Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 23, 1998, 525–548

QUASICONFORMALITY AND QUASISYMMETRY IN METRIC MEASURE SPACES

Jeremy Tyson

University of Michigan, Department of Mathematics
Ann Arbor, MI 48109, U.S.A.; jttyson@math.lsa.umich.edu

Abstract. A homeomorphism $f\colon X \to Y$ between metric spaces is called quasisymmetric if it satisfies the three-point condition of Tukia and Väisälä. It has been known since the 1960's that when $X = Y = \mathbf{R}^n$ $(n \geq 2)$, the class of quasisymmetric maps coincides with the class of quasiconformal maps, i.e. those homeomorphisms $f\colon \mathbf{R}^n \to \mathbf{R}^n$ which quasipreserve the conformal moduli of all families of curves. We prove that quasisymmetry implies quasiconformality in the case that the metric spaces in question are locally compact and connected and have Hausdorff dimension $Q > 1$ quantitatively. The main conceptual tool in the proof is a discrete version of the conformal modulus due to Pansu.

1. Introduction

Let X and Y be metric spaces and let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $f\colon X \to Y$ is said to be η-*quasisymmetric* if

$$(1.1) \qquad |x-a| \leq t|x-b| \quad \Rightarrow \quad |fx - fa| \leq \eta(t)|fx - fb|$$

for every $t > 0$ and $x, a, b \in X$. We use the Polish notation $|x-y|$ for the distance function in any metric space. Quasisymmetric maps on the real line were first introduced by Beurling and Ahlfors [BA], who characterized them as the boundary values of quasiconformal self-maps of the upper half-plane. The general definition is due to Tukia and Väisälä, who laid the foundation for a general study of quasisymmetric maps in [TV].

For $n \geq 2$, we say that a homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is K-*quasiconformal* $(K \geq 1)$ if

$$(1.2) \qquad \frac{1}{K} \operatorname{Mod}_n \Gamma \leq \operatorname{Mod}_n f\Gamma \leq K \operatorname{Mod}_n \Gamma$$

for every family Γ of curves in $\mathbf{R}^n$. Here $\operatorname{Mod}_n \Gamma$ denotes the conformal modulus of the curve family Γ (see Section 2). A fundamental property of Euclidean n-space $(n \geq 2)$ is that the classes of quasiconformal and quasisymmetric maps

---

1991 Mathematics Subject Classification: Primary 30C65.

The author is supported under a U. S. National Science Foundation Graduate Research Fellowship.

coincide. Indeed, it suffices merely to require that (1.2) hold for those curve families $\Gamma = (E, F)$ which consist of the curves joining a pair of disjoint, nondegenerate continua E and F in $\mathbf{R}^n$. (By a *continuum* we mean a compact, connected set.)

The standard proof that quasiconformal maps are quasisymmetric is geometric in nature. Key ingredients are a quantitative (positive) lower bound for the conformal modulus of curve families of the type $\Gamma = (E, F)$ and an explicit calculation of the modulus in the particular case of a spherical shell. The former was known classically in dimension 2, while the corresponding result in higher dimensions was established in a qualitative form in 1959 by Loewner [L] and in a quantitative form in the 1960's by Gehring [G1]. In [HK3], Heinonen and Koskela define a *Loewner space* to be a metric measure space where a similar positive lower bound holds for the conformal modulus. The same argument as in the case of Euclidean space then shows that quasiconformal maps between Ahlfors Q-regular metric measure spaces ($Q > 1$) are quasisymmetric, provided that the source is a Loewner space and the target satisfies a (necessary) quantitative connectivity condition. (Note that the use of the term "quasiconformal" in [HK1]–[HK3] differs from its use in this paper.)

Recall that we call a metric space X endowed with a Borel measure μ an *Ahlfors regular space of dimension* Q (for short, a Q*-regular space*) if there exists a constant $C_0 \geq 1$ so that

$$C_0^{-1} r^Q \leq \mu(B_r) \leq C_0 r^Q \tag{1.3}$$

for every ball B_r in X with radius $r < \operatorname{diam} X$. It is easy to see that if a locally compact metric space X satisfies (1.3) for some Borel measure μ then in fact it satisfies it for Hausdorff Q-measure $\mathscr{H}_Q$ (possibly with a different constant C_0) and any measure satisfying (1.3) is comparable to $\mathscr{H}_Q$. For this reason, we will often say that a metric space X is Q-regular without specifying the measure in question. Q-regularity of a metric space X can be regarded as a quantitative and scale-invariant version of the qualitative statement that X has Hausdorff dimension Q.

The classical proof of the converse implication (quasisymmetry implies quasiconformality) relies on more specialized properties of Euclidean space. The original proof, due to Gehring [G2], uses nontrivial analytic properties of these maps—absolute continuity on lines (ACL) and absolute continuity in measure—to perform change of variables in the integrals defining the conformal modulus. In [HK3], the authors prove that the same result holds in locally compact Q-regular spaces ($Q > 1$), provided the source satisfies an inequality of Poincaré type similar to (but weaker than) that which holds in Euclidean space. Their proof, however, is still a variant of the classical one. Indeed, they show that in this situation quasisymmetric maps are absolutely continuous in measure and absolutely continuous on almost every curve (in the sense of the conformal modulus), from which quasiconformality follows via the same analytic change of variables argument.

Our principal result is the following theorem.

1.4. Theorem. *Let X and Y be locally compact, connected, Q-regular metric spaces ($Q > 1$) and let $f\colon X \to Y$ be an η-quasisymmetric homeomorphism. There exists a constant C depending only on η, Q and the regularity constants of X and Y so that*

$$\frac{1}{C}\operatorname{Mod}_Q \Gamma \le \operatorname{Mod}_Q f\,\Gamma \le C\operatorname{Mod}_Q \Gamma \tag{1.5}$$

for all curve families Γ in X.

The proof of Theorem 1.4 avoids the analytic machinery of [HK3]. We use instead a discrete version of the conformal modulus due to Pansu [P1], [P2] which is "intrinsically quasisymmetrically invariant". Propositions 4.5 and 4.7 establish relations between Pansu's modulus and the classical modulus in locally compact and connected Q-regular spaces.

Note that we do not require *a priori* that the metric spaces X and Y admit any rectifiable curves. We deduce that if X admits enough rectifiable curves so that $\operatorname{Mod}_Q \Gamma > 0$ for some curve family Γ in X, then $\operatorname{Mod}_Q f\,\Gamma > 0$ and hence Y also admits some rectifiable curves. The following corollary to Theorem 1.4 expresses this in quantitative terms (see Section 2 for definitions). It answers in the affirmative a conjecture of Heinonen and Koskela [HK3, Section 8.7].

1.6. Corollary. *Let X and Y be locally compact Q-regular metric spaces ($Q > 1$). If $f\colon X \to Y$ is quasisymmetric and X is a Loewner space, then Y is also a Loewner space.*

Theorem 1.4 can also be used to address the problem of classifying spaces up to quasisymmetric equivalence. Corollary 1.7 provides a result along these lines.

1.7. Corollary. *Let X be a locally compact and connected Q-regular metric space which admits a curve family Γ in X with $\operatorname{Mod}_Q \Gamma > 0$ and let Y be a locally compact and connected Q'-regular metric space with $Q > Q' \ge 1$. Then there does not exist a quasisymmetric homeomorphism mapping X onto Y.*

It is an open question whether a quasisymmetric map between locally compact and Q-regular spaces satisfies either of the analytic properties described above (absolute continuity in measure and absolute continuity along Mod_Q-almost every curve), even if the spaces in question are assumed to be Loewner. Combining Proposition 4.5 with [P1, Lemma 4.6] leads to the following corollary.

1.8. Corollary. *Let (X,μ) and (Y,ν) be locally compact and connected Q-regular metric measure spaces and let $f\colon (X,\mu) \to (Y,\nu)$ be a quasisymmetric homeomorphism which is absolutely continuous. Then f is absolutely continuous along Mod_Q-almost every curve γ in X.*

Recall that f is said to be *absolutely continuous along* γ if $f|_\gamma$ is absolutely continuous as a map from the measure space $(\gamma, \mathscr{H}_1)$ to $(f\gamma, \mathscr{H}_1)$. Here $\mathscr{H}_1$ denotes the Hausdorff 1-dimensional measures on γ and $f\gamma$.

1.9. Remark. The proof of Theorem 1.4 will show that in fact it would suffice to assume only that there exists a constant $\delta_0 > 0$ so that X and Y satisfy (1.3) for all balls B_r in X with radius $r < \delta_0$. In this case, the constant C which appears in (1.5) does not depend on δ_0.

Outline. In Section 2, we review the classical theory of the conformal modulus and describe in more detail some of the results cited above, particularly the corollaries. Section 3 is devoted mainly to definitions. We introduce the generalized modulus of Pansu and show that it is quasisymmetrically invariant (in a suitable sense). Section 4 is devoted to the proof of the main theorem.

Acknowledgement. I wish to thank my advisor, Juha Heinonen, for introducing me to this subject and for his numerous helpful comments regarding my work. I also wish to acknowledge Seppo Rickman, Eero Saksman and Mario Bonk for an informative discussion on the topics of this paper. I am particularly grateful to Mario Bonk for the ideas of Section 3.14, which led to a proof of the main theorem in its current form.

2. Conformal modulus in metric spaces

2.1. Notation and standing assumptions. In an arbitrary metric space X, we will denote the open ball about x of radius r by $B(x,r)$ and the closed ball by $\overline{B(x,r)}$. We use the notation CB to denote the *dilated ball* $B(x,Cr)$. Recall that the center and radius of a ball in an arbitrary metric space need not be unique.

The *diameter* of a set $E \subset X$ is the quantity $\operatorname{diam} E = \sup_{x,y\in E}|x-y|$ and the *distance* between $A,B \subset X$ is $\operatorname{dist}(A,B) = \inf_{x\in A,\ y\in B}|x-y|$. The ε*-neighborhood* of a set $E \subset X$ is the open set $N_\varepsilon E = \bigcup_{x\in E} B(x,\varepsilon)$.

All measures in this paper will be assumed to be Borel regular and to assign finite measure to each bounded set.

2.2. Definitions. A *curve* γ is a continuous mapping of an interval $I \subset \mathbf{R}$ into X. When I_0 is a subinterval of I, we will write $\gamma_0 = \gamma|_{I_0}$. If the mapping is one-to-one, we call γ an *arc*. (To avoid certain technical problems, we will always exclude the case when the mapping γ is constant.)

For closed intervals $I = [a,b]$, we define the *length* of γ to be

$$\operatorname{length}(\gamma) = \sup_\pi \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|, \tag{2.3}$$

the supremum being taken over all partitions $\pi = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of I. In general, we set $\operatorname{length}(\gamma)$ to be the supremum of the values $\operatorname{length}(\gamma_0)$ over all closed subintervals $I_0 \subset I$. We call the curve γ *rectifiable* if $\operatorname{length}(\gamma) < \infty$. An arbitrary curve $\gamma\colon I \to X$ is called *locally rectifiable* if its restriction to every closed subinterval $I_0 \subset I$ is rectifiable.

Every rectifiable curve $\gamma\colon [a,b] \to X$ with $\mathrm{length}(\gamma) = L$ admits a (unique) *reparametrization by arc length* $\tilde{\gamma}\colon [0,L] \to X$. This reparametrization is characterized by the relation

$$\mathrm{length}(\tilde{\gamma}|_{[s,t]}) = t - s$$

for all $0 \le s < t \le L$.

If $\varrho\colon X \to \mathbf{R}$ is a Borel function and γ is a rectifiable curve with compact parametrizing interval, we define the *line integral of* ϱ *along* γ to be

$$\int_\gamma \varrho\, ds = \int_0^L \varrho\big(\tilde{\gamma}(t)\big)\, dt$$

where $\tilde{\gamma}$ is the arc length reparametrization of γ. When ϱ is continuous, this equals

$$\lim_{\|\pi\|\to 0} \sum_{k=1}^{m} \varrho\big(\tilde{\gamma}(t_k)\big) \cdot |\tilde{\gamma}(t_k) - \tilde{\gamma}(t_{k-1})|, \tag{2.4}$$

where the limit is taken over partitions $\pi = \{0 = t_0 < t_1 < \cdots < t_m = L\}$ of I such that the *mesh* $\|\pi\| = \max\{t_k - t_{k-1} : k = 1,2,\ldots,m\}$ tends to zero.

For an arbitrary locally rectifiable curve $\gamma\colon I \to X$, we define $\int_\gamma \varrho\, ds$ to be the supremum of the values of $\int_{\gamma_0} \varrho\, ds$ over all closed subintervals $I_0 \subset I$ for which γ_0 is rectifiable. We do not define line integrals over curves which are not locally rectifiable.

2.5. Modulus of a curve family. Let (X,μ) be a metric measure space and let Γ be a family of (nonconstant) curves in X. We say that a Borel function $\varrho\colon X \to [0,\infty]$ is *admissible* for Γ if

$$\int_\gamma \varrho\, ds \ge 1 \tag{2.6}$$

for all locally rectifiable curves γ in Γ.

Let $p > 0$. The *p-modulus* of Γ is defined as

$$\mathrm{Mod}_p\, \Gamma = \inf \int_X \varrho^p\, d\mu, \tag{2.7}$$

the infimum being taken over all nonnegative functions ϱ which are admissible for Γ. Note in particular that the modulus of the collection of all non-locally rectifiable curves is zero.

2.8. Remarks. (1) When the metric space X is Ahlfors regular of dimension Q, we call the Q-modulus the *conformal modulus.* The term arises because the n-modulus is a conformal invariant in the n-regular space $\mathbf{R}^n$. We will also use the term *classical modulus* to distinguish it from the generalized modulus defined in Section 3.20.

(2) For any $p > 0$, Mod_p is an outer measure on the collection of all curve families in X, that is

$$\mathrm{Mod}_p\,\emptyset = 0; \tag{2.9}$$

$$\mathrm{Mod}_p\,\Gamma_1 \le \mathrm{Mod}_p\,\Gamma_2 \qquad \text{if } \Gamma_1 \subset \Gamma_2; \tag{2.10}$$

$$\mathrm{Mod}_p \bigcup_i \Gamma_i \le \sum_i \mathrm{Mod}_p\,\Gamma_i. \tag{2.11}$$

It is thus reasonable to say that a property holds for Mod_p-*almost every curve* if it holds for every curve except for a collection Γ with $\mathrm{Mod}_p\,\Gamma = 0$.

(3) A detailed discussion of the modulus in $\mathbf{R}^n$ (including proofs of the preceding results) can be found in [V]; for a treatment of the theory in a more abstract setting see [HK3, Section 2]. We mention one elementary result which does not appear in either of these references. Using the uniform convexity of the Banach space $L^p(X,\mu)$ ($1 < p < \infty$), Ziemer [Z] has shown that the modulus is well-behaved with respect to increasing unions of curve families. Specifically, he has shown that

$$\mathrm{Mod}_p\biggl(\bigcup_{n=1}^{\infty} \Gamma_n\biggr) = \lim_{n\to\infty} \mathrm{Mod}_p\,\Gamma_n \tag{2.12}$$

whenever $\Gamma_1 \subset \Gamma_2 \subset \cdots$ is an increasing sequence of curve families and $p > 1$. This result will be used in the proof of Proposition 4.5.

(4) The choice of Borel functions in the definition of admissibility for a curve family is necessary for a satisfactory theory. For example, with the above definition, the n-modulus of the family of all nonconstant curves in $\mathbf{R}^n$ which pass through a fixed point is zero. This is not the case, however, if we restrict our admissible functions to be continuous. If X is locally compact we may require that our admissible functions be lower semicontinuous by the Vitali–Carathéodory theorem: any function $f \in L^p(X,\mu)$ ($p \ge 1$) may be approximated in $L^p(X,\mu)$ by a lower semicontinuous function g with $g \ge f$ [R, Theorem 2.25].

(5) The conformal modulus is a useful tool only when the metric space in question supports locally rectifiable curves. However, there are important examples where this is not the case. As an example, we may begin with a metric space (X,d) and perturb the metric to get the *snowflaked* metric space (X,d^ε) ($0 < \varepsilon < 1$). The resulting metric space never admits rectifiable curves. One advantage of Pansu's definition of the generalized modulus which we will consider in Section 3 is that it makes no distinction between locally rectifiable and non-locally rectifiable curves. See Remark 4.10(2).

2.13. Loewner spaces and Poincaré inequalities. Let E and F be continua in $\mathbf{R}^n$ $(n \geq 2)$ and let (E,F) denote the collection of curves joining E to F. A fundamental property of the conformal modulus $\operatorname{Mod}_n(E,F)$ is that it is positive if and only if the continua E and F are nondegenerate, i.e. do not reduce to a point. In fact, a quantitative and scale-invariant version of this statement holds: the conformal modulus $\operatorname{Mod}_n(E,F)$ is bounded below by a positive constant which depends only on the *relative separation*

$$\Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}. \tag{2.14}$$

Loewner [L] used a qualitative version of this result to establish the quasiconformal inequivalence of $\mathbf{R}^n$ with any proper subset. The quantitative form appears in [G1]. The importance of this feature of the conformal modulus in Euclidean space motivated the definition of a Loewner space in [HK3].

2.15. Definition. We call a connected metric measure space (X,μ) a *Loewner space* if the modulus $\operatorname{Mod}_Q(E,F)$ between two nondegenerate continua E and F admits a positive lower bound depending only on $\Delta(E,F)$.

The class of regular Loewner spaces is known to be quite large. Particular examples include the so-called *Carnot groups* as well as the *strong A_∞ geometries* of David and Semmes [DS], [S]. Philosophically, such spaces may be viewed as admitting a “large number” of “relatively short” rectifiable curves joining any two pieces of X. Indeed, Loewner spaces satisfy several quantitative connectivity conditions which we will not discuss here.

We now define inequalities of Poincaré type in arbitrary metric spaces, which are closely related to the Loewner condition.

2.16. Definition. Let $p \geq 1$. We say that a metric measure space (X,μ) supports a $(1,p)$-*Poincaré inequality* if there exist constants $C, C_0 \geq 1$ so that

$$\fint_B |u - u_B| \, d\mu \leq C(\operatorname{diam} B)\biggl(\fint_{C_0 B} \varrho^p \, d\mu\biggr)^{1/p} \tag{2.17}$$

for every pair of real-valued functions (u,ϱ) on X (where u is continuous and ϱ is Borel) satisfying

$$\bigl|u\bigl(\gamma(a)\bigr) - u\bigl(\gamma(b)\bigr)\bigr| \leq \int_\gamma \varrho \, ds \tag{2.18}$$

for every rectifiable curve $\gamma\colon [a,b] \to X$. Here we use the (relatively standard) notation $v_A = \fint_A v \, d\mu = \mu(A)^{-1} \int_A v \, d\mu$ for subsets A of X.

2.19. Remarks. (1) If $X = \mathbf{R}^n$ and u is a Lipschitz function, then u is differentiable a.e. by the theorem of Rademacher–Stepanov and the gradient ∇u may be appropriately redefined a.e. so that the pair $(u, |\nabla u|)$ satisfies (2.18). In this situation, a $(1,1)$-Poincaré inequality holds.

(2) By Hölder's inequality, condition (2.17) becomes weaker as p increases, i.e. if X supports a $(1,p)$-Poincaré inequality, then it satisfies a $(1,q)$-Poincaré inequality for every $q > p$. The converse need not hold; for each value $1 < p \le n$, Koskela [K] has given an example of an n-regular domain in $\mathbf{R}^n$ supporting a $(1,p)$-Poincaré inequality which does not support a $(1,q)$-Poincaré inequality for any $q < p$.

(3) Under suitable (weak) topological conditions, a Q-regular space X is Loewner if and only if it supports a $(1,Q)$-Poincaré inequality [HK3, Corollary 5.12].

(4) We describe the manner in which inequalities of this type transform under quasisymmetric mappings. Let X and Y be locally compact Q-regular spaces and let $f\colon X \to Y$ be a quasisymmetric map. Assume that X supports a $(1,p)$-Poincaré inequality for some $p \ge 1$.

– When $X = \mathbf{R}^n$ and $p = 1$, Y also supports a $(1,1)$-Poincaré inequality [DS].

– For arbitrary X and $p < Q$, there exists $q \in [p, Q)$ so that Y supports a $(1,q)$-Poincaré inequality [KM, Theorem 2.1].

– When $p > Q$, there exist examples where Y supports no Poincaré inequality of this type [KM, Section 3].

The borderline case $p = Q$ is particularly interesting. The proof in [KM, Theorem 2.1] applies also to this situation (with $q = Q$) provided that the map f is absolutely continuous in measure and induces an A_∞-weight on X (in the sense of Muckenhoupt). However, this additional caveat is unnecessary. Indeed, combining Corollary 1.6 with the previous remark yields

– When $p = Q$ and X and Y satisfy the topological conditions of remark (3), Y also supports a $(1,Q)$-inequality.

The relative separation (2.14) is controlled by quasisymmetric maps; if $f\colon X \to Y$ satisfies (1.1) and E and F are disjoint continua in X, then

$$\Delta(fE, fF) \le \eta\bigl(2\Delta(E,F)\bigr).$$

Thus Corollary 1.6 follows directly from Theorem 1.4.

2.20. Further remarks. Finally, let us discuss the proof of Corollary 1.7. Given metric spaces X and Y satisfying the hypotheses of that corollary, consider the snowflaked metric space $Y' = (Y, d^\varepsilon)$ where $\varepsilon = Q'/Q < 1$ (see Remark 2.8(5)). The identity map id: $Y \to Y'$ is quasisymmetric and Y' is Ahlfors regular of dimension $Q'/\varepsilon = Q$. If a quasisymmetric map $f\colon X \to Y$ existed, then $h = \mathrm{id} \circ f$ would be a quasisymmetric map of X onto Y'. By Theorem 1.4, $\mathrm{Mod}_Q\, h\,\Gamma$ would be positive. But this contradicts the fact that Y' admits no rectifiable curves.

3. Pansu's generalized modulus

This section introduces the basic concepts needed for the proof of Theorem 1.4. Much of the theory in this section is drawn from [P1], [P2], although some new theory is developed. We begin by defining the notion of a "ring" in a metric space and giving an alternative characterization of quasisymmetry in terms of rings. We next discuss a construction of measures due to Carathéodory which generalizes the concept of Hausdorff measure and a similar construction which corresponds to the concept of line integrals. This leads naturally to our treatment of the generalized modulus. We describe two versions of this concept, both the original definition due to Pansu and a new definition. Finally, we prove a fundamental property of these generalized moduli (quasisymmetric invariance) in arbitrary metric spaces.

We begin with an easy lemma.

3.1. Lemma. *Let $f\colon X \to Y$ be an η-quasisymmetric homeomorphism of metric spaces. If $A, \tilde{A} \subset X$ satisfy*

$$B \subset A \subset \tilde{A} \subset \overline{lB}, \tag{3.2}$$

for some ball $B = B(x,r)$ in X and some $l \geq 1$, then there exists a ball $B' = B'(fx, m)$ in Y so that

$$B' \subset fA \subset f\tilde{A} \subset \overline{\eta(l)B'}. \tag{3.3}$$

Proof. Assume we have data $B, A, \tilde{A}, l$ satisfying (3.2). If $B = X$, then X is bounded and hence Y is also. In this case $Y = B'$ for some ball B' and the proof is done.

If B is a proper subset of X, define

$$M = \sup_{y \in \overline{lB}} |fy - fx|, \qquad m = \inf_{z \in X \setminus B} |fz - fx|.$$

Then $M \leq \eta(l)m$ and $B' = B'(fx, m)$ satisfies (3.3). □

Note that the conclusion of Lemma 3.1 involves the metrics on X and Y only through pairs of subsets which satisfy conditions of the form (3.2) or (3.3). This motivates the following definition.

3.4. Definition. Let $l \geq 1$ and let A and $\tilde{A}$ be subsets of X satisfying $A \subset \tilde{A}$. We call the pair $(A, \tilde{A})$ an *l-ring* if there exists a ball B so that (3.2) holds. (If the choice of l does not matter, we merely call $(A, \tilde{A})$ a *ring*.) If the ball $B = B(x, r)$ we call x a *center* and r a *radius* of $(A, \tilde{A})$. (Again, these need not be unique.)

We let $\mathscr{R}_l(X, x)$ denote the collection of all l-rings in X centered at x, $\mathscr{R}_l(X)$ denote the collection of all l-rings in X and $\mathscr{R}(X)$ denote the collection of all rings in X. Any homeomorphism $f\colon X \to Y$ which maps bounded sets to bounded sets induces an obvious map from $\mathscr{R}(X)$ to $\mathscr{R}(Y)$ which we also denote by f.

We can characterize quasisymmetric maps as follows:

3.5. Proposition. *Let $f\colon X \to Y$ be a homeomorphism of metric spaces. The following are quantitatively equivalent:*

(1) *f is η-quasisymmetric;*

(2) *there exists an increasing homeomorphism $\eta_1\colon [1, \infty) \to [H, \infty)$ satisfying $\eta_1(t) \ge t$ for all $t \ge 1$ so that*

$$f\colon \mathscr{R}_l(X, x) \to \mathscr{R}_{\eta_1(l)}(Y, fx) \qquad \text{for all } x \in X \tag{3.6}$$

and

$$f^{-1}\colon \mathscr{R}_l(Y, y) \to \mathscr{R}_{\eta_1(l)}(X, f^{-1}y) \qquad \text{for all } y \in Y. \tag{3.7}$$

Proof. (1) $\Rightarrow$ (2) By [TV, Theorem 2.2], f^{-1} is η'-quasisymmetric with $\eta'(t) = 1/\eta^{-1}(1/t)$. The result follows from Lemma 3.1 with

$$\eta_1(t) = \max\{\eta(t), \eta'(t), t\}.$$

(2) $\Rightarrow$ (1) Define

$$\eta_2(t) = \begin{cases} \eta_1(t), & \text{for } t \ge 1; \\ H, & \text{for } 1/H \le t < 1; \\ 1/\eta_1^{-1}(1/t), & \text{for } t < 1/H, \end{cases}$$

and choose a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that $\eta \ge \eta_2$.

For $x, a, b \in X$ set $t = |x - a|/|x - b|$ and $t' = |fx - fa|/|fx - fb|$. It will suffice to show that $t' \le \eta_2(t)$. We deduce this from the following four results:

(i) $t \ge 1 \Rightarrow t' \le \eta_1(t)$.

(ii) $t < 1 \Rightarrow t' \le H$.

(iii) $t' > 1 \Rightarrow t \ge 1/H$.

(iv) $t' \le 1 \Rightarrow 1/t \le \eta_1(1/t')$.

(i) Define $A = B(x, |b - x|)$ and $\tilde{A} = \overline{B(x, |a - x|)}$. Then $a \in \tilde{A}$ and $b \notin A$ (recall that balls are open). By hypothesis, there exists $B' = B'(fx, m)$ so that $B' \subset fA \subset f\tilde{A} \subset \overline{\eta_1(t)B'}$. Since $fa \in \overline{\eta_1(t)B'}$ and $fb \notin B'$, we conclude that $|fx - fa| \le \eta_1(t)m \le \eta_1(t)|fx - fb|$.

(ii) In this case, consider the ball $B = B(x, |b-x|)$. We have $a \in B$ (since $t < 1$) and $b \notin B$. By hypothesis, there exists $B' = B'(fx, m)$ so that $B' \subset fB \subset \overline{HB'}$. Then $fa \in \overline{HB'}$ and $fb \notin B'$ and we conclude that $|fx - fa| \le Hm \le H|fx - fb|$.

(iii) and (iv) follow by applying (ii) and (i) (respectively) with the triples x, a, b and fx, fa, fb replaced by fx, fb, fa and x, b, a. □

3.8. Remarks. (1) In [P1], [P2], Pansu studies homeomorphisms f which satisfy condition (2) locally, i.e. for which there exists $\delta_1 > 0$ and η_1 as above so that

$$f\colon \mathscr{R}_l^{\delta_1}(X, x) \to \mathscr{R}_{\eta_1(l)}(Y, fx)$$

for all $x \in X$ and vice versa, where $\mathscr{R}_l^{\delta}(X, x)$ denotes the collection of l-rings $(A, \tilde{A})$ in X centered at x with $\operatorname{diam} \tilde{A} < \delta$. A straightforward modification of the proof of Proposition 3.5 establishes that this definition is quantitatively equivalent to the requirement that f be *uniformly locally η-quasisymmetric,* i.e. that there exist $\delta > 0$ so that f is η-quasisymmetric on every open set $U \subset X$ with $\operatorname{diam} U \le \delta$. See [P2, Remark 4.2].

(2) If $(A, \tilde{A})$ is an l-ring for which $A = \tilde{A}$, we call A an *l-round* set. We define the classes $\mathscr{B}_l(X, x)$, $\mathscr{B}_l(X)$ and $\mathscr{B}(X)$ for round sets as we did for rings in Definition 3.4. These are the classes of sets which we will use to define the generalized modulus in Section 3.20.

We state (without proof) a variation of a standard covering lemma expressed in the language of the classes $\mathscr{B}_k(X)$ ($k \ge 1$). See [Fe, Section 2.8] or [M, Section 2.1] for a proof when $k = 1$.

3.9. Lemma. *Let X be a metric space and let $\{A_i\} \subset \mathscr{B}_k(X)$. There exists a subcollection $\{A_j\}$ consisting of disjoint sets and to each set A_j in the subcollection there corresponds a ball B_j so that $(A_j, B_j) \in \mathscr{R}_{5k}(X)$ and*

$$\bigcup_i A_i \subset \bigcup_j B_j. \tag{3.10}$$

In this paper, we will be interested in set functions $\varphi\colon \mathscr{B}(X) \to [0, \infty]$. For a completely arbitrary function of this type, the utility of the covering Lemma 3.9 is not clear. Indeed, the values of φ on the expanded sets B_j may be completely unrelated to its values on the original sets A_j. This is potentially troublesome since it will be important for us to allow for complete generality in our choice of φ.

We address this issue by defining a new set function derived from φ. For $l \ge 1$, define $\widetilde{\varphi}_l\colon \mathscr{B}_l(X) \to [0, \infty]$ by

$$\widetilde{\varphi}_l(A) = \sup \varphi(\tilde{A}), \tag{3.11}$$

where the supremum is taken over all $\tilde{A} \in \mathscr{B}_l(X)$ such that $(A, \tilde{A}) \in \mathscr{R}_l(X)$.

Note that $\widetilde{\varphi}_l(A)$ is not defined for arbitrary round sets A but merely for the l-round sets.

The construction in (3.11) will allow us to use the covering Lemma 3.9 in conjunction with φ; after passing to the subcover $\{B_j\}$ we will compensate by using the derived set function $\widetilde{\varphi}_{5k}$ in place of φ.

3.12. A construction of measures (after Carathéodory). Let X be a metric space and let $\mathscr{F}$ be a family of subsets of X. For $\delta > 0$, define

$$\mathscr{F}^\delta = \{A \in \mathscr{F} : \operatorname{diam} A < \delta\}.$$

We make the technical assumption in what follows that $\mathscr{F}^\delta$ covers X for each $\delta > 0$. (We will only use this notion with the choice $\mathscr{F} = \mathscr{B}_k(X)$ for some $k \geq 1$ in which case this condition is satisfied.)

We would like to associate to an arbitrary function $\psi\colon \mathscr{F} \to [0,\infty]$ an outer measure on X which retains some properties of ψ. The following construction, due to Hausdorff [H], generalizes the linear measures of Carathéodory [C].

For $S \subset X$, set

$$\Psi(S) = \lim_{\delta\to 0} \Psi^\delta(S)$$

where

$$\Psi^\delta(S) = \inf \sum_i \psi(A_i),$$

the infimum being taken over all countable covers $\{A_i\} \subset \mathscr{F}^\delta$ of S.

3.13. Remarks. (1) For any choice of $\mathscr{F}$ and ψ, the set function Ψ is a Borel outer measure on X [Fe, Section 2.10.1]. We call Ψ the *Carathéodory measure* associated with ψ.

(2) The choice $\psi(A) = (\operatorname{diam} A)^\alpha$ $(\alpha > 0)$ for A in $\mathscr{F} = \mathscr{B}_1(X)$ yields the *Hausdorff measure* $\mathscr{H}_\alpha$ on X. (Actually, this measure is called spherical Hausdorff measure in the literature, while the term Hausdorff measure is used when $\mathscr{F}$ is the collection of all subsets of X. However, these two measures are always comparable, and thus—at least for this paper—indistinguishable.)

(3) Carathéodory's construction respects scalar multiplication, i.e. the measure associated to the set function $a\cdot\psi$ $(a \geq 0)$ is $a\cdot\Psi$. Also $\psi_1 \leq \psi_2$ implies $\Psi_1 \leq \Psi_2$. However, it is not clear that the construction respects addition, i.e. that the measure Ψ associated to the set function $\psi = \psi_1 + \psi_2$ is related to Ψ_1 and Ψ_2. For this reason, it is unclear if the generalized modulus defined below is an outer measure on the collection of all families of subsets of X, cf. Remark 3.23(2).

3.14. A Carathéodory-type construction for line integrals. In our proof of the main theorem in Section 4, certain measures Ψ of the type constructed in the previous section will be naturally associated to the volume integrals $\int_X \varrho^Q\,d\mu$ of (2.7). However, the corresponding line integrals $\int_\gamma \varrho\,ds$ will not correspond in the same way to measures of this type. Indeed, it is clear that the measures described in 3.12 compute only the size of the image set $\gamma(I)$ and are unrelated to the actual parametrizing map $\gamma\colon I \to X$. In order to model these line integrals, we will introduce new "Carathéodory-type" constructions which compute quantities analogous to the length of the curve. These constructions make use of the notion of a *parametrized cover* of a curve[1].

We begin with the definition for curves for which the parametrizing interval is compact. Let $\gamma\colon I \to X$ be such a curve and let $\mathscr{F}$ be a collection of subsets of X as in Section 3.12. Consider a collection $\{(I_\lambda, A_\lambda)\}$ indexed by λ in some countable indexing set Λ, where $I_\lambda \subset I$ is an interval and $A_\lambda \in \mathscr{F}$. (We allow here the possibility of repetitions, that is, a given set $A \in \mathscr{B}(X)$ may appear as A_λ for several different values of $\lambda \in \Lambda$.) We call such a collection a *parametrized* (or *indexed*) *cover* of γ if $I = \bigcup_\lambda I_\lambda$ and $\gamma(I_\lambda) \subset A_\lambda$ for all $\lambda \in \Lambda$.

Given a function $\psi\colon \mathscr{F} \to [0,\infty]$, we define the *$\psi$-length* of γ as follows:

$$\Psi\text{-length}(\gamma) = \lim_{\delta \to 0} \Psi\text{-length}^\delta(\gamma) \tag{3.15}$$

where

$$\Psi\text{-length}^\delta(\gamma) = \inf \sum_\lambda \psi(A_\lambda),$$

the infimum being taken over all parametrized covers $\{A_\lambda\}$ of γ drawn from $\mathscr{F}^\delta$.

For general curves $\gamma\colon I \to X$ (when I is not necessarily compact), we set

$$\Psi\text{-length}(\gamma) = \sup_{I_0} \Psi\text{-length}(\gamma_0) \tag{3.16}$$

where the supremum is taken over all compact subintervals $I_0 \subset I$.

3.17. Remarks. (1) Definition (3.15) makes sense for all curves (regardless of whether the parametrizing interval is compact or not) and one may ask if the two definitions (3.15) and (3.16) agree in general. Although this seems plausible, we have been unable to give a proof for completely general choices of the set function ψ. The choice of (3.16) as the definition of the ψ-length of a curve with noncompact parametrizing interval is necessary for the proof of Proposition 3.24.

(2) The quantity Ψ-length(γ) does not depend on the actual choice of parametrization of the curve. Indeed, if $\gamma'\colon I' \to X$ is a reparametrization of γ, we can always find corresponding intervals $\{I'_\lambda\}$ for which $I' = \bigcup_\lambda I'_\lambda$ and $\gamma(I_\lambda) = \gamma'(I'_\lambda)$. For this reason, when the curve γ is in fact rectifiable, we will always assume it is parametrized by arc length.

(3) For $\gamma\colon I \to X$, we always have the inequality $\Psi\text{-length}(\gamma) \geq \Psi\bigl(\gamma(I)\bigr)$, generalizing the fact that $\text{length}(\gamma) \geq \mathscr{H}_1\bigl(\gamma(I)\bigr)$. This is immediate since any parametrized cover is (in particular) a cover of the image set.

The following lemma provides additional evidence for the claim that the ψ-length of a curve γ generalizes the classical notion of a line integral along γ.

[1] The author wishes to express his thanks to Mario Bonk for this definition.

3.18. Lemma. *Let* $\gamma\colon I \to X$ *be a curve and let* $\mathscr{F} = \mathscr{B}_k(X)$ *for any* $k \ge 1$. *Let* $\psi(A) = \operatorname{diam} A$. *Then* $\Psi\text{-length}(\gamma) = \operatorname{length}(\gamma)$.

Proof. It suffices to prove the result in the case that the parametrizing interval $I = [a, b]$ is compact.

We prove first that $\Psi\text{-length}(\gamma) \le \operatorname{length}(\gamma)$. If γ is not rectifiable there is nothing to prove. If γ is rectifiable and $L = \operatorname{length}(\gamma)$, then (without loss of generality) we may assume that $\gamma\colon [0, L] \to X$ is parametrized by arc length. Let N be large and set $\delta = L/N$. For each $n = 1, 2, \dots, N$, let J_n be the interval $[(n-1)\delta, n\delta]$, let t_n be the midpoint of J_n and let A_n be the closed ball $\overline{B\bigl(\gamma(t_n), \delta/2\bigr)}$. Then $\{(J_n, A_n) : n = 1, 2, \dots, N\}$ is a parametrized cover of $\gamma\colon [0, L] \to X$ in $\mathscr{B}_k^\delta(X)$ and hence

$$\Psi\text{-length}^\delta(\gamma) \le \sum_{n=1}^{N} \operatorname{diam} A_n \le N\delta = L.$$

Letting $\delta \to 0$, we deduce $\Psi\text{-length}(\gamma) \le L$.

Conversely, we show that $\operatorname{length}(\gamma) \le \Psi\text{-length}(\gamma)$. Choose a partition

$$\pi = \{a = t_0 < t_1 < \cdots < t_m = b\}$$

of I. By removing terms if necessary, we may assume that $\gamma(t_k) \ne \gamma(t_{k-1})$ for all k. Let δ_0 denote the minimum of the quantities $|\gamma(t_k) - \gamma(t_{k-1})|$ for $1 \le k \le m$. If $\delta < \delta_0$ and $\{(J_\lambda, A_\lambda)\}$ is an arbitrary parametrized cover of γ with the sets A_λ drawn from $\mathscr{B}_k^\delta(X)$, then any one of the intervals J_λ contains at most one of the t_k. Using the triangle inequality, we may estimate

$$\sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le \sum_{\lambda} \operatorname{diam} A_\lambda + (m+1)\delta \tag{3.19}$$

where the second term on the right hand side is necessary to account for the sets covering each of the points $t_0, t_1, \dots, t_m$ (note that at these points, we may have to allow a given A_λ to be counted twice on the right hand side of (3.19)). Therefore

$$\sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le \Psi\text{-length}^\delta(\gamma) + (m+1)\delta.$$

Letting $\delta \to 0$ and taking the supremum over partitions π finishes the proof. □

3.20. Pansu's generalized modulus. In [P1], [P2], Pansu introduces a generalized version of the modulus which is naturally adapted to the study of quasisymmetric mappings. We give his original definition as well as a new definition suited to our needs.

First, some notation. Let $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ and let $p > 0$, $l \geq k \geq 1$. Since $\mathscr{B}_k(X) \subset \mathscr{B}_l(X)$, the value of $\widetilde{\varphi}_l(A)$ is well-defined for $A \in \mathscr{B}_k(X)$ (recall the remarks following (3.11)). We will denote the Carathéodory measure associated with the set function φ^p and $\mathscr{F} = \mathscr{B}_k(X)$ by $\Phi_{p,k}$ and the corresponding measure associated with the set function $\widetilde{\varphi}_l{}^p$ and $\mathscr{F} = \mathscr{B}_k(X)$ by $\widetilde{\Phi}_{p,k,l}$. We will also denote the φ-length of a curve γ measured with respect to the family $\mathscr{F} = \mathscr{B}_k(X)$ by Φ-$\mathrm{length}_k(\gamma)$.

Let Γ be a collection of subsets of a metric space X and let $m \geq 1$ and $l \geq k \geq 1$. The *generalized p-modulus of Γ (with parameters k, l, m)* is the value

$$\operatorname{mod}_{p,k,l,m} \Gamma = \inf \widetilde{\Phi}_{p,k,l}(X), \tag{3.21}$$

where the infimum is taken over all set functions $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ for which $\Phi_{1,m}(\gamma) \geq 1$ for all $\gamma \in \Gamma$.

When Γ consists of a family of curves, we will write $\operatorname{mod}_{p,k,l,m} \Gamma$ for the generalized modulus of the collection of images $\{\gamma(I) : \gamma \in \Gamma,\ \gamma\colon I \to X\}$. In this case we also define the *modified generalized p-modulus of Γ* to be

$$\operatorname{mod}^*_{p,k,l,m} \Gamma = \inf \widetilde{\Phi}_{p,k,l}(X), \tag{3.22}$$

where the infimum is now taken over all set functions $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ for which Φ-$\mathrm{length}_m(\gamma) \geq 1$ for all $\gamma \in \Gamma$.

3.23. Remarks. (1) The number of parameters involved in these definitions may seem daunting at first sight. Let us discuss the dependence of the generalized modulus on these various parameters.

The parameters k and m describe the amount of freedom we allow in measuring both the curves in Γ and the total space X. Their role in most applications is minor. It is easy to verify that (all other parameters being held constant), the value of $\operatorname{mod}_{p,k,l,m} \Gamma$ (or $\operatorname{mod}^*_{p,k,l,m} \Gamma$) decreases as k increases but increases as m increases.

The introduction of the parameter l is necessary (as mentioned before) in order to apply Lemma 3.9. Again, it is easy to see that both of the generalized moduli increase as l increases.

(2) For fixed values of p, k, l, m, the generalized moduli satisfy the analogues of (2.9) and (2.10). However, as mentioned in Remark 3.13(3), the operation $\varphi \mapsto \Phi$ is not necessarily subadditive and hence we do not know if (2.11) holds and hence if the generalized modulus $\operatorname{mod}_{p,k,l,m} \Gamma$ (respectively $\operatorname{mod}^*_{p,k,l,m} \Gamma$) is an outer measure on the collection of all subsets (respectively curve families) of X. Nevertheless, we still define a notion of generalized modulus zero. We say that a property holds for mod_p*-almost every set* (respectively mod^*_p*-almost every curve*) if the family Γ of exceptions may be written as a countable union $\Gamma = \bigcup_n \Gamma_n$, where $\operatorname{mod}_{p,k,l,m} \Gamma_n$ (respectively $\operatorname{mod}^*_{p,k,l,m} \Gamma_n$) is zero for some $k \geq 1$ and all $l \geq k$, $m \geq 1$, and $n \in \mathbf{N}$.

We next establish that both definitions of the generalized modulus are (in a certain sense) quasisymmetrically invariant. Equation (3.26) has been shown by Pansu [P2, Proposition 2.6]; the proof is similar to our proof of (3.25).

3.24. Theorem. *Let $f\colon X \to Y$ be an η-quasisymmetric homeomorphism between metric spaces. There exists a homeomorphism $\eta_1\colon [1,\infty) \to [H,\infty)$ which depends only on η with the following properties: $\eta_1(t) \ge t$ for all $t \ge 1$ and, for all triples $k,l,m \ge 1$ satisfying $l \ge \eta_1(k)$, we have*

$$\operatorname{mod}^*_{p,\eta_1(k),l,m} f\,\Gamma \le \operatorname{mod}^*_{p,k,\eta_1(l),\eta_1(m)} \Gamma \tag{3.25}$$

for all curve families Γ in X and

$$\operatorname{mod}_{p,\eta_1(k),l,m} f\,\Gamma \le \operatorname{mod}_{p,k,\eta_1(l),\eta_1(m)} \Gamma. \tag{3.26}$$

for all families Γ of subsets of X.

Proof. We will give a proof of (3.25). Choose η_1 depending on η as in Proposition 3.5. Let $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ satisfy $\Phi\text{-length}_{\eta_1(m)}(\gamma) \ge 1$ for all $\gamma \in \Gamma$. Define $\psi\colon \mathscr{B}(Y) \to [0,\infty]$ by

$$\psi(A') = \varphi(f^{-1}A').$$

We claim that

(i) $\widetilde{\Psi}_{p,\eta_1(k),l}(fE) \le \widetilde{\Phi}_{p,k,\eta_1(l)}(E)$ for each Borel set $E \subset X$, and

(ii) $\Phi\text{-length}_{\eta_1(m)}(\gamma) \le \Psi\text{-length}_m(f\gamma)$ for any curve $\gamma\colon I \to X$.

We first establish (i). Note that

$$\tilde{\psi}_l(A) \le \widetilde{\varphi}_{\eta_1(l)}(f^{-1}A)$$

for any $A \in \mathscr{B}_l(Y)$ by (3.7). Since $\widetilde{\Psi}_{p,k,l}$ is a Borel measure, we may reduce to the case when E is bounded by writing E as an increasing union of bounded subsets. Choose a point $x_0 \in E$ and let $U = B(x_0, 2\operatorname{diam} E)$. By [TV, Theorem 2.5], the image fU is also bounded and for any $S \subset U$,

$$\frac{\operatorname{diam} fS}{\operatorname{diam} fU} \le \eta\left(\frac{2\operatorname{diam} S}{\operatorname{diam} U}\right). \tag{3.27}$$

Let $\delta > 0$ and let $\{A_i\}$ be a cover of E drawn from $\mathscr{B}^\delta_k(X)$ each of whose sets meets E. Since $\operatorname{dist}(X \setminus U, E) > 0$, by taking δ sufficiently small, we may arrange that $A_i \subset U$. In this case the sets fA_i are contained in fU and (3.6) and (3.27) imply that they are also elements of $\mathscr{B}^{\delta'}_{\eta_1(k)}(Y)$, where

$$\delta' = \eta(2\delta/\operatorname{diam} U) \cdot \operatorname{diam} fU.$$

Note that δ' tends to zero as δ does. We conclude that

$$\widetilde{\Psi}^{\delta'}_{p,\eta_1(k),l}(fE) \le \sum_i \tilde{\psi}_l(fA_i)^p \le \sum_i \widetilde{\varphi}_{\eta_1(l)}(A_i)^p$$

and (i) follows upon taking the infimum over all such covers and the limit as $\delta \to 0$.

The same argument establishes (ii) when we reverse the roles of X and Y. Note that the definition (3.16) allows us to reduce to the case when the parametrizing interval I is compact (and hence the set $E' = f(\gamma(I)) \subset Y$ is bounded). It suffices to note that if $\{(I_\lambda, A'_\lambda)\}$ is a parametrized cover of $f\circ\gamma\colon I \to Y$ and we define $A_\lambda = f^{-1}A'_\lambda$, then $\{(I_\lambda, A_\lambda)\}$ is a parametrized cover of $\gamma\colon I \to X$. The rest of the proof is the same.

To complete the proof, we note that (ii) implies $\Psi\text{-length}_m(\gamma) \ge 1$ for each $\gamma \in \Gamma$ and so (using (i) with $E = X$)

$$\operatorname{mod}^*_{p,\eta_1(k),l,m} f\,\Gamma \le \widetilde{\Psi}_{p,\eta_1(k),l}(Y) \le \widetilde{\Phi}_{p,k,\eta_1(l)}(X).$$

The proof is completed by taking the infimum over all such φ. □

4. Proof of the main theorem

We begin with a definition.

4.1. Definition. A Borel measure μ on a metric space X is said to be *doubling* if there exists a constant $C_\mu \ge 1$ so that

$$\mu(2B) \le C_\mu \cdot \mu(B) \tag{4.2}$$

for all balls B in X.

4.3. Remarks. (1) If μ is a doubling measure on X, then the measure of an l-ring is essentially well-defined; if $(A,\tilde{A}) \in \mathscr{R}_l(X)$ then $\mu(\tilde{A}) \le {C_\mu}^s \mu(A)$ where $s = 1 + \log_2 l$.

(2) If (X,μ) is Q-regular, then μ is doubling. In fact, for all $(A,\tilde{A})$ in $\mathscr{R}_l(X)$ we have $\mu(\tilde{A}) \le {C_0}^2 l^Q \mu(A)$ where C_0 is the constant of (1.3).

The case $X = \mathbf{R}^n$ of the following lemma can be found in [B, Lemma 4.2]; we have used the terminology of this paper in our formulation. The proof uses the boundedness of the Hardy–Littlewood maximal operator on the dual of $L^p(X,\mu)$ and Hölder's inequality.

4.4. Lemma. *Let X be a metric space endowed with a doubling measure μ with constant C_μ. Let $\{(A_i,\tilde{A}_i)\}$ be a countable collection of elements of $\mathscr{R}_l(X)$ $(l \ge 1)$ and let $\{c_i\}$ be a corresponding collection of nonnegative real numbers. For any $p \ge 1$, there exists a constant $C = C(l,p,C_\mu)$ such that*

$$\int_X \Bigl(\sum_i c_i \chi_{\tilde{A}_i}\Bigr)^p d\mu \le C \int_X \Bigl(\sum_i c_i \chi_{A_i}\Bigr)^p d\mu.$$

We state two propositions which describe the relationship between the classical modulus and the generalized moduli of the previous section. (Recall that "curve" for us always means nonconstant curve.)

4.5. Proposition. *Let X be a connected metric space and let μ be a doubling measure on X with constant C_μ which also satisfies the upper mass bound in* (1.3) *with constant C_0 and $Q>1$. Let $k,m\geq 1$ and $l\geq 10k$. There exists $C=C(Q,k,C_0,C_\mu)$ so that for all families Γ of curves in X,*

$$\operatorname{Mod}_Q\Gamma \leq C\ \operatorname{mod}^*_{Q,k,l,m}\Gamma \leq C\ \operatorname{mod}_{Q,k,l,m}\Gamma. \tag{4.6}$$

4.7. Proposition. *Let X be a locally compact, connected metric space with a Borel measure μ which satisfies the lower mass bound in* (1.3) *with constant C_0 and $Q\geq 1$. Let $k,l,m\geq 1$. There exists $C=C(Q,l,C_0)$ so that*

$$\operatorname{mod}^*_{Q,k,l,m}\Gamma \leq C\ \operatorname{Mod}_Q\Gamma \tag{4.8}$$

for all families Γ of curves in X and

$$\operatorname{mod}_{Q,k,l,m}\Gamma \leq C\ \operatorname{Mod}_Q\Gamma \tag{4.9}$$

for all families Γ of arcs in X.

4.10. Remarks. (1) By Remark 3.23(1), the generalized moduli decrease as k increases and increase as l and m increase. Thus it will suffice to establish (4.8) with $l=10k$ and $m=1$ and to establish (4.8) and (4.9) with $k=1$. Note that the second inequality in Proposition 4.5 follows by Remark 3.17(4)

(2) Recall that the classical modulus $\operatorname{Mod}_Q\Gamma$ depends only on the locally rectifiable curves in Γ while the generalized moduli make use of all of the elements of Γ. We thus obtain as a corollary of Proposition 4.7 that in a locally compact and connected metric measure space which satisfies the lower mass bound, mod^*_Q-a.e. curve (respectively mod_Q-a.e. arc) is locally rectifiable, which is not at all obvious from the definitions.

(3) Theorem 1.4 clearly follows by combining the estimates found in Propositions 4.5 and 4.7 for the modified generalized modulus $\operatorname{mod}^*_{p,k,l,m}\Gamma$ (for a suitable choice of k, l and m) with Theorem 3.24. Let us pause here to discuss our rationale for including Pansu's original definition of the generalized modulus $\operatorname{mod}_{p,k,l,m}\Gamma$. Strictly speaking, this concept is not needed for the proof of the main theorem. Indeed, the estimates for Pansu's generalized modulus suffice only to establish the main theorem for arc families Γ. We have decided to include both definitions here to better illustrate the role of Pansu's modulus in the theory.

(4) In [HK1], Heinonen and Koskela introduce another discrete version of the modulus in order to establish their "infinitesimal-to-global" principle for quasiconformal mappings in Carnot groups (recall that their use of the term quasiconformal differs from that used in this paper). Necessary for their proof is the fact that their discrete modulus serves as an upper bound for the classical modulus. The proof of this result [HK1, Lemma 2.4] is our model for the proof of Proposition 4.5. The crucial definition (4.13) "reassembles" an admissible function ϱ for the classical modulus from an admissible φ for the discrete modulus.

Proof of Proposition 4.5. Let $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ satisfy $\Phi\text{-length}_1(\gamma) \ge 1$ for all $\gamma \in \Gamma$. Let Γ_n consist of all those curves $\gamma\colon I \to X$ in Γ with the following property: there exists a compact subinterval $I_0 \subset I$ for which the set $\gamma(I_0)$ has diameter at least $1/n$ and

$$\Phi\text{-length}_1^{1/2n}(\gamma_0) \ge \tfrac{1}{2} \tag{4.11}$$

Since $\Gamma = \bigcup_n \Gamma_n$ and $Q > 1$, it suffices by (2.12) to show that

$$\operatorname{Mod}_Q \Gamma_n \le C\widetilde{\Phi}_{Q,k,10k}(X) \tag{4.12}$$

where C is independent of n.

Fixing n, we choose $\delta > 0$ and a covering $\{A_i\} \subset \mathscr{B}_k^\delta$ of X. Using the covering Lemma 3.9, we find a disjoint subfamily $\{A_j\}$ and, for each A_j, a corresponding ball B_j of radius r_j so that $(A_j, B_j) \in \mathscr{R}_{5k}$ and the collection $\{B_j\}$ still covers X. Our goal is to construct a parametrized cover of each of the subcurves γ_0 out of the balls $\{B_j\}$.

First, note that (provided δ is chosen sufficiently small) the diameter of $2B_j$ is bounded by $1/2n$ for each j and hence no curve γ_0 lies entirely in any ball $2B_j$. An easy calculation shows in this case that $\operatorname{length}(\gamma_0|_J) \ge r_j$ for any open interval $J \subset I_0$ such that $\gamma(J)$ meets both B_j and the boundary of $2B_j$.

Define a Borel function ϱ by

$$\varrho(x) = 2\sum_j \frac{\varphi(2B_j)}{r_j}\chi_{2B_j}(x). \tag{4.13}$$

Let $\gamma \in \Gamma_n$ be a locally rectifiable curve and let γ_0 be the associated subcurve. For each j such that B_j meets $\gamma_0(I_0)$, set

$$\Lambda_j = \big\{J\colon J \text{ is a maximal open subinterval of } I_0 \\ \text{such that } \gamma(J) \subset 2B_j \text{ and } \gamma(J) \cap B_j \ne \emptyset\big\}.$$

Our index set Λ is defined to be the disjoint union of the collections Λ_j. Recall that to each element of Λ, we must associate two sets, the first a subinterval of I and the second an element of $\mathscr{B}_1^\delta(X)$. Let $\lambda = J$ be an element of Λ. Suppose $\lambda \in \Lambda_j$. We associate to λ the pair (I_λ, A_λ), where $I_\lambda = J$ and $A_\lambda = 2B_j$. The collection $\{(I_\lambda, A_\lambda)\}$ is a parametrized cover of γ_0 with $\operatorname{diam} A_\lambda \le 1/2n$ for all λ and hence $1 \le 2\sum_\lambda \varphi(A_\lambda)$. Thus

$$\begin{aligned}\int_\gamma \varrho\, ds \ge \int_{\gamma_0} \varrho\, ds &= 2\sum_j \frac{\varphi(2B_j)}{r_j}\operatorname{length}(2B_j \cap \gamma_0)\\ &\ge 2{\sum_j}' \frac{\varphi(2B_j)}{r_j}\operatorname{length}(2B_j \cap \gamma_0)\end{aligned}$$

(where the second sum is over all j for which B_j meets $\gamma(I_0)$)

$$\geq 2{\sum_j}' \frac{\varphi(2B_j)}{r_j} \sum_{J\subset \Lambda_j} \mathrm{length}(\gamma_0|_J) \geq 2\sum_{\lambda\in\Lambda} \varphi(A_\lambda) \geq 1.$$

Thus ϱ is admissible for Γ_n and we conclude that

$$\mathrm{Mod}_Q\,\Gamma_n \leq \int_X \varrho^Q\,d\mu \leq C(Q)\int_X \Bigl(\sum_j a_j\chi_{2B_j}\Bigr)^Q\,d\mu$$

where $a_j = \varphi(2B_j)/r_j$. Lemma 4.4 and the upper mass bound imply that

$$\mathrm{Mod}_Q\,\Gamma_n \leq C\int_X \Bigl(\sum_j a_j\chi_{A_j}\Bigr)^Q\,d\mu = C\sum_j a_j^Q\mu(A_j) \leq C\sum_j \varphi(2B_j)^Q$$

where $C = C(Q,k,C_0,C_\mu)$. Finally, $\varphi(2B_j) \leq \widetilde{\varphi}_{10k}(A_j)$ since $(A_j, 2B_j)$ is a $10k$-ring and so

$$\mathrm{Mod}_Q\,\Gamma_n \leq C\sum_j \widetilde{\varphi}_{10k}(A_j)^Q \leq C\sum_i \widetilde{\varphi}_{10k}(A_i)^Q.$$

(4.12) follows by taking the infimum over all such coverings $\{A_i\}$ and the limit as $\delta\to 0$. □

The proof of Proposition 4.7 is (in some sense) dual to that of Proposition 4.5. Here the basic step is the "discretization" procedure (4.14) which takes admissible functions ϱ for the classical modulus to admissible set functions φ for Pansu's modulus. The introduction of the integrable function h in (4.14) is necessary to deal with the case when the space X has infinite measure.

Proof of 4.8. Let ϱ be an admissible function for $\mathrm{Mod}_p\,\Gamma$. By Remark 2.8(4), we may assume that ϱ is lower semicontinuous. In order to deal with the non-locally rectifiable curves in Γ, we would like to be able to assume in addition that ϱ is bounded away from zero. In general, however, we cannot make this assumption unless the space X has finite measure. We deal with this problem by making a slight modification in our definition of φ in (4.14).

Choose a positive Borel function $h\in L^Q(X,\mu)$ for which $\inf_{x\in K} h(x) > 0$ for each bounded set $K\subset X$. For example, we may take

$$h(x) = \sum_{n=0}^\infty c_n\chi_{A_n}(x)$$

where $A_n = \{x\in X\colon n\leq |x-x_0| < n+1\}$ ($x_0\in X$ a fixed basepoint) and $\{c_n\}$ is a sequence of positive constants for which the sum $\sum_{n=0}^\infty c_n^Q\mu(A_n)$ converges (recall our standing assumption that μ assigns finite measure to each bounded set of X).

For $\tau > 0$, define a set function $\varphi\colon \mathscr{B}(X) \to [0,\infty]$ by

$$\varphi(A) = \inf_{x\in A}\bigl(\varrho(x) + \tau h(x)\bigr) \cdot \operatorname{diam} A. \tag{4.14}$$

(We suppress the dependence of φ on τ for simplicity.) We wish to show that $\Phi\text{-length}_m(\gamma) \ge 1$ for all curves $\gamma \in \Gamma$.

Suppose first that γ is not locally rectifiable. Then there exists a subinterval $I_0 \subset I$ for which the associated subcurve γ_0 is not rectifiable. Now $\gamma(I_0)$ is contained in some bounded subset $K \subset X$. Setting $h_0 = \inf_{x\in K} h(x) > 0$, we see from Lemma 3.18 that

$$\Phi\text{-length}_m(\gamma) \ge \Phi\text{-length}_m(\gamma_0) \ge \tau h_0 \operatorname{length}(\gamma_0) = \infty \ge 1.$$

On the other hand, suppose γ is locally rectifiable. In this case, we will prove that

Claim 1. $\int_\gamma \varrho\, ds \le \Phi\text{-length}_m(\gamma)$.

Hence $\Phi\text{-length}_m(\gamma) \ge 1$ for all $\gamma \in \Gamma$ and so

$$\operatorname{mod}^*_{Q,1,l,m}\Gamma \le \widetilde{\Phi}_{Q,1,l}(X).$$

Now the associated set function $\widetilde{\varphi}_l$ is equivalent to φ. Indeed, since X is connected, it follows that

$$\widetilde{\varphi}_l(A) \le 2l\varphi(A). \tag{4.15}$$

Therefore

$$\operatorname{mod}^*_{Q,1,l,m}\Gamma \le C\Phi_{Q,1}(X)$$

where $C = C(l,Q)$. Now we will prove

Claim 2. There exists $C = C(Q, C_0)$ so that

$$\Phi_{Q,1}(X) \le C\int_X (\varrho^Q + \tau^Q h^Q)\, d\mu.$$

Using the result of this claim, we find that

$$\operatorname{mod}^*_{Q,1,l,m}\Gamma \le C\biggl(\int_X \varrho^Q\, d\mu + \tau^Q\int_X h^Q\, d\mu\biggr)$$

for all τ, where $C = C(l,Q,C_0)$. Since $h \in L^Q(X,\mu)$, we may pass to the limit as $\tau \to 0$ and then take the infimum over ϱ to complete the proof of (4.8). □

4.16. Remark. To show (4.9), we note that for arcs $\gamma\colon I \to X$, the line integral $\int_\gamma \varrho\, ds$ agrees with the integral $\int_{\gamma(I)} \varrho\, d\mathscr{H}_1$, where $\mathscr{H}_1$ denotes Hausdorff linear (i.e. one-dimensional) measure on X. In this case, Claim 1 is replaced by

$$\int_{\gamma(I)} \varrho\, d\mathscr{H}_1 \le \Phi_{1,m}(\gamma(I)).$$

The proof of this fact, as well as the rest of the proof of (4.9), is similar to the proof we give here.

Finally, we establish the two claims.

Proof of Claim 1. Since $\int_\gamma \varrho\, ds$ is the supremum of the values $\int_{\gamma_0} \varrho\, ds$ over all closed subintervals $I_0 \subset I$ for which γ_0 is rectifiable, we may assume without loss of generality that γ is rectifiable with compact parametrizing interval. Since ϱ is lower semicontinuous, we have

$$\int_\gamma \varrho\, ds = \sup \int_\gamma \varrho'\, ds$$

(where the supremum is taken over all nonnegative continuous functions $\varrho' \le \varrho$) and thus we may in addition assume that ϱ is continuous. (Note that throughout the proof of this claim, the curve γ is fixed.) Our proof of Claim 1 will be a modification of the argument we used for Lemma 3.18.

Without loss of generality, we assume that $\gamma\colon I = [0, L] \to X$ is parametrized by arc length. Since $\gamma(I)$ is compact and X is locally compact, we may choose $\varepsilon > 0$ so that the ε-neighborhood $U = N_\varepsilon \gamma(I)$ has compact closure in X. Recall in this case we have $h_0 = \inf_{x \in \overline{U}} h(x) > 0$.

Let $0 < \eta < \tau h_0$. Since ϱ is uniformly continuous on $\overline{U}$, we may assume (making ε smaller if necessary) that $|\varrho(x) - \varrho(y)| < \eta$ whenever $x, y \in U$ satisfy $|x - y| < \varepsilon$.

Using (2.4), we choose a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_m = L\}$ of I with mesh $\|\pi\| \le \frac{1}{2}\varepsilon$ and

$$\int_\gamma \varrho\, ds - \eta \le \sum_{k=1}^{m} \varrho(\gamma(t_k)) \cdot |\gamma(t_k) - \gamma(t_{k-1})|.$$

By removing terms if necessary, we may assume that $\gamma(t_k) \ne \gamma(t_{k-1})$ for all k. Let δ_0 denote the minimum of the $(m+1)$ quantities $|\gamma(t_k) - \gamma(t_{k-1})|$ (for $1 \le k \le m$) and $\frac{1}{2}\varepsilon$ and let I_k denote the interval $[t_{k-1}, t_k] \subset I$. Let $0 < \delta < \delta_0$ be arbitrary.

We prove that if $A \in \mathscr{B}_m^\delta(X)$ meets $\gamma(I_k)$, then

$$\varrho(\gamma(t_k)) \cdot \operatorname{diam} A < \varphi(A). \tag{4.17}$$

Indeed, let $\gamma(t) \in A \cap \gamma(I_k)$. For any $a \in A$,

$$|a - \gamma(t_k)| \le |a - \gamma(t)| + |\gamma(t) - \gamma(t_k)| < \delta + |t - t_k| < \tfrac{1}{2}\varepsilon + |I_k| \le \varepsilon$$

and so (since $A \subset U$) $\varrho\big(\gamma(t_k)\big) < \varrho(a) + \eta$.

Now recall that we chose $\eta < \tau h_0 < \tau \cdot \inf_{a\in A} h(a)$. Hence

$$\varrho\big(\gamma(t_k)\big) \cdot \operatorname{diam} A < \inf_{a\in A}\big(\varrho(a) + \tau h(a)\big) \cdot \operatorname{diam} A = \varphi(A).$$

Returning to the proof of Claim 1, suppose that $\{(J_\lambda, A_\lambda)\}$ is an arbitrary parametrized cover of γ with sets A_λ drawn from $\mathscr{B}^\delta_m(X)$. Consider the quantity $\sum_{k=1}^m \varrho\big(\gamma(t_k)\big) \cdot |\gamma(t_k) - \gamma(t_{k-1})|$. As in the proof of Lemma 3.18, each of the intervals J_λ contains at most one of the t_k. Using the triangle inequality, we bound each expression $|\gamma(t_k - \gamma(t_{k-1})|$ by a sum of terms of the form $\operatorname{diam} A_\lambda$. By (4.17),

$$\sum_{k=1}^m \varrho\big(\gamma(t_k)\big) \cdot |\gamma(t_k) - \gamma(t_{k-1})| \le \sum_\lambda \varphi(A_\lambda) + (m+1)M\delta$$

where $M = \sup\{\varrho(x) : x \in \gamma(I)\}$ is finite. Here (as in Lemma 3.18) the second term on the right hand side is necessary to account for the sets covering each of the points $t_0, t_1, \dots, t_m$. We conclude that

$$\int_\gamma \varrho\, ds - \eta \le \Phi\text{-length}^\delta_m(\gamma) + (m+1)M\delta$$

and Claim 1 follows in the limit $\delta, \eta \to 0$. □

Proof of Claim 2. For each value of $\delta > 0$, we use the covering Lemma 3.9 to find a disjoint collection of balls $\{B_j\}$ for which $X = \bigcup_j 5B_j$ and $5B_j \in \mathscr{B}^\delta_1(X)$ for all j. Then

$$\Phi^\delta_{Q,1}(X) \le \sum_j \varphi(5B_j)^Q \le C \sum_j \varphi(B_j)^Q$$

where the latter inequality follows from (4.15) with $C = C(C_1)$. But for any ball B in X we have

$$\begin{aligned}\varphi(B)^Q &= \inf_{x\in B}\big(\varrho(x) + \tau h(x)\big)^Q \cdot (\operatorname{diam} B)^Q \\ &\le C_0 \inf_{x\in B}\big(\varrho(x) + \tau h(x)\big)^Q \mu(B) \le C_0 \int_B (\varrho + \tau h)^Q\, d\mu\end{aligned}$$

(using the lower mass bound for X) and so

$$\Phi^\delta_{Q,1}(X) \le C \sum_j \int_{B_j} (\varrho + \tau h)^Q\, d\mu \le C \int_X (\varrho + \tau h)^Q\, d\mu$$

where $C = C(C_0, C_1, Q)$. Now take the supremum over δ. □

References

[B] BOJARSKI, B.: Remarks on Sobolev imbedding inequalities. - In: Proc. of the Conf. on Complex Analysis, Joensuu 1987, Lecture Notes in Math. 1351, Springer-Verlag, 1988.

[BA] BEURLING, A., and L.V. AHLFORS: The boundary correspondence under quasiconformal mappings. - Acta Math. 96, 1956, 125–142.

[C] CARATHÉODORY, C.: Über das lineare Maß von Punktmengen—eine Verallgemeinerung des Längenbegriffs. - Nachr. Ges. Wiss. Göttingen, 1914, 404–426.

[DS] DAVID, G., and S. SEMMES: Strong A_∞-weights, Sobolev inequalities, and quasiconformal mappings. - In: Analysis and Partial Differential Equations, Lecture Notes in Pure and Appl. Math. 122, Marcel Dekker, 1990.

[Fe] FEDERER, H.: Geometric Measure Theory. - Springer-Verlag, Berlin–Heidelberg–New York, 1969.

[G1] GEHRING, F.W.: Symmetrization of rings in space. - Trans. Amer. Math. Soc. 101, 1961, 499–519.

[G2] GEHRING, F.W.: Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103, 1962, 353–393.

[H] HAUSDORFF, F.: Dimension und äußeres Maß. - Math. Ann. 79, 1919, 157–179.

[HK1] HEINONEN, J., and P. KOSKELA: Definitions of quasiconformality. - Invent. Math. 120, 1995, 61–79.

[HK2] HEINONEN, J., and P. KOSKELA: From local to global in quasiconformal structures. - Proc. Nat. Acad. Sci. U.S.A. 93, 1996.

[HK3] HEINONEN, J., and P. KOSKELA: Quasiconformal maps in metric spaces with controlled geometry. - Preprint.

[K] KOSKELA, P.: Removable singularities and integration by parts. - Preprint.

[KM] KOSKELA, P. and P. MACMANUS: Quasiconformal mappings and Sobolev spaces. - Preprint.

[L] LOEWNER, L.: On the conformal capacity in space. - J. Math. Mech. 8, 1959, 411–414.

[M] MATTILA, P.: Geometry of Sets and Measures in Euclidean Spaces. - Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, 1995.

[P1] PANSU, P.: Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. - Ann. Math. 129, 1989, 1–60.

[P2] PANSU, P.: Dimension conforme et sphére á l'infini des variétés á courbure négative. - Ann. Acad. Sci. Fenn. Ser. A I Math. 14, 1989, 177–212.

[R] RUDIN, W.: Real and Complex Analysis, 2nd Edition. - McGraw-Hill, 1974.

[S] SEMMES, S.: Bilipschitz mappings and strong A_∞-weights. - Ann. Acad. Sci. Fenn. Ser. A I Math. 18, 1993, 211–248.

[TV] TUKIA, P., and J. VÄISÄLÄ: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 97–114.

[V] VÄISÄLÄ, J.: Lectures on n-dimensional Quasiconformal Mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[Z] ZIEMER, W.P.: Extremal length and p-capacity. - Mich. Math. J. 16, 1969, 43–51.

Received 3 March 1997