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ON THE ZEROS AND HYPER-ORDER OF MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the exponent of convergence of the zeros and the hyper-order of meromorphic solutions of higher-order homogeneous linear differential equations with transcendental coefficients.

1. Introduction and results

Throughout the presentation, f will denote a transcendental meromorphic function in the finite complex plane, and the standard notations of the Nevanlinna theory will be employed (see e.g. [8]). In addition, we will use notations $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the exponent of convergence of the zeros and distinct zeros of $f(z)$, respectively; $\bar{\lambda}(1/f)$ will be used to denote the exponent of convergence of the distinct poles of meromorphic function $f(z)$, and $\sigma(f)$ will denote the order of growth of $f(z)$ and $\mu(f)$ the lower order of $f(z)$. The hyper-order of $f, \sigma_2(f)$, is defined to be $\overline{\lim}_{r\to\infty} \log\log T(r,f)/\log r$. The following two results were proved by S. Bank and I. Laine in [1] and [2], respectively.

Theorem A. *Let $A(z)$ be a transcendental entire function of order σ $(0 < \sigma \le \infty)$, with $\lambda(A) < \sigma$. Then any solution f of*

$$f'' + Af = 0 \tag{1.1}$$

satisfies $\lambda(f) \ge \sigma$.

Theorem B. *Let $A(z)$ be a transcendental meromorphic function of order σ, where $0 < \sigma \le +\infty$, and assume that $\bar{\lambda}(A) < \sigma$. Then if $f(z) \not\equiv 0$ is a meromorphic solution of* (1.1)*, we have*

$$\max\{\bar{\lambda}(f), \bar{\lambda}(1/f)\} \ge \sigma.$$

In this paper, the exponent of convergence of the distinct zeros of an arbitrary solution of higher-order linear equations will be considered.

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Theorem 1. *Let $A_0, \dots, A_{k-1}$ be entire functions such that*

$$\max\{\sigma(A_1), \sigma(A_2), \dots, \sigma(A_{k-1}), \lambda(A_0)\} < \sigma(A_0) = \sigma \qquad (0 < \sigma \le \infty), \tag{1.2}$$

and that A_0 has at least one zero whose multiplicity is not a multiple of k. Then every solution f of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0 \tag{1.3}$$

satisfies $\bar{\lambda}(f) \ge \sigma$. Moreover, if A_0 is transcendental with $\sigma(A_0) = 0$ having at least one zero whose multiplicity is not a multiple of k, and $A_1, \dots, A_{k-1}$ are polynomials, every solution f of (1.3) has infinitely many zeros.

We will give some further estimates on the growth of infinite order solutions of the equation (1.3). It is well known that all solutions of (1.3) are entire functions, and when some of the coefficients of (1.3) are transcendental, (1.3) has at least one solution f with $\sigma(f) = \infty$.

Recently, Ki-Ho Kwon obtained the following result in [8] for the second-order linear differential equation

$$f'' + A(z) f' + B(z) f = 0. \tag{1.4}$$

Theorem C. *Let $A(z)$ and $B(z)$ be entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < \frac{1}{2}$. Then every solution $f \not\equiv 0$ of (1.4) satisfies*

$$\varlimsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} \ge \max\{\sigma(A), \sigma(B)\}.$$

We shall investigate the above types of problems as generalizations for higher-order homogeneous linear differential equations, with one coefficient whose growth rate is the dominant one. The reader is referred to [9] and [11] for some of the results that relate to Theorem 2.4 below.

Theorem 2. *Suppose that $A_0, \dots, A_{k-1}$ are entire functions and there exists one A_s $(0 \le s \le k-1)$ satisfying for $j \ne s$ such that*

$$\sigma(A_j) < \sigma(A_s).$$

Then the equation (1.3) has at least one solution f satisfying either $\lambda(f) \ge \sigma(A_s)$ or $\sigma_2(f) = \sigma(A_s)$.

Theorem 3. *Suppose that $A_0, \dots, A_{k-1}$ are entire functions such that*

$$\max\{\sigma(A_j) : j = 1, \dots, k-1\} < \sigma(A_0).$$

Then every solution $f \not\equiv 0$ of (1.3) satisfies $\sigma_2(f) \ge \sigma(A_0)$. Furthermore, if $\lambda(f) < +\infty$, then $\sigma_2(f) = \sigma(A_0)$.

Theorem 4. *Suppose that $A_0, \dots, A_{k-1}$ are entire functions such that $A_0 \not\equiv 0$ and*

$$\max\{\sigma(A_j) : j = 0, 2, \dots, k-1\} < \sigma(A_1) < \tfrac{1}{2}.$$

Then every solution $f \not\equiv 0$ of (1.3) satisfies $\sigma_2(f) \ge \sigma(A_1)$. Moreover, if $\lambda(f) < +\infty$, then $\sigma_2(f) = \sigma(A_1)$.

2. Proof of Theorem 1

Assume that $f \not\equiv 0$ is a solution of (1.3). It is easy to see that f is entire with $\sigma(f) = \infty$. Now we assume that the assertion of Theorem 1 is false. That is,

$$\bar{\lambda}(f) < \sigma. \tag{2.1}$$

Set $f'/f = g$. Then, by an elementary computation, it follows that

$$f^{(j)}/f = g^j + \tfrac{1}{2} j(j-1) g^{j-2} g' + H_{j-2}(g), \tag{2.2}$$

where $H_{j-2}(g)$ is a differential polynomial in g and its derivatives with constant coefficients, and the degree of $H_{j-2}(g)$ is no greater than $j-2$. Substituting (2.2) in (1.3), we get

$$-A_0 = g^k + \tfrac{1}{2} k(k-1) g^{k-2} g' + A_{k-1} g^{k-1} + P_{k-2}(g), \tag{2.3}$$

where $P_{k-2}(g)$ is a differential polynomial in g and its derivatives with coefficients that are linear combinations of $A_1, \ldots, A_{k-2}$ with constant coefficients, and the degree of $P_{k-2}(g)$ is no greater than $k-2$. Since the poles of $g = f'/f$ can only occur at the zeros of f, and all poles of g are simple, it follows from (2.1) that

$$\sigma\big(N(r,g)\big) = \bar{\lambda}(f) < \sigma. \tag{2.4}$$

By (2.3), we have

$$T(r, A_0) \le M\Bigg(T(r,g) + \sum_{j=1}^{k-1} T(r, A_j)\Bigg) + S(r,g) \qquad (r \notin E), \tag{2.5}$$

where $E \subset (0, +\infty)$ with a finite linear measure and M (> 0) is a constant. By (1.2) and hence because $\mu(A_0) = \sigma(A_0)$ (see e.g. [5, Corollary 6.1]), we have $\sigma(A_j) < \mu(A_0) = \sigma(A_0) = \sigma$ for $j = 1, \ldots, k-1$. It follows that the inequality

$$M \sum_{j=1}^{k-1} T(r, A_j) < \tfrac{1}{2} T(r, A_0) \tag{2.6}$$

holds for $r \notin E$ and sufficiently large r. By (2.5) and (2.6), we have

$$T(r, A_0) \le \big(2M + o(1)\big) T(r,g) \tag{2.7}$$

for $r \notin E$ and sufficiently large r. Hence, again by [5] as well as by (2.4), (1.2), and (2.7), we have $\mu(g) = \sigma(g) \ge \sigma$,

$$T(r, A_j) = S(r,g) \qquad (j = 1, \ldots, k),$$

and, for $r \notin E$,

$$N(r,g) = o(1)T(r,g) = S(r,g).$$

Thus by (1.2) and $\mu(g) \geq \sigma$, $N(r, 1/A_0) = S(r,g)$ holds. Therefore (2.3) satisfies all the conditions of the Tumura–Clunie theorem [8, p. 69], and it follows that

$$-A_0 = h^k(z), \qquad h(z) = g(z) + \frac{1}{k}a(z),$$

where h satisfies

$$\begin{aligned} h^{k-1}a &= \tfrac{1}{2}k(k-1)h^{k-2}h' + A_{k-1}h^{k-1} \\ a &= \tfrac{1}{2}k(k-1)\frac{h'}{h} + A_{k-1}, \qquad \frac{h'}{h} = \frac{1}{k}\frac{A_0'}{A_0}. \end{aligned}$$

Hence

$$-A_0 = \left(\frac{f'}{f} + \frac{k-1}{2k}\frac{A_0'}{A_0} + \frac{A_{k-1}}{k}\right)^k.$$

This contradicts the assumption that A_0 has at least one zero whose multiplicity is not a multiple of k. Thus we must have $\bar{\lambda}(f) \geq \sigma$, and Theorem 1 is proved.

3. Lemmas needed for the proof of Theorem 2

Lemma 1 ([6, Lemma 3]). *Let the differential equation*

$$w^{(k)} + a_{k-1}w^{(k-1)} + \cdots + a_0 w = 0 \tag{3.1}$$

be satisfied in the complex plane by linearly independent meromorphic functions $f_1, \dots, f_k$. Then the coefficients a_j $(j = 0, \dots, k-1)$ are meromorphic in the plane satisfying the properties

$$m(r, a_j) = O\bigl\{\log\bigl[\max\bigl(T(r, f_s) : s = 1, \dots, k\bigr)\bigr]\bigr\}. \tag{3.2}$$

Using the same reasoning as in the proof of the Tumura–Clunie theorem, we can obtain the following lemma.

Lemma 2. *Let $g^{n+1} = P_n(g)$, and let $P_n(g)$ be a differential polynomial in the transcendental entire function $g(z)$ of total degree at most n, with the coefficients in $w'/w, \dots, w^{(n)}/w$, and $A_0, \dots, A_{k-1}$ (where w is an entire function with $\sigma(w) < \infty, A_0, \dots, A_{k-1}$, satisfying the additional hypotheses of Theorem 2). Then $\sigma(g) \leq \sigma(A_s)$.*

Proof. Since

$$m(r,g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_{\varepsilon_1} \log^+ |g(re^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_{\varepsilon_2} \log^+ |g(re^{i\theta})|\, d\theta, \tag{3.3}$$

where $\varepsilon_1 = \{\theta : |g(re^{i\theta})| < 1\} \cap [0, 2\pi]$, $\varepsilon_2 = [0, 2\pi] - \varepsilon_1$, we have

$$\int_{\varepsilon_1} \log^+ |g(re^{i\theta})|\, d\theta = 0. \tag{3.4}$$

On ε_2, we have $|g(re^{i\theta})| \geq 1$. By $g = P_n(g)/g^n$ and the hypothesis that $P_n(g)$ is the sum of a finite number of terms of the type

$$a(z)g^{l_0} \cdot (g')^{l_1} \cdots (g^{(\nu)})^{l_\nu},$$

where $l_0, \ldots, l_\nu$ are non-negative integers and $\sum_{t=0}^{\nu} l_t = n$, $a(z)$ is a combination of the addition, subtraction and multiplication operations of $w'/w, \cdots, w^{(n)}/w$, $A_0, \ldots, A_{k-1}$. We have

$$\frac{1}{2\pi}\int_{\varepsilon_2} \log^+ |g(re^{i\theta})|\, d\theta \leq M\left(\sum m(r,a) + \sum_{t=1}^{\nu} m\left(\frac{r, g^{(t)}}{g}\right)\right) \leq M\left(\sum_{j=1}^{n} m\left(r, \frac{w^{(j)}}{w}\right) + \sum_{j=0}^{k-1} m(r, A_j) + S(r,g)\right) \tag{3.5}$$

for $|z| = r$ outside a set E of finite linear measure, where M is some positive constant. By the fact that $m(r, w^{(j)}/w) = O(\log r)$ $(j = 1, \ldots, n)$ and (3.3)–(3.5), we have

$$m(r,g) \leq M\left(\sum_{j=0}^{k-2} m(r, A_j) + S(r,g)\right), \qquad r \notin E. \tag{3.6}$$

By $S(r,g)/m(r,g) \to 0$ $(r \to \infty)$ and the additional hypotheses, we get $\sigma(g) \leq \sigma(A_s)$ from (3.6).

Lemma 3 [12, Theorem 1.45]. *Let $h(z)$ be a nonconstant entire function with $\sigma(h) = \sigma$. If $f(z) = \exp\{h(z)\}$, then $\sigma_2(f)$, the hyper-order of $f(z)$, satisfies $\sigma_2(f) = \sigma$.*

4. Proof of Theorem 2

Proof. By Lemma 1 and $\sigma(A_j) < \sigma(A_s)$ $(j \neq s)$, it follows that the equation (1.3) has at least one solution f with $\sigma(f) = \infty$. Now assume that $\lambda(f) < \sigma(A_s)$. Then f can be expressed in the form $f = we^h$ such that $\sigma(w) < \sigma(A_s)$ and h is a transcendental entire function. By mathematical induction, we can prove for $n = 1, \dots, k$

$$f^{(n)}(z) = we^h(h')^n + we^h P_{n-1}(h'), \tag{4.1}$$

where $P_{n-1}(h')$ is a differential polynomial in $h', \dots, h^{(n)}$ of total degree $n-1$ with its coefficients being polynomials in $w'/w, \cdots, w^{(n)}/w$. Substituting (4.1) in (1.3), we obtain

$$(h')^k + P_{k-1}(h') + A_{k-1}[(h')^{k-1} + P_{k-2}(h')] + \cdots + A_0 = 0, \tag{4.2}$$

i.e.,

$$(h')^k = P^*_{k-1}(h'), \tag{4.3}$$

where $P^*_{k-1}(h')$ is a differential polynomial in $h', \dots, h^{(k)}$ of total degree $k-1$ with its coefficients being polynomials in $w'/w, \dots, w^{(k)}/w, A_0, \dots, A_s, \dots, A_{k-1}$. By (4.3) and Lemma 2, we have $\sigma(h) = \sigma(h') \le \sigma(A_s)$. Since $(h')^s + P_{s-1}(h') = f^{(s)}/f \not\equiv 0$ and (4.2), we get $\sigma(h') \ge \sigma(A_s)$. Hence $\sigma(h) = \sigma(A_s)$.

By Lemma 3 and $f = we^h$ with $\sigma(w) < \infty$, it follows that $\sigma_2(f) = \sigma(h) = \sigma(A_s)$.

5. Proof of Theorem 3

Set $\max\{\sigma(A_j) : j = 1, \dots, k-1\} = \rho, < \sigma(A_0) = \alpha$. We can rewrite (1.3) as

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}. \tag{5.1}$$

Hence the inequality

$$m(r, A_0) \le \sum_{j=1}^{k-1} m(r, A_j) + O\{\log T(r, f) + \log r\} \tag{5.2}$$

holds for all r outside a set $E \subset (0, +\infty)$ with a linear measure $mE = \delta < +\infty$. Since $\sigma(A_0) = \alpha$, there exists $\{r'_n\}$ $(r'_n \to \infty)$ such that

$$\lim_{r'_n \to \infty} \frac{\log m(r'_n, A_0)}{\log r'_n} = \alpha. \tag{5.3}$$

By $mE = \delta < \infty$, there exists a point $r_n \in [r'_n, r'_n + \delta + 1] - E$. From

$$\frac{\log m(r_n, A_0)}{\log r_n} \geq \frac{\log m(r'_n, A_0)}{\log(r'_n + \delta + 1)} = \frac{\log m(r'_n, A_0)}{\log r'_n + \log\big(1 + (\delta + 1)/r'_n\big)}$$

we get

$$\underline{\lim}_{r_n \to \infty} \frac{\log m(r_n, A_0)}{\log r_n} \geq \alpha. \tag{5.4}$$

So, for any given ε $(0 < 2\varepsilon < \alpha - \rho)$, and for $j = 1, \ldots, k-1$,

$$m(r_n, A_j) \leq r_n^{\rho+\varepsilon} \qquad \text{and} \qquad m(r_n, A_0) > r_n^{\alpha-\varepsilon}$$

hold for any sufficiently large r_n. Therefore,

$$\sum_{j=1}^{k-1} m(r_n, A_j) \leq \tfrac{1}{2} m(r_n, A_0) \tag{5.5}$$

holds for a sufficiently large r_n. By (5.2) and (5.5), we get for a sufficiently large r_n

$$m(r_n, A_0) \leq M\{\log T(r_n, f) + \log r_n\}. \tag{5.6}$$

By (5.4) and (5.6), we have $\sigma_2(f) \geq \sigma(A_0)$. By this and Theorem 2, we easily conclude that $\sigma_2(f) = \sigma(A_0)$ if $\lambda(f) < +\infty$.

6. Proof of Theorem 4

Our proof depends mainly on the following four known results.

Lemma 4 ([6]). *Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E_1 \subset (1, \infty)$ that has finite logarithmic measure and a constant $B > 0$ that depends only on α and $j = 1, \ldots, k$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$ and for $j = 2, \ldots, k$, we have*

$$\left|\frac{f^{(j)}(z)}{f'(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\right)^{k-j}.$$

Lemma 5 ([10]). *Let $f(z)$ be a nonconstant entire function. Then there exists a real number R such that for all $r \geq R$ there is some corresponding z_r with $|z_r| = r$ satisfying*

$$|f(z_r)/f'(z_r)| \leq r.$$

Lemma 6 ([3]). *Let $f(z)$ be entire of order $\sigma(f)=\rho<\frac{1}{2}$. Denote $A(r)=\inf_{|z|=r}\log|f(z)|$, $B(r)=\sup_{|z|=r}\log|f(z)|$. If $\rho<\alpha<1$, then*

$$\underline{\log\text{dens}}\{r:A(r)>(\cos\pi\alpha)B(r)\}\geq 1-\rho/\alpha,$$

where

$$\underline{\log\text{dens}}(H)=\underline{\lim}_{r\to\infty}\{lm(H\cap[1,r])/\log r\}$$

and

$$\overline{\log\text{dens}}(H)=\overline{\lim}_{r\to\infty}\{lm(H\cap[1,r])/\log r\}.$$

Lemma 7 ([4]). *Let $f(z)$ be entire with $\mu(f)=\mu<\frac{1}{2}$ and $\mu<\rho=\sigma(f)$. If $\mu\leq\delta<\min(\rho,\frac{1}{2})$ and $\delta<\alpha<\frac{1}{2}$, then*

$$\overline{\log\text{dens}}\{r:A(r)>(\cos\pi\alpha)B(r)>r^{\delta}\}>C(\rho,\delta,\alpha),$$

where $C(\rho,\delta,\alpha)$ is a positive constant depending only on ρ,δ, and α.

Proof of Theorem 4. Suppose that ρ and α are real numbers satisfying

$$\max\{\sigma(A_j):j=0,2,\ldots,k-1\}<\rho<\alpha<\sigma(A_1).$$

It is easy to see that every solution $f\not\equiv 0$ of (1.3) is transcendental and one can have from (1.3)

$$|A_1|\leq\left|\frac{f^{(k)}}{f'}\right|+|A_{k-1}|\left|\frac{f^{(k-1)}}{f'}\right|+\cdots+\left|A_2\frac{f''}{f'}\right|+|A_0|\cdot\left|\frac{f}{f'}\right|. \tag{6.1}$$

By Lemma 4, there exists a set $E_1\subset(1,\infty)$ with the logarithmic measure $lmE_1<\infty$, such that for $j=2,\ldots,k$, and for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f'(z)}\right|\leq\log^{2k}r\cdot[T(2r,f')]^{2k}. \tag{6.2}$$

By Lemma 6 (if $\mu(A_1)=\sigma(A_1)$) or Lemma 7 (if $\mu(A_1)<\sigma(A_1)$), there exists a set $H\subset(1,+\infty)$ with $lmH=\infty$, such that for all z satisfying $|z|=r\in H$ we have

$$|A_1(z)|\geq\exp\{r^{\alpha}\} \tag{6.3}$$

and

$$|A_j(z)|\leq\exp\{r^{\rho}\}\qquad(j=0,2,\ldots,k-1). \tag{6.4}$$

By Lemma 5, there exists a number $R > 0$ such that for all $r \geq R$ there is a corresponding z_r with $|z_r| = r$ satisfying

$$|f(z_r)/f'(z_r)| \leq r. \tag{6.5}$$

By (6.1)–(6.5), it follows that there is a sequence $\{z_r\}$ with $|z_r| = r \in H - \big([0,R) \cup E_1\big)$ $(z_r \to \infty)$ such that

$$\exp\{r^\alpha\} \leq M \exp\{r^\rho\} r \log^{2k} r [T(2r, f')]^{2k},$$

where M (> 0) is a suitable constant. Thus

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r,f')}{\log r} \geq \alpha.$$

Since α is arbitrary and

$$\begin{aligned} T(r,f') = m(r,f') + N(r,f') &\leq m(r,f) + m\left(r, \frac{f'}{f}\right) + 2N(r,f) \\ &\leq 2T(r,f) + M\{\log rT(r,f)\} \\ &\leq (2+M)T(r,f) + M \log r \qquad (r \notin E), \end{aligned}$$

where $mE < +\infty$, we get $\sigma_2(f) \geq \sigma(A_1)$. By Theorem 2, we easily conclude that $\sigma_2(f) = \sigma(A_1)$ if $\lambda(f) < +\infty$.

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