)$. By removing the branch point, one obtains a surface S_0 of signature $(1,2;\infty,\infty)$. Let x_1 and x_2 denote the two punctures.

Let G_τ be as above. Consider the universal covering $\mathbf{C}$ of the torus $\overline{S_0}$. The preimages of the pair of punctures (x_1,x_2) form two lattices. Let us denote $(\tilde{x}_1,\tilde{x}_2)$ one pair of preimages of (x_1,x_2). Any fundamental region for G_τ contains exactly one pair of preimages of (x_1,x_2). By composing with an Euclidean motion of $\mathbf{C}$ if necessary, one may assume that $(\tilde{x}_1,\tilde{x}_2)$ are symmetric about the origin. In this case a fundamental domain (parallelogram) D for G_τ can be chosen so that D is symmetric with respect to the origin and one pair of sides of D is parallel to the x-axis. See Figure 3(A).

Let $c_3 = D \cap \{x\text{-axis}\}$. Without loss of generality, we assume that $\tilde{x}_1$ and $\tilde{x}_2$ are not in c_3. Otherwise we choose $D \cap \{\tilde{c}_3\}$ as c_3, where $\tilde{c}_3$ is drawn in Figure 3(A). Again, the map $j\colon z \mapsto -z$ interchanges the two punctures $\tilde{x}_1$ and $\tilde{x}_2$ in

D and satisfies (4.1). Hence, S_0 admits a hyperelliptic involution j interchanging the two punctures.

Observe that $S_0/\langle j\rangle$ is an orbifold of signature $(0,5;2,\dots,2,\infty)$. S_0 can be drawn in Figure 3(B), where c_1', c_2', come from the boundary of D by "pasting procedure" and c_3' comes from c_3 or $\tilde{c}_3$ by identifying the endpoints (see Figure 3(A)). One sees that $j(c_i')=c_i'$, for $i=1,2,3$.

Define $f = h_{c_3'}\circ h_{c_2'}\circ h_{c_1'}$. By Lemma 4.3, f is isotopic to a lift of σ (described in (4.3)). By Magnus' result, σ^4 is isotopic to id. So f^4 is isotopic to id or j. Obviously, f^4 does not interchange the two punctures. It follows that f^4 is isotopic to id on S_0. Clearly, f extends to a self-map of an orbifold of signature $(1,2;2,\infty)$. This completes the proof of the lemma. □

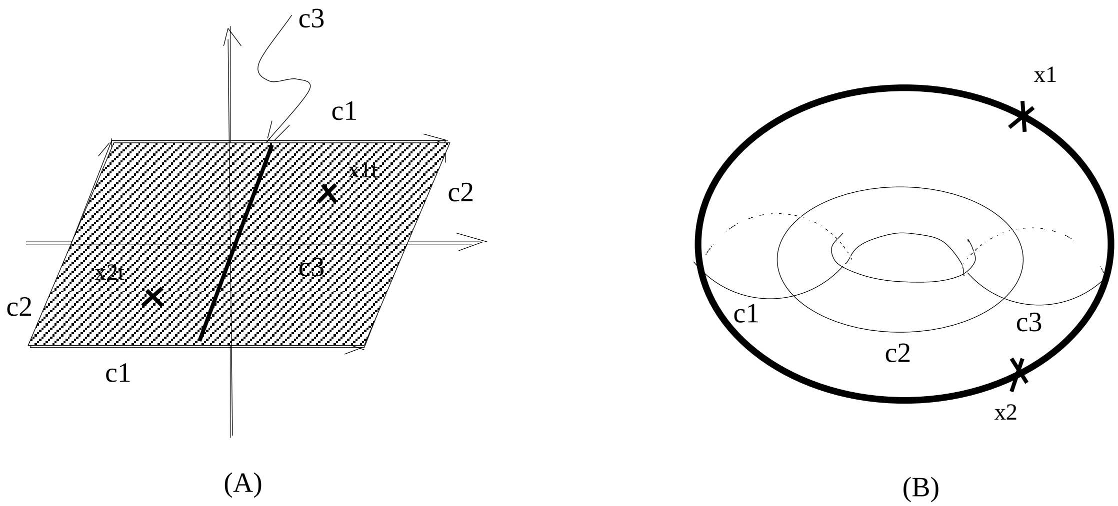

Figure 3.

5. Periodic automorphisms of Bers fiber spaces

Let Γ be a finitely generated Fuchsian group of the first kind which acts on U and contains at least one elliptic element, and let f be a self-map of U/Γ in the sense of orbifolds. Then f can be lifted to a quasiconformal self-map of U. See Birman–Hilden [8] for an exposition.

The map f induces a modular transformation χ_f on $T(\Gamma)$. Let s denote a canonical section of $\pi\colon F(\Gamma)\to T(\Gamma)$. Then s induces a map

$$s_*\colon \operatorname{Mod}\Gamma \to \operatorname{Aut} s\big(T(\Gamma)\big)$$

defined by the formula

$$s_*(\chi_f)(x) = s\circ\chi_f\circ\pi(x) \qquad \text{for } x\in s\big(T(\Gamma)\big).$$

It is easy to check that $s_*(\chi_f)\colon s\big(T(\Gamma)\big)\to s\big(T(\Gamma)\big)$ is a holomorphic automorphism. Unfortunately, it is not true that for an arbitrary s and an arbitrary f,

$s_*(\chi_f)$ can be extended holomorphically to the whole Bers fiber space. Our objective is to choose a specific self-map f of U/Γ and a specific canonical section s of $\pi\colon F(\Gamma) \to T(\Gamma)$ so that the automorphism $s_*(\chi_f)$ of $s\big(T(\Gamma)\big)$ is the restriction of a global holomorphic automorphism.

Let f be a self-map of U/Γ which fixes the branch point $\hat{z}_0$ determined by an elliptic element e of Γ, and let s be the canonical section of $\pi\colon F(\Gamma) \to T(\Gamma)$ which is determined by the fixed point of e in U. Let $s\big(T(\Gamma)\big)^{s_*(\chi_f)}$ denote the set of all points in $s\big(T(\Gamma)\big)$ which are fixed by $s_*(\chi_f)$. Similarly, we denote by $F(\Gamma)^\chi$ the set of all points in $F(\Gamma)$ which are fixed by an element $\chi \in \operatorname{mod}\Gamma$. We have

Lemma 5.1. (1) $s_*(\chi_f) \in \operatorname{Aut} s\big(T(\Gamma)\big)$ *can be extended to an element* χ *of* $\operatorname{mod}\Gamma$.

(2) $\langle \chi, e\rangle$ *is an abelian subgroup of* $\operatorname{mod}\Gamma$.

(3) $F(\Gamma)^\chi = F(\Gamma)^{e\circ\chi} = s\big(T(\Gamma)\big)^{s_*(\chi_f)}$.

Proof. Lift f to a map $\hat{f}$ on U so that $\hat{f}$ fixes the fixed point z_0 of e and makes the following diagram commutative:

$$\begin{array}{ccc} U & \xrightarrow{\hat{f}} & U \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U/\Gamma & \xrightarrow{f} & U/\Gamma. \end{array}$$

Thus we have $\hat{f} \circ e \circ \hat{f}^{-1} = e$. The equivalence class $[\hat{f}\,]$ of $\hat{f}$ (that is, all $\hat{f} \in Q(\Gamma)$ which lie in the normalizer of Γ with the property that $\hat{f}|_{\mathbf{R}} = \hat{f}'|_{\mathbf{R}}$) is an element of $\operatorname{mod}\Gamma$. Let $\chi = [\hat{f}\,]$. We claim that χ is an extension of $s_*(\chi_f)\colon s\big(T(\Gamma)\big) \to s\big(T(\Gamma)\big)$. To see this, first we note that

$$\chi_f([\mu]) = [\text{Beltrami coefficient of } (w_\mu \circ \hat{f}^{-1})]$$

for $[\mu] \in T(\Gamma)$. We see that the diagram

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\chi} & F(\Gamma) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T(\Gamma) & \xrightarrow{\chi_f} & T(\Gamma) \end{array} \tag{5.1}$$

commutes. Let $[\nu] = \chi_f([\mu])$. The diagram (5.1) shows that χ maps the fiber $w^\mu(U)$ over $[\mu]$ to the fiber $w^\nu(U)$ over $[\nu]$. In particular, χ maps the fiber that the point $s([\mu]) \in s\big(T(\Gamma)\big)$ lies in to the fiber that the point $s([\nu]) \in s\big(T(\Gamma)\big)$ lies in. But we have

$$(5.2) \qquad \begin{aligned} s([\nu]) &= s \circ \chi_f([\mu]) = s \circ \chi_f \circ \pi\big([\mu], w^\mu(z_0)\big) \\ &= s_*(\chi_f)\big([\mu], w^\mu(z_0)\big) \in s\big(T(\Gamma)\big). \end{aligned}$$

To prove that $\chi \in \operatorname{mod}\Gamma$ is an extension of $s_*(\chi_f)$ (that is, $\chi|_{s(T(\Gamma))} = s_*(\chi_f)$), we only need to show

$$\chi\big(s([\mu])\big) = s_*(\chi_f)\big([\mu], w^\mu(z_0)\big) \in s\big(T(\Gamma)\big).$$

Now by (5.2), it remains to show that

$$(5.3) \qquad \chi\big(s([\mu])\big) = s([\nu]).$$

Note that the action of χ on $F(\Gamma)$ is given by $\chi([\mu], z) = ([\nu], \hat{z})$, for $z \in w^\mu(U)$, where $\hat{z} = w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)$. Recall that s is the canonical section of π determined by the fixed points of $e \in \operatorname{mod}\Gamma$. Clearly, $s([\mu]) = \big([\mu], w^\mu(z_0)\big) \in s\big(T(\Gamma)\big)$ is the fixed point of $e^\mu = w^\mu \circ e \circ (w^\mu)^{-1}$, and $s([\nu]) = \big([\nu], w^\nu(z_0)\big) \in s\big(T(\Gamma)\big)$ is the fixed point of $e^\nu = w^\nu \circ e \circ (w^\nu)^{-1}$. Thus, (5.3) is equivalent to the statement that $e^\mu(z) = z$ implies that $e^\nu(\hat{z}) = \hat{z}$. Now let $z = s([\mu])$. We get

$$(5.4) \qquad \begin{aligned} e^\nu(\hat{z}) &= e^\nu \circ \big(w^\nu \circ \hat{f} \circ (w^\mu)^{-1}\big)(z) = w^\nu \circ e \circ \hat{f} \circ (w^\mu)^{-1}(z) \\ &= w^\nu \circ \hat{f} \circ e \circ (w^\mu)^{-1}(z) = w^\nu \circ \hat{f} \circ (w^\mu)^{-1} \circ e^\mu(z) \\ &= w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z) = \hat{z}. \end{aligned}$$

It follows that $\hat{z} = s_*(\chi_f)(z)$ equals $s([\nu])$.

To prove (2), we use a computation similar to (5.4). In fact, to each point $([\mu], z) \in F(\Gamma)$, we have

$$\begin{aligned} \chi \circ e([\mu], z) &= \chi\big([\mu], e^\mu(z)\big) = \big([\nu], w^\nu \circ \hat{f} \circ (w^\mu)^{-1} \circ e^\mu(z)\big) \\ &= \big([\nu], w^\nu \circ \hat{f} \circ e \circ (w^\mu)^{-1}(z)\big) = \big([\nu], w^\nu \circ e \circ \hat{f} \circ (w^\mu)^{-1}(z)\big) \\ &= \big([\nu], e^\nu \circ w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)\big) = e\big([\nu], w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)\big) \\ &= e \circ \chi([\mu], z). \end{aligned}$$

(3) We only prove that $F(\Gamma)^\chi = s\big(T(\Gamma)\big)^{s_*(\chi_f)}$; the proof of the other equality is the same. Since χ is a fiber-preserving extension of $s_*(\chi_f)$, it is trivial that

$$s\big(T(\Gamma)\big)^{s_*(\chi_f)} \subset F(\Gamma)^\chi.$$

Suppose now that there is a point $x \in F(\Gamma)^{\chi}$ which is not in $s\bigl(T(\Gamma)\bigr)^{s_*(\chi_f)}$, and that $x \in s\bigl(T(\Gamma)\bigr)$. Since χ is an extension of $s_*(\chi_f)$, we have $x = \chi(x) = s_*(\chi_f)(x)$. This implies that $x \in s\bigl(T(\Gamma)\bigr)^{s_*(\chi_f)}$, a contradiction. We conclude that $x \notin s\bigl(T(\Gamma)\bigr)$. Therefore, in the fiber $\pi^{-1}\bigl(\pi(x)\bigr)$, there are at least two points, x and the intersection $\pi^{-1}\bigl(\pi(x)\bigr) \cap s\bigl(T(\Gamma)\bigr)$, which are fixed by χ. But the restriction of χ to the fiber $\pi^{-1}\bigl(\pi(x)\bigr)$ is a conformal automorphism. It follows that $\chi = \mathrm{id}$ on $\pi^{-1}\bigl(\pi(x)\bigr)$.

We need to investigate the action of χ on $\pi^{-1}\bigl(\pi(y)\bigr)$ for any $y \in F(\Gamma)$. Following Bers [5], let h_μ be defined by

$$w_\mu = h_\mu \circ w^\mu \mid U$$

for $\mu \in M(\Gamma)$. Then $h_\mu\colon w^\mu(U) \to U$ is a conformal map keeping $0,\ 1,\ \infty$ fixed. It is easy to see that h_μ depends only on $[\mu]$. For each $x = ([\mu], z) \in F(\Gamma)$, we have $\chi([\mu], z) = ([\nu], \hat{z})$, where $\hat{z} = w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)$. Recall that $w_\nu = \alpha \circ w_\mu \circ \hat{f}^{-1}$, where $\alpha \in \mathrm{PSL}\,(2, \mathbf{R})$ is such that $\alpha \circ w_\mu \circ \hat{f}^{-1}$ is normalized. We thus have

$$\begin{aligned} \hat{z} &= w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z) = (h_\nu)^{-1} \circ w_\nu \circ \hat{f} \circ (w_\mu)^{-1} \circ h_\mu(z) \\ &= (h_\nu)^{-1} \circ \alpha \circ w_\mu \circ \hat{f}^{-1} \circ \hat{f} \circ (w_\mu)^{-1} \circ h_\mu(z) = (h_\nu)^{-1} \circ \alpha \circ h_\mu(z). \end{aligned} \tag{5.5}$$

Set $[\mu] = \pi(x) \in T(\Gamma)$. χ must keep the fiber $\pi^{-1}\bigl(\pi(x)\bigr)$ invariant. This means that $[\mu] = [\nu]$ and $\chi([\mu], z) = \bigl([\mu], (h_\mu)^{-1} \circ \alpha \circ h_\mu(z)\bigr)$. By the above argument, the restriction of χ to $\pi^{-1}\bigl(\pi(x)\bigr)$ is id. We see from (5.5) that $\alpha = \mathrm{id}$, and hence that $\hat{f}$ is normalized. Since $w_\mu = w_\mu \circ \hat{f}^{-1}$, $\hat{f}$ restricts to id on $\mathbf{R}$. It follows that $\hat{f}$ commutes with all elements of Γ, which in turn implies that f is isotopic to id on $U/\Gamma - \{\text{all branch points}\}$, which leads to a contradiction. Therefore, $F(\Gamma)^{\chi} = s\bigl(T(\Gamma)\bigr)^{s_*(\chi_f)}$. Lemma 5.1 is proved. □

The following lemma shows that there are periodic automorphisms of the Bers fiber spaces $F(\Gamma)$ for certain special Fuchsian groups Γ. We also see that the dimensions of the fixed point sets of these automorphisms are computable. This will lead to the settlement of most cases of Theorem 2. Again, let Γ be of signature $\mathrm{sig} = (0, n; \nu_1, \dots, \nu_n)$, where ν_i is 2 or ∞, and let $e \in \Gamma$ denote an elliptic element of order 2.

Lemma 5.2. *Associated to $e \in \Gamma$ there is a non-trivial automorphism $\chi \in \operatorname{mod}\Gamma$, $\chi \neq e$, so that $\langle \chi, e\rangle$ is an abelian subgroup of $\operatorname{mod}\Gamma$. Further, we have*

(1) $\chi^3 = \mathrm{id}$, *and* $\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = 1$ *if*

$$\mathrm{sig} = (0, 8; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{8-m}), \qquad m = 1, 2, 4, 5, 7, 8,$$

(2) χ^2 *is either* id *or* e, *and* $\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = 2$ *if*

$$\text{sig} = (0,7;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{7-m}), \qquad m = 1,3,5,7,$$

(3) χ^4 *is either* id *or* e. *Furthermore,* $\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = 0$, *and* $\dim F(\Gamma)^{\chi^2} = \dim F(\Gamma)^{e\circ\chi^2} = 1$ *if*

$$\text{sig} = (0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m}), \qquad m = 1,2,5,$$

(4) *if* sig $= (0,5;2,\infty,\ldots,\infty)$ *or* $(0,5;2,\ldots,2)$, *then there are two automorphisms* χ_1, $\chi_2 \in \operatorname{mod}\Gamma$, *both* $\langle\chi_1, e\rangle$ *and* $\langle\chi_2, e\rangle$ *are abelian, such that* χ_1^4 *is either* id *or* e, *and* $\chi_2^3 = \text{id}$. *Further,* $\dim F(\Gamma)^{\chi_2} = \dim F(\Gamma)^{e\circ\chi_2} = 0$,

(5) *if* sig $= (0,5;2,2,\infty,\infty,\infty)$, *then* $\chi^3 = \text{id}$. *Furthermore,* $\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = \dim F(\Gamma)^{\chi^2} = \dim F(\Gamma)^{e\circ\chi^2} = 0$,

(6) χ^2 *is either* id *or* e, *and* $\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = 1$ *if*

$$\text{sig} = (0,5;2,2,2,\infty,\infty),$$

(7) $\chi^3 = \text{id}$ *if* sig $= (0,4;2,\infty,\infty,\infty)$,

(8) $\chi^3 = \text{id}$ *if* sig $= (1,1;2)$, *and*

(9) χ^4 *is either* id *or* e, *and* $\dim F(\Gamma)^{\chi^2} = \dim F(\Gamma)^{e\circ\chi^2} = 1$ *if*

$$\text{sig} = (1,2;2,2) \text{ or } (1,2;2,\infty).$$

Proof. Notice that Lemma 4.1(1) provides a self-map f of the orbifold U/Γ which has order 3 (up to isotopy) and fixes the branch point x_1 of order 2. See Figure 1(a). Let $e \in \Gamma$ be the elliptic element corresponding to the branch point $x_1 \in U/\Gamma$, and s the canonical section of $\pi\colon F(\Gamma) \to T(\Gamma)$ determined by the fixed point of e in U. By using Lemma 5.1(1) and (2), we conclude that there exists a $\chi \in \operatorname{mod}\Gamma$ with the following properties:

(i) χ leaves invariant the set $s\big(T(\Gamma)\big)$ which is isomorphic to $T(\Gamma)$;

(ii) χ commutes with e; and

(iii) χ^3 restricts to id on $s\big(T(\Gamma)\big)$.

From (iii) we see that $q(\chi^3) \in \operatorname{Mod}\Gamma$ is id, where $q\colon \operatorname{mod}\Gamma \to \operatorname{Mod}\Gamma$ is the quotient homomorphism. Thus χ^3 lies in the kernel of q. Since $\operatorname{Mod}\Gamma$ is defined by the quotient group $\operatorname{mod}\Gamma/\Gamma$, $\chi^3 \in \Gamma$. Now χ and hence χ^3 leave invariant the set $s\big(T(\Gamma)\big)$. It follows that χ^3 is either id or e. If $\chi^3 = \text{id}$, we are done. If $\chi^3 = e$, then we take $\chi_0 = e\circ\chi$. χ_0 has properties (i), (ii) and (iii) above. Thus $\chi_0^3 = \text{id}$. We see that χ_0 is the required automorphism.

The proof of existence of automorphisms in each case (2)–(9) is omitted. To calculate the dimensions of the fixed point sets, we apply the theorem of Kravetz [13] described in Proposition 3.1. We know that for any elliptic modular transformation χ of $T(\Gamma)$, the set $T(\Gamma)^{\chi}$ of the fixed points of χ is identified with another Teichmüller space $T(g^*, n^*)$, where g^* and n^* are defined in the proof of Proposition 3.1. By definition, to each $x \in s\big(T(\Gamma)\big)$, we have $s_*(\chi_f)(x) = s \circ \chi_f \circ \pi(x)$. Thus

$$T(\Gamma)^{\chi_f} = s\big(T(\Gamma)\big)^{s\circ\chi_f\circ\pi} = s\big(T(\Gamma)\big)^{s_*(\chi_f)}.$$

In particular, since each $\chi_f \in \operatorname{Mod}\Gamma$ is elliptic, the set $s\big(T(\Gamma)\big)^{s_*(\chi_f)}$ is not empty. Further, we have dim $s\big(T(\Gamma)\big)^{s_*(\chi_f)} = \dim T(\Gamma)^{\chi_f}$. From Lemma 5.1(3), we see that

$$\dim F(\Gamma)^{\chi} = \dim T(\Gamma)^{\chi_f}.$$

Now the rest of proof of Lemma 5.2 only involves rather simple computations.

Computations of dimensions. Lemma 5.2(1). Notice that $f = A^2$ and that $S/\langle A^2\rangle$ is a Riemann sphere with 4 distinguished points (Figure 1(a)). Thus,

$$\dim F(\Gamma)^{e\circ\chi} = \dim F(\Gamma)^{\chi} = \dim T(S/\langle A^2\rangle) = \dim T(0,4) = 1.$$

Lemma 5.2(2). In this case $f = A^3$ and $S/\langle A^3\rangle$ is a Riemann sphere with 5 distinguished points (Figure 1(b)). This yields

$$\dim F(\Gamma)^{e\circ\chi} = \dim F(\Gamma)^{\chi} = \dim T(S/\langle A^3\rangle) = \dim T(0,5) = 2.$$

Lemma 5.2(3). From Lemma 4.1(3), we see that $f = A$ and $S/\langle A\rangle$ is a Riemann sphere with 3 distinguished points (Figure 1(c)). Hence,

$$\dim F(\Gamma)^{e\circ\chi} = \dim F(\Gamma)^{\chi} = \dim T(S/\langle A\rangle) = \dim T(0,3) = 0.$$

Similarly, we have

$$\dim F(\Gamma)^{e\circ\chi^2} = \dim F(\Gamma)^{\chi^2} = \dim T(S/\langle A^2\rangle) = \dim T(0,4) = 1.$$

Lemma 5.2(4). To compute the dimension of the fixed point set of χ_2, we refer to Figure 1(d). Observe that $f = A$ and $S/\langle A\rangle$ is a Riemann sphere with 3 distinguished points. It turns out that

$$\dim F(\Gamma)^{e\circ\chi_2} = \dim F(\Gamma)^{\chi_2} = \dim T(S/\langle A\rangle) = \dim T(0,3) = 0.$$

Lemma 5.2(5). The computation for dim $F(\Gamma)^{\chi}$ is the same as (4). We refer to Figure 1(e). Similarly, we obtain

$$\dim F(\Gamma)^{e\circ\chi^2} = \dim F(\Gamma)^{\chi^2} = \dim T(S/\langle A^2\rangle) = \dim T(0,3) = 0.$$

Lemma 5.2(6). In this case, $f = A^2$ and $S/\langle A^2\rangle$ is a Riemann sphere with signature $(0,4;2,2,4,\infty)$ (Figure 1(d)). Thus we obtain

$$\dim F(\Gamma)^{e\circ\chi} = \dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = \dim T(S/\langle A^2\rangle) = 1.$$

Lemma 5.2(9). f is constructed in Lemma 4.4. Since $\chi_f \in \operatorname{Mod}\Gamma$ is elliptic, χ_f has a fixed point in $T(1,2)$. We assume that f^2 is conformal on the orbifold S of signature $(1,2;2,\infty)$. f^2 has order 2. By Lemma 5.1(3), we have

$$\dim F(\Gamma)^{e\circ\chi^2} = \dim F(\Gamma)^{\chi^2} = \dim T(\Gamma)^{\chi_{f^2}} = \dim T(S/\langle f^2\rangle).$$

Let g' denote the genus of $S/\langle f^2\rangle$, and let n' denote the number of fixed points of f^2 on $\overline{S}$. Clearly, $n' \geq 2$ since the branch point and the puncture on S are fixed by f^2. Now the Riemann–Hurwitz formula gives us

$$0 = 2(2g'-2) + n'.$$

So $n' = 4$. It follows that

$$\dim T(S/\langle f^2\rangle) = 3g' - 3 + n' = 1.$$

Therefore, $\dim F(\Gamma)^{e\circ\chi^2} = \dim F(\Gamma)^{\chi^2} = 1$.

This completes the proof of Lemma 5.2. ▫

6. Elliptic transformations on Teichmüller spaces

In this section, we calculate the dimensions of the fixed point sets of some elliptic modular transformations of Teichmüller spaces in some low dimensional cases. With the aid of the periodic automorphisms constructed in the previous section, certain elliptic modular transformations of Teichmüller spaces are defined whose orders are known. Our purpose is to show that, under the condition of Lemma 5.2, the dimensions of the fixed point sets of these elliptic modular transformations are actually different from those we obtained from Lemma 5.2.

Throughout this section we assume that Γ is a torsion free finitely generated Fuchsian group of the first kind whose type is (g,n). Let $\chi \in \operatorname{Mod}\Gamma$ be an elliptic element. As before, let $T(\Gamma)^{\chi}$ denote the non-empty fixed point set of χ in $T(\Gamma)$.

Let J denote the element of $\operatorname{Mod}\Gamma$ which is induced by a hyperelliptic involution on a surface of type (g,n).

Lemma 6.1. *In each of the following cases we assume that there is an elliptic element* $\chi \in \operatorname{Mod}\Gamma$, $\chi \neq J$, *such that* $\langle \chi, J\rangle$ *is an abelian subgroup of* $\operatorname{Mod}\Gamma$.

(1) *If* Γ *is of type* $(3,0)$ *and* $\chi^3 = \mathrm{id}$, *then* $\dim T(\Gamma)^{\chi} = 2$.

(2) *If* Γ *is of type* $(2,2)$ *and* $\chi^2 = \mathrm{id}$ *or* J, *then either* $\dim T(\Gamma)^{\chi} = 3$ *or* $\dim T(\Gamma)^{J\circ\chi} = 3$.

(3) *If* Γ *is of type* $(1,4)$ *and* $\chi^4 = \mathrm{id}$ *or* J, *then either* $\dim T(\Gamma)^{\chi} \geq 1$, *or* $\dim T(\Gamma)^{J\circ\chi} \geq 1$, *or* $\dim T(\Gamma)^{J\circ\chi^2} \geq 2$.

(4) *If* Γ *is of type* $(0,6)$ *and* $\chi^3 = \mathrm{id}$, *then* $\dim T(\Gamma)^{\chi} = 1$.

(5) *If* Γ *is of type* $(0,6)$ *and* $\chi^2 = \mathrm{id}$ *or* J, *then either* $\dim T(\Gamma)^{\chi} = 2$, *or* $\dim T(\Gamma)^{J\circ\chi} = 2$.

Proof. Since $\langle \chi, J\rangle$ is abelian, χ leaves invariant the component of hyperelliptic locus in $T(\Gamma)$ which is determined by J. From Kravetz's theorem [13], there is a fixed point of χ in this component of the hyperelliptic locus. We conclude that there exists a common fixed point $x \in T(\Gamma)$ of J and χ.

Let S denote the hyperelliptic Riemann surface of type (g,n) which represents x, let h be the conformal automorphism of S which induces χ, and let j be the hyperelliptic involution of S which induces J.

Let k the number of the fixed points of h on the compactification $\overline{S}$. The symbol B stands for the total branch number of the corresponding branched covering: $\overline{S} \to \overline{S}/\langle h\rangle$, and g^* stands for the genus of the orbifold $S^* = S/\langle h\rangle$.

(1) From the Riemann–Hurwitz formula, we have

$$4 = 3(2g^* - 2) + B = 3(2g^* - 2) + 2k. \tag{6.1}$$

If $g^* = 1$, then from (6.1), one sees that $k = 2$. By Kravetz's theorem [13], one obtains

$$\dim T(\Gamma)^{\chi} = \dim T(S^*) = 3g^* - 3 + k = 2.$$

The second equality holds because S is compact and the number of the fixed points of h on S is the number of the branch points of S.

If $g^* = 0$, then $k = 5$ and again, it is easy to see that $\dim T(\Gamma)^{\chi} = 2$. This proves (1).

(2) By hypothesis, χ^2 is either id or J. Observe that χ^2 is induced by the self-map h^2. Since h^2 fixes the two punctures on S, χ^2 cannot be J. Hence $\chi^2 = \mathrm{id}$. Now the Riemann–Hurwitz formula tells us that

$$2 = 2(2g^* - 2) + k. \tag{6.2}$$

If $g^* = 0$, then $k = 6 = 2g + 2$ and χ is another hyperelliptic involution of S. Observe also that the hyperelliptic involution on S is unique. Hence, $h = j$, contradicting our hypothesis. Therefore, by (6.2), the only possibility is that $g^* = 1$, and $k = 2$. There are two cases to consider.

Case I. h fixes the two punctures. In this case, h has no other fixed points. Since χ commutes with J, h can be projected to a conformal automorphism h' of the orbifold $S_0 = S/\langle j\rangle$ in the sense of orbifolds. h' is an elliptic Möbius transformation. This means that h' has two fixed points a and b, one of which, say a, comes from the projection of the punctures. The set $\{\zeta^{-1}(b)\}$ (where $\zeta\colon S \to S/\langle j\rangle$ is the two-sheeted branched covering) must contain exactly 2 points, otherwise $\{\zeta^{-1}(b)\}$ would be a fixed point of h. This is impossible. It follows that h must interchange the two points $\{\zeta^{-1}(b)\}$.

Consider the modular transformation $J \circ \chi$ which is induced by $j \circ h$. Since h commutes with j, the self-map $j \circ h$ is of order 2. Moreover, $j \circ h$ has the property that it fixes $\{\zeta^{-1}(b)\}$ pointwise, and interchanges the two punctures. Now we apply the formula (6.2) for the map $j \circ h$ to conclude that $S/\langle j \circ h\rangle$ is of signature $(1,3;2,2,\infty)$ and that $\dim T(\Gamma)^{J\circ\chi} = \dim T(S/\langle j\circ h\rangle) = 3$.

Case II. h interchanges the two punctures. In this case, h has two fixed points elsewhere which are symmetric with respect to j (by assumption h commutes with j). It is rather easy to see that these two points cannot be Weierstrass points on the compactification $\overline{S}$ (otherwise, h' would have three fixed points). Thus, $S/\langle h\rangle$ is of signature $(1,3;2,2,\infty)$ and we obtain $\dim T(\Gamma)^{\chi} = \dim T(S/\langle h\rangle) = 3$. This proves (2).

(3) Notice that h^4 fixes all of the punctures (which are denoted by x_1, x_2, x_3, x_4), χ^4 is not a hyperelliptic involution. Hence $\chi^4 = \mathrm{id}$. Assume that S is a hyperelliptic Riemann surface and that x_1, x_2, x_3, and x_4 are arranged so that $j(x_1) = x_2$ and $j(x_3) = x_4$. By hypothesis, h commutes with j, h can be projected to a conformal automorphism h' of the orbifold $S_0 = S/\langle j\rangle$ in the sense of orbifolds. Hence, h' is a Möbius transformation. Observe that S_0 is of signature $(0,6;2,2,2,2,\infty,\infty)$. Let x_1', x_2' denote the two punctures, and let x_3', x_4', x_5', x_6' denote the four branch points of order 2 on S_0. Then h' either fixes both x_1' and x_2', or interchanges these two punctures.

Case I. h' interchanges x_1' and x_2'. In this case, since h' is an elliptic Möbius transformation, it has two fixed points a' and b'. If $\{a',b'\} \subset \{x_3',x_4',x_5',x_6'\}$, then ${h'}^2$ fixes the set $\{x_3',x_4',x_5',x_6'\}$ pointwise, which implies that ${h'}^2 = \mathrm{id}$, a contradiction. If $\{a',b'\}\cap\{x_3',x_4',x_5',x_6'\}$ is a' or b', then h' induces a permutation of the three points in $\{x_3',x_4',x_5',x_6'\}-\{a',b'\}$, contradicting the fact that ${h'}^4 = \mathrm{id}$. Finally, if $\{a',b'\}$ and $\{x_3',x_4',x_5',x_6'\}$ are disjoint, then ${h'}^2$ fixes $\{a',b',x_1',x_2'\}$, and ${h'}^2 = \mathrm{id}$. So the case that h' interchanges x_1' and x_2' cannot occur.

Case II. h' fixes both x_1' and x_2'. In this case, there are three possibilities:

(i) h fixes all punctures x_1, x_2, x_3, and x_4,

(ii) $h(x_i) = j(x_i)$ for $i = 1,2,3,4$,

(iii) $h(x_i) = j(x_i)$ for $i = 1,2$, and $h(x_i) = x_i$ for $i = 3,4$.

If h fixes all punctures x_1, x_2, x_3, and x_4, then $S/\langle h\rangle$ has at least 4 distinguished points, which means that $\dim T(\Gamma)^{\chi} = \dim T(S/\langle h\rangle) \geq 1$.

If $h(x_i) = j(x_i)$ for $i = 1,2,3,4$, then $j\circ h$ has order 4 and fixes all the punctures. It follows that $S/\langle j\circ h\rangle$ has at least 4 distinguished points coming from the fixed points (punctures) of $j\circ h$. This implies that $\dim T(\Gamma)^{J\circ\chi} = \dim T(S/\langle j\circ h\rangle) \geq 1$.

If $h(x_i) = j(x_i)$ for $i = 1,2$, and $h(x_i) = x_i$ for $i = 3,4$, then again, h^2 fixes all the punctures x_1, x_2, x_3, and x_4. In this case, $j\circ h^2$ has order 2 and fixes no punctures, and therefore, by the Riemann–Hurwitz formula, the argument is reduced to discuss two subcases. If the genus g' of $S/\langle j\circ h^2\rangle$ is one and k' (k' is the number of the fixed points of $j\circ h^2$ on $\overline{S}$) is zero, then $S/\langle j\circ h^2\rangle$ is of signature $(1,2;\infty,\infty)$, which implies that $\dim T(\Gamma)^{J\circ\chi^2} = 2$. If $g' = 0$ and

$k'=4$, then $S/\langle j\circ h^2\rangle$ is a Riemann sphere with 6 distinguished points including the two punctures x_1' and x_2'. It follows that $\dim T(\Gamma)^{J\circ\chi^2}=3$. This proves (3).

(4) By hypothesis, χ is of order 3, so h is also of order 3. If h fixes two punctures, then h permutes the remaining 4 punctures, which is impossible since h is of order 3. If h fixes one puncture, then again, h permutes the remaining 5 punctures, but this case cannot happen either. It remains to consider the case that h fixes no punctures. In this case, all 6 punctures of S must be divided into two orbits under the iteration of h. It follows that the surface $S/\langle h\rangle$ has 4 distinguished points; more precisely, $S/\langle h\rangle$ has signature $(0,4;2,2,\infty,\infty)$. Therefore, we obtain $T(\Gamma)^\chi = T(S/\langle h\rangle)=T(0,4)$. This proves (4).

(5) h is an elliptic Möbius transformation; it has two fixed points. Obviously, h cannot fix one puncture and one regular point; otherwise the number of remaining punctures would be 5, contradicting that h has order 2 or 4. We assume first that h fixes two punctures, say x_1 and x_2. In this case, h has order 2. Since h commutes with j, x_1 and x_2 must be j-symmetric; that is, we have $j(x_1)=x_2$. Also, it is easily seen that h interchanges the two fixed points (not punctures) of j. Observe that the remaining 4 punctures cannot be a single orbit under the iteration of h since $h^2=\mathrm{id}$. The map h can be projected to a self-map h' of $S_0=S/\langle j\rangle$ which is of signature $(0,5;2,2,\infty,\infty,\infty)$. Moreover, h' fixes one puncture, interchanges the other two punctures and the two branch points. There is one more fixed point y' of h'. This implies that h interchanges the two points $\{\zeta^{-1}(y')\}$.

Consider the conformal automorphism $j\circ h$ of S. Since j commutes with h, $j\circ h$ is of order 2 and thus divides the 6 punctures into 3 orbits. There are also 2 branch points on $S/\langle j\circ h\rangle$, which come from the fixed points $\{\zeta^{-1}(y')\}$. We see that $S/\langle j\circ h\rangle$ is of signature $(0,5;2,2,\infty,\infty,\infty)$. Therefore, we obtain $\dim T(\Gamma)^{J\circ\chi}=2$.

Next, if h fixes two regular points, then h cannot be of order 4 unless $h=\mathrm{id}$ (since h defines a permutation of the 6 punctures). It follows that the 6 punctures are divided into three orbits under the iteration of h, which means that $S/\langle h\rangle$ is of signature $(0,5;2,2,\infty,\infty,\infty)$. In particular, $\dim T(\Gamma)^\chi=2$. This proves (5) and hence the proof of Lemma 6.1 is complete. ▫

Lemma 6.2. *We assume in each of the following cases that there is a non-trivial elliptic element* $\chi\in\mathrm{Mod}\,\Gamma$, $\chi\neq J$, *so that* $\langle\chi,J\rangle$ *is an abelian subgroup of* $\mathrm{Mod}\,\Gamma$.

(1) *If* Γ *is of type* $(1,3)$, *then* $\chi^3=\mathrm{id}$ *or* J.

(2) *If* Γ *is of type* $(0,5)$, *then* $\chi^2=\mathrm{id}$ *or* J.

Proof. (1) First we assume that Γ is of type $(1,3)$. Let h, j, and S be as in Lemma 6.1. Then clearly, $h\circ j\circ h^{-1}=j$ on S. This means that h can be projected to a Möbius transformation h' of $S_0=S/\langle j\rangle$ in the sense of orbifolds. Note that S_0 is an orbifold of signature $(0,5;2,2,2,\infty,\infty)$, where one puncture comes from the puncture of S fixed by h (since S is of type $(1,3)$ and h commutes with j). It follows that h' fixes the two punctures pointwise. Since h' is periodic, either h' or ${h'}^3$ is id. In the first case, h is either id or j, both of which cannot occur. If ${h'}^3 = \mathrm{id}$, then either h^3 or h^6 is id.

(2) Once again, h can be projected to an elliptic Möbius transformation h' of $S_0 = S/\langle j\rangle$ in the sense of orbifolds. Observe that S_0 is an orbifold of signature $(0,4;2,\infty,\infty,\infty)$, where one puncture comes from the fixed point (puncture) of j. It follows that h' fixes both the branch point and the puncture coming from the fixed point of j. So ${h'}^2 = \mathrm{id}$ on S_0. Therefore, either h^2 or h^4 is id. □

7. Proofs of the theorems

Let Γ and Γ' be finitely generated Fuchsian groups of the first kind. Assume that Γ contains at least one elliptic element and is of signature $(g,n;\nu_1,\dots,\nu_n)$. By applying the theorem of Bers–Greenberg [7], we may assume, without loss of generality, that Γ' is torsion free. Let Γ' be of type (g',n').

Suppose that there is an isomorphism $\varphi\colon F(\Gamma)\to T(\Gamma')$. Then by Theorem 1, the pair $\big((g,n),(g',n')\big)$ lies in Table A in the introduction. Assume that $(g,n)\neq(0,3)$. Theorem 1 also asserts that every elliptic element e of Γ is of order 2. Let s be the canonical section of $\pi\colon F(\Gamma)\to T(\Gamma)$ determined by the fixed point of e. Lemma 3.5 says that $l' = \varphi\circ s\big(T(\Gamma)\big)$ must be a component of the hyperelliptic locus in $T(\Gamma')$. Further, l' is the fixed point set of the hyperelliptic involution $J' = \varphi\circ e\circ\varphi^{-1}\in\operatorname{Mod}\Gamma'$. More precise information can be captured by the following lemma.

Lemma 7.1. *Suppose that* $(g,n;\nu_1,\dots,\nu_n)$ *is in Lemma* 4.1 *and that there is an isomorphism* $\varphi\colon F(\Gamma)\to T(\Gamma')$. *Then the Teichmüller modular group* $\operatorname{Mod}\Gamma'$ *contains an abelian subgroup which is generated by an elliptic element* χ' *and a hyperelliptic involution* J'.

Proof. The assertion follows immediately from Lemma 3.5, Lemma 5.2, Royden's theorem [19] as well as its generalization (Earle–Kra [9]). □

Proof of 26 *cases of Theorem* 2. (1) The signature $(g,n;\nu_1,\dots,\nu_n)$ is in Lemma 4.1(1). By Theorem 1, $(g',n')=(3,0)$. Now Lemma 5.2(1) tells us that we can find a $\chi\in\operatorname{mod}\Gamma$ so that $\chi^3=\mathrm{id}$ and $\dim F(\Gamma)^\chi = 1$. But Lemma 7.1 and Lemma 6.1(1) assert that $\dim T(\Gamma')^{\varphi\circ\chi\circ\varphi^{-1}} = 2$. This is a contradiction.

(2) $(g,n;\nu_1,\dots,\nu_n)$ is in Lemma 4.1(2). In this case, $(g',n')=(2,2)$. Similarly, the assertion follows from Lemma 5.2(2), Lemma 7.1 and Lemma 6.1(2).

(3) $(g,n;\nu_1,\dots,\nu_n)$ is in Lemma 4.1(3). In this case (g',n') is either $(2,1)$, or $(1,4)$. Our method leads to no contradiction if $(g',n')=(2,1)$. We only assume that $(g',n')=(1,4)$. Again, by using Lemma 7.1, one sees at once that Lemma 5.2(3) contradicts Lemma 6.1(3).

(4) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.1(4). In this case, (g', n') is either $(0,6)$, or $(2,0)$, or $(1,3)$. We do not need to deal with the case that $(g', n') = (2,0)$ since $T(0,6) \cong T(2,0)$ (see Section 2). We first assume that $(g', n') = (1.3)$. By Lemma 5.2(4), there is a $\chi \in \operatorname{mod}\Gamma$, whose order is 4 or 8, such that χ commutes with an elliptic element $e \in \Gamma$. By Lemma 7.1 and Lemma 6.2(1), we see that this case cannot happen.

If $(g', n') = (0,6)$, then by Lemma 5.2(4), there is another $\chi \in \operatorname{mod}\Gamma$ (with order 3) so that $\dim F(\Gamma)^\chi = 0$. But Lemma 6.1(4) excludes this possibility.

(5) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.1(5). Then again $(g', n') = (0,6), (2,0)$, or $(1,3)$. Our method does not work for the case that $(g', n') = (1,3)$. We only assume that (g', n') is $(0,6)$ or $(2,0)$. But this case can be treated in the same way as (4).

(6) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.1(6). If $(g', n') = (1,3)$, by using Lemma 7.1, one sees that Lemma 6.2(1) contradicts Lemma 5.2(6). If $(g', n') = (0,6)$, then Lemma 6.1(5) contradicts Lemma 5.2(6). The case that $(g', n') = (2,0)$ can also be settled since $T(0,6) \cong T(2,0)$.

(7) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.1(7). Then $(g', n') = (0,5)$ or $(1,2)$. We only need to treat the case that $(g', n') = (0,5)$ since $T(0,5) \cong T(1,2)$. The assertion follows from Lemma 6.2(2) and Lemma 5.2(7).

(8) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.2. Then $(g', n') = (1,2)$ or $(0,5)$. Again, it suffices to consider the case that $(g', n') = (0,5)$. The assertion follows from Lemma 6.2(2) and Lemma 5.2(8).

(9) $(g, n; \nu_1, \ldots, \nu_n)$ is in Lemma 4.4. Then $(g', n') = (1,3), (0,6)$, or $(2,0)$. Once again, we do not need to consider the case that $(g', n') = (2,0)$.

Case I. $(g', n') = (0,6)$. Then Lemma 4.4 and Lemma 7.1 guarantee that there exists a $\chi' \in \operatorname{Mod}\Gamma'$ so that ${\chi'}^4$ is either id or a hyperelliptic involution J'. If ${\chi'}^4 = \mathrm{id}$, then Lemma 6.1(5) contradicts Lemma 5.2(9) since $F(\Gamma)^\chi \subset F(\Gamma)^{\chi^2}$.

If ${\chi'}^4 = J'$, then ${\chi'}^2$ satisfies the condition of Lemma 6.1(5). Hence, $\dim T(\Gamma')^{{\chi'}^2} = 2$. But this contradicts Lemma 5.2(9).

Case II. $(g', n') = (1,3)$. Then Lemma 4.4 implies that there is $\chi' \in \operatorname{Mod}\Gamma'$ so that ${\chi'}^4$ is either id or equal to J'. This contradicts Lemma 6.2(1). ▫

Proof of Theorem 3. First we prove the *if* part. If Γ is of type $(0,3)$, then the assertion follows from (1.2) in the introduction. If Γ is torsion free, then we have (1.1). If Γ is of signature $(0,4;2,2,2,\infty)$ or $(0,4;2,2,\infty,\infty)$, then from (1.5) of the introduction, one sees that $F(0,4;\infty,\infty,\infty,\infty) \cong F(0,4;2,2,2,\infty) \cong F(0,4;2,2,\infty,\infty) \cong T(0,5)$.

Now we prove the *only if* part. Assume that Γ contains elliptic elements, whose type is not $(0,3)$, and whose signature is neither $(0,4;2,2,\infty,\infty)$ nor $(0,4;2,2,2,\infty)$. Also assume that $F(\Gamma)$ is isomorphic to a Teichmüller space $T(\Gamma')$ for some Fuchsian group Γ' of type $(g, n+1)$. The hypothesis implies that the condition of Theorem 1 is satisfied. By using Theorem 1, one finds that all elliptic elements of Γ must be of order 2. Moreover, by examining all entries of Table B in the introduction, one sees at once that several cases are possible, which are:

(1) $(g,n;\nu_1,\dots,\nu_n) = (0,5;\underbrace{2,\dots,2}_{k},\underbrace{\infty,\dots,\infty}_{5-k})$, $0 < k \le 5$ and $(g',n') = (0,6)$;

(2) $(g,n;\nu_1,\dots,\nu_n) = (1,2;2,2)$ or $(1,2;2,\infty)$ and $(g',n') = (1,3)$;

(3) $(g,n;\nu_1,\dots,\nu_n) = (1,1;2)$, and $(g',n') = (1,2)$;

(4) $(g,n;\nu_1,\dots,\nu_n) = (0,4;2,\infty,\infty,\infty)$ and $(g',n') = (0,5)$.

But all of these cases are excluded by Theorem 2. Hence, the proof of Theorem 3 is complete. □

8. Continuation of the proof of Theorem 2

Let Γ_1 be a finitely generated Fuchsian group of the first kind whose signature is $(2,0;\_\_)$. Γ_1 is a normal subgroup of a Fuchsian group Γ_0 with Γ_0 of signature $(0,6;2,\dots,2)$. See Section 2 for an exposition. Note that $\Gamma_1 \triangleleft \Gamma_0$ is of index 2 and $T(\Gamma_0) \cong T(\Gamma_1)$. Let Φ denote this isomorphism. The considerations of Section 5 in Earle–Kra [9] lead to an isomorphism:

$$\lambda\colon F(\Gamma_0) \to F(\Gamma_1)$$

defined by sending $([\mu],z) \in F(\Gamma_0)$ to $\big(\Phi([\mu]),z\big) \in F(\Gamma_1)$.

To see that λ is well defined, we observe that for every $\mu \in M(\Gamma_0)$, there corresponds to a $\nu \in M(\Gamma_1)$ with $[\nu] = \Phi([\mu])$ and vise versa (since Φ is an isomorphism). This implies that the set $w^{\mu'}(U)$ coincides with $w^{\nu'}(U)$ for $\mu' \sim \mu$ and $\nu' \sim \nu$.

To see that λ is holomorphic, note first that Φ is a biholomorphic map. Next, when $[\mu'] \in T(\Gamma_0)$ lies in a sufficiently small neighborhood of $[\mu] \in T(\Gamma_0)$, z stays in $w^{\mu'}(U)$. It is trivial that λ is biholomorphic.

Let $\pi_0\colon F(\Gamma_0) \to T(\Gamma_0) \cong T(0,6)$ and $\pi_1\colon F(\Gamma_1) \to T(\Gamma_1) \cong T(2,0)$ be the natural projections. Since Γ_0 is of signature $(0,6;2,\dots,2)$, all canonical sections s_0 of π_0 are determined by elliptic elements of Γ_0. Let $\mathscr{S}_0$ denote the set of all images $s_0\big(T(\Gamma_0)\big)$. We first prove:

Lemma 8.1. *Let $\theta \in \operatorname{mod}\Gamma_1$ and $\theta_0 = \lambda^{-1}\circ\theta\circ\lambda$. Then $\theta_0 \in \operatorname{mod}\Gamma_0$. Furthermore, θ_0 keeps the set $\mathscr{S}_0$ invariant; that is, for any canonical section s_0 of π_0, $\theta_0\big(s_0\big(T(\Gamma_0)\big)\big)$ is the image of a canonical section of π_0.*

Proof. Let $\theta \in \operatorname{mod}\Gamma_1$ be induced by a self-map f of U, and f_1 the projection of f to the surface U/Γ_1. Note that U/Γ_1 is a compact Riemann surface of genus 2 which is, of course, hyperelliptic.

A theorem of Lickorish [14] tells us that f_1 is isotopic to a self-map f_1' which is a product of Dehn twists about the curves belonging to the set of Lickorish's generators. But the set of Lickorish's generators are invariant under the hyperelliptic involution j. It follows that any Dehn twist about a Lickorish's generator is isotopic to a twist about that generator which commutes with j. This implies that f_1' is isotopic to a map (still called f_1') which commutes with j as well. It follows that f_1' projects to a self-map $f_0\colon U/\Gamma_0 \to U/\Gamma_0$ in the sense of orbifolds.

Now lift the self-map f_1' of U/Γ_1 to the map $f'\colon U \to U$. Since f_1 is isotopic to f_1' on U/Γ_1, we can choose a lift so that f' is isotopic to f. On the other hand, f' is also a lift of f_0; that is, $f' \in N(\Gamma_0)$, the normalizer of Γ_0 in $Q(\Gamma_0)$. Hence, by definition of λ, the geometric isomorphism of Γ_0 induced by f' is exactly θ_0. It follows that $\theta_0 \in \operatorname{mod}\Gamma_0$. Since f_0 is a self-map in the sense of orbifolds (all branch points here are of order 2), it sends a branched point to a branched point. This implies that θ_0 sends an image of a canonical section to an image of a canonical section. The lemma is proved. □

We need to study the relationships among components of the hyperelliptic locus in $T(2,1)$. Observe that every component of the hyperelliptic locus of $T(2,1)$ is the fixed point set of an elliptic modular transformation induced by a hyperelliptic involution of a marked Riemann surface in $T(2,1)$; it is a connected, closed submanifold of $T(2,1)$.

Lemma 8.2. *Any two components l_1, l_2 of the hyperelliptic locus in $T(2,1)$ are modular equivalent; that is, there exists a $\chi \in \operatorname{Mod}(2,1)$ so that $\chi(l_1) = l_2$. Furthermore, if we denote by $J_1, J_2 \in \operatorname{Mod}(2,1)$ the hyperelliptic involutions corresponding to l_1 and l_2, respectively, we have $\chi \circ J_1 \circ \chi^{-1} = J_2$.*

Proof. See the appendix. □

Let Γ_0 and Γ_1 be as above. We choose a torsion free group Γ' of type $(2,1)$, and let $\psi\colon F(\Gamma_1) \to T(\Gamma')$ be the Bers isomorphism.

Let Γ be a finitely generated Fuchsian group of the first kind whose signature is $(0,6;2,\dots,2,\infty)$. Suppose that there is an isomorphism $\varphi\colon F(\Gamma) \to T(\Gamma')$. It turns out that $\psi' = \psi \circ \lambda\colon F(\Gamma_0) \to T(\Gamma')$ is an isomorphism, we thus obtain an equivalence $\omega = \psi'^{-1} \circ \varphi\colon F(\Gamma) \to F(\Gamma_0)$. For convenience we exhibit these isomorphisms in the following diagram:

(8.1)
$$\begin{array}{ccc} F(\Gamma_0) & \xrightarrow{\ \lambda\ } & F(\Gamma_1) \\ \uparrow \omega & \searrow \psi' & \downarrow \psi \\ F(\Gamma) & \xrightarrow{\ \varphi\ } & T(\Gamma'). \end{array}$$

Note that the diagram (8.1) is commutative. Let $\mathscr{S}$ denote the set of all images $s\bigl(T(\Gamma)\bigr)$ under canonical sections s of $\pi\colon F(\Gamma) \to T(\Gamma) \cong T(0,6)$. We have

Lemma 8.3. (1) *ω carries $\mathscr{S}$ into $\mathscr{S}_0$.*

(2) *Let e denote an elliptic element of Γ (e can be thought of as an element of $\operatorname{mod}\Gamma$). Then $\omega\circ e\circ\omega^{-1}\in\operatorname{mod}\Gamma_0$. Furthermore, $\omega\circ e\circ\omega^{-1}$ is defined by an elliptic element of Γ_0.*

Caution. The isomorphism ω need not be fiber-preserving.

Proof. Let $e_0\in\Gamma_0$ be an elliptic element of order 2, let s and s_0 be the canonical sections of $\pi\colon F(\Gamma)\to T(\Gamma)\cong T(0,6)$ and $\pi_0\colon F(\Gamma_0)\to T(\Gamma_0)\cong T(0,6)$ corresponding to the fixed points of e and e_0, respectively. We define $l'=\varphi\big(s\big(T(\Gamma)\big)\big)$ and $l'_0=\psi'\big(s_0\big(T(\Gamma_0)\big)\big)$. Lemma 3.5 asserts that both l' and l'_0 are the components of the hyperelliptic locus in $T(\Gamma')$ and that both $e'=\varphi\circ e\circ\varphi^{-1}$ and $e'_0=\psi'\circ e_0\circ{\psi'}^{-1}$ are the corresponding hyperelliptic involutions. By Lemma 8.2, we see that there is a modular transformation $\chi'\in\operatorname{Mod}\Gamma'$ such that $\chi'(l')=l'_0$, and $\chi'\circ e'\circ{\chi'}^{-1}=e'_0$. From a theorem of Bers (Theorem 10 of Bers [5]), we know that $\theta=\psi^{-1}\circ\chi'\circ\psi$ is a modular transformation of $F(\Gamma_1)$. Lemma 8.1 then says that $\theta_0=\lambda^{-1}\circ\theta\circ\lambda$ keeps the set of the images of canonical sections invariant. We claim that $\omega\big(s\big(T(\Gamma)\big)\big)=\theta_0^{-1}\big(s_0\big(T(\Gamma_0)\big)\big)$.

Indeed, from the diagram (8.1) we can obtain

$$\begin{aligned}\omega\big(s\big(T(\Gamma)\big)\big)&={\psi'}^{-1}\circ\varphi\big(s\big(T(\Gamma)\big)\big)={\psi'}^{-1}(l')\\&={\psi'}^{-1}\circ{\chi'}^{-1}(l'_0)={\psi'}^{-1}\circ{\chi'}^{-1}\circ\psi'\big(s_0\big(T(\Gamma_0)\big)\big)\\&=\lambda^{-1}\circ\psi^{-1}\circ\psi\circ\theta^{-1}\circ\psi^{-1}\circ\psi\circ\lambda\big(s_0\big(T(\Gamma_0)\big)\big)\\&=\lambda^{-1}\circ\theta^{-1}\circ\lambda\big(s_0\big(T(\Gamma_0)\big)\big)=\theta_0^{-1}\big(s_0\big(T(\Gamma_0)\big)\big).\end{aligned}$$

To prove the second statement of this lemma, we note that $e\in\operatorname{mod}\Gamma$ fixes $s\big(T(\Gamma)\big)$ pointwise. Hence, $\omega\circ e\circ\omega^{-1}$ fixes $\theta_0^{-1}\big(s_0\big(T(\Gamma_0)\big)\big)$ pointwise as well. By Lemma 3.5, we see that

$$\psi'\big(\theta_0^{-1}\big(s_0\big(T(\Gamma_0)\big)\big)\big)=l''_0$$

is a component of the hyperelliptic locus in $T(\Gamma')$. Now $\psi'\circ(\omega\circ e\circ\omega^{-1})\circ{\psi'}^{-1}\in\operatorname{Mod}\Gamma'$ has the property that its restriction to l''_0 is id. By Proposition 3.1 and Corollary 2 to Proposition III.7.9 of Farkas–Kra [11], we conclude that $\psi'\circ(\omega\circ e\circ\omega^{-1})\circ{\psi'}^{-1}$ is either id or the hyperelliptic involution e''_0 corresponding to l''_0. But evidently, $\psi'\circ(\omega\circ e\circ\omega^{-1})\circ{\psi'}^{-1}$ is not id. Thus

$$\psi'\circ(\omega\circ e\circ\omega^{-1})\circ{\psi'}^{-1}=e''_0. \tag{8.2}$$

On the other hand, if we denote by e_0 the elliptic element of Γ_0 corresponding to $\theta_0^{-1}\big(s_0\big(T(\Gamma_0)\big)\big)$, by Proposition 3.1, $\psi'\circ e_0\circ{\psi'}^{-1}=e''_0$. It follows from (8.2) that $e_0=\omega\circ e\circ\omega^{-1}$. This completes the proof of the lemma. □

Proof of the case $\big((g,n;\nu_1,\dots,\nu_n),(g',n')\big) = \big((0,6;2,2,2,2,2,\infty),(2,1)\big)$. Assume that U is the central fiber of both $F(\Gamma_0)$ and $F(\Gamma_1)$ (by a central fiber we mean the fiber $\pi_0^{-1}([0])$). By investigating the diagram (8.1), we know that the restriction of φ to U is a holomorphic map into $T(\Gamma')$ (here U is also thought of as the central fiber of $F(\Gamma)$). Hence

$$d\big(\varphi([0],x),\varphi([0],y)\big) \le \varrho\big(([0],x),([0],y)\big) \qquad \text{for all } x,y\in U, \tag{8.3}$$

where d is the Kobayashi metric on $T(\Gamma')$. By Royden's theorem [19], the Kobayashi metric is the same as the Teichmüller metric. Therefore,

$$\langle\varphi([0],x),\varphi([0],y)\rangle \le \varrho\big(([0],x),([0],y)\big). \tag{8.4}$$

Unfortunately, there is no guarantee that $\omega(U)$ is a fiber in $F(\Gamma_0)$. To get rid of this difficulty, let $e_1,\dots,e_5$, and e_∞ be a set of generators of Γ, where e_i, $i=1,\dots,5$, are elliptic Möbius transformations of order 2, and e_∞ is a parabolic Möbius transformation. These generators may be chosen so as to satisfy the following relation:

$$e_5\circ\cdots\circ e_1 = e_\infty. \tag{8.5}$$

Choose a point $x\in U$ so that

$$\varrho\big(([0],x),e_\infty([0],x)\big) < \varepsilon \tag{8.6}$$

for an arbitrarily small positive number ε. This is possible because e_∞ is parabolic. Observe that $\omega([0],x)\in F(\Gamma_0)$. Let $\pi_0^{-1}([\mu])\in F(\Gamma_0)$ be the fiber to which the point $\omega([0],x)$ belongs. Then we construct a Bers' allowable mapping of $F(\Gamma_0)$ onto another isomorphic Bers fiber space, this Bers' allowable mapping can be defined by carrying the fiber $\pi_0^{-1}([\mu])$ to the central fiber of the new Bers fiber space. In this regard, we may assume, without loss of generality, that $\omega([0],x)\in\omega(U)\cap U\subset F(\Gamma_0)$, and hence also that $\lambda\circ\omega([0],x)\in U\subset F(\Gamma_1)$. Let $\tau'=\varphi([0],x)\in T(\Gamma')$, and let $\chi'_\infty=\varphi\circ e_\infty\circ\varphi^{-1}$. By Royden's theorem [19] (and its generalization [9]), $\chi'_\infty\in\operatorname{Mod}\Gamma'$. Moreover, (8.4) and (8.6) imply that

$$\begin{aligned}\langle\tau',\chi'_\infty(\tau')\rangle &= \langle\varphi([0],x),\varphi\circ e_\infty\circ\varphi^{-1}(\tau')\rangle\\ &= \big\langle\varphi([0],x),\varphi\circ\big(e_\infty([0],x)\big)\big\rangle \le \varrho\big(([0],x),e_\infty([0],x)\big) < \varepsilon.\end{aligned} \tag{8.7}$$

As a holomorphic automorphism, $e_\infty\in\operatorname{mod}\Gamma$ has no fixed point in $F(\Gamma)$, thus χ'_∞ has no fixed point in $T(\Gamma')$ either. It turns out that χ'_∞ is a parabolic modular transformation of $T(\Gamma')$.

From Theorem 6 of Bers [5], χ'_∞ is induced by a reducible self-map f'_∞ of a Riemann surface S' of type $(2,1)$ (the puncture is denoted by x'). Let

$S' = \varphi([0], x)$, and let $c' = \{c'_1, \ldots, c'_r\}$, $r \geq 1$, be the corresponding admissible system of curves on S' which completely reduces f'_∞.

Since χ'_∞ is parabolic, the restriction of f'_∞ to all parts of $S' - N(c')$ is either trivial or periodic (but not necessarily componentwise), where $N(c')$ is an arbitrary small neighborhood of $c' = \{c'_1, \ldots, c'_r\}$. A basic observation shows that if all restrictions of f'_∞ to $S' - N(c')$ are trivial, then f'_∞ must be isotopic to some product of Dehn twists about $c'_1, \ldots, c'_r$ (see Abikoff [1]).

We need the following lemma.

Lemma 8.4. *Suppose that χ'_∞ is defined as above. Then $r = 2$, and either χ'_∞ or ${\chi'_\infty}^2$ is actually induced by a spin about x'; that is, either χ'_∞ or ${\chi'_\infty}^2$ is induced by a self-map of S' which is a power of the composition $h_{c'_2}^{-1} \circ h_{c'_1}$, where c'_1 and c'_2 bounds a cylinder which contains the puncture x'.*

Proof. Recall that χ'_∞ is induced by f'_∞, and f'_∞ fixes the puncture x'. From the proof of Lemma 8.1, $\lambda^{-1} \circ \psi^{-1} \circ \chi'_\infty \circ \psi \circ \lambda = {\psi'}^{-1} \circ \chi'_\infty \circ \psi'$ is an element of mod Γ_0. In particular, ${\psi'}^{-1} \circ \chi'_\infty \circ \psi'$ is a fiber-preserving automorphism of $F(\Gamma_0)$. Consider the following commutative diagram

$$\text{(8.8)} \qquad \begin{array}{ccc} T(\Gamma') & \xrightarrow{\chi'_\infty} & T(\Gamma') \\ \downarrow{\scriptstyle \pi'_1} & & \downarrow{\scriptstyle \pi'_1} \\ T(\Gamma_1) & \xrightarrow{\chi'} & T(\Gamma_1) \end{array}$$

where $\pi'_1 = \pi_1 \circ \psi^{-1}$, and as before, $\psi \colon F(\Gamma_1) \to T(\Gamma')$ is the Bers isomorphism. Note that U/Γ_1 is a surface of type $(2, 0)$ which is always hyperelliptic. The map χ' is defined by the formula $\pi'_1 \circ \chi'_\infty = \chi' \circ \pi'_1$. Since π'_1 is defined by forgetting the puncture x', χ' is defined by f'_∞ by filling in the puncture x'. By $\overline{f_\infty}'$ we denote the self-map on $\overline{S'}$ (type $(2, 0)$) inducing χ'. Suppose that χ' is neither induced by id nor induced by the hyperelliptic involution. The action of χ' on $T(\Gamma_1)$ is non-trivial. We see from (8.8) that there is a fiber which is sent by χ'_∞ to a different fiber. This implies that ${\psi'}^{-1} \circ \chi'_\infty \circ \psi'$ sends a fiber of $F(\Gamma_0)$ to another different fiber of $F(\Gamma_0)$. We may assume that ${\psi'}^{-1} \circ \chi'_\infty \circ \psi' \circ ({\psi'}^{-1}(\tau'))$ and ${\psi'}^{-1}(\tau')$ lie in different fibers.

On the other hand, the group Γ is generated by $e_1, \ldots, e_5$, and e_∞. From Lemma 8.3, the ω-image of Γ is a subgroup of Γ_0. It follows that

$$\text{(8.9)} \qquad {\psi'}^{-1}(\tau') = {\psi'}^{-1} \circ \varphi([0], x) = \omega([0], x).$$

We also have

$$\text{(8.10)} \qquad \begin{aligned} {\psi'}^{-1} \circ \chi'_\infty \circ \psi' &= {\psi'}^{-1} \circ \varphi \circ e_\infty \circ \varphi^{-1} \circ \psi' = \omega \circ e_\infty \circ \omega^{-1} \\ &= \omega \circ (e_5 \circ \cdots \circ e_1) \circ \omega^{-1} = e_{0,5} \circ \cdots \circ e_{0,1}, \end{aligned}$$

where $e_{0,i}$, $i = 1, \dots, 5$, are ω-images of e_i in Γ_0. From (8.9) and (8.10) we conclude that

$$\psi'^{-1} \circ \chi'_\infty \circ \psi' \circ (\psi'^{-1}(\tau')) = e_{0,5} \circ \cdots \circ e_{0,1} \circ (\omega([0], x)). \tag{8.11}$$

Since $\Gamma_0 \triangleleft \operatorname{mod}\Gamma_0$ keeps all fibers of $F(\Gamma_0)$ invariant, and since $x \in U$, by (8.11), we see that $\psi'^{-1} \circ \chi'_\infty \circ \psi' \circ (\psi'^{-1}(\tau'))$ and $\psi'^{-1}(\tau')$ lie in the same fiber U. This is a contradiction.

We conclude that χ' is either induced by id or induced by the hyperelliptic involution. If χ' is induced by id, the lemma is proved by using the fact that the map (reduced by c') f'_∞ is a power of a spin if and only if $\overline{f_\infty}'$ is isotopic to id. If χ' is induced by the hyperelliptic involution, then χ'^2 is induced by id. From a similar argument as above, we conclude that ${\chi'_\infty}^2$ is induced by a spin.

This completes the proof of the lemma. □

By taking squares of maps if necessary, without loss of generality we may assume that χ'_∞ is induced by a spin throughout this section.

Observe that the spin described in the above lemma defines a closed c'_0 curve in $\overline{S'}$ passing through x'. In fact, c'_0 is freely homotopic to both c'_1 and c'_2 in $\overline{S'}$. Thus the homotopy class of c'_0 in $\overline{S'}$ determines an element of the fundamental group $\pi_1(\overline{S'}, x')$, and hence corresponds to an element $\gamma_1 \in \Gamma_1$ which is hyperbolic.

To proceed, we need to construct a certain automorphism on $F(\Gamma)$ which does not act on the corresponding $T(\Gamma')$. The method was introduced in Sections 4 and 5.

Let S be the orbifold drawn as as in Figure 1(b); except that instead of putting $x_2, \dots, x_7$ on Σ, we place five (order 2) branch points $x_2, \dots, x_6$ on Σ. Let x_1 be a puncture, and let A denote the rotation about the z axis with rotation angle $2\pi/5$. Clearly, A is an automorphism of S in the sense of orbifolds.

Lemma 8.5. *Suppose that* Γ *is of signature* $(0, 6; 2, 2, 2, 2, 2, \infty)$. *There is a non-trivial automorphism* $\delta \in \operatorname{mod}\Gamma - \Gamma$ (*not unique*) *with the property that* $\delta^5 = e_\infty \in \Gamma$ *is parabolic.*

Proof. We may assume that Γ is a Fuchsian group so that $U/\Gamma = S$ is the surface described above. There is a self-map A of S in the sense of orbifolds such that A fixes x_1 and A^5 is isotopic to the identity. Moreover, A can always be lifted to self-maps $\hat{A}$ of U. See Birman–Hilden [8].

Let $e_\infty \in \Gamma$ be a parabolic Möbius transformation corresponding to the puncture x_1. Since A fixes x_1, $\hat{A}$ can be chosen so that $\hat{A}$ fixes the fixed point of e_∞ (which lies in $\mathbf{R}$). We obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\hat{A}} & U \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho} \\ U/\Gamma & \xrightarrow{A} & U/\Gamma, \end{array}$$

where ϱ is the natural projection. Now $\hat{A}$ induces an element $[\hat{A}]$, which consists of those quasiconformal self-maps of U which lie in the normalizer of Γ and are isotopic to $\hat{A}$, in $\operatorname{mod}\Gamma$. Let $\delta = [\hat{A}]$. Clearly, $\zeta(\delta) \in \operatorname{Mod}\Gamma$ is induced by the self-map A. Since A is not isotopic to the identity, δ is not in Γ. To prove that $\delta^5 = e_\infty$, we note that $\zeta(\delta^5) = \zeta(\delta)^5 = A^5 = \mathrm{id}$. It turns out that δ^5 lies in the kernel of $\zeta\colon \operatorname{mod}\Gamma \to \operatorname{Mod}\Gamma$.

Assume that the group Γ is normalized so that x_1 corresponds to the parabolic transformation e_∞ which is of the form $e_\infty(z) = z+1$. The local coordinate chart near the puncture x_1 can be written as $\xi = e^{2\pi i z}$, where $z = x + iy$ and y is sufficiently large. Take a small loop c around x_1 in the coordinate patch, and choose $x \in c$. Choose $\tilde{x} \in U$ so that $\varrho(\tilde{x}) = x$. The loop c can be lifted to a horizontal segment

$$\tilde{c} = \{z = x + iy \in \mathbf{C} : \operatorname{Re}\tilde{x} \le x \le \operatorname{Re}\tilde{x} + 1 \text{ and } y = \operatorname{Im}\tilde{x}\}.$$

Since A leaves the loop c invariant, $\hat{A}^5$ sends $\tilde{x}$ to $\tilde{x}+1$. In particular, δ^5 is not the identity. It follows that the restriction of δ^5 to each fiber is a parabolic Möbius transformation. □

Recall that f'_∞ induces $\chi'_\infty = \varphi \circ e_\infty \circ \varphi^{-1}$. From Lemma 8.5, f'_∞ is a spin about the puncture x'. This in particular implies that f'_∞ is a reducible map with respect to the system $c' = \{c'_1, c'_2\}$ and that the restrictions of f'_∞ to each modular part of $S' - \{N(c')\}$ is id. On the other hand, $\chi''_\infty = \varphi \circ \delta \circ \varphi^{-1}$ is another modular transformation with ${\chi''_\infty}^5 = \chi'_\infty$. Let χ''_∞ be induced by the map f'_δ. It is easy to see that f'_∞ and f'_δ share the common system $c' = \{c'_1, c'_2\}$ of admissible curves.

We need to prove that there is at least one component S_i of $S - \{N(c')\}$ on which f'_δ is non-trivial. Suppose that both f'_δ and f'_∞ are products of Dehn twists about the system c'. Since $\gamma \in \Gamma$ is parabolic, we can choose a point $[\mu] \in T(\Gamma)$ but not in $T(\Gamma)^{\zeta(\delta)}$, and a sequence $\{[\mu], z_m)\}$ in $w^\mu(U)$ tending to the fixed point of γ^μ so that

$$\varrho\big(([\mu], z_m), \gamma([\mu], z_m)\big) \to 0.$$

Note that $\varphi|_{w^\mu(U)}\colon w^\mu(U) \to T(\Gamma')$ is distance non-increasing. If necessary, a convergent subsequence of $\{[\mu], z_m)\}$ may be chosen, and we may assume that the sequence $y_m = \varphi([\mu], z_m)$ also converges and satisfies

$$\langle y_m, \chi'_\infty(y_m)\rangle \to 0, \qquad \to \infty.$$

We can write $f'_\delta = \tau_1^{\alpha_1} \circ \tau_2^{\alpha_2}$, and $f'_\infty = \tau_1^{5\alpha_1} \circ \tau_2^{5\alpha_2}$, where α_i are non-trivial integers and τ_i are the modular transformations induced by the Dehn twists about c'_i, $i = 1, 2$.

We claim that

$$\langle y_m, \chi''_\infty(y_m)\rangle \to 0, \qquad \text{as } m \to \infty. \tag{8.12}$$

Indeed, for sufficiently large m, the surfaces S_m representing y_m have long thin disjoint cylinders with central curves c_1' and c_2'. These cylinders are twisted (defined by f_∞') about these central curves, but with small distortion of twist factors. It follows that for the sequence $\{S_m\}$, $\langle\chi_\infty''(y_m), y_m >\rangle = \langle\chi_\infty''^{(1-5)} \circ \chi_\infty''^{5}(y_m), y_m\rangle$ is also arbitrarily small for large m.

We conclude that (8.12) holds. Consider the holomorphic map $\psi = \pi \circ \varphi^{-1}$ of $T(\Gamma')$ onto $T(\Gamma)$. ψ is distance non-increasing. So

$$\langle\pi(\varphi^{-1}(\chi_\infty''(y_m))), \pi(\varphi^{-1}(y_m))\rangle \to 0.$$

On the other hand, $\varphi^{-1}(y_m)$ and $\varphi^{-1}(\chi_\infty''(y_m))$ lie in two different fibers (by construction) of $\pi\colon F(\Gamma) \to T(\Gamma)$. More precisely, $\varphi^{-1} \circ \chi_\infty''(y_m)$ lies in the fiber over $\zeta(\delta)([\mu])$, and we must have

$$\langle\pi(\varphi^{-1}(\chi_\infty''(y_m))), \pi(\varphi^{-1}(y_m))\rangle = \langle\zeta(\delta)(\pi(y_m)), \pi(y_m)\rangle,$$

which does not depend on m. So it is a constant. This is a contradiction. The same argument also shows that $\chi_\infty''^{j}$, $j = 2, 3, 4$, are non-trivial on at least one S_i. It follows that χ_∞'' is of order 5 on S_i.

There remain two cases to consider.

Case I. c_1' (and hence c_2') is a dividing curve. A geometric observation shows that $S' - \{N(c')\} = S_1' + S_2' + S_3'$, where S_1' is of type $(1,1)$, S_2' is of type $(1,1)$, and S_3' is of type $(0,3)$. If f_δ' acts on $S' - \{N(c')\}$ componentwise, it follows from the above argument that at least one of the maps $f_\delta'|_{S_1'}$, $f_\delta'|_{S_2'}$ is non-trivial. Suppose that $f_\delta'|_{S_1'}$ is non-trivial. It is easy to see that $f_\delta'|_{S_1'}$ can be extended to a periodic map of the corresponding finite-analytic type surface and hence to a periodic map on the corresponding closed surface. Thus an automorphism of order 5 on a torus is defined, which is absurd. If f_δ' interchanges S_1' and S_2', so does $f_\delta'^{5}$, which contradicts the fact that $f_\delta'^{5}$ is a spin about x'.

Case II. c_1' (and hence c_2') is a non-dividing curve. In this case, $S' - \{N(c')\} = S_1' + S_2'$, where S_1' is of type $(1,2)$, and S_3' is of type $(0,3)$. Once again, from the above argument, $f_\delta'|_{S_1'}$ is non-trivial, and the map can be easily extended to a periodic map (order 5) of the corresponding closed surface which is once again a torus. This also leads to a contradiction.

This completes the proof of Theorem 2.

9. Appendix

The purpose of this section is to prove Lemma 8.2. It is well known that for any Teichmüller space of compact Riemann surface, any two components of the hyperelliptic locus are modular equivalent. Our argument is similar to [16].

Let S be a hyperelliptic Riemann surface (with the hyperelliptic involution j) of type $(g,1)$. Then the puncture (denoted by x_{2g+2}) is a Weierstrass point of $\overline{S}$. From a discussion in Section 2, we know that $\overline{S}/\langle j\rangle$ is an orbifold with signature $(0, 2g+2; 2, \ldots, 2)$. Let f be a self-map of $S/\langle j\rangle$ in the sense of orbifolds. f always lifts to a self-map $\tilde{f}\colon S \to S$ (see Birman–Hilden [8] for a construction) so that the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{\tilde{f}} & S \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta} \\ S/\langle j\rangle & \xrightarrow{f} & S/\langle j\rangle, \end{array}$$

where $\zeta\colon S \to S/\langle j\rangle$ is the natural projection. It is easy to see that $\tilde{f}$ and $j\circ\tilde{f}$ are all possible lifts of f.

Lemma 9.1. *f is isotopic to* id *on $S/\langle j\rangle - \{$all branch points$\}$ if and only if either $\tilde{f}$ or $j\circ\tilde{f}$ (but not both) is isotopic to* id *on S.*

Proof. See Birman–Hilden [8]. □

The geometric intersection number of two unoriented non-separating, simple closed curves α and β of S, denoted by $i(\alpha,\beta)$, is the minimal number of intersections of $\tilde{\alpha}$ and $\tilde{\beta}$ as $\tilde{\alpha}$ and $\tilde{\beta}$ run over free homotopy classes of α and β, respectively. By definition, a canonical curve α satisfies the condition that $j(\alpha)=\alpha$. Since $j\colon S\to S$ is considered a 180° rotation around the x-axis, α must contain only two fixed points. Moreover, α can be parametrized as $\alpha(\theta)$, $0\le\theta\le 2\pi$, so that $\alpha(\theta)$ contains only two Weierstrass points $\alpha(0)$ and $\alpha(\pi)$.

A *chain for* x_{2g+2} is a $2g$-tuple $(\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g)$ of canonical curves on S with the following properties:

(i) $i(\alpha_j,\alpha_k)=0$, $i(\beta_j,\beta_k)=0$, $i(\alpha_j,\beta_j)=1$, for $1\le j,k\le g$;

(ii) $i(\beta_j,\alpha_{j+1})=1$, for $1\le j\le g-1$;

(iii) $i(\alpha_j,\beta_k)=0$, for $1\le j,k\le g$, $j\ne k, k+1$.

Figure 4 below shows a chain for x_{12} in the case of $g=5$.

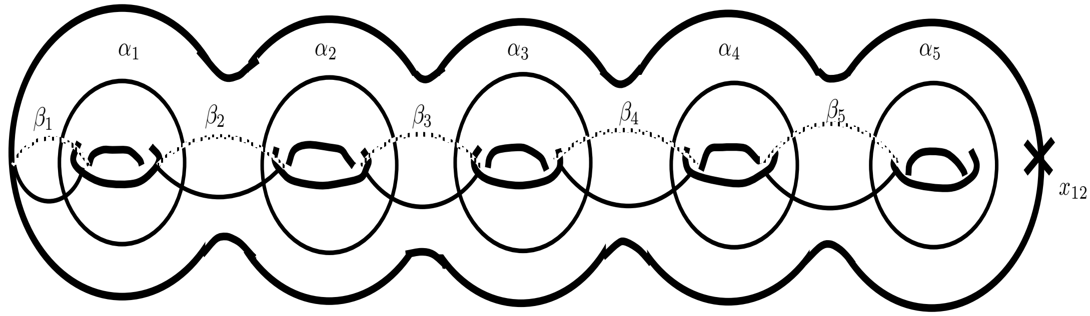

Figure 4.

As in Section 2, let h_c denote the Dehn twist about the simple closed curve c on S. Let $\mathscr{C} = (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ be a chain for x_{2g+2}. Define

$$h = h_{\beta_g} \circ h_{\alpha_g} \circ \cdots \circ h_{\beta_1} \circ h_{\alpha_1}. \tag{9.1}$$

From Lemma 4.3, one sees immediately that h is isotopic to a lift of σ (defined by (4.2) and (4.3)). By Magnus [15], σ^{2g+1} is isotopic to id on $S/\langle j \rangle - \{2g + 1 \text{ branch points}\}$. By Lemma 9.1, h^{2g+1} is either isotopic to j, or isotopic to id. But it is easy to see that h^{2g+1} reverses the orientation of the chain for x_{2g+2} of canonical curves (give arbitrarily an orientation to the curves before doing the Dehn twists). We see that h^{2g+2} is isotopic to j. We thus have

Lemma 9.2. *As a self-map of S, the hyperelliptic involution j is isotopic to the product of Dehn twists $(h_{\beta_g} \circ h_{\alpha_g} \circ \cdots \circ h_{\beta_1} \circ h_{\alpha_1})^{2g+1}$.* □

The set of components of the hyperelliptic locus in $T(2,1)$ is one-to-one correspondent with the set of isotopy classes of orientation-preserving self-maps represented by hyperelliptic involutions. Let l be a component of the hyperelliptic locus in $T(2,1)$. By using Lemma 9.2, we see that the corresponding hyperelliptic involution j is isotopic to $(h_{\beta_2} \circ h_{\alpha_2} \circ h_{\beta_1} \circ h_{\alpha_1})^5$ for a chain $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. Two chains for x_6 are called *equivalent* if they are invariant under the same hyperelliptic involution. From (9.1), one observes that h is uniquely determined by the chain $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. This leads to the following lemma.

Lemma 9.3. *There is a bijection between the set of components of the hyperelliptic locus in $T(2,1)$ and the set of chains for x_6 modulo the equivalence relation.* □

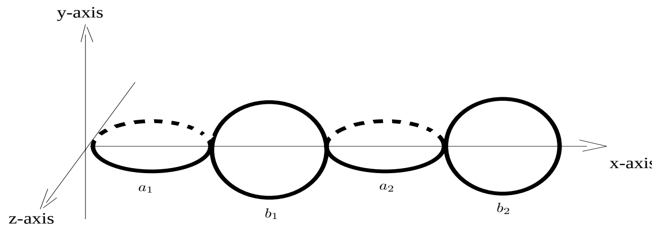

Figure 5.

Now S is considered a hyperelliptic Riemann surface of type $(2,1)$. Our purpose is to show that any chain for x_6 is homeomorphic (as a set of points) to a standard chain drawn in Figure 5, where

$$a_1 = \{(x,y,0) \in \mathbf{R}^3 : (x-1)^2 + y^2 = 1\},$$
$$b_1 = \{(x,0,z) \in \mathbf{R}^3 : (x-3)^2 + z^2 = 1\},$$
$$a_2 = \{(x,y,0) \in \mathbf{R}^3 : (x-5)^2 + y^2 = 1\},$$

and

$$b_2 = \{(x,0,z) \in \mathbf{R}^3 : (x-7)^2 + z^2 = 1\}.$$

Indeed, the desired homeomorphism w can be easily obtained by gluing 4 simple maps $w_1, \ldots, w_4$ together, where $w_1\colon \alpha_1 \to a_1$, $w_2\colon \beta_1 \to b_1$, $w_3\colon \alpha_2 \to a_2$, and $w_4\colon \beta_2 \to b_2$ are only defined on curves and can be constructed as follows. First, w_1 is defined as a homeomorphism of α_1 to a_1 with $w_1(\alpha_1 \cap \beta_1) = a_1 \cap b_1$; w_2 is a homeomorphism of β_1 to b_1 with the properties that $w_2(\alpha_1 \cap \beta_1) = a_1 \cap b_1$ and $w_2(\beta_1 \cap \alpha_2) = b_1 \cap a_2$. Since both $\beta_1 - \{\alpha_1 \cap \beta_1, \beta_1 \cap \alpha_2\}$ and $b_1 - \{a_1 \cap b_1, b_1 \cap a_2\}$ consist of two open intervals, w_2 can be easily constructed so as to satisfy the above properties. The constructions of w_3 and w_4 are similar to those of w_1 and w_2, respectively. We thus have

Lemma 9.4. *Any two chains for x_6 are homeomorphic in the sense of one dimension.* ▫

We are in the position to prove Lemma 8.2. Lemma 9.3 asserts that one can choose two chains $\mathscr{C}_1$ and $\mathscr{C}_2$ for x_6 on S corresponding to l_1 and l_2, respectively. By Lemma 9.4, there is a homeomorphism $w\colon \mathscr{C}_1 \to \mathscr{C}_2$. w extends to a homeomorphism w_0 of a tubular neighborhood $N(\mathscr{C}_1)$ onto a tubular neighborhood $N(\mathscr{C}_2)$, where $N(\mathscr{C}_1)$ is drawn in Figure 7. Hence, w determines a homeomorphism (call it w also) of $\partial N(\mathscr{C}_1)$ onto $\partial N(\mathscr{C}_2)$. The Euler characteristic of $\overline{S}$ is $2-2g=-2$. On the other hand, by looking at the chain $\mathscr{C}_1$, we see that the number V of vertices of $\overline{S} - \mathscr{C}_1$ is 3, the number E of edges of $\overline{S} - \mathscr{C}_1$ is 6. Let F denote the number of faces of $\overline{S} - \mathscr{C}_1$. By computing the Euler characteristic via the formula $V+F-E$, one obtains

$$-2 = F + 3 - 6.$$

So $F=1$. In particular, we conclude that $S-N(\mathscr{C}_1)$ is conformally equivalent to the punctured disk $\dot{\Delta}=\{z:0<|z|<1\}$. We denote by ξ_1 this conformal map. Similarly, there is a conformal map ξ_2 of $S-N(\mathscr{C}_2)$ onto $\dot{\Delta}$.

Now $\bar{S}-N(\mathscr{C}_1)$ and $\bar{S}-N(\mathscr{C}_2)$ are considered polygons. Then ξ_1 and ξ_2 can be extended to the closed polygons of $\bar{S}-N(\mathscr{C}_1)$ and $\bar{S}-N(\mathscr{C}_2)$, respectively. Notice that w establishes a boundary correspondence between the two polygons. It follows that the map $\xi_2\circ w\circ\xi_1^{-1}$ is a homeomorphism of $\mathbf{S}^1$ onto $\mathbf{S}^1$. Clearly, $\xi_2\circ w\circ\xi_1^{-1}$ can be extended to a self-map η of the closed unit disk by the radial extension. It turns out that

$$f(x)=\begin{cases}\xi_2^{-1}\circ\eta\circ\xi_1(x), & \text{if } x\in S-N(\mathscr{C}_1);\\ w_0(x), & \text{if } x\in N(\mathscr{C}_1)\end{cases}$$

is a self-map of $\bar{S}$ which fixes x_6, and hence defines a self-map of S which carries $\mathscr{C}_1$ to $\mathscr{C}_2$. Let $\mathscr{C}_1=(\alpha_1,\beta_1,\alpha_2,\beta_2)$ and $\mathscr{C}_2=(\alpha_1',\beta_1',\alpha_2',\beta_2')$. By (4.3) and (9.1), we obtain

$$f\circ(h_{\beta_2}\circ h_{\alpha_2}\circ h_{\beta_1}\circ h_{\alpha_1})^5\circ f^{-1}=(h_{\beta_2'}\circ h_{\alpha_2'}\circ h_{\beta_1'}\circ h_{\alpha_1'})^5.$$

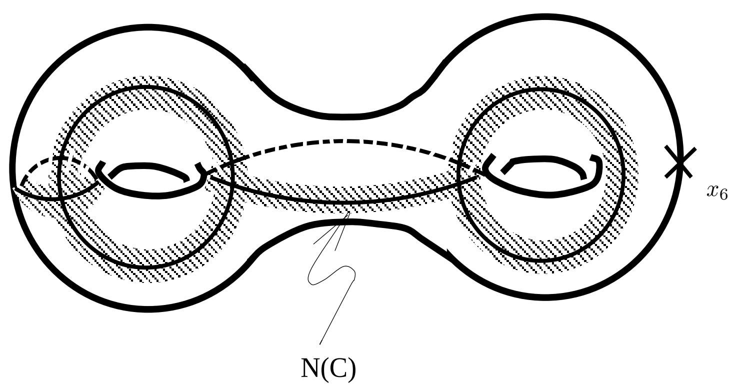

Figure 6.

It follows from Lemma 9.2 that $f\circ j\circ f^{-1}$ is isotopic to j'. This implies that $\chi\circ j\circ\chi^{-1}=j'$, where $\chi\in\operatorname{Mod}(2,1)$ is induced by f. This completes the proof. □

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