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# DISTANCE BETWEEN DOMAINS IN THE SENSE OF LEHTO IS NOT A METRIC

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**Abstract.** In this paper we consider the question whether the quotient of the set of domains conformally equivalent to a halfplane by the group of Möbius transformations with distance in the sense of Lehto is a metric space. The answer is shown to be negative in the general case. However, restricted to analytic domains the question has an affirmative answer.

## 1. Introduction

Let $\mathscr{D}$ denote the set of complex domains conformally equivalent to the upper half-plane $H = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Throughout this paper we shall denote elements of $\mathscr{D}$ by $D, \widetilde{D}, \widetilde{\widetilde{D}}$ . The Poincaré density of the hyperbolic metric of $D$ is defined by

$$\eta_D(z) = \frac{|\pi_D'(z)|}{\operatorname{Im} \pi_D(z)},$$

where $\pi_D \colon D \to H$ is a conformal mapping onto $H$.

The Schwarzian derivative, or Schwarzian, of a conformal mapping $f \colon D \to \widetilde{D}$ is defined in $D$ as the holomorphic function

$$S(f,z) = S_f(z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2.$$

The following two properties of the Schwarzian derivative are well known:
(1) $S_f \equiv 0$ if and only if $f$ is a Möbius transformation.
(2) Cayley's formula

$$S(g \circ f, z) = S\bigl(g, f(z)\bigr) f'(z)^2 + S(f,z)$$

holds for conformal mappings $f \colon D \to \widetilde{D}$ and $g \colon \widetilde{D} \to \widetilde{\widetilde{D}}$.

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Furthermore, if a weighted sup-norm is defined for functions $\varphi$ holomorphic in $D$ by

$$\|\varphi\|_D = \sup_{z\in D} |\varphi(z)|\eta_D(z)^{-2},$$

we have the identities (see e.g. [3])

$$\|S_{g\circ f^{-1}}\|_{\widetilde{D}} = \|S_g - S_f\|_D, \qquad \|S_f\|_D = \|S_{f^{-1}}\|_{\widetilde{D}}. \tag{3}$$

Two domains $D$ and $\widetilde{D}$ are said to be Möbius equivalent if there is a Möbius transformation of $D$ onto $\widetilde{D}$. We denote by $\Omega$ the quotient set of the set $\mathscr{D}$ by the group of Möbius transformations. The Lehto distance between two domains $D$ and $\widetilde{D}$ in $\mathscr{D}$ is defined by

$$\delta(D,\widetilde{D}) = \inf\{\|S_f\|_D : f\colon D \to \widetilde{D} \text{ conformal}\}.$$

The above properties of the Schwarzian derivative imply that

$$\delta(D,\widetilde{D}) = \delta(\widetilde{D},D),$$
$$\delta(D,\widetilde{\widetilde{D}}) \le \delta(D,\widetilde{D}) + \delta(\widetilde{D},\widetilde{\widetilde{D}})$$

for all domains $D, \widetilde{D}$ and $\widetilde{\widetilde{D}}$ in $\mathscr{D}$. Because $\delta(D,\widetilde{D}) = 0$ for Möbius equivalent domains $D$ and $\widetilde{D}$, the Lehto distance $\delta$ defines a pseudometric in the quotient set $\Omega$. We shall see in the sequel that for the subset of $\Omega$ consisting of domains with analytic boundaries the Lehto distance does, indeed, define a metric, but that for an arbitrary domain $D \in \mathscr{D}$, the equality $\delta(D,\widetilde{D}) = 0$ does not imply the Möbius equivalence of $D$ and $\widetilde{D}$.

## 2. Reducing the problem to quadratic differentials

We can consider the space

$$Q = \{\varphi : \varphi \text{ holomorphic in } H,\ \|\varphi\|_H < \infty\}$$

as the space of quadratic differentials of bounded norm corresponding to the universal Teichmüller space $T(H)$; cf. [3]. If $D$ is a domain in $\mathscr{D}$ and $f\colon H \to D$ a conformal mapping of the upper half-plane $H$ onto $D$, the Kraus–Nehari theorem implies the norm inequality $\|\varphi\|_H \le \frac{3}{2}$. Thus every conformal mapping $f\colon H \to D$ of a domain $D \in \mathscr{D}$ determines a corresponding quadratic differential $\varphi = S_f \in Q$. For another domain $\widetilde{D} \in \mathscr{D}$ and conformal mapping $\tilde{f}\colon H \to \widetilde{D}$ we denote the respective quadratic differential by $\widetilde{\varphi} = S_{\tilde{f}} \in Q$. The vanishing $\delta(D,\widetilde{D}) = 0$ of the Lehto distance is equivalent to the existence of a sequence $g_n\colon D \to \widetilde{D}$ of conformal mappings so that $\|S_{g_n}\|_D \to 0$ when $n \to \infty$. For each $g_n$ there is a Möbius transformation $A\colon H \to H$ such that $g_n = \tilde{f} \circ {A_n}^{-1} \circ f^{-1}$. Now by (3) we have the identity $\|S_{g_n}\|_D = \|S_{\tilde{f}} - S_{f\circ A_n}\|_H$, and thus we get the following

**Lemma 1A.** *The vanishing $\delta(D,\widetilde{D}) = 0$ of the Lehto distance between the domains $D,\widetilde{D} \in \mathscr{D}$ is equivalent to the existence of a sequence of Möbius transformations $A_n\colon H \to H$ such that $\|\varphi - \varphi_n\|_H \to 0$ when $n \to \infty$, where the sequence of quadratic differentials $\varphi_n \in Q$ is defined by $\varphi_n(z) = \varphi(A_n z)A_n'(z)^2 = S_{f\circ A_n}$.*

Similarly we have for the Möbius equivalence of domains

**Lemma 1B.** *Two domains $D,\widetilde{D} \in \mathscr{D}$ are Möbius equivalent, if and only if there exists a Möbius transformation $A\colon H \to H$ such that $\widetilde{\varphi} = \varphi(Az)A'(z)^2$.*

For any quadratic differential $\varphi \in Q$ we define the subset $N(\varphi) \subset Q$ by

$$N(\varphi) = \{(\varphi \circ A){A'}^2 : A\colon H \to H \text{ is a Möbius transformation}\}.$$

As an immediate consequence of Lemmas 1A and 1B we have

**Lemma 1C.** *Let $f\colon H \to D$ be a conformal mapping. The set of all $\widetilde{D} \in \mathscr{D}$ which satisfy $\delta(D,\widetilde{D}) = 0$ is equal to the set of domains Möbius equivalent to $D$ if and only if $N(S_f)$ is closed in $Q$.*

Finally we prove

**Lemma 1D.** $(\Omega,\delta)$ *is a metric space, if and only if $N(\varphi)$ is closed in $Q$ for all $\varphi \in Q$.*

*Proof.* Should $N(\varphi)$ be closed in $Q$ for each $\varphi \in Q$, then $\delta(D,\widetilde{D}) = 0$ if and only if $D$ and $\widetilde{D}$ are Möbius equivalent by Lemma 1C, so that $(\Omega,\delta)$ would be a metric space. Conversely any $\varphi \in Q$ with $\|\varphi\|_H, < \frac{1}{2}$ is the Schwarzian of a conformal mapping of the upper half-plane $H$ by the Ahlfors–Weill theorem (cf. [2]). So should $\delta$ define a metric in $\Omega$, then $N(\varphi)$ would by Lemma 1C be closed wherever $\|\varphi\|_H < \frac{1}{2}$, and thus actually for all $\varphi \in Q$. □

## 3. A metric for analytic domains

A domain $D \in \mathscr{D}$ is analytic, if the boundary of $D$ is an analytic curve, i.e. the image of a circle $K$ under a conformal mapping defined in a neighbourhood of $K$. We denote by $C_0(H)$ the subspace of $C(H)$ consisting of continuous functions $f$ vanishing on the boundary of $H$, and define a subspace $Q_0$ of $Q$ by

$$Q_0 = \{\varphi \in Q : \varphi\eta_H^{-2} \in C_0(H)\}.$$

**Lemma 2.** *$S_f \in Q_0$ when $f\colon H \to D$ is a conformal mapping onto an analytic domain $D$.*

*Proof.* Let $M\colon H \to U$ be a Möbius transformation of $H$ onto the unit disc $U$, and $g = f \circ M^{-1}\colon U \to D$. Because $D$ is an analytic domain, the mapping $g$ extends to a conformal mapping defined in a neighbourhood of the closed unit disk $\overline{U}$. Having a holomorphic extension into a neighbourhood of $\overline{U}$, the Schwarzian $S_g$ remains bounded on $\overline{U}$, so that $\lim_{|w|\to 1} S_g(w)\eta_U^{-2}(w) = 0$. Thus we also have $\lim_{z\to r} S_f(w)\eta_H^{-2}(z) = 0$ for every boundary point $r \in \overline{\mathbf{R}}$ of $H$, so that the Schwarzian $S_f$ belongs to $Q_0$ when $D$ is an analytic domain. □

**Lemma 3.** *$N(\varphi)$ is closed in $Q$ for every $\varphi \in Q_0$.*

*Proof.* Let $\varphi \in Q_0$ and $A_n$ a sequence of Möbius automorphisms of $H$ such that $\varphi_n = (\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi}$ in $Q$ when $n \to \infty$. By the compactness principle (e.g. [4]) we may suppose that the sequence $A_n$ converges locally uniformly in $H$ either to a Möbius automorphism $A\colon H \to H$, or to a constant $c \in \overline{\mathbf{R}}$. Should the sequence $A_n$ converge to a constant function, we would have for all $z \in H$

$$\widetilde{\varphi}(z)\eta_H(z)^{-2} = \lim_{n\to\infty} \varphi(A_n z)\eta_H(A_n z)^{-2} = 0$$

because $\varphi \in Q_0$. Thus $\widetilde{\varphi}$ vanishes identically, and we have $\|\widetilde{\varphi}\|_H = 0$. But $\varphi_n \to \widetilde{\varphi}$ in $Q$, and as $\|\varphi_n\|_H = \|\varphi\|_H$ for all $n$, and we have $\varphi = \widetilde{\varphi} = 0 \in Q_0$, so that $\widetilde{\varphi} = \varphi \in N(\varphi)$. When the sequence $A_n$ converges to a Möbius automorphism $A$, we have $\widetilde{\varphi} = (\varphi \circ A){A'}^2$, which obviously belongs to $N(\varphi)$. Thus $N(\varphi)$ is a closed subset of $Q$ for all $\varphi \in Q_0$. □

Denoting by $\Omega_A$ the quotient of the set of analytic domains by the group of Möbius transformations we get as an immediate consequence of Lemmas 2, 3 and 1D.

**Theorem 1.** *$(\Omega_A, \delta)$ is a metric space.*

## 4. $(\Omega, \delta)$ is not a metric space

We are going to give here three slightly different examples of quadratic differentials $\varphi \in Q$ for which $N(\varphi)$ is not closed in $Q$. To do this we construct a quadratic differential $\varphi \in Q$ and determine a sequence of Möbius transformations $A_n\colon H \to H$ so that the sequence $\varphi_n = (\varphi \circ A_n){A'}_n^2$ converges in $Q$ towards a quadratic differential $\widetilde{\varphi}$ not in $N(\varphi)$. Particularly, if we choose $\varphi$ so that $\|\varphi\|_H < \frac{1}{2}$, then by the Ahlfors–Weill theorem there are conformal mappings $f\colon H \to D$ and $\tilde{f}\colon H \to \widetilde{D}$ with $S_f = \varphi$, $S_{\tilde{f}} = \widetilde{\varphi}$, so that we have $\delta(D, \widetilde{D}) = 0$ for the image domains $D$ and $\widetilde{D}$, although $D$ and $\widetilde{D}$ are not Möbius equivalent.

To begin with let us note that for all $a > 0$ the function $e^{iaz}$ is in $Q$ with the norm $\|e^{iaz}\|_H = (2/ae)^2$.

**Example 1.** Let $\varphi(z) = e^{2\pi iz} + e^{iz}$, $\widetilde{\varphi}(z) = e^{2\pi iz} - e^{iz}$ and $A_k(z) = z + n_k$, where $n_k = 2\pi m_k + \pi + o(1)$ when $k \to \infty$ ($n_k, m_k \in \mathbf{N}$). To establish the existence of such a sequence $n_k$ we notice that the convergents $P_s/Q_s, P_s, Q_s \in \mathbf{N}$, of the continued fraction expansion of $\pi$ satisfy

$$|\pi - P_s/Q_s| < \frac{1}{Q_s Q_{s+1}} < \frac{1}{Q_s^2}, \qquad P_s Q_{s-1} - Q_s P_{s-1} = (-1)^s$$

with a strictly increasing sequence of denominators $Q_s$ (see e.g. [5]). Thus by choosing an appropriate subsequence we have $P_{s_k} = n_k$, $Q_{s_k} = 2m_k + 1$ with $n_k$, $m_k$ satisfying the above conditions.

It is not difficult to see that $(\varphi \circ A_k){A'}_k^2 \to \widetilde{\varphi}$ in $Q$. It remains to be shown that

$$\widetilde{\varphi} = (\varphi \circ A){A'}^2 \tag{4}$$

holds for no Möbius transformation $A\colon H \to H$. Now $\varphi$ has zeros at $z_k = (2k+1)\pi/(2\pi - 1)$ and $\widetilde{\varphi}$ at $\tilde{z}_k = 2k\pi/(2\pi - 1)$, $k \in \mathbf{Z}$. So should (4) hold for a Möbius transformation, then $A$ would be a translation by an odd integral multiple of $\pi/2\pi - 1$. Now for any $c \in \mathbf{R}$ the coefficients of $e^{2\pi iz}$ and $e^{iz}$ in the expansion

$$\varphi(z + c) = e^{2\pi ic} e^{2\pi iz} + e^{ic} e^{iz}$$

are uniquely determined. However, the equations $e^{2\pi ic} = 1$ and $e^{ic} = -1$ cannot be simultaneously satisfied for any $c$, so that $\widetilde{\varphi} \neq (\varphi \circ A)A^2$ holds for all Möbius automorphisms $A$ of the upper half-plane $H$.

**Example 2.** Suppose that the sequence $\lambda_k$ satisfies $\sum_{k=1}^{\infty} 3^{2k} e^{-2} \pi^{-2} |\lambda_k| < +\infty$ and that $\lambda_k \neq 0$ for infinitely many $k \in \mathbf{N}$. We define $\varphi$ and $\widetilde{\varphi}$ by

$$\varphi(z) = \frac{e^{i\pi z}}{(1 - e^{i\pi z})^2} + \sum_{k=1}^{\infty} \lambda_k e^{2i\pi z/3^k},$$
$$\widetilde{\varphi}(z) = \frac{-e^{i\pi z}}{(1 + e^{i\pi z})^2} + \sum_{k=1}^{\infty} \lambda_k e^{2i\pi z/3^k}.$$

Since $\|e^{i\pi z}/(1 - e^{i\pi z})^2\|_H = 1/\pi^2$ and $\|e^{2i\pi z/3^k}\|_H = 3^{2k}/e^2\pi^2$, we conclude that $\varphi, \widetilde{\varphi} \in Q$. Setting $A_n(z) = z + 3^n$ it is easy to see that $(\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi}$ in $Q$. It remains to be shown that $\widetilde{\varphi} \notin N(\varphi)$, i.e., that (4) cannot hold for any Möbius automorphism $A$ of $H$. Now we notice that the limit

$$\lim_{z \to r} \varphi(z) \eta_H^{-2}(z)$$

is 0 for all $r \in \mathbf{R} \setminus 2\mathbf{Z}$ and does not exist if $r \in 2\mathbf{Z}$. Similarly, the expression $\widetilde{\varphi}(z)\eta_H^{-2}(z)$ equals 0 at all boundary points $r \in \mathbf{R}$ except for the set $2\mathbf{Z}+1$ of odd integers. Thus a Möbius transformation $A$ satisfying (4) would be a translation by an odd integer. But in the expansion

$$\varphi(z+2l+1) - \widetilde{\varphi}(z) = \sum_{k=1}^{\infty} \lambda_k (e^{2i\pi(2l+1)/3^k} - 1) e^{2i\pi z/3^k}$$

the coefficients of $e^{i\pi z/3^k}$ are uniquely determined, so that $\varphi(z+2l+1) = \widetilde{\varphi}(z)$ cannot hold for any $2l+1 \in \mathbf{Z}$.

**Example 3.** We now define $\varphi$ and $\widetilde{\varphi}$ by

$$\varphi(z) = (e^{2\pi iz} - e^{-\pi})\biggl(\sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi iz/2^k}\biggr),$$
$$\widetilde{\varphi}(z) = (e^{2\pi iz} - e^{-\pi})\biggl(\sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi i/2^k(z+(4^k-1)/3)}\biggr),$$

and set $A_n(z) = z + \frac{1}{3}(4^n - 1)$. Since $\|e^{\pi iz/2^k}\|_H = 4^{k+1}/e^2\pi^2$, the series

$$s(z) = \sum_{k=1}^{\infty} \frac{1}{10^k} e^{\pi iz/2^k}$$

converges in $Q$. Because $e^{2\pi iz} - e^{-\pi}$ is bounded in $H$, both functions $\varphi$ and $\widetilde{\varphi}$ are in $Q$, and it is not difficult to see that $(\varphi \circ A_n)A_n'^2 \to \widetilde{\varphi}$ in $Q$.

For $\mathrm{Im}(z) \le 1$ we have

$$\biggl|\frac{1}{10^k} e^{\pi iz/2^k}\biggr| > 2\biggl|\frac{1}{10^{k+1}} e^{\pi iz/2^{k+1}}\biggr|,$$

so that the series $s(z)$ has no zeros with $\mathrm{Im}(z) < 1$. Since all zeros of $e^{2\pi iz} - e^{-\pi}$ are lying on the line $\mathrm{Im}(z) = \frac{1}{2}$, we see that the functions $\varphi$ and $\widetilde{\varphi}$ do not have any zeros in the horizontal strip $\{z \in H : \mathrm{Im}(z) < \frac{1}{2}\}$. On the other hand, the functions $\varphi$ and $\widetilde{\varphi}$ have zeros on the line $\mathrm{Im}(z) = \frac{1}{2}$ at the points $\frac{1}{2}i + l$ with $l \in \mathbf{Z}$. We conclude that a Möbius automorphism of $H$ satisfying $\widetilde{\varphi} = (\varphi \circ A)A'^2$ would be a translation by an integer. Now for any $l \in \mathbf{Z}$ we have an expansion

$$\varphi(z+l) - \widetilde{\varphi}(z) = (e^{2\pi iz} - e^{-\pi}) \sum_{k=1}^{\infty} c_n e^{\pi iz/2^k}$$

with uniquely determined coefficients $c_n$. Hence, for $\varphi(z+l) - \widetilde{\varphi}(z)$ to vanish identically we should have

$$l \equiv \tfrac{1}{3}(4^k - 1) = 1 + 4 + \cdots + 4^{k-1} \pmod{2^{k+1}}$$

for every $k = 1, 2, \dots$. This is obviously impossible for any fixed $l \in \mathbf{Z}$, so that $\widetilde{\varphi} \ne (\varphi \circ A)A'^2$ for all Möbius transformations $A$.

By Lemma 1D and the above three counterexamples we get thus the following theorem.

**Theorem 2.** $(\Omega, \delta)$ *is not a metric space.*

## 5. Further results and discussion

Having shown in the previous section that $N(\varphi)$ is not closed for some $\varphi \in Q$ we now ask how large the set of such $\varphi$ actually is.

**Theorem 3.** *The set of quadratic differentials $\varphi \in Q$ with $N(\varphi)$ not closed is nowhere dense in $Q$.*

*Proof.* We proved above that $N(\varphi)$ is not closed in $Q$ if and only if there is a sequence of Möbius transformations $A_n$ of $H$ such that $(\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi}$ and $A_n \to c \in \overline{\mathbf{R}}$. We shall show that this is possible only for quadratic differentials $\varphi$ in a nowhere dense subset of $Q$.

Let $\Psi_0 \in Q$ be an arbitrary quadratic differential. We shall show that within any ball $B \subset Q$ of radius $\varepsilon > 0$ centered at $\Psi_0$ there is a ball $B'$ of radius $\frac{1}{4}\varepsilon$ such that $N(\varphi)$ is closed for every $\varphi \in B'$. First choose a quadratic differential $\Phi_0 \in Q_0$ with $\|\Phi_0\|_H = 1$ and a point $z_0 \in H$ such that $|\Psi_0(z_0)|\eta_H(z_0)^{-2} > \|\Psi_0\|_H - \frac{1}{4}\varepsilon$, and further a point $z_1 \in H$ with $|\Phi_0(z_1)|\eta_H(z_1)^{-2}$ sufficiently close to 1 and a Möbius transformation $A\colon H \to H$ mapping $z_0$ to $z_1$ so that

$$\big|\tfrac{3}{4}\varepsilon\Phi_0(Az_0)A'(z_0)^2 e^{i\theta} + \Psi_0(z_0)\big|\eta_H(z_0)^{-2} > \|\Psi_0\|_H + \tfrac{1}{2}\varepsilon.$$

Thus $\|\Psi_0 + \Phi\|_H > \|\Psi_0\|_H + \frac{1}{2}\varepsilon$ when $\Phi$ is defined by

$$\Phi = \tfrac{3}{4}\varepsilon(\Phi_0 \circ A){A'}^2 e^{i\theta}.$$

For any $\|\Phi_1\|_H < \frac{1}{4}\varepsilon$, the quadratic differential $\varphi = (\Psi_0 + \Phi) + \Phi_1$ is contained in the ball $B$ of radius $\varepsilon$ centered at $\Psi_0$ with a norm satisfying $\|\varphi\|_H > \|\Psi_0\|_H + \frac{1}{4}\varepsilon$. Because $\Phi$, too, belongs to the subspace $Q_0$, we have $(\Phi \circ A_n){A'}_n^2 \to 0$ in $H$ for any sequence $A_n$ of Möbius automorphisms of $H$ converging to a constant. Assuming that $(\varphi \circ A_n){A'}_n^2 \to \widetilde{\varphi} \in Q$ when $n \to +\infty$ we would thus also have $\big((\Psi_0 + \Phi_1) \circ A_n\big){A'}_n^2 \to \widetilde{\varphi}$ in $H$ as $n \to +\infty$. But $\|\widetilde{\varphi}\|_H = \|\varphi\|_H > \|\Psi_0\|_H + \frac{1}{4}\varepsilon$, so that for some $z \in H$ and $n > n_0$ we would have

$$\big|(\Psi_0 + \Phi_1)(A_n z)A_n'(z)^2\big|\eta_H(z)^{-2} = \big|(\Psi_0 + \Phi_1)(A_n z)\big|\eta_H\big(A_n(z)\big)^{-2} > \|\Psi_0\|_H + \tfrac{1}{4}\varepsilon,$$

contradicting the inequality $\|\Psi_0 + \Phi_1\|_H < \|\Psi_0\|_H + \frac{1}{4}\varepsilon$. □

Let us finally discuss the background of our examples, especially Examples 3 and 2.

Let $\Gamma_j$ be a decreasing sequence of Fuchsian groups and denote by $Q(\Gamma_j)$ the increasing sequence of subspaces of $Q$ consisting of all quadratic differentials $\varphi$ satisfying $\varphi = (\varphi \circ A){A'}^2$ for all $A \in \Gamma_j$. Let $P = \overline{\bigcup_{j=1}^{\infty} Q(\Gamma_j)}$. Consider now sequences $A_n$ of Möbius transformations such that for every $j$, the transformations $A_n$ are eventually in the same right coset of the group $\Gamma_j$. Thus for every $j$ there is $N_j$ such that $(\varphi \circ A_m){A'}_m^2 = (\varphi \circ A_n){A'}_n^2$ for all $\varphi \in Q(\Gamma_j)$ whenever $n, m > N_j$. Such a sequence $A_n$ obviously induces a mapping $F\colon P \to Q$ when $F(\varphi)$ is defined by $F(\varphi) = \lim_{n\to\infty} (\varphi \circ A_n){A'}_n^2 \in Q$. Any fixed Möbius transformation $A\colon H \to H$ determines a mapping $F_A\colon P \to Q$, $\widetilde{\varphi}_A(\varphi) = (\varphi \circ A){A'}^2$ corresponding to the constant sequence $A_n = A$.

If $P \neq \bigcup_{j=1}^{\infty} Q(\Gamma_j)$, there can be mappings $F$ which are not equal to $F_A$ for any Möbius transformation $A$. This can be clearly seen from Example 3. There $\Gamma_j$ is the group of translations by integral multiples of $2^j$, and the mapping $F$ corresponding to the sequence $A_n$ acts as if it were a "translation" by the 2-adic number $1 + 4 + 4^2 + \cdots$. We could choose any other 2-adic number and "translate" any element of $P$, which is here the smallest closed subspace of $Q$ containing all functions $e^{m\pi iz/2^n}$, $m, n \in \mathbf{N}$.

We end this paper by posing the natural problem of characterizing all $\varphi \in Q$ such that $N(\varphi)$ is closed in $Q$.

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