Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 23, 1998, 151–168

QUASISYMMETRICALLY THICK SETS

Susan G. Staples and Lesley A. Ward

Texas Christian University, Department of Mathematics
Fort Worth, TX 76129, U.S.A.; sstaples@gamma.is.tcu.edu
Rice University, Department of Mathematics
Houston, TX 77005, U.S.A.; lesley@math.rice.edu

Abstract. A subset of the real line is called quasisymmetrically thick if all its images under quasisymmetric self-mappings of the real line have positive Lebesgue measure. We establish two sufficient conditions for a set to be quasisymmetrically thick, give an example distinguishing the conditions, and show that one of these conditions, which applies to sets with a Cantor-type structure, is sharp. We give the analogues of these conditions for sets all of whose K-quasisymmetric images have positive measure, for fixed K. These results are related to Wu's work on sets all of whose quasisymmetric images have measure zero [Wu]. We also prove a result about when a Cantor set of positive measure cannot be mapped quasisymmetrically to a set of zero measure; for instance, a middle-interval Cantor set of positive measure, constructed in the usual way, cannot be mapped quasisymmetrically to the ternary Cantor set.

1. Introduction

An increasing self-homeomorphism f of the real line $\mathbf{R}$ is called *quasisymmetric* if there is some $K \geq 1$ such that

$$\frac{1}{K} \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq K, \tag{1.1}$$

for all $x \in \mathbf{R}$, $t > 0$. We also say f is K-*quasisymmetric*, when we wish to emphasize a particular value of K. The central question of this paper is the following: When does a quasisymmetric function map a set of positive Lebesgue measure in the real line to a set which also has positive measure?

A strong formulation of this question is to ask which sets are mapped to sets of positive measure by *all* quasisymmetric maps. In other words, which sets are quasisymmetrically thick, according to the following definition:

Definition 1.1. A subset E of the real line is *quasisymmetrically thick* if for every quasisymmetric self-homeomorphism f of the real line, the set $f(E)$ has positive Lebesgue measure.

For example, any set which contains an interval is quasisymmetrically thick. Quasisymmetrically thick sets have positive Lebesgue measure.

1991 Mathematics Subject Classification: Primary 30C62; Secondary 28A12.

We begin with two types of sufficient conditions for a set to be quasisymmetrically thick. For convenience we consider subsets of the unit interval $[0,1]$ and quasisymmetric maps $f\colon [0,1] \to [0,1]$. Our first result applies to sets formed by successively removing intervals, which may overlap, from $[0,1]$. If these intervals decrease in size sufficiently fast, then the set which remains is quasisymmetrically thick. (Throughout the paper, $|\cdot|$ denotes Lebesgue measure on $\mathbf{R}$.)

Theorem 1.2. *Let $E = [0,1] \setminus \bigcup_m I_m$, where the I_m are open subintervals of $[0,1]$ such that $\sum_m |I_m|^p < \infty$ for all $p > 0$, and E has positive measure. Then E is quasisymmetrically thick.*

Our second sufficient condition is for sets with a Cantor-type property which we call being $\{\alpha_n\}$-thick.

Definition 1.3. Given a sequence $\{\alpha_n\}$ with $0 < \alpha_n < 1$, a set $E \subset \mathbf{R}$ is called *$\{\alpha_n\}$-thick* if there is a sequence of sets $\mathscr{E}_n = \{E_{n,j}\}$, where for each n the $E_{n,j}$ are intervals with mutually disjoint interiors, such that

$$\sup_j |E_{n,j}| \to 0 \qquad \text{as} \quad n \to \infty, \tag{1.2}$$

and each $E_{n,j} \setminus E$ contains an interval $J_{n,j}$ so that the following conditions hold:

$$|J_{n,j}| \le \alpha_n \, |E_{n,j}|, \tag{1.3}$$

$$\bigcup_{\mathscr{E}_n} (E_{n,j} \setminus J_{n,j}) \subset \bigcup_{\mathscr{E}_{n+1}} E_{n+1,k}, \tag{1.4}$$

$$\bigcup_{\mathscr{E}_{n+1}} (E_{n+1,k} \setminus J_{n+1,k}) \subset \bigcup_{\mathscr{E}_n} (E_{n,j} \setminus J_{n,j}), \tag{1.5}$$

$$\bigcap_n \bigcup_j (E_{n,j} \setminus J_{n,j}) \subset E. \tag{1.6}$$

An $\{\alpha_n\}$-thick set E contains a Cantor-like set $\bigcap_n \bigcup_j (E_{n,j} \setminus J_{n,j})$, with the $J_{n,j}$ lying in the complement of E, which has 'small gaps': condition (1.3) says that at level n, the lengths of the removed intervals $J_{n,j}$ are at most α_n times the lengths of the intervals $E_{n,j}$. An $\{\alpha_n\}$-thick set may have positive or zero measure.

The ternary Cantor set is $\{\frac{1}{3}\}$-thick. More generally, any $\{\alpha_n\}$-regular Cantor set (Definition 1.9 below) is $\{\alpha_n\}$-thick. One can vary the standard Cantor construction by first dividing the intervals remaining after the n^{th} stage into subintervals I, and then removing an open interval from each of these. If the open intervals removed from subintervals I all have length at most $\alpha_n |I|$, then this also yields an $\{\alpha_n\}$-thick set.

We give a sufficient condition, on the sequence $\{\alpha_n\}$, for an $\{\alpha_n\}$-thick set to be quasisymmetrically thick.

Theorem 1.4. *Let E be an $\{\alpha_n\}$-thick set with $\sum_1^\infty \alpha_n^p < \infty$ for all $p > 0$. Then E is quasisymmetrically thick.*

We also show that this condition is sharp:

Theorem 1.5. *If $\{\alpha_n\}$ is a decreasing sequence with $\alpha_n \to 0$, $0 < \alpha_n < 1$, and $\sum_1^\infty \alpha_n^p = \infty$ for some $p > 4\alpha_1$, then there exists a perfect set E which is $\{\alpha_n\}$-thick and satisfies $|f(E)| = 0$ for some quasisymmetric map f.*

We remark that this $\{\alpha_n\}$-thick set E will itself have positive measure, if also $\sum \alpha_n < \infty$.

A little more generally, given $K \geq 1$ one can ask which sets are mapped to sets of positive measure by all K-quasisymmetric mappings. In Corollaries 4.1 and 4.2, we give sufficient conditions for this to happen. In these analogues of Theorems 1.2 and 1.4, the summation conditions are assumed to hold only for sufficiently large values of the exponent p. We give relations between the quasisymmetry constant K and the critical exponent of convergence.

Our next result is of a different type. We first recall the Cantor set construction.

Definition 1.6. Given a sequence $\{\alpha_n\}$ with $0 < \alpha_n < 1$, we construct an $\{\alpha_n\}$*-regular middle-interval Cantor set* E in $[0,1]$. Let $E_{1,1} = [0,1]$. First remove the open middle interval $J_{1,1}$ of length $\alpha_1|E_{1,1}|$ from $E_{1,1}$, leaving two closed intervals $E_{2,1}$ and $E_{2,2}$ of equal length. At stage two, remove the two open middle intervals $J_{2,1}$ and $J_{2,2}$, both of length $\alpha_2|E_{2,1}|$, from the intervals $E_{2,1}$ and $E_{2,2}$ respectively. At the n^{th} stage, remove the 2^{n-1} open middle intervals $J_{n,j}$, each of length $\alpha_n|E_{n,1}|$, from the intervals $E_{n,j}$. The Cantor set is $E = \bigcap_{n=1}^\infty \bigcup_{j=1}^{2^{n-1}} E_{n,j}$. The Lebesgue measure of E is $|E| = \prod_{n=1}^\infty (1-\alpha_n)$, which is positive if and only if $\sum_{n=1}^\infty \alpha_n < \infty$.

By a *regular Cantor set* we mean one in which the 2^{n-1} open intervals removed at the n^{th} stage all have the same length, but need not be middle intervals.

Suppose E is an $\{\alpha_n\}$-regular middle-interval Cantor set of positive measure. We give a sufficient condition on a quasisymmetric map f which ensures that the image $f(E)$ also has positive measure.

Theorem 1.7. *Let E be a regular middle-interval Cantor set of positive measure in $[0,1]$. Then E cannot be mapped to a set of measure zero by any quasisymmetric map $f\colon [0,1] \to [0,1]$ which satisfies the condition $|f(J_{n,j})| \overset{A}{\sim} |f(J_{n,l})|$ for all n, j, and l with $1 \leq j$, $l \leq 2^{n-1}$, for a constant A independent of n, j, and l.*

The notation $x \overset{A}{\sim} y$ means that $(1/A)\,y \leq x \leq A\,y$; we say that x and y are *comparable* with constant A. The case $A = 1$ gives the following result:

Corollary 1.8. *A regular middle-interval Cantor set of positive Lebesgue measure cannot be mapped quasisymmetrically to a regular Cantor set of measure zero.*

For example, the $\{(n+1)^{-2}\}$-regular middle-interval Cantor set cannot be mapped quasisymmetrically to the ternary Cantor set.

Of course, the homeomorphic image $f(E)$ of a Cantor set also has a Cantor-like structure, but f may strongly distort the geometry of E. Our condition on f says that f does not distort the geometry too much, in the sense that the gaps in $f(E)$ corresponding to intervals removed from E at a given stage all have about the same length, and that this holds at each stage of the construction of E. This need not be true for a general quasisymmetric map f.

Much of this paper was inspired by J.-M. Wu's work [Wu] on sets which are null with respect to all doubling measures on the real line. We call these sets *quasisymmetrically null*, in view of the relationship between doubling measures and quasisymmetric maps, recalled below. A set E is quasisymmetrically null if and only if $f(E)$ has measure zero for all quasisymmetric maps f. Quasisymmetrically null sets are at the opposite extreme from quasisymmetrically thick sets E, for which all quasisymmetric images $f(E)$ have positive measure.

Our $\{\alpha_n\}$-thick sets are related to Wu's $\{\alpha_n\}$-porous sets, which she defines as follows: Given a sequence $\{\alpha_n\}$ with $0<\alpha_n<1$, a set $E\subset \mathbf{R}$ is called $\{\alpha_n\}$-*porous* if there is a sequence of covers $\mathscr{E}_n=\{E_{n,j}\}$ of E, by intervals $E_{n,j}$ with mutually disjoint interiors, such that $\sup_j |E_{n,j}|\to 0$ as $n\to\infty$, and each $E_{n,j}\setminus E$ contains an interval $J_{n,j}$ such that $|J_{n,j}|\ge \alpha_n |E_{n,j}|$ and $\bigcup_{\mathscr{E}_n}(E_{n,j}\setminus J_{n,j})\supset \bigcup_{\mathscr{E}_{n+1}} E_{n+1,k}$. In contrast with $\{\alpha_n\}$-thick sets, an $\{\alpha_n\}$-porous set is *covered* by a Cantor-like set $\bigcap_n\bigcup_j(E_{n,j}\setminus J_{n,j})$, with the $J_{n,j}$ lying in the complement of E, which has *large* gaps: at level n, the lengths of the removed intervals $J_{n,j}$ are at *least* α_n times the lengths of the intervals $E_{n,j}$. Cantor-type sets provide examples of $\{\alpha_n\}$-porous sets, and the porous sets studied by Martio ([M]; see also [HKM]) in relation to $\mathscr{A}$-harmonic measure are $\{c\}$-porous for some $c>0$.

Wu gives a sufficient condition for an $\{\alpha_n\}$-porous set to be quasisymmetrically null, and shows that this condition is sharp:

Theorem 1.9 [Wu, Theorem 1]. *Suppose that E is an $\{\alpha_n\}$-porous set with $\sum_1^\infty \alpha_n^p=\infty$ for all $p\ge 1$. Then E is quasisymmetrically null.*

Theorem 1.10 [Wu, Theorem 2]. *If $0<\alpha_n<\frac14$ is a decreasing sequence with $\sum_1^\infty \alpha_n^p<\infty$ for some $p\ge 1$, then there exists a perfect set E which is $\{\alpha_n\}$-porous and satisfies $|f(E)|>0$ for some quasisymmetric map f.*

Our Theorems 1.4 and 1.5 are analogues of these results for $\{\alpha_n\}$-thick sets and quasisymmetrically thick sets. To summarize, if E is $\{\alpha_n\}$-thick with small enough gaps, then E is quasisymmetrically thick, while if E is $\{\alpha_n\}$-porous with large enough gaps, then E is quasisymmetrically null; and the examples in Theorems 1.5 and 1.10 show that these results are sharp.

We also give a K-dependent version of [Wu, Theorem 1] (see Corollary 4.3).

Theorem 1.7 has some overlap with [Wu, Theorem 1]. Her result implies that an $\{\alpha_n\}$-regular Cantor set of measure zero, for which the sum $\sum_n \alpha_n^p$ diverges for every $p \geq 1$, cannot be mapped quasisymmetrically to any set of positive measure. Theorem 1.7 implies that no $\{\alpha_n\}$-regular Cantor set of measure zero can be mapped quasisymmetrically to any regular Cantor set of positive measure.

A positive measure μ on $\mathbf{R}$ is a *doubling measure* with constant λ, written $\mu \in \mathscr{D}(\lambda)$, if for every pair of neighboring intervals I and J of $\mathbf{R}$ of the same length, the condition $(1/\lambda)\,\mu(J) \leq \mu(I) \leq \lambda\,\mu(J)$ holds. If $\mu \in \mathscr{D}(\lambda)$, then the function f defined by $f(x) - f(0) = \int_0^x d\mu$ is λ-quasisymmetric. Conversely, if f is K-quasisymmetric, then the measure μ defined by $\mu([a,b]) = f(b) - f(a)$ for intervals $[a,b]$ is doubling with constant K. Clearly $\mu(E) = 0$ if and only if $|f(E)| = 0$. If a measure $\mu \in \mathscr{D}(\lambda)$ is given by $\mu(E) = \int_E w(x)\,dx$, where $w(x)$ is a locally integrable, non-negative, real-valued function on $\mathbf{R}$, we call $w(x)$ a *doubling weight function* for μ.

In Section 2 of the paper we prove Theorems 1.2 and 1.4. We also construct a set which satisfies the hypotheses of Theorem 1.4 but not the criterion of Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.5. Section 4 contains the K-dependent versions of Theorems 1.2, 1.4, and 1.9; an example linking these to Theorem 1.7; and the proof of Theorem 1.7.

It is a pleasure for the second author to thank her advisor, Peter Jones, for his patient guidance and encouragement, one small part of which was his suggestion of Theorem 1.2 above. Theorems 1.2 and 1.7 appeared in the second author's thesis [W].

2. Sufficient conditions for quasisymmetrically thick sets

We begin with a 'gap sum' sufficiency condition which gives a precise version of the following idea: a set formed by successively removing subintervals of the unit interval will be quasisymmetrically thick if the lengths of these removed 'gaps' decrease sufficiently fast. Note that the subintervals need not be removed in any prescribed pattern, and they need not be disjoint.

Theorem 1.2. *Let $E = [0,1] \setminus \bigcup_{m=1}^{\infty} I_m$, where the I_m are open subintervals of $[0,1]$ such that*

$$\sum_{m=1}^{\infty} |I_m|^p < \infty \qquad \text{for all} \quad p > 0, \tag{2.1}$$

and E has positive measure. Then E is quasisymmetrically thick.

Remarks. For example, (2.1) holds if $|I_m| = 2^{-m}$. If E contains an interval, the result is immediate. If, for instance, the midpoints of the I_m's are dense in $[0,1]$, then E contains no interval.

Proof. Let φ: $[0,1] \to [0,1]$ be K-quasisymmetric. We show that $|\varphi(E)| > 0$ by establishing a Hölder condition $|\varphi(E)| \geq c\,|E|^K$.

Order the I_m so that $|I_1| \geq |I_2| \geq \cdots$. Let $c_p = \sum_m |I_m|^p$ for each $p > 0$. Then $|I_m| \leq (c_p/m)^{1/p}$ for all m. Otherwise there is some k such that

$$\sum_{m=1}^{k} |I_m|^p \geq k\,\frac{c_p}{k} = \sum_{m=1}^{\infty} |I_m|^p; \tag{2.2}$$

therefore E is a finite union of intervals and $|\varphi(E)| > 0$.

Let N be a large integer, and let I be the largest subinterval of $F = [0,1] \setminus \bigcup_{m=1}^{N-1} I_m$. Since F has at most N components, and $F \supset E$, we have

$$|I| \geq \frac{|F|}{N} \geq \frac{|E|}{N}. \tag{2.3}$$

Let $\widetilde{\varphi} = g \circ \varphi \circ f$, where $f\colon [0,1] \to I$ and $g\colon \varphi(I) \to [0,1]$ are linear bijections. Then $\widetilde{\varphi}$ has the same quasisymmetry constant K as φ. Therefore $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ are Hölder continuous with exponent $1/K$. Let $\tilde{I}_m = f^{-1}(I_m \cap I)$. Then

$$|\tilde{I}_m| = \frac{|I_m \cap I|}{|I|} \leq \frac{|I_m|}{|I|} \leq \frac{N}{|E|}\Big(\frac{c_p}{m}\Big)^{1/p}, \tag{2.4}$$

and $|\tilde{I}_m| = 0$ for $1 \leq m < N$. Set $p = 1/(2K)$; then

$$\begin{aligned} \Big|\widetilde{\varphi}\Big(\bigcup_{m=0}^{\infty} \tilde{I}_m\Big)\Big| &\leq c \sum_{m=0}^{\infty} |\tilde{I}_m|^{1/K} \leq c \sum_{m=N}^{\infty} \left[\frac{N}{|E|}\Big(\frac{c_p}{m}\Big)^{1/p}\right]^{1/K} \\ &= c\,(|E|,K)\, N^{1/K} \sum_{m=N}^{\infty} \frac{1}{m^2} = c\,(|E|,K)\, N^{(1/K)-1}, \end{aligned} \tag{2.5}$$

which tends to zero as $N \to \infty$, since $K > 1$. Choose $N = N(K)$ large enough that $\big|\widetilde{\varphi}\big(\bigcup_{m=N}^{\infty} \tilde{I}_m\big)\big| \leq \frac{1}{2}$. Then $\big|\varphi\big(\bigcup_m I_m \cap I\big)\big| \leq \frac{1}{2}|\varphi(I)|$, and so

$$|\varphi(E)| \geq |\varphi(E \cap I)| \geq \frac{|\varphi(I)|}{2} \geq \frac{c\,|I|^K}{2} \geq \frac{c}{2}\Big(\frac{|E|}{N}\Big)^K > 0. \tag{2.6}$$

Therefore E is quasisymmetrically thick. □

Next we show that if the gaps in an $\{\alpha_n\}$-thick set are small enough, then the set is quasisymmetrically thick.

Theorem 1.4. *Let E be an $\{\alpha_n\}$-thick set with $\sum_1^\infty \alpha_n^p < \infty$ for all $p > 0$. Then E is quasisymmetrically thick.*

The proof of Theorem 1.4 relies on Lemma 1 of [Wu], which we rephrase here:

Lemma 2.1. *Let μ be any doubling measure on $[0,1]$ with doubling constant λ and let I be any subinterval of $[0,1]$. Then*

$$\frac{1}{(1+\lambda)^2}|I|^{\log_2(1+\lambda)}\mu([0,1]) \le \mu(I) \le 4|I|^{\log_2((1+\lambda)/\lambda)}\mu([0,1]). \tag{2.7}$$

Proof of Theorem 1.4. Assume that $E \subset [0,1]$, and let $\mathscr{E}_n = \bigcup_j E_{n,j}$ be the sequence of sets and $J_{n,j}$ the subintervals of $E_{n,j} \setminus E$ given in the $\{\alpha_n\}$-thick definition. Let μ be any doubling measure with constant λ, and set $p = \log_2((1+\lambda)/\lambda)$. Rescaling Lemma 2.1 and using condition (1.3), we have

$$\mu(J_{n,j}) \le 4\alpha_n^p \mu(E_{n,j}). \tag{2.8}$$

This gives

$$\mu(E_{n,j} \setminus J_{n,j}) \ge (1 - 4\alpha_n^p)\mu(E_{n,j}). \tag{2.9}$$

The right hand side of (2.9) is positive for all sufficiently large n, say $n \ge N_0$, because

$$\sum_1^\infty 4\alpha_n^p < \infty, \tag{2.10}$$

by hypothesis.

Using condition (1.4) and summing over j, we obtain

$$\sum_k \mu(E_{n+1,k}) \ge (1 - 4\alpha_n^p) \sum_j \mu(E_{n,j}). \tag{2.11}$$

From (2.9) it follows recursively that for $n \ge N_0$,

$$\mu\Big(\bigcup_j (E_{n,j} \setminus J_{n,j})\Big) \ge \Big(\prod_{m=N_0}^{n} (1 - 4\alpha_m^p)\Big) \sum_j \mu(E_{N_0,j}), \tag{2.12}$$

so that

$$\mu(E) \ge \Big(\prod_{n \ge N_0} (1 - 4\alpha_n^p)\Big) \sum_j \mu(E_{N_0,j}). \tag{2.13}$$

The convergence of the sum in (2.10) guarantees that this infinite product is positive, and we conclude that $\mu(E) > 0$. Since μ was an arbitrary doubling measure, it follows that E is quasisymmetrically thick. □

We give an example of a quasisymmetrically thick set which meets the criterion of Theorem 1.4, but does not satisfy the gap sum condition of Theorem 1.2.

Example 2.2. The set $E \subset [0,1]$ is the $\{\alpha_n\}$-regular, middle-interval Cantor set with $\{\alpha_n\} = \{2^{-n}\}$. With this choice of $\{\alpha_n\}$, $|E| > 0$.

The hypothesis of Theorem 1.4 holds, since $\sum_n \alpha_n^p = \sum_n 2^{-np} < \infty$ for all $p > 0$.

However, the gap sum for this particular Cantor set diverges when the exponent p is small. To compute the gap sum, note that we removed one interval of length α_1, two intervals of length $2^{-1}(1-\alpha_1)(\alpha_2)$, and similarly 2^n intervals of length $2^{-n}\big[\prod_1^n(1-\alpha_i)\big](\alpha_{n+1})$, for $n \geq 1$. Thus,

$$\sum_{m=1}^{\infty} |I_m|^p = \alpha_1^p + \sum_{n=1}^{\infty} 2^n \cdot 2^{-np}\left(\prod_{i=1}^{n}(1-\alpha_i)\right)^p \alpha_{n+1}^p \geq \sum_{n=1}^{\infty} 2^{n(1-p)}|E|^p \alpha_{n+1}^p, \tag{2.14}$$

since $\prod_{i=1}^n(1-\alpha_i)$ decreases to $|E|$ as n tends to infinity. For this specific example with $\{\alpha_n\} = \{2^{-n}\}$,

$$\sum_{m=1}^{\infty} |I_m|^p \geq |E|^p \sum_{n=1}^{\infty} 2^{n(1-p)} 2^{-(n+1)p}, \tag{2.15}$$

which diverges for $p \leq \frac{1}{2}$. □

3. Sharpness of Theorem 1.4

In this section we show that the condition in Theorem 1.4 for an $\{\alpha_n\}$-thick set to be quasisymmetrically thick is sharp.

Theorem 1.5. *If $\{\alpha_n\}$ is a decreasing sequence with $\alpha_n \to 0$, $0 < \alpha_n < 1$, and $\sum_1^\infty \alpha_n^p = \infty$ for some $p > 4\alpha_1$, then there exists a perfect set E which is $\{\alpha_n\}$-thick and satisfies $|f(E)| = 0$ for some quasisymmetric map f.*

Proof. We construct both the set E and the measure μ. The set E is a minor modification of the set constructed by Wu to prove [Wu, Theorem 2] above. For completeness, we include here the full description of E. The method of construction of E parallels the Cantor construction; the main difference is that after removing open intervals, we subdivide the closed intervals which remain into many subintervals $E_{n,j}$, and then remove open intervals from each of these $E_{n,j}$. The measure μ involves a class of periodic doubling weight functions $h_n(x)$; μ will be a weak limit of finite products $f_n(x)$ of dilations of these weights. Since we shall want both the α_n portion that is removed in the n^{th} level of construction of E and the portion which remains to represent an integral number of periods for the doubling weight $h_n(x)$, we slightly alter the original sequence $\{\alpha_n\}$.

Below we inductively define a sequence $\{N_n\}$ which determines the period M_n of the doubling weight function $f_n(x)$ by the formula $M_n = \prod_1^n N_j$. For now it suffices to say that $\{N_n\}$ is a rapidly increasing sequence of odd integers such that $N_1 = 1$ and $N_{n+1} \geq 3(\alpha_n/2)^{-1} = 6\alpha_n^{-1}$ for $n \geq 1$. A simple arithmetic argument shows that we may replace $\frac{1}{2}\alpha_n$ by a number β_n such that

(i) $\frac{1}{2}\alpha_n \leq \beta_n \leq \alpha_n$; and

(ii) $\beta_n = m_n/N_{n+1}$, where m_n is an odd integer.

We now construct E. First remove the open middle interval of length $\beta_1 = m_1/N_2$ from $[0,1]$, leaving two intervals each of length $(N_2 - m_1)/2N_2$. This number is an integer multiple of $(N_2)^{-1}$, since both N_2 and m_1 are odd. Therefore we may subdivide these two intervals into a total of $N_2 - m_1$ subintervals of equal length N_2^{-1}. Let P_1 denote this collection of subintervals.

Continue this process inductively. At the n^{th} stage, P_{n-1} consists of $(N_2 - m_1)\cdots(N_n - m_{n-1})$ intervals I, each of length $(N_2\cdots N_n)^{-1}$. From each such I, remove the open middle interval of length $\beta_n|I| = (m_n/N_{n+1})|I|$. The resulting closed intervals each have length

$$\frac{1-\beta_n}{2}|I| = \frac{N_{n+1}-m_n}{2N_{n+1}}\frac{1}{N_2\cdots N_n}, \tag{3.1}$$

which is an integer multiple of $(N_2\cdots N_{n+1})^{-1}$. Therefore we may subdivide these intervals evenly into subintervals of length $(N_2\cdots N_{n+1})^{-1}$. Let P_n denote the collection of these subintervals. Define

$$E = \bigcap_n \bigcup_{I\in P_n} I. \tag{3.2}$$

This set E is $\{\alpha_n\}$-thick; the intervals in P_{n-1} serve as the $E_{n,j}$. The sequences $\{N_n\}$ and $\{\beta_n\}$ will be prescribed below.

Next we construct a doubling weight h which will be the main tool in the construction of the doubling measure μ. The map h is a modification of the function in [Wu, Lemma 2] which in turn is based on the example of Beurling and Ahlfors [BA, Theorem 3].

Lemma 3.1. *Given $0<\alpha<\frac14 p$ and $0<p<\frac17$, there exists a function $h(x)$ such that $h(x)$ is continuous on $\mathbf{R}$ with period 1;*

$$\int_0^1 h(x)\,dx = 1; \tag{3.3}$$

$$h(x) = \tfrac14\alpha^{p-1} \quad \text{on} \quad \left[\tfrac12(1-\alpha), \tfrac12(1+\alpha)\right]; \tag{3.4}$$

and $h(x)$ is a doubling weight with constant $\lambda=\lambda(p)$ which depends only on p.

Proof. We define a suitable function on $[0,1]$. Let h be a symmetric function, $h(x) = h(1-x)$, defined piecewise on five subintervals of $\left[0,\frac12\right]$: constant on the first interval as dictated by (3.4), linear and decreasing by a factor of $p^{3/2}$ on the next interval, a shifted power function with power $p-1$ on the middle interval, linear and increasing on the fourth interval and constant on the last interval. The constant d on the fifth interval is chosen last to make h have mean value one. It turns out that d should be

$$d = (4 - 2\alpha^p - p^{3/2}\alpha^p - 2p^{p+1/2} + 2p^{1/2}\alpha^p - p^{p+3/2})/(4 - 4\alpha - 12p).$$

Note that the quantity d is uniformly bounded away from 0 and ∞ for $0<p<\frac{1}{7}$ and $0<\alpha<\frac{1}{4}p$; in particular $\frac{1}{4}<d<\frac{5}{2}$. The salient features of h are that the first two intervals where h is comparable to α^{p-1} are of size α and that h is comparable to α^{p-1} for intervals of size comparable to α on the left hand side of the middle interval, but bounded independently of α at the right hand endpoint of the middle interval. Our function h is

(1) $h(x)=\frac{1}{4}\alpha^{p-1}$ on $I_1=\left[\frac{1}{2}(1-\alpha),\frac{1}{2}(1+\alpha)\right]$;

(2) $h(x)$ is linear on $I_2=\left[\frac{1}{2}(1+\alpha),\frac{1}{2}(1+3\alpha)\right]$, with $h\left(\frac{1}{2}(1+\alpha)\right)=\frac{1}{4}\alpha^{p-1}$ and $h\left(\frac{1}{2}(1+3\alpha)\right)=\frac{1}{4}p^{3/2}\alpha^{p-1}$;

(3) $h(x)=\frac{1}{4}\left(p^{3/2}\left(\left(x-\frac{1}{2}(1+\alpha)\right)^{p-1}\right)\right)$ on $I_3=\left[\frac{1}{2}(1+3\alpha),\frac{1}{2}(1+\alpha)+p\right]$;

(4) $h(x)$ is linear on $I_4=\left[\frac{1}{2}(1+\alpha)+p,\frac{1}{2}(1+\alpha)+2p\right]$, with $h\left(\frac{1}{2}(1+\alpha)+p\right)=\frac{1}{4}p^{p+1/2}$ and $h\left(\frac{1}{2}(1+\alpha)+2p\right)=d$;

(5) $h(x)=d$ on $I_5=\left[\frac{1}{2}(1+\alpha)+2p,1\right]$.

We verify the desired properties of $h(x)$. The continuity is evident from the definition of $h(x)$. Property (3.3) holds by our choice of d.

To confirm that $h(x)$ is indeed a doubling weight whose constant λ depends only on p, one compares $\int_I h(x)\,dx$ to $\int_J h(x)\,dx$ for adjacent intervals I and J of equal length. If $I=[a,b]$ the notation $I+c=[a+c,b+c]$ denotes the translate of I by c. The lengths of the intervals used in the definition of $h(x)$ satisfy $|I_1|=|I_2|=\alpha$, $|I_3|=p-\alpha$, $|I_4|=p$, and $|I_5|=\frac{1}{2}(1-\alpha)-2p$. Note that $|I_3|=p-\alpha>3\alpha$, $\frac{3}{4}|I_4|\le|I_3|<|I_4|$, and $|I_5|>|I_4|$, since $0<\alpha<\frac{1}{4}p$ and $0<p<\frac{1}{7}$. For the intervals of critical concern, we check that the ratios of the corresponding integrals can be uniformly bounded away from 0 and ∞ independent of α. For intervals of length $|I|>1$, this follows from the periodicity of h. When $\alpha\le|I|\le 1$, the bounds follow from the following computations over key intervals:

$$\frac{1+p^{3/2}}{2}=\frac{\int_{I_2}h(x)\,dx}{\int_{I_1}h(x)\,dx}\le 1\,, \tag{3.5}$$

$$\frac{2p^{1/2}(2^p-1)}{1+p^{3/2}}=\frac{\int_{I_2+\alpha}h(x)\,dx}{\int_{I_2}h(x)\,dx}\le 1\,, \tag{3.6}$$

$$\frac{p^{1/2}(4^p-1)}{2+p^{3/2}}=\frac{\int_{(I_2+\alpha)\cup(I_2+2\alpha)\cup(I_2+3\alpha)}h(x)\,dx}{\int_{I_1\cup I_2\cup(I_1-\alpha)}h(x)\,dx}\le 1, \tag{3.7}$$

$$\frac{p(p^{p+1/2}+4d)}{2\left((\frac{1}{4}p)^p+p^{p+1/2}\right)}\le\frac{\int_{I_4}h(x)\,dx}{\int_{I_2\cup I_3}h(x)\,dx}\le\frac{p(p^{p+1/2}+4d)}{2p^{p+1/2}},\qquad\text{and} \tag{3.8}$$

$$\frac{2d}{\frac{1}{4}p^{p+1/2}+d}=\frac{\int_{I_4+p}h(x)\,dx}{\int_{I_4}h(x)\,dx}. \tag{3.9}$$

For smaller intervals of size $|I|<\alpha$, the computations of interest are for adjacent intervals on the border of I_2 and I_3, and for adjacent intervals at the left hand end of I_3. If we set $|I| = c\alpha$, $0 < c < 1$, the calculations can be checked more easily and similar bounds independent of α can be established. For small intervals on the border of I_3 and I_4 the uniform boundedness of d controls the ratio of the integrals. This completes the proof of Lemma 3.1. □

We continue the proof of Theorem 1.5; it remains to construct the measure μ. We are given a decreasing sequence $\alpha_n \to 0$ with $0 < \alpha_n < p/4$, satisfying $\sum \alpha_n^p = \infty$ for some $p > 0$. We may assume that $p < \frac{1}{7}$, for if the sum diverges for an exponent p_0, it diverges for $p < p_0$ as well. If by choosing $p < \frac{1}{7}$ we no longer have $\alpha_n < p/4$ for some range $1 \le n \le N_0$, but know that $\alpha_n < p/4$ for $n > N_0$, note that the proof below can be modified by taking $h_n(x) = x$ for $1 \le n \le N_0$ and by appropriately shifting the starting indices in both the induction and the infinite product used to generate the upper bound for $\mu(E)$.

For each pair $(\frac{1}{2}\alpha_n, p)$, fix a function $h_n(x)$ which satisfies the criteria of Lemma 3.1 with $\alpha = \frac{1}{2}\alpha_n$. We define

$$M_n = \prod_{j=1}^{n} N_j \quad \text{and } f_n(x) = \prod_{j=1}^{n} h_j(M_j x), \tag{3.10}$$

$$A_n = \bigcup_{i=-\infty}^{\infty} \bigl[i, i + \bigl(\tfrac{1}{4}(2-\alpha_n)\bigr)\bigr] \cup \bigl[i + \bigl(\tfrac{1}{4}(2+\alpha_n)\bigr), i+1\bigr], \tag{3.11}$$

$$B_n = \bigcup_{i=-\infty}^{\infty} \bigl[i, i + \bigl(\tfrac{1}{2}(1-\beta_n)\bigr)\bigr] \cup \bigl[i + \bigl(\tfrac{1}{2}(1+\beta_n)\bigr), i+1\bigr], \tag{3.12}$$

$$F_{n+1} = \{x \in F_n : M_{n+1}x \in A_{n+1}\}, \qquad F_0 = [0,1], \tag{3.13}$$

$$G_{n+1} = \{x \in G_n : M_{n+1}x \in B_{n+1}\} = \bigcup_{I \in P_{n+1}} I, \qquad G_0 = [0,1]. \tag{3.14}$$

Note that $E = \bigcap_n G_n$ and $G_n \subset F_n$. We now explain how to prescribe $\{N_n\}$ so that the measure μ defined as a weak limit point of $\{f_n\}$ satisfies $\mu(E) = 0$. Choose N_{n+1} inductively according to the following scheme. (The induction process here and in [Wu] must be done more carefully than as originally stated in [Wu]. For example, (1.1) in [Wu] (the condition corresponding to (3.16) here) is dependent on knowing the "modified α_n" (β_n in our notation) which is defined in terms of N_{n+1}. Condition (1.1) is also dependent on knowing what we call G_n in our notation; this already requires that there are an integral number of intervals in P_n of size $(M_{n+1})^{-1}$. However, the induction proceeds by finding N_{n+1}. The problem can be avoided by defining and working with the four sets A_n, B_n, F_n, and G_n.)

Set $N_1 = 1$. For the inductive step, assume that odd integers $N_2, N_3, \dots, N_n$ have been chosen such that

$$N_j > 6(\alpha_{j-1})^{-1} \qquad \text{for} \quad 2 \le j \le n, \tag{3.15}$$

$$\int_{F_j} f_j(x)\,dx \le \prod_{i=1}^{j} \Bigl(1 - \frac{\alpha_i^p}{8(2^p)}\Bigr) \qquad \text{for} \quad 1 \le j \le n, \tag{3.16}$$

and whenever $|x - x'| \le M_j^{-1}$ for $2 \le j \le n$, we have both

$$\frac{j-1}{j} < \frac{h_{j-1}(M_{j-1}x)}{h_{j-1}(M_{j-1}x')} < \frac{j+1}{j} \qquad \text{and} \tag{3.17}$$

$$\frac{j-1}{j} < \frac{f_{j-1}(x)}{f_{j-1}(x')} < \frac{j+1}{j}. \tag{3.18}$$

The goal is to select an odd integer $N_{n+1} > N_n$ so that the conditions (3.15)–(3.18) also hold for the index $j = n+1$. Clearly condition (3.15) can be satisfied by taking an odd integer N_{n+1} large enough. By the uniform continuity of $h_n(x)$ and $f_n(x)$, conditions (3.17) and (3.18) will hold for the index $j = n+1$ for sufficiently large N_{n+1} whenever $|x - x'| < (M_{n+1})^{-1}$.

It remains to show that (3.16) for $j = n+1$ will also be satisfied if N_{n+1} is sufficiently large. We approximate by simple functions. Since F_n is measurable, $\chi_{F_n} f_n$ can be written as the pointwise a.e. and L^1-limit of a decreasing sequence of simple functions of the form $g_k = \sum_{j=1}^{n_k} a_j \chi_{I_j}$, where the a_j are constants and the intervals $\{I_j\} = \{[c_j, d_j]\}$ are finitely many disjoint open intervals with rational endpoints.

Since

$$\int_{F_{n+1}} f_{n+1}(x)\,dx = \int_0^1 \chi_{F_n}(x) f_n(x) \chi_{A_{n+1}}(M_{n+1}x) h_{n+1}(M_{n+1}x)\,dx, \tag{3.19}$$

we consider the limit as $M \to \infty$ of

$$\int_0^1 \chi_{F_n}(x) f_n(x) \chi_{A_{n+1}}(Mx) h_{n+1}(Mx)\,dx. \tag{3.20}$$

Replacing $\chi_{F_n} f_n$ with a fixed step function g_k, this becomes

$$\sum_{j=1}^{n_k} a_j \int_{I_j} \chi_{A_{n+1}}(Mx) h_{n+1}(Mx)\,dx. \tag{3.21}$$

To simplify this we make the following observations and computations. For a large enough M each of the rational numbers c_j, d_j for $1 \le j \le n_k$ can be expressed as an integer multiple of $1/M$. For such an M each I_j can be decomposed evenly into an integral number of periods of the function $\chi_{A_{n+1}}(Mx) h_{n+1}(Mx)$. Then,

$$\begin{aligned} \int_{I_j} \chi_{A_{n+1}}(Mx) h_{n+1}(Mx)\,dx &= M|I_j| \int_0^{1/M} \chi_{A_{n+1}}(Mx) h_{n+1}(Mx)\,dx \\ &= |I_j| \int_0^1 \chi_{A_{n+1}}(x) h_{n+1}(x)\,dx \\ &= |I_j| \int_{[0,1]\cap A_{n+1}} h_{n+1}(x)\,dx = |I_j| \Big(1 - \frac{\alpha_{n+1}^p}{4(2^p)}\Big), \end{aligned} \tag{3.22}$$

by (3.4). For the step function g_k and large enough M we thus have

$$\int_0^1 g_k(x)\chi_{A_{n+1}}(Mx)h_{n+1}(Mx)\,dx = \Big(1-\frac{\alpha_{n+1}^p}{4(2^p)}\Big)\sum a_j|I_j|. \tag{3.23}$$

It follows that

$$\lim_{M\to\infty}\int_0^1 \chi_{F_n}(x)f_n(x)\chi_{A_{n+1}}(Mx)h_{n+1}(Mx)\,dx = \Big(1-\frac{\alpha_{n+1}^p}{4(2^p)}\Big)\int_{F_n} f_n(x)\,dx. \tag{3.24}$$

Thus (3.16) can be satisfied by choosing N_{n+1} sufficiently large.

With the induction now complete, let μ be a weak limit point of $\{f_n\}$. We verify that μ is a doubling measure, and then show that $\mu(E)=0$. Let I and I' be any two neighboring intervals satisfying $|I|=|I'|$. We seek upper and lower bounds for the ratio $\big(\int_{I'} f_m(x)\,dx\big)/\big(\int_I f_m(x)\,dx\big)$ as $m\to\infty$. Since the period of $f_m(x)$ is 1 for all m, we may assume that $|I|\le 1$. There exists an n such that $(M_{n+1})^{-1}\le |I|\le M_n^{-1}$. Write $f_m(x)$ as

$$f_m(x) = f_{n-1}(x)h_n(M_nx)\frac{f_m(x)}{f_n(x)}, \tag{3.25}$$

where $m\ge n+1$. For $m\ge n+1$, the function $f_m(x)/f_n(x)$ has period $P=(M_{n+1})^{-1}$. Thus the interval I contains at least one full period of $f_m(x)/f_n(x)$. Write $I=[x_1,x_1+aP+\varepsilon]$ and $I'=[x_1+aP+\varepsilon, x_1+2aP+2\varepsilon]$, where $a\ge 1$ is an integer and $0\le\varepsilon<P$. Then,

$$\frac{\int_{I'} f_m(x)\,dx}{\int_I f_m(x)\,dx} \le \frac{\int_{x_1+aP}^{x_1+4aP} f_{n-1}(x)h_n(M_nx)\big(f_m(x)/f_n(x)\big)\,dx}{\int_{x_1}^{x_1+aP} f_{n-1}(x)h_n(M_nx)\big(f_m(x)/f_n(x)\big)\,dx}. \tag{3.26}$$

On the entire interval $[x_1,x_1+4aP]$, condition (3.18) guarantees that

$$f_{n-1}(x_1)\Big(\frac{n-1}{n}\Big)^4 \le f_{n-1}(x)\le f_{n-1}(x_1)\Big(\frac{n+1}{n}\Big)^4, \tag{3.27}$$

since $4aP\le 4M_n^{-1}$.

Consider the denominator of (3.26):

$$\int_{x_1}^{x_1+aP} f_m(x)\,dx \ge f_{n-1}(x_1)\Big(\frac{n-1}{n}\Big)^4\Big(\int_{x_1}^{x_1+aP} h_n(M_nx)\frac{f_m(x)}{f_n(x)}\,dx\Big). \tag{3.28}$$

On intervals of length $P=(M_{n+1})^{-1}$, the function $h_n(M_nx)$ is essentially constant by condition (3.17). Therefore, dividing $[x_1,x_1+aP]$ into subintervals of length P, and letting $A=\int_{x_1}^{x_1+P}(f_m(x)/f_n(x))\,dx$, we have

$$\int_{x_1}^{x_1+P} h_n(M_nx)\frac{f_m(x)}{f_n(x)}\,dx \ge h_n(M_nx_1)\Big(\frac{n-1}{n}\Big)A, \tag{3.29}$$

and so on up to

$$\int_{x_1+(a-1)P}^{x_1+aP} h_n(M_n x)\frac{f_m(x)}{f_n(x)}\,dx \geq h_n\big(M_n(x_1+(a-1)P)\big)\Big(\frac{n-1}{n}\Big)A. \tag{3.30}$$

Using (3.17) again, it follows that

$$\int_{x_1}^{x_1+aP} f_m(x)\,dx \geq c_1 f_{n-1}(x_1)AP^{-1}\int_{x_1}^{x_1+aP} h_n(M_n x)\,dx, \quad c_1 = c_1(n). \tag{3.31}$$

Similarly, we can bound the numerator of (3.26) by

$$\int_{x_1+aP}^{x_1+4aP} f_m(x)\,dx \leq c_2 f_{n-1}(x_1)AP^{-1}\int_{x_1+aP}^{x_1+4aP} h_n(M_n x)\,dx, \quad c_2 = c_2(n). \tag{3.32}$$

Since $h_n(M_n x)$ is a doubling weight with constant $\lambda = \lambda(p)$, we have the bound

$$\frac{\int_{I'} f_m(x)\,dx}{\int_I f_m(x)\,dx} \leq \frac{c_2}{c_1}(\lambda+\lambda^2+\lambda^3). \tag{3.33}$$

In the same way, we can find an upper bound for $(\int_I f_m(x)\,dx)/(\int_{I'} f_m(x)\,dx)$. We conclude that μ is a doubling measure.

To see that $\mu(E)=0$, note first that by the hypothesis $\sum_i\big(\alpha_i^p/8(2^p)\big)=\infty$. Combining this with the bound $\mu(E)\leq\prod_{i=1}^{\infty}\big(1-\big(\alpha_i^p/8(2^p)\big)\big)$, which follows from (3.16), gives the desired result. □

4. K-dependent results, and regular middle-interval Cantor sets

We make explicit the dependence of our sufficient conditions on the constant K of quasisymmetry. First we relate the exponent p in the gap sum of Theorem 1.2 to K; this corollary follows immediately from the proof of the theorem.

Corollary 4.1. *Let $E=[0,1]\setminus\bigcup_m I_m$, where the I_m are open subintervals of $[0,1]$ such that $\sum_m |I_m|^{1/(1+K)}<\infty$, and E has positive measure. Then the measure $|f(E)|>0$ for all K'-quasisymmetric mappings f such that $K'<K$.*

The exponent p in the hypotheses of Theorems 1.4 and 1.9 can be linked to K as well. We can conclude that certain $\{\alpha_n\}$-porous sets of measure zero cannot be mapped K-quasisymmetrically onto sets of positive measure and certain $\{\alpha_n\}$-thick sets of positive measure cannot be mapped K-quasisymmetrically onto sets of zero measure. The next two results are direct consequences of Lemma 2.1 and the ideas of Theorem 1.4.

Corollary 4.2. *Let E be an $\{\alpha_n\}$-thick set for which $\sum \alpha_n^{\log_2(1+K/K)}<\infty$. Then the measure $|f(E)|>0$ for all K-quasisymmetric mappings f.*

Corollary 4.3. *Let E be an $\{\alpha_n\}$-porous set for which $\sum \alpha_n^{\log_2(1+K)} = \infty$. Then the measure $|f(E)| = 0$ for all K-quasisymmetric mappings f.*

If E is an $\{\alpha_n\}$-Cantor set, that is $|J_{n,j}| = \alpha_n |E_{n,j}|$ for all n and j, then in the case $K = 1$, such as with the identity map, Corollaries 4.2 and 4.3 together give the standard result that $|E| = 0$ if and only if $\sum \alpha_n = \infty$.

We illustrate our results in relation to a natural example.

Example 4.4. Consider the $\{(n+1)^{-2}\}$-regular middle-interval Cantor set E. Is this set quasisymmetrically thick? E is $\{(n+1)^{-2}\}$-thick, but the summability condition of Theorem 1.4 does not hold. Nor does the gap sum condition of Theorem 1.2. For this set E, Corollary 4.2 asserts that $|f(E)| > 0$ for all K-quasisymmetric functions f such that $1 \le K < 1 + \sqrt{2}$, but says nothing for larger values of K.

Theorem 1.5 asserts that there is some perfect $\{(n+1)^{-2}\}$-thick set which is not quasisymmetrically thick. However, in contrast to E, the example constructed in our proof of Theorem 1.5 is not a regular middle-interval Cantor set, so this does not imply anything about E.

Corollary 1.8, which is a special case of Theorem 1.7 below, implies that E cannot be mapped quasisymmetrically to any regular Cantor set of zero measure. However, we still do not know whether E is quasisymmetrically thick. □

Given an $\{\alpha_n\}$-regular middle-interval Cantor set E, the next theorem gives a condition on a quasisymmetric map f which implies that the image $f(E)$ also has positive measure. In contrast to the results discussed above, this condition does not depend on the quasisymmetry constant K of f.

Theorem 1.7. *Let E be a regular middle-interval Cantor set of positive measure in $[0,1]$. Then E cannot be mapped to a set of measure zero by any quasisymmetric map $f\colon [0,1] \to [0,1]$ which satisfies the condition $|f(J_{n,j})| \overset{A}{\sim} |f(J_{n,l})|$ for all n, j, and l with $1 \le j$, $l \le 2^{n-1}$, for a constant A independent of n, j, and l.*

Proof. Let $E = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} E_{n,j}$ be an $\{\alpha_n\}$-regular middle-interval Cantor set as described in Section 1, with $\{\alpha_n\}$ such that the measure $|E| = \prod_{n=1}^{\infty}(1-\alpha_n)$ of E is positive. We refer to the closed intervals $E_{n,j}$ which appear at the n^{th} stage of the construction of E as *closed construction intervals* of E, and to the open middle intervals $J_{n,j}$ which are removed from the $E_{n,j}$ as *open construction intervals* of E. Define

$$D_n = |E_{n,j}|, \qquad L_n = |J_{n,j}| = \alpha_n D_n, \qquad \text{for} \quad 1 \le j \le 2^{n-1}. \tag{4.1}$$

In particular,

$$D_n = \tfrac{1}{2}(1-\alpha_{n-1})D_{n-1} \ge \tfrac{1}{2}|E| D_{n-1}, \tag{4.2}$$

and D_n is comparable to 2^{-n}:

$$\frac{|E|}{2^{n-1}} \leq D_n \leq \frac{1}{2^{n-1}}. \tag{4.3}$$

For instance, if $\alpha_n = 1/(n+1)^2$, then

$$D_n = \frac{1}{2^n}\frac{n+1}{n}, \quad L_n = \frac{1}{2^n}\frac{1}{n(n+1)}, \quad \text{and} \quad |E| = \frac{1}{2}.$$

Suppose there is a K-quasisymmetric increasing homeomorphism $f\colon [0,1] \to [0,1]$ such that $|f(E)| = 0$. Suppose further that the images under f of the gaps at any given level of E are all of equal length:

$$|f(J_{n,j})| = |f(J_{n,l})| \qquad \text{for all} \quad n,\ 1 \leq j,\ l \leq 2^{n-1}. \tag{4.4}$$

This means that $f(E)$ is also a regular Cantor set. (At the end of the proof we consider the more general case where the gaps at any given level of $f(E)$ are all of comparable but not necessarily equal length, with a constant A which is uniform for all levels.)

We show that the quasisymmetry of f at the endpoints of the construction intervals of E implies that f cannot be a homeomorphism.

Let x be the left endpoint of $J_{1,1}$. Take n large enough that $D_n < L_1$. Since $|f(E)| = 0$, we have $|f(B)| = \sum_{n,j} |f(B \cap J_{n,j})|$ for any measurable set B in $[0,1]$. Let $E_{n,j}$ denote the closed construction interval $[x - D_n, x]$. Counting the open construction intervals in $E_{n,j}$, and using (4.4), we see that

$$\bigl|f\bigl((x - D_n, x)\bigr)\bigr| = \sum_{m=n}^{\infty} 2^{m-n} |f(J_{m,k_m})|, \tag{4.5}$$

where J_{m,k_m} is the first m^{th} level open construction interval to the left of x. Fix $m \geq n$, and let y be the left endpoint of J_{m,k_m}. Let n_m be the unique integer such that

$$D_{n_m+1} < L_m \leq D_{n_m}. \tag{4.6}$$

Then

$$\begin{aligned} |f(J_{m,k_m})| &= \bigl|f\bigl((y, y + L_m)\bigr)\bigr| \leq \bigl|f\bigl((y, y + D_{n_m})\bigr)\bigr| \\ &\leq K\bigl|f\bigl((y - D_{n_m}, y)\bigr)\bigr| = K\bigl|f\bigl((x - D_{n_m}, x)\bigr)\bigr|, \end{aligned} \tag{4.7}$$

since $[y - D_{n_m}, y]$ and $[x - D_{n_m}, x]$ are both closed construction intervals at level n_m, and so they contain the same numbers and sizes of open construction intervals.

Therefore, by (4.5), (4.7), and the quasisymmetry of f at x,

$$\begin{aligned}\big|f\big((x,x+D_n)\big)\big| &\le K\big|f\big((x-D_n,x)\big)\big| = K\sum_{m=n}^{\infty}2^{m-n}|f(J_{m,k_m})|\\ &\le K^2\sum_{m=n}^{\infty}2^{m-n}\big|f\big((x-D_{n_m},x)\big)\big|\\ &\le K^3\sum_{m=n}^{\infty}2^{m-n}\big|f\big((x,x+D_{n_m})\big)\big|.\end{aligned} \tag{4.8}$$

Let $h(u)=\big(f(x+u)-f(x)\big)/u$ for $0<u<L_1$. Then (4.8) becomes

$$D_n h(D_n)\le K^3\sum_{m=n}^{\infty}2^{m-n}D_{n_m}h(D_{n_m}). \tag{4.9}$$

Now by (4.2), (4.3), and the definitions of n_m and L_m,

$$\begin{aligned}h(D_n) &\le \frac{K^3}{2^{n-1}D_n}\sum_{m=n}^{\infty}2^{m-1}D_{n_m}\,h(D_{n_m})\\ &\le \frac{K^3}{|E|}\sum_{m=n}^{\infty}2^{m-1}\frac{2}{|E|}D_{n_m+1}h(D_{n_m})\le\frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}2^{m-1}L_m h(D_{n_m})\\ &= \frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}2^{m-1}\alpha_m D_m h(D_{n_m})\le\frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}\alpha_m h(D_{n_m}).\end{aligned} \tag{4.10}$$

A similar calculation shows that

$$h(D_n)\ge\frac{|E|^2}{2K^3}\sum_{m=n}^{\infty}\alpha_m\,h(D_{n_m+1}). \tag{4.11}$$

The quasisymmetry of f, and (4.2), imply that $h(D_{n_m+1})\ge ch(D_{n_m})$, where c is a constant depending on $|E|$ and K. So (4.11) implies that

$$h(D_n)\ge\frac{c\,|E|^2}{2K^3}\sum_{m=n}^{\infty}\alpha_m h(D_{n_m}). \tag{4.12}$$

In particular the sum on the right hand side of (4.12) is finite. Then by (4.10) the $h(D_n)$ are bounded independently of n. Write $\|h\|_\infty=\sup_m h(D_{n_m})$. Now iterate (4.10):

$$\begin{aligned}h(D_n) &\le \frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}\alpha_m h(D_{n_m})\\ &\le \frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}\alpha_m\left[\frac{2K^3}{|E|^2}\sum_{r=n_m}^{\infty}\alpha_r h(D_{n_r})\right]\le\|h\|_\infty\left[\frac{2K^3}{|E|^2}\sum_{m=n}^{\infty}\alpha_m\right]^2,\end{aligned} \tag{4.13}$$

since $n_m \geq m \geq n$. For each $k = 2, 3, \dots$, we obtain

$$h(D_n) \leq \|h\|_\infty \left[\frac{2K^3}{|E|^2} \sum_{m=n}^{\infty} \alpha_m \right]^k. \tag{4.14}$$

Choose n large enough that $(2K^3/|E|^2) \sum_{m=n}^{\infty} \alpha_m < 1$. Then the right hand side of (4.14) goes to zero as $k \to \infty$, so $h(D_n) = 0$ for all sufficiently large n. Hence f is not one-to-one. Therefore f is not a homeomorphism, contradicting our assumption.

Finally, suppose that the gaps at any given level of $f(E)$ are all of comparable but not necessarily equal length, with a constant A which is uniform for all levels. Then (4.4) is replaced by

$$|f(J_{n,j})| \overset{A}{\sim} |f(J_{n,l})|, \qquad \text{for all } n,\ 1 \leq j,\ l \leq 2^{n-1}, \tag{4.15}$$

where A is independent of n, j, and l. Earlier, we used (4.4) to establish (4.5) and (4.7); now (4.15) gives their analogues

$$\bigl|f\bigl((x - D_n, x)\bigr)\bigr| \overset{A}{\sim} \sum_{m=n}^{\infty} 2^{m-n} |f(J_{m,k_m})| \tag{4.16}$$

and

$$|f(J_m, k_m)| \leq KA \bigl|f\bigl((x - D_{n_m}, x)\bigr)\bigr|, \tag{4.17}$$

and the result follows as before. □

References

[BA] Beurling, A., and L. Ahlfors: The boundary correspondence under quasiconformal mappings. - Acta. Math. 96, 1956, 125–142.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993.

[M] Martio, O.: Harmonic measures for second order nonlinear partial differential equations. - In: Function Spaces, Differential Operators and Nonlinear Analysis, edited by L. Päivärinta, Longman Scientific and Technical Publishers, 1989.

[W] Ward, L.: Fuchsian groups, quasiconformal groups, and conical limit sets. - Ph.D. Thesis, Yale University, 1994.

[Wu] Wu, J.-M.: Null sets for doubling and dyadic doubling measures. - Ann. Acad. Sci. Fenn. Ser. A I Math. 18, 1993, 77–91.

Received 13 August 1996