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# ON THE NONCOMPACTNESS OF DAVID CLASSES

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**Abstract.** Certain classes of plane homeomorphisms with general measure-theoretic restrictions on the local dilatation are considered. The noncompactness of such classes is proved. In particular, the noncompactness of all David's classes with exponential bounding functions is obtained from this. It is also established that the closure of such a class cannot be a class of the same type.

## 1. Definitions and preliminary considerations

In view of the well-known Gehring–Lehto theorem (see e.g. [LV, pp. 10, 134]), an arbitrary sense-preserving homeomorphism of the complex plane $f\colon \mathbf{C} \to \mathbf{C}$, having the first partial derivatives almost everywhere, satisfies the Beltrami equation

$$f_{\bar z} = \mu(z) f_z \qquad \text{a.e.,} \tag{1}$$

where $\mu\colon \mathbf{C} \to \mathbf{C}$ is a measurable function with $|\mu(z)| \le 1$ and, as usual, $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$, $f_z = \frac{1}{2}(f_x - i f_y)$, $z = x + iy$. If $f_z = f_{\bar z} = 0$, we set $\mu(z) = 0$. The functions $\mu(z)$ and

$$p(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

are called the complex dilatation and, simply, the dilatation of the mapping $f$ at the point $z$, respectively.

In particular, if $f \in W^1_{1,\mathrm{loc}}$ and $p(z) \le Q$ a.e. for some $Q \in [1, \infty)$, the homeomorphism $f$ is called a $Q$-quasiconformal mapping (cf. e.g. [LV, p. 176], [A2, p. 24, 33]). In what follows, $\mathfrak{F}_Q$ denotes the class of all $Q$-quasiconformal self-mappings of the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ normalized in the following way:

$$f(0) = 0, \qquad f(1) = 1, \qquad f(\infty) = \infty. \tag{3}$$

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As is well known (see [LV, p. 76]), the class $\mathfrak{F}_Q$ is sequentially compact with respect to the locally uniform convergence, i.e. firstly, every sequence $(f_n) \subset \mathfrak{F}_Q$ has a subsequence converging locally uniformly to some mapping $f_0$ and, secondly, $f_0 \in \mathfrak{F}_Q$. It is customary to call the first property the precompactness and the second the completeness of a class.

Various classes of mappings with unbounded dilatations have been studied by many authors. The recent paper [D] by Guy David is of a special interest. There he proved a new theorem on the existence and uniqueness of homeomorphic solutions $f \in W^1_{1,\mathrm{loc}}$ normalized by (3) for the Beltrami equation (1) with the coefficient $\mu(z)$ satisfying the following restrictions:

$$\text{(4)} \qquad \operatorname{meas}\{z \in \mathbf{C} : |\mu(z)| > 1 - \varepsilon\} \le C_0 e^{-\alpha/\varepsilon}$$

for all $\varepsilon \le \varepsilon_0$ and for some fixed $\varepsilon_0 \in (0,1]$, $\alpha > 0$, $C_0 > 0$.

A study of compactness properties for David homeomorphisms was begun by P. Tukia [T], who also wrote the condition (4) in the following equivalent and more convenient form:

$$\text{(5)} \qquad \operatorname{meas}\{z \in \mathbf{C} : p(z) > t\} \le c e^{-\gamma t}$$

for all $t \ge T$, where $T \ge 1$, $c > 0$, $\gamma > 0$.

G. David [D] established the locally uniform boundedness, the local equicontinuity and openness of such a class of homeomorphisms. Thus, in view of the Arzela–Ascoli theorem (see e.g. [DS]), the above class of homeomorphisms is precompact in the space of all plane homeomorphisms.

For precompact classes, it is clear that sequential compactness is equivalent to completeness. In this respect, P. Tukia [T] succeeded only in establishing that all the locally uniform limit functions of David homeomorphisms with the restriction (5) are also David homeomorphisms but, in general, with other constants $T$, $\gamma$ and $c$. In this connection, questions have also arisen on sequential compactness and completeness for the classes with measure-theoretic restrictions of the general form

$$\text{(6)} \qquad \operatorname{meas}\{z \in \mathbf{C} : p(z) > t\} \le \varphi(t),$$

where $\varphi\colon [1,\infty) \to [0,\infty]$ is an arbitrary function.

In what follows, $H(\varphi)$ denotes the class of all sense-preserving homeomorphisms $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ in $W^1_{1,\mathrm{loc}}$ normalized by (3) and satisfying the restriction (6) for the dilatation.

G. David [D] has given examples showing that, for power functions $\varphi$, the classes $H(\varphi)$ may not be equicontinuous and, consequently, not sequentially compact nor precompact.

Our main goal is to show that the lack of sequential compactness and the lack of completeness are common properties for the classes $H(\varphi)$, with the exception of degenerate cases. In particular, David classes with the exponential restriction (5) are not closed and, consequently, not sequentially compact, although all of these classes are precompact.

Note that the classes $H(\varphi)$ are always nonempty, because they contain the identity $I$. The following statement shows that they are nontrivial, i.e., they contain many mappings, under a natural condition on $\varphi$.

**Proposition 1.** *Let $\varphi$: $[1,\infty) \to [0,\infty]$ be an arbitrary function. Then $H(\varphi) \setminus \{I\} \neq \emptyset$ if and only if*

$$\liminf_{t\to 1} \varphi(t) = \varepsilon > 0. \tag{7}$$

*Moreover, in this case, the cardinality of the collection $\mathfrak{M}(\varphi)$ of all complex dilatations $\mu$: $\mathbf{C} \to \Delta = \{\nu \in \mathbf{C} : |\nu| < 1\}$ of the mappings from $H(\varphi)$ coincides with the cardinality of the class of all measurable functions $\omega$: $\mathbf{C} \to \Delta$.*

Thus, in spite of the lack of an existence theorem in the general case, the classes $H(\varphi)$ remain a noteworthy object for investigation.

It is also important for what follows that the function $\varphi$ in (6) may always be replaced by another function with additional properties and without changing the class $H(\varphi)$. More precisely, the following statement on a *regular change* holds.

**Proposition 2.** *For every function $\varphi$: $[1,\infty) \to [0,\infty]$ there exists a unique non-increasing function $\varphi_r$: $[1,\infty) \to [0,\infty]$ that is continuous from the right and such that*

$$H(\varphi) = H(\varphi_r). \tag{8}$$

*Moreover,*

$$\varphi_r(t) = \lim_{b\to t+0} \left\{ \inf_{1\le a\le b} \varphi(a) \right\} \tag{9}$$

*for all $t \geq 1$.*

The nonnegative non-increasing functions that are continuous from the right are sometimes called *measure functions* in view of the well-known role of such functions in general measure theory. In what follows, we will often use this terminology.

**Corollary 1.** *Let $\varphi_1$ and $\varphi_2$: $[1,\infty) \to [0,\infty]$ be measure functions. Then*

$$H(\varphi_1) = H(\varphi_2) \tag{10}$$

*if and only if $\varphi_1(t) \equiv \varphi_2(t)$.*

*Proof of Proposition* 1. 1) The necessity of the condition (7) is obvious by the well-known Weyl lemma (see, e.g., [A2, p. 33]) because $f_{\bar z} = 0$ a.e. if $\varepsilon = 0$.

2) If (7) holds, there exists $\delta > 0$ such that $\varphi(t) \ge \frac{1}{2}\varepsilon$ for all $t \in (1, 1+\delta)$. For such $\delta$, the class $H(\varphi)$ includes the mapping

$$f(z) = \begin{cases} z|z|^{\delta}, & z \in \Delta_{\varrho}, \\ cz, & z \in \overline{\mathbf{C}} \setminus \Delta_{\varrho}, \end{cases}$$

if $\varrho \ge 1$ and $f(z)/c$ if $\varrho \le 1$, where $\Delta_{\varrho} = \{z \in \mathbf{C} : |z| < \varrho\}$, $\varrho = \sqrt{\varepsilon/2\pi}$, $c = \varrho^{\delta}$, because its dilatation is given by

$$p(z) = \begin{cases} 1+\delta, & z \in \Delta_{\varrho}, \\ 1, & z \in \mathbf{C} \setminus \Delta_{\varrho}. \end{cases}$$

Note that $f$ is even quasiconformal in view of the quasiconformal removability of analytic arcs (see [LV, p. 47]).

3) Finally, let us show that, under the condition (7), $\mathfrak{M}(\varphi)$ has the cardinality of the class of all measurable functions $\omega$: $\mathbf{C} \to \Delta$.

By the existence theorem for the class $\mathfrak{F}_Q$, $Q = 1+\delta$, for every measurable function $\alpha(z)$: $\Delta_{\varrho} \to \mathbf{R}$ there exists a mapping $f \in H(\varphi) \cap \mathfrak{F}_Q$ with the complex dilatation

$$\mu(z) = \begin{cases} \tau e^{i\alpha(z)}, & z \in \Delta_{\varrho}, \\ 0, & z \in \mathbf{C} \setminus \Delta_{\varrho}, \end{cases}$$

where $\tau = \delta/(2+\delta)$ and $\delta$ is as in the previous part of the proof. However, the formula $\alpha(z) = \gamma\big(\kappa(z)\big)$, where $\kappa(z) = (\varrho - |z|)^{-1} z/|z|$, provides a one-to-one correspondence between all measurable functions $\gamma$: $\mathbf{C} \to \mathbf{R}$ and all measurable functions $\alpha$: $\Delta_{\varrho} \to \mathbf{R}$. Moreover, the collection of all measurable functions $\Omega$: $\mathbf{C} \to \mathbf{C}$ has a natural one-to-one correspondence to all pairs $(\gamma_1, \gamma_2)$ of measurable functions $\gamma_1 = \operatorname{Re}\Omega$: $\mathbf{C} \to \mathbf{R}$, $\gamma_2 = \operatorname{Im}\Omega$: $\mathbf{C} \to \mathbf{R}$.

It is well known that the cardinality of the Cartesian product $M \times M$ for each infinite set $M$ coincides with the cardinality of $M$ (see e.g. [AP, p. 30]). Consequently, the cardinality of $\mathfrak{M}(\varphi)$ is equal to or greater than the cardinality of all measurable functions $\Omega$: $\mathbf{C} \to \mathbf{C}$. On the other hand, as a subset of the latter, $\mathfrak{M}(\varphi)$ cannot have a greater cardinality. Hence their cardinalities coincide. Further, the formula $\Omega(z) = \beta\big(\omega(z)\big)$, where $\beta$: $\Delta \to \mathbf{C}$, $\beta(\zeta) = (1-|\zeta|)^{-1}\zeta/|\zeta|$, gives us a one-to-one correspondence between all measurable functions $\Omega$: $\mathbf{C} \to \mathbf{C}$ and all measurable functions $\omega$: $\mathbf{C} \to \Delta$.

Thus the proof of Proposition 1 is complete.

*Proof of Proposition* 2. 1) Consider first the function

$$\omega(b) = \inf_{1 \le a \le b} \varphi(a), \qquad b \in [1, \infty). \tag{11}$$

By the construction, $\omega$ is nonincreasing and $\omega(b) \leq \varphi(b)$ for all $b \in [1, \infty)$. Hence

$$H(\omega) \subseteq H(\varphi). \tag{12}$$

On the other hand, every measurable function $p$: $\mathbf{C} \to [1, \infty)$ satisfying the inequality

$$\operatorname{meas}\{z \in \mathbf{C} : p(z) > a\} \leq \varphi(a)$$

for all $a \geq 1$, will, by (11), also satisfy the inequality

$$\operatorname{meas}\{z \in \mathbf{C} : p(z) > b\} \leq \omega(b)$$

for all $b \geq 1$, i.e.

$$H(\varphi) \subseteq H(\omega). \tag{13}$$

Comparing (12) and (13) we have

$$H(\varphi) = H(\omega). \tag{14}$$

2) Now consider the function

$$\kappa(t) = \lim_{b \to t+0} \omega(b). \tag{15}$$

By the construction, this function is continuous from the right, non-increasing and $\kappa(t) \leq \omega(t)$ for all $t \in [1, \infty)$. Consequently, the inclusion

$$H(\kappa) \subseteq H(\omega) \tag{16}$$

holds. Moreover, for every measurable function $p$: $\mathbf{C} \to [1, \infty)$,

$$\operatorname{meas}\{z \in \mathbf{C} : p(z) > t\} = \lim_{n \to \infty} \operatorname{meas}\left\{z \in \mathbf{C} : p(z) > t + \frac{1}{n}\right\}$$

in view of the countable additivity of the Lebesgue measure. Hence

$$H(\omega) \subseteq H(\kappa). \tag{17}$$

Comparing (11)–(17) we obtain (8) and (9).

3) Finally, let us prove the uniqueness of the regular change $\varphi_r$.

Indeed, let us assume that there exist two different non-increasing functions $\varphi_1$ and $\varphi_2$: $[1, \infty) \to [0, \infty]$, which are continuous from the right and such that (10) holds. Let

$$\Phi_2 = \varphi_2(t) > \varphi_1(t) = \Phi_1$$

for some $t \in [1, \infty)$ and let $\Delta = \Phi_2 - \Phi_1 > 0$. Then there exists $\tau > t$ such that $0 < \varphi_2(t) - \varphi_2(\tau) < \frac{1}{3}\Delta$ and $0 < \varphi_1(t) - \varphi_1(\tau) < \frac{1}{3}\Delta$. Consider the mapping

$$f(z) = \begin{cases} z|z|^{\tau-1}, & |z| \leq \varrho, \\ cz, & |z| \geq \varrho, \end{cases}$$

if $\varrho \geq 1$ or $f(z)/c$ if $\varrho \leq 1$, where $\varrho = \sqrt{\varphi_2(\tau)/\pi}$, $c = \varrho^{\tau-1}$. It has the dilatation

$$p(z) = \begin{cases} \tau, & |z| < \varrho, \\ 1, & |z| > \varrho, \end{cases}$$

and thus $f$ or $f/c \in H(\varphi_2) \setminus H(\varphi_1)$. This contradicts the assumption (10).

## 2. Main results

In view of Proposition 2 on the regular change in the classes $H(\varphi)$, we may assume without loss of generality that the functions $\varphi\colon [1,\infty) \to [0,\infty]$ are non-increasing and continuous from the right, i.e., they are measure functions. Thus, the criterion below gives the final solution to the compactness problem for the classes under consideration.

**Theorem 1.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be an arbitrary measure function. Then the class $H(\varphi)$ is sequentially compact if and only if $\varphi$ has the form*

$$\varphi(t) = \begin{cases} \infty, & 1 \le t < Q, \\ 0, & t \ge Q, \end{cases} \tag{18}$$

*for some $Q \in [1,\infty)$.*

In other words, there exist no compact classes among the classes $H(\varphi)$, with the exception of the well-known classes $\mathfrak{F}_Q$, $Q \ge 1$, consisting of all $Q$-quasiconformal mappings $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ normalized by (3).

**Corollary 2.** *There exist no compact classes among David classes. There exist no closed classes among David classes.*

Indeed, for David classes,

$$\varphi(t) = \begin{cases} \infty, & 1 \le t < T, \\ ce^{-\gamma t}, & t \ge T, \end{cases} \tag{19}$$

where $T \ge 1$, $c > 0$ and $\gamma > 0$. It is easy to see that the given functions are non-increasing and continuous from the right. By Theorem 1, the classes $H(\varphi)$ cannot be sequentially compact. Moreover, by David's theorem, the given classes of homeomorphisms are locally uniformly bounded and equicontinuous and, consequently, precompact by the Arzela–Ascoli theorem. Thus, the lack of sequential compactness in these classes is equivalent to the lack of completeness.

**Corollary 3.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be a measure function. Then the class $H(\varphi)$ is closed if and only if $\varphi$ has the form (18).*

Thus, there exist no closed classes $H(\varphi)$ with the exception of $\mathfrak{F}_Q$.

To verify this, apply Theorem 1 to the precompact classes $H(\varphi) \cap \mathfrak{F}_K$, in other words, to the classes $H(\varphi_K) \subseteq \mathfrak{F}_K$, where for all $K > 1$

$$\varphi_K(t) = \begin{cases} \varphi(t), & 1 \le t < K, \\ 0, & t \ge K. \end{cases} \tag{20}$$

In this connection, the problem of describing the closure of a class $H(\varphi)$ arises. The following theorem shows that the problem is nontrivial.

**Theorem 2.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be a measure function that is exponentially decreasing at $\infty$, with the exception of the form* (18). *Then*

$$\overline{H(\varphi)} \neq H(\Psi) \tag{21}$$

*for all functions $\Psi\colon [1,\infty) \to [0,\infty]$.*

Thus, the closure of a class $H(\varphi)$ cannot be a class of the same type except for the degenerate cases (18).

With respect to compactness and closure the situation of the classes $H(\varphi)$ radically differs from that of the classes $H^{\Phi}$ of all sense-preserving homeomorphisms $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ in $W^1_{1,\mathrm{loc}}$ normalized by (3) and satisfying integral restrictions

$$\iint_{\mathbf{C}} \Phi\bigl(p(z)\bigr)\,dx\,dy \leq 1. \tag{22}$$

The classes $H^{\Phi}$ have a very strong theory as regards compactness and closure (see [R4]).

First of all, note that $H^{\Phi}$ is nonempty if and only if $\inf \Phi = 0$. If the function $\Phi$ has exponential growth at $\infty$, i.e., $\Phi(t) \geq Be^{\beta t}$, $t \geq T$, *the existence theorem* holds by David's theorem because (22) implies (5) under the given growth condition. Under these conditions $H^{\Phi}$ is compact if and only if the function $\Phi$ is non-decreasing, convex and continuous from the left at the point

$$K = \sup\{t \in [1,\infty) : \Phi(t) < \infty\}. \tag{23}$$

We note that, for such $\Phi$,

$$H^{\Phi} \subseteq H(\varphi), \tag{24}$$

where $\varphi = 1/\Phi$. However, as the above criteria for compactness show, the converse inclusion is valid if and only if $\varphi$ has the form (18). Thus, the classes $H^{\Phi}$ and $H(\varphi)$ practically never coincide.

The closure of $H^{\Phi}$ is always a class $H^{\Phi_0}$ of a similar type, where $\Phi_0$ is the so-called lower envelope of $\Phi$. More precisely,

$$\Phi_0(t) = \sup_{\omega \in \Omega} \omega(t), \qquad t \in [1,\infty), \tag{25}$$

where $\Omega$ is the family of all continuous non-decreasing convex functions $\omega\colon [1,\infty) \to [0,\infty)$ such that $\omega(t) \leq \Phi(t)$ for all $t \in [1,\infty)$. In other words, the lower envelope of a function $\Phi\colon [1,\infty) \to [0,\infty]$ is the greatest non-decreasing convex function $\Phi_0\colon [1,\infty) \to [0,\infty]$ with its graph lying below the graph of $\Phi$ and continuous from the left at the point (23) (see e.g. [B]).

## 3. Main lemma

The construction in the following lemma is due to [R1, p. 9].

**Lemma 1.** *Let $t_1, t_2 \in [1,Q]$, $t_1 \neq t_2$, $\lambda \in [0,1]$, be arbitrary numbers. Then there exists a sequence of $Q$-quasiconformal mappings $f_n\colon \mathbf{C} \to \mathbf{C}$, whose dilatations $p_n$ take only two values, $t_1$ and $t_2$ a.e., and which converges locally uniformly to a $Q$-quasiconformal mapping $f_0\colon \mathbf{C} \to \mathbf{C}$ with the dilatation*

$$p_0(z) \equiv t_0 = \lambda t_1 + (1-\lambda)t_2. \tag{26}$$

*Moreover, for every measurable set $E \subseteq \mathbf{C}$ with $0 < \operatorname{meas} E < \infty$, the limit relation*

$$\lim_{n\to\infty} \frac{\operatorname{meas}\{z \in E : p_n(z) = t_1\}}{\operatorname{meas} E} = \lambda \tag{27}$$

*holds.*

**Remark 1.** By the construction, $f_0$ will be simply a stretching along an imaginary axis and $f_n$ are obtained by the method of gluing of such stretchings together.

*Proof.* First of all, consider $Q$-quasiconformal affine mappings

$$\zeta = g_s(z) = \tfrac{1}{2}(1+t_s)z + \tfrac{1}{2}(1-t_s)\bar{z} = x + iyt_s$$

that are stretchings along the $y$-axis, $z = x + iy$, with distortion coefficients $t_s$, $s = 0,1,2$.

Let us fix a positive integer $n \in \{1,2,\dots,\}$ and make use of the lines

$$l_{mn} = \Big\{x + i\frac{m}{2^n} : x \in \mathbf{R}\Big\},$$

where $m = 0, \pm1, \pm2, \dots$, to partition the whole plane $\mathbf{C}$ into strips that are parallel to the real axis. In turn, each such strip is partitioned into two strips by lines of the form

$$l^*_{mn} = \Big\{x + i\Big(\frac{m}{2^n} + \frac{\lambda}{2^n}\Big) : x \in \mathbf{R}\Big\}.$$

We set $f_n(z) = g_1(z)$ between the lines $l_{0n}$ and $l^*_{0n}$. In all the remaining strips we define $f_n(z) = g_s(z) + C^s_{mn}$, where $s = 1$ or $2$ (we take $s = 1$ between the lines $l_{mn}$ and $l^*_{mn}$, and $s = 2$ between the lines $l^*_{mn}$ and $l_{m+1,n}$) and the constants $C^s_{mn}$ are found for every fixed $n$ by induction on $m$ on both sides of zero ($m = 0, \pm1, \pm2, \dots$) from the gluing condition.

Note that the mappings $f_n$ are $Q$-quasiconformal in view of the removability of analytic arcs (see e.g. [LV, p. 47]). Moreover, in the strips between the lines

$l_{mn}$ and $l^*_{mn}$, we have $p_n(z) = t_1$, while in the strips between the lines $l^*_{mn}$ and $l_{m+1,n}$ we have $p_n(z) = t_2$.

Further, when increments of the independent variable $\Delta z_s = ih_s$, $h_s > 0$, are purely imaginary, we have purely imaginary increments of the functions

$$\Delta g_s = ih_s t_s, \qquad s = 0, 1, 2.$$

Thus by the construction $f_n$ we obtain

$$\Delta f_n = i\frac{1}{2^n}\{\lambda t_1 + (1-\lambda)t_2\} = i\frac{t_0}{2^n} = \Delta g_0$$

for $\Delta z = i2^{-n}$. To verify this, take above $h_0 = 2^{-n}$, $h_1 = \lambda 2^{-n}$, $h_2 = (1-\lambda)2^{-n}$. Hence

$$f_n(z) = g_0(z) \tag{28}$$

on the lines $l_{mn}$, $m = 0, \pm 1, \pm 2, \ldots$, $n = 1, 2, \ldots$ in view of the initial date $f_n(z) = g_1(z) = z = g_0(z)$ on the real axis.

It is thus obvious that on the lines $l_{mn}$, which constitute a dense set in $\mathbf{C}$, the sequence $f_n(z)$ converges to the mapping $f_0(z) \equiv g_0(z)$ and, consequently, it converges to $f_0(z)$ locally uniformly in the whole plane $\mathbf{C}$ (see [LV, p. 76]).

Finally, the relation (27) follows immediately from the measure partitioning among the values $t_1$ and $t_2$ in the dilatations $p_n$.

**Corollary 4.** *For all* $t_1$, $t_2 \in [1, Q]$, $t_1 \neq t_2$, $1 < Q < \infty$, $\lambda \in [0, 1]$, $E \subset \mathbf{C}$ *with* $0 < \operatorname{meas} E < \infty$, *there exists a sequence of mappings* $f_n \in \mathfrak{F}_Q$ *with dilatations*

$$p_n(z) = \begin{cases} t_1, & z \in E_n, \\ t_2, & z \in E \setminus E_n, \\ 1, & z \in \mathbf{C} \setminus E, \end{cases}$$

*a.e. where* $E_n \subseteq E$, *such that* $f_n \to f_0 \in \mathfrak{F}_Q$ *locally uniformly,*

$$\lim_{n\to\infty} \operatorname{meas} E_n / \operatorname{meas} E = \lambda,$$

*and*

$$p_0(z) = \begin{cases} t_0, & z \in E, \\ 1, & z \in \mathbf{C} \setminus E, \end{cases}$$

*where* $t_0 = \lambda t_1 + (1-\lambda)t_2$.

Corollary 4 is immediately obtained from Lemma 1, the sequential compactness of the class $\mathfrak{F}_Q$ and the following comparison lemma (see [GR], [R2]).

**Lemma.** *Let $(f_n)$ and $(g_n)$, $n = 1,2,\ldots$, be sequences of $Q$-quasiconformal mappings with complex dilatations $\mu_n$ and $\nu_n$ that converge locally uniformly to $Q$-quasiconformal mappings $f_0$ and $g_0$ with complex dilatations $\mu_0$ and $\nu_0$, respectively. If $E \subset \mathbf{C}$ is measurable and*

$$\operatorname{meas}\{z \in E \mid \mu_n(z) \neq \nu_n(z)\} \to 0, \tag{29}$$

*then*

$$\mu_0(z) = \nu_0(z) \tag{30}$$

*almost everywhere in $E$.*

Here we have to stress that, generally speaking, in spite of (29), $\mu_n(z)$ and $\nu_n(z)$ may be nonconverging in measure to $\mu_0(z)$ and $\nu_0(z)$, respectively. Furthermore, there exist examples of $Q$-quasiconformal mappings $f_n$, where the corresponding $\mu_n$ do not converge even weakly to $\mu_0$ in $L^p_{\text{loc}}$, $p \geq 1$, although $f_n \to f_0$ locally uniformly (see [LV, p. 195], [R1, p. 14]).

## 4. Proof of the compactness criterion

The sufficiency of the condition (18) for the compactness of $H(\varphi)$ is obvious, because in this case $H(\varphi) = \mathfrak{F}_Q$. The proof of the necessity of the condition (18) is reduced to the proof of the following series of simple statements.

**Proposition 3.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be an arbitrary measure function. If the class $H(\varphi)$ is sequentially compact, then*

$$Q_1 = \sup_{\varphi(t)=\infty} t < \infty. \tag{31}$$

Here the supremum is taken over $t \in [1,\infty)$. If there exist no points $t \in [1,\infty)$ such that $\varphi(t) = \infty$, we let $Q_1 = 1$.

**Proposition 4.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be an arbitrary measure function. If the class $H(\varphi)$ is sequentially compact, $\varphi$ has the form*

$$\varphi(t) = \begin{cases} \infty, & t \in [1, Q_1), \\ c, & t \in [Q_1, Q_2), \\ 0, & t \in [Q_2, \infty), \end{cases} \tag{32}$$

*where $0 < c < \infty$, $Q_1$ is given by (31) and*

$$Q_2 = \inf_{\varphi(t)=0} t. \tag{33}$$

Here the infimum is taken over $t \in [1,\infty)$. If there exist no points $t \in [1,\infty)$ such that $\varphi(t) = 0$, we set $Q_2 = \infty$.

**Proposition 5.** *Let $\varphi\colon [1,\infty) \to [0,\infty]$ be an arbitrary measure function. If the class $H(\varphi)$ is sequentially compact, then*

$$Q_1 = Q_2 \tag{34}$$

*where $Q_1$ and $Q_2$ are given by* (31) *and* (33), *respectively.*

*Proof of Proposition* 3. Let us assume that $\varphi \equiv \infty$, $t \in [1,\infty)$. Consider the sequence of quasiconformal mappings

$$f_n(z) = z|z|^{t_n-1} = z^{(t_n+1)/2}\,\bar{z}^{(t_n-1)/2},$$

$z \in \mathbf{C}$, where $t_n \in [1,\infty)$ is an arbitrary sequence such that $t_n \to \infty$ as $n \in \infty$. It is easy to see that

$$\mu_n(z) = \frac{z}{\bar{z}}\,\frac{t_n-1}{t_n+1} \qquad \text{a.e.,}$$

i.e., $p_n(z) = t_n$ a.e. and $f_n \in H(\varphi)$ for all $n = 1, 2, \dots$.

On the other hand, for every fixed $|z| > 1$ we have

$$\lim_{n\to\infty} |f_n(z)| = \lim_{n\to\infty} |z|^{t_n} = \infty.$$

Hence there exists no subsequence $(f_{n_k})$ of $(f_n)$ converging locally uniformly to a mapping $f_0 \in H(\varphi)$. This contradicts the sequential compactness of $H(\varphi)$.

*Proof of Proposition* 4. If $Q_1 = Q_2$, we automatically have the form (32).

Let $Q_2 > Q_1$. Then, by the definitions of $Q_1$ and $Q_2$, $0 < \varphi(t) < \infty$ for all $t \in (Q_1, Q_2)$. We show first that if $\varphi(t_*) = c > 0$ for some $t_* \in (Q_1, Q_2)$, then $\varphi(t) = c$ for all $t \in (t_*, Q_2)$, too. Since $t_* \in (Q_1, Q_2)$ is arbitrary, this immediately leads us to (32).

Set $t^* = \sup_{\varphi(t)=c} t \geq t_*$ and let us assume that $t^* < Q_2$. Then there exists a point $\tau \in (t^*, Q_2)$ where $0 < \varphi(\tau) < c$. Taking $t_1 = \tau$, $t_2 = t^*$, $\lambda = \varphi(\tau)/c$ and

$$E = \{z = x + iy : 0 \leq x \leq 1,\ 0 \leq y \leq c - \varepsilon\},$$

$\varepsilon \in (0, c)$, in Corollary 4, we get a sequence $(f_n) \subset H(\varphi) \cap \mathfrak{F}_\tau$ converging locally uniformly to a mapping $f_0 \in H(\varphi) \cap \mathfrak{F}_\tau$ with the dilatation

$$p_0(z) = \begin{cases} t_0, & z \in E, \\ 1, & z \in \mathbf{C} \setminus E, \end{cases}$$

where $t_0 = \lambda\tau + (1-\lambda)t^*$, i.e. $t^* < t_0 < \tau$. Moreover,

$$\operatorname{meas}\{z \in \mathbf{C} : p_0(z) = t_0\} = c - \varepsilon.$$

Thus, for every $t \in (t^*, t_0)$, we have the lower bound $\varphi(t) \geq c - \varepsilon$ and, since $\varepsilon \in (0, c)$ was arbitrary, $\varphi(t) \geq c = \varphi(t_*)$. Since the function $\varphi$ is non-increasing,

$$\varphi(t) \equiv \varphi(t_*) = c, \qquad t \in (t^*, t_0).$$

However, the last statement contradicts the definition of the point $t^*$. Thus, the assumption $t^* < Q_2$ was false and, consequently, $\varphi(t) = c$ in the whole interval $(t_*, Q_2)$. From this we finally come to the relation (32).

*Proof of Proposition* 5. Let us assume that $Q_2 > Q_1$. Then there exists a point $\tau \in (Q_1, Q_2)$. Letting $t_1 = \tau$, $t_2 = Q_1$,

$$E = \{z = x + iy : 0 \le x \le 1,\ 0 \le y \le N\}, \qquad \lambda = \frac{(c-\varepsilon)}{N},$$

in Corollary 4, where $\varepsilon \in (0, c)$ and $N > c = \varphi(\tau) > 0$, we get a sequence $(f_n) \subset H(\varphi) \cap \mathfrak{F}_\tau$ converging locally uniformly to some mapping $f_0 \in H(\varphi) \cap \mathfrak{F}_\tau$ with the dilatation

$$p_0(z) = \begin{cases} t_0, & z \in E, \\ 1, & z \in \mathbf{C} \setminus E, \end{cases}$$

where $t_0 = \lambda\tau + (1-\lambda)Q_1$. Hence $Q_1 < t_0 < \tau < Q_2$. Moreover, for $t \in (Q_1, t_0)$,

$$\varphi(t) \ge \operatorname{meas}\{z \in \mathbf{C} : p_0(z) = t_0\} = N > c = \varphi(\tau).$$

This, however, contradicts (32). Thus, the above assumption was false.

## 5. Proof of the closure theorem

By the David theorem, the class $H(\varphi)$ and, consequently, also $\overline{H(\varphi)}$ is locally bounded and locally equicontinuous. Hence $\overline{H(\varphi)}$ is precompact by the Arzela–Ascoli theorem. Since the class $\overline{H(\varphi)}$ is closed, it is sequentially compact.

By Theorem 1 the equality

$$\overline{H(\varphi)} = H(\Psi) \tag{35}$$

would hold only for the function $\Psi$ of the form

$$\Psi(t) = \begin{cases} \infty, & 1 \le t < Q, \\ 0, & t \ge Q, \end{cases}$$

for some $Q \in [1, \infty)$.

Since (35) implies the inclusion $H(\varphi) \subseteq H(\Psi) = \mathfrak{F}_Q$, it is clear that $\inf_{\varphi(t)=0} t \le Q$. Moreover,

$$\varphi(Q-0) < \infty, \tag{36}$$

because $\varphi(t)$ is not of the form (18).

Consider the mapping $f \in H(\Psi)$,

$$f(z) = \frac{z + q\bar{z}}{1+q},$$

where $q = (Q-1)/(Q+1)$. By (35) there exists a sequence $(f_n) \subset H(\varphi)$ such that $f_n \to f$ locally uniformly as $n \to \infty$.

By the convergence theorem in [R3],

$$\liminf_{n\to\infty} \iint_E p_n(z)\,dx\,dy \ge \iint_E p(z)\,dx\,dy \tag{37}$$

for every measurable set $E \subset \mathbf{C}$ with $0 < \operatorname{meas} E < \infty$; here $p_n$ and $p$ are the dilatations of $f_n$ and $f$, respectively. However, $p(z) \equiv Q$ and, by (36), $\varphi(t) \le c < \infty$ for some $\varepsilon > 0$ and $c > 0$ and all $t \in [Q-\varepsilon, Q]$. Choosing

$$E = \Delta_r = \{z \in \mathbf{C} : |z| < r\},$$

where

$$r = \sqrt{\frac{\alpha}{\pi}}, \qquad \alpha = \frac{2cQ}{\varepsilon},$$

we obtain

$$\limsup_{n\to\infty} \iint_E p_n(z)\,dx\,dy \le (Q-\varepsilon)\alpha + Qc = Q\alpha - Qc < Q\alpha = \iint_E p(z)\,dx\,dy,$$

i.e., a contradiction with inequality (37).

Thus (35) is impossible and the proof of Theorem 2 is complete.

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