Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 23, 1998, 317–388

ITERATION, LEVEL SETS, AND ZEROS OF DERIVATIVES OF MEROMORPHIC FUNCTIONS

A. Hinkkanen

University of Illinois at Urbana–Champaign, Department of Mathematics
Urbana, Illinois 61801, U.S.A.; aimo@math.uiuc.edu

Abstract. Let f be a meromorphic non-entire function in the plane, and suppose that for every $k \geq 0$, the derivative $f^{(k)}$ has only real zeros. We have proved that then $f(az+b) = P(z)/Q(z)$ for some real numbers a and b where $a \neq 0$, where $Q(z) = z^n$ or $Q(z) = (z^2+1)^n$, n is a positive integer, and P is a polynomial with only real zeros such that $\deg P \leq \deg Q + 1$; or $f(az+b) = C(z-i)^{-n}$ or $f(az+b) = C(z-\alpha)/(z-i)$ where α is real and C is a non-zero complex constant.

In this paper we provide part of the proof of this theorem, by obtaining the following result. Let f be given by $f(z) = g(z)/(z^2+1)^n$ where g is a real entire function of finite order with $g(i)g(-i) \neq 0$ and n is a positive integer. If f, f', and f'' have only real zeros then g is a polynomial of degree at most $2n+1$.

Conversely, if f is of this form where g is a polynomial of degree at most $2n$ with only real zeros, then $f^{(k)}$ has only real zeros for all $k \geq 0$. If the degree of g is $2n+1$ then $f^{(k)}$ has only real zeros for all $k \geq 0$ if, and only if, f and f' have only real zeros.

1. Introduction and results

In this paper we develop a theory that ties together complex dynamical properties of meromorphic functions and the location of the zeros of their first few derivatives. The specific assumptions made in this paper are that the function is real on the real axis, is of finite order, and has only finitely many poles. The methods used, however, would undoubtedly be of greater applicability, yielding different conclusions if coupled with different assumptions. In this paper we assume that the zeros of f and its first few derivatives are real and obtain results concerning domains in the upper (and, by symmetry, lower) half plane. Under more general circumstances, similar results could be obtained in regions which are assumed or known not to contain such zeros.

The immediate motivation for this study is to complete the answer to the following problem. Let f be a function meromorphic in the complex plane $\mathbf{C}$. We consider the question of under what circumstances all the derivatives of f,

---

1991 Mathematics Subject Classification: Primary 30D05.

Research partially supported by the Alfred P. Sloan Foundation, by the U.S. National Science Foundation grant DMS 94-00999, and by the U.S. National Security Agency grant MDA904-95-H-1014. Part of this research was conducted while the author was visiting University College London, U.K. He wishes to thank the Department of Mathematics for its kind hospitality.

including f itself, can have only real zeros. We may and will assume that f is not a polynomial so that none of the derivatives $f^{(k)}$ vanishes identically. We write $f^{(0)} = f$. We say that f is *real* if $f(z)$ is real or $f(z) = \infty$ whenever z is real. If f is not a constant multiple of a real function, then f is called *strictly non-real.* We have proved the following result. The proof is given partly in this paper and partly in the two companion papers [H2], [H3].

Theorem A. *Let f be a non-entire meromorphic function in the complex plane, and suppose that for every integer $k \geq 0$, the derivative $f^{(k)}$ has only real zeros. Then there are real numbers a and b where $a \neq 0$, and a polynomial P with only real zeros, such that*

(i) $f(az+b) = P(z)/Q_0(z)$, where $Q_0(z) = z^n$ or $Q_0(z) = (z^2+1)^n$, n is a positive integer, and $\deg P \leq \deg Q_0 + 1$; or

(ii) $f(az+b) = C(z-i)^{-n}$ where C is a non-zero complex constant; or

(iii) $f(az+b) = C(z-\alpha)/(z-i)$, where α is a real number and C is a non-zero complex constant.

Conversely, if f is as in (i) with $\deg P \leq \deg Q_0$, or if f is as in (ii) or (iii), then $f^{(k)}$ has only real zeros for all $k \geq 0$. If f is as in (i) with $\deg P = \deg Q_0 + 1$ then $f^{(k)}$ has only real zeros for all $k \geq 0$ if, and only if, f' (or, equivalently, $zP'(z) - nP(z)$ or $(z^2+1)P'(z) - 2nzP(z)$) has only real zeros.

If f is as in (i) of Theorem A and $\deg P = \deg Q_0 + 1$ then there are polynomials P with only real zeros for which f' has only real zeros, and other polynomials P for which f' has at least two non-real zeros (compare [H2]).

For entire f, the corresponding problem has been solved by Hellerstein and Williamson [HW1], [HW2], and by Hellerstein, Shen and Williamson [HSW]. They proved, among other things, that if f is entire and $f^{(k)}$ has only real zeros for all k with $0 \leq k \leq 3$, then f is in the Laguerre–Pólya class, or $f(z) = Ae^{Bz}$, or $f(z) = A(e^{icz} - e^{id})$, where $A, B \in \mathbf{C}$, c, d are real, and $ABcd \neq 0$, and then $f^{(k)}$ has only real zeros for all $k \geq 0$.

The complete proof of Theorem A is long, and is divided into three papers (this paper and [H2], [H3]). In this paper we prove the following result (stated as [H2, Theorem 3] without proof) that deals with real functions of finite order.

Theorem 1.1. *Let f be given by*

$$f(z) = \frac{g(z)}{(z^2+1)^n} \tag{1}$$

where g is a real entire function of finite order with $g(i)g(-i) \neq 0$ and n is a positive integer. If f, f', and f'' have only real zeros then g is a polynomial of degree at most $2n+1$.

Conversely, if f is as in (1) where g is a polynomial of degree at most $2n$ with only real zeros, then $f^{(k)}$ has only real zeros for all $k \geq 0$. If the degree of g is $2n+1$ then $f^{(k)}$ has only real zeros for all $k \geq 0$ if, and only if, f and f' have only real zeros.

The last paragraph of Theorem 1.1 has been proved in [H2, Theorem 8(ii)]. When stating [H2, Theorem 3], I had the extra assumption that if $g \in \mathcal{U}_0$ then also f''' is to have only real zeros. I later on managed to improve my methods so as to obtain the result without that assumption.

2. Set-up and auxiliary results

2.1. Definitions. We write $\operatorname{sgn} x = x/|x|$ if $x \neq 0$, and $\operatorname{sgn} 0 = 0$. We define some classes of functions (compare [HW1, pp. 227–228]). We say that $f \in \mathcal{V}_{2p}$ where p is an integer with $p \geq 0$ if f is of the form

$$f(z) = g(z) \exp\{-az^{2p+2}\}$$

where $a \geq 0$ and g is a constant multiple of a real entire function with genus not exceeding $2p+1$ and with only real zeros. We set $\mathcal{U}_0 = \mathcal{V}_0$ and $\mathcal{U}_{2p} = \mathcal{V}_{2p} \setminus \mathcal{V}_{2p-2}$ for $p \geq 1$. The class $\mathcal{U}_0$ is the so-called Laguerre–Pólya class. We have $f \in \mathcal{U}_0$ if, and only if, there are real polynomials P_n with only real zeros such that $P_n \to f$ locally uniformly in $\mathbf{C}$. Also, $f \in \mathcal{U}_0$ if, and only if, we may write

$$f(z) = cz^m e^{-az^2+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

where c is a non-zero complex constant, m is a non-negative integer, $a \geq 0$, b is a real number, $z_n \in \mathbf{R} \setminus \{0\}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} z_n^{-2} < \infty$. Here $\mathbf{R}$ denotes the real axis; we write $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ for the extended real axis. Thus if $f \in \mathcal{U}_0$ then $f^{(k)} \in \mathcal{U}_0$ and so $f^{(k)}$ has only real zeros, for all $k \geq 0$ if f is transcendental, and for $0 \leq k \leq d$ if f is a polynomial of degree d.

The elements of $\mathcal{U}_{2p}$ are, by definition, constant multiples of real functions, but not necessarily real. For our purposes the possibly non-real multiplicative constant does not matter. Hence, for simplicity, but without any loss of generality, we shall assume in the rest of this paper that each element of any class $\mathcal{U}_{2p}$ is real.

If $b \in \mathbf{C}$ is a zero of f of order $m \geq 1$, we shall write $\operatorname{ord}(f, b) = m$. If $f(b) \notin \{0, \infty\}$, we set $\operatorname{ord}(f, b) = 0$. If b is a pole of f of order $m \geq 1$, we write $\operatorname{ord}(f, b) = -m$.

To prove Theorem 1.1, we shall develop techniques based on the Fatou–Julia iteration theory, the use of iteration of analytic functions taking a half plane into itself, and level sets of harmonic functions. We note that the idea of studying level sets of harmonic functions in the connection of investigating the number of non-real zeros of the second derivative of an entire function has been successfully applied by Sheil-Small [S]. In connection with problems on the reality of zeros of f and its derivatives, iteration theory, in the form of considering a parabolic fixed point of $z - f/f'$ at infinity when $z - f/f'$ is rational [E], or similar behaviour when $z - f/f'$ is transcendental [H1], has been applied by Eremenko [E] and by the author [H1]. The methods that we develop here will probably be of greater applicability than only the proof of Theorem 1.1 in the area of studying the reality of the zeros of the first few derivatives of a real meromorphic function. Therefore, in the interest of obtaining a more general theory, we shall formulate and prove various auxiliary results for meromorphic functions f of the form

$$f(z) = g(z)/\Phi(z) \tag{2}$$

where g is a real entire function of finite order with only real zeros and Φ is a non-constant real polynomial with leading coefficient 1. We always assume that g and Φ have no common zeros. Mostly, we make additional assumptions on Φ, such as assuming that Φ has only non-real zeros, and finally we specialize to the case $\Phi(z) = (z^2+1)^n$. We denote by Ψ a polynomial with leading coefficient 1 and with only simple zeros that vanishes exactly at the zeros of Φ. Then Ψ has real coefficients. For example, if $\Phi(z) = (z^2+1)^n$, then $\Psi(z) = z^2+1$.

For the purpose of taking the boundary of a set, we consider every set, including the boundary, to be a subset of the extended complex plane or Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. For example, $\partial H^+ = \overline{\mathbf{R}}$, where $H^+ = \{w : \operatorname{Im} w > 0\}$ is the upper half plane.

We shall use the standard notation and the basic results of Nevanlinna's theory of value distribution. For details, we refer to [H].

2.2. The degree of an auxiliary polynomial. Suppose that $f(z) = g(z)/\Phi(z)$, where g is a real entire function of finite order and Φ is as described above. Write

$$L(z) = \frac{f'}{f}(z), \qquad Q(z) = z - \frac{1}{L(z)} = z - \frac{f}{f'}(z) \tag{3}$$

so that f, L, and Q are real. If L is constant, then $f(z) = e^{az+b}$, and if Q is constant, then $f(z) = A(z-B)$. Neither case is possible here so that both L and Q are non-constant. We have

$$Q' = \frac{ff''}{(f')^2}. \tag{4}$$

Thus if f, f' and f'' have only real zeros, then $Q'(z) \notin \{0, \infty\}$ for $z \in H^+ = \{w : \operatorname{Im} w > 0\}$. We write $H^- = \{w : \operatorname{Im} w < 0\}$. Note that for any zero z_0 of Φ we have $Q'(z_0) \neq \infty, 0$. Indeed $Q'(z_0) = 1 + \nu^{-1}$, where $\nu \geq 1$ is the order of the zero of Φ at z_0. Write

$$\Lambda = \{z \in H^+ : \operatorname{Im} L(z) > 0\}, \qquad \Lambda^- = \{z \in H^+ : \operatorname{Im} L(z) < 0\}, \tag{5}$$

$$K = \{z \in H^+ : \operatorname{Im} Q(z) > 0\}, \qquad K^- = \{z \in H^+ : \operatorname{Im} Q(z) < 0\} \tag{6}$$

so that

$$\Lambda \subset K \subset H^+ \quad \text{and} \quad K^- \subset \Lambda^- \subset H^+. \tag{7}$$

Note that for any zero z_0 of Φ we have

$$L(z_0) = \infty, \qquad Q(z_0) = z_0, \tag{8}$$

so that if $z_0 \in H^+$ then

$$z_0 \in \partial\Lambda \quad \text{and} \quad z_0 \in K. \tag{9}$$

In particular, if $f(z) = g(z)/(z^2+1)^n$, then

$$L(i) = \infty, \qquad Q(i) = i,$$

so that

$$i \in \partial\Lambda \quad \text{and} \quad i \in K.$$

If $c \in \mathbf{R}$ we seek a Levin representation for $L(z) - c$. Analogously to [S, p. 181, (2.4)] and [HW1, (2.5), p. 231] we seek to prove that we can write

$$L(z) - c = \psi(c,z)\frac{\varphi(c,z)}{\Psi(z)\Psi_1(z)} \tag{10}$$

where $\Psi_1(z)$ is a polynomial to be specified, and in the case $\Phi(z) = (z^2+1)^n$,

$$L(z) - c = \psi(c,z)\frac{\varphi(c,z)}{z^2+1}$$

where $\psi(z) = \psi(c,z)$ is a real meromorphic function of z with only real zeros such that

$$\operatorname{Im}\psi(c,z) > 0 \text{ for all } z \in H^+, \text{ unless } \psi \equiv 1, \tag{11}$$

and where $\varphi(z) = \varphi(c,z)$ is a polynomial in z whose degree can be estimated in terms of the order of g and $\deg\Psi$ and can be more precisely determined by using further information on f. It then follows that L takes the value c at most $\deg\varphi(c,z)$ times in $\mathbf{C}\setminus\mathbf{R}$, hence at most $\deg\varphi(c,z)/2$ times in H^+.

In order to find a Levin representation for $L(z)-c$, it suffices to consider the case $c = 0$ since the logarithmic derivative of $f(z)e^{-cz}$ is equal to $L(z) - c$. So suppose that $c = 0$ and suppress c in all notation. Let the distinct zeros of f (or g) be $a_k \in \mathbf{R}$ labelled so that (compare [HW1, p. 230])

$$\cdots < a_{k-1} < a_k < a_{k+1} < \cdots \tag{12}$$

where

$$(13) \qquad -\infty \le \tau \le k \le \omega \le +\infty, \qquad k \text{ finite.}$$

Without further comment, we argue as in [HW1, pp. 230–243]. By Rolle's theorem, f' has at least one zero in each (a_k, a_{k+1}) that does not contain any pole of f, that is, any zero of Ψ. We denote by $\mathscr{E}$ the set of the integers k such that (a_k, a_{k+1}) contains a zero of f'. For $k \in \mathscr{E}$, we choose one zero of f' in (a_k, a_{k+1}) and call it b_k. For the purposes of estimating $\deg \varphi(c, z)$, it does not matter which zero of f' is chosen as b_k. However, it will be helpful, in connection with Lemmas 2.2 and 5.1 below, to agree that b_k is taken to be the smallest or largest zero of f' on (a_k, a_{k+1}) (the choice between those two can be allowed to depend on k) so that the remaining zeros of f' on (a_k, a_{k+1}), which will be zeros of the function φ below, will not be separated by b_k. (If b_k is a multiple zero of f', it will still be only a simple zero of the function ψ defined below, and will therefore also be a zero of φ.) Thus

$$(14) \qquad a_k < b_k < a_{k+1}.$$

We also adjust the numbering so that

$$(15) \qquad b_{-1} < 0 < a_1.$$

If $f(0) = 0$, set $a_0 = 0$. If some b_k must be equal to 0, set $b_0 = 0$. This is consistent with (15).

It would make some estimates for a function f defined in terms of a general polynomial Ψ slightly more accurate, if we were to choose similarly a zero β_k of f' on each interval between consecutive real poles α_k and α_{k+1} of f (if there are any) that does not contain any zeros of f, and if we were to use the pair (α_k, β_k) in the definition of ψ below in the same way as we will use each pair (a_k, b_k). However, since f has only finitely many poles, this will not matter for our purposes. Therefore we shall not use this device.

We denote by $\mathscr{E}'$ the set of integers k as in (13) such that f' has no zero on (a_k, a_{k+1}). By Rolle's theorem, this can be the case only if (a_k, a_{k+1}) contains a pole of f, that is, a zero of Φ. Thus the set $\mathscr{E}'$ is finite. We denote by Ψ_1 the polynomial with leading coefficient 1 whose zeros are all simple and occur exactly at the points a_k for $k \in \mathscr{E}'$. Thus $\Psi_1 \equiv 1$ if $\mathscr{E}'$ is empty. In particular, if Φ has no real zeros (for example, if $\Phi(z) = (z^2+1)^n$) then $\Psi_1 \equiv 1$. Note that the set of the distinct zeros of f is equal to $\{a_k : k \in \mathscr{E}\} \cup \{a_k : k \in \mathscr{E}'\}$.

If f has no zeros, set $\psi(z) \equiv 1$. If f has exactly one zero (of some multiplicity), call it a_0 and set $\psi(z) = -1/(z - a_0)$. Otherwise, write

$$(16) \qquad \psi(z) = \frac{z - b_0}{z - a_0} \prod_{\substack{k \ne 0 \\ k \in \mathscr{E}}} \frac{1 - (z/b_k)}{1 - (z/a_k)} \qquad \text{if } \omega = +\infty,$$

$$\psi(z) = \frac{z-b_0}{(z-a_0)(a_\omega - z)} \prod_{\substack{k\neq 0,\omega \\ k\in\mathscr{E}}} \frac{1-(z/b_k)}{1-(z/a_k)} \qquad \text{if } \omega < +\infty. \tag{17}$$

If $0 \notin \mathscr{E}$, then the factor $(z-b_0)/(z-a_0)$ is to be omitted in (16) and (17). If $\tau > 0$ then we omit $(z-b_0)/(z-a_0)$ in (16) and (17). In all cases, ψ satisfies (11) (compare [HW1, p. 230]). Now we may clearly write

$$\frac{f'}{f} \equiv L = \frac{\psi\varphi}{\Psi\Psi_1} \tag{18}$$

where φ is a real entire function. Once we specialize to prove Theorem 1.1, we take $\Phi(z) = (z^2+1)^n$ so that f has no real poles and the technical complications associated with the set $\mathscr{E}$ not being the set of all integers in (13), or with Ψ_1 not being the constant 1, do not arise. The reader is welcome to think about this special case only.

We want to show that φ is a polynomial. It is *only in this argument* that we consider the case where Ψ_1 might be non-constant (this more general conclusion will be used in the proof of Lemma 4.1). It is useful to record this result separately.

Lemma 2.1. *Let f be given by (2), where Φ is a non-constant real polynomial and $g \in \mathscr{U}_{2p}$, so that f has only real zeros while we do not make any a priori assumption concerning the reality of the zeros of f' or f''. Let the real polynomials Ψ and Ψ_1 be as defined above, and let the function ψ be given by (16) or (17), as appropriate. Then the function φ defined by (18) is a polynomial.*

Proof of Lemma 2.1. We have

$$\varphi = \Psi\Psi_1 \frac{f'}{f}\frac{1}{\psi} \tag{19}$$

so that

$$T(r,\varphi) = m(r,\varphi) \leq O(\log r) + m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{1}{\psi}\right). \tag{20}$$

Since f has finite order, we have

$$\left(r, \frac{f'}{f}\right) = O(\log r) \tag{21}$$

as $r \to \infty$. If ψ is not constant, then, since $\psi(H^+) \subset H^+$, there is a constant $C_0 \geq 1$ such that

$$\frac{|\sin\theta|}{C_0 r} \leq |\psi(re^{i\theta})| \leq C_0 \frac{r}{|\sin\theta|} \tag{22}$$

when $r \geq 1$ and $\sin\theta \neq 0$, by a theorem of Carathéodory. Thus

$$m\left(r, \frac{1}{\psi}\right) = O(\log r). \tag{23}$$

By (20)–(23) we get $T(r,\varphi) = O(\log r)$ so that φ is a polynomial, as asserted. This completes the proof of Lemma 2.1.

The zeros of φ are called the *extraordinary* zeros of f'. All other zeros of f' are called *ordinary* zeros of f' (those b_k whose multiplicity $= 1$ and those a_k whose multiplicities are > 1 as zeros of f). Thus a multiple zero of f'/f might count as an ordinary zero of f' (if it is among the b_k) and will count as an extraordinary zero of f'.

From now on, we shall assume that Φ has no real zeros. Thus $\Psi_1 \equiv 1$ and we need not worry about the set $\mathscr{E}$. The reason for this is that even though arguments similar to those that we shall give, would go through also if Φ has some real zeros, it would require extra bookkeeping in an already very long paper to explain the details. Therefore we feel that it is better to skip that case here. An industrious reader should be able to fill in the details and see what modifications should be made in the proofs and the conclusions in the case of a general Φ.

Now we seek to estimate or determine the degree, $\deg\varphi$, of φ. Suppose that g is real as before and $g \in \mathscr{U}_{2p}$ where $p \geq 0$. Suppose that $z = 0$ is a zero of g of order $m_0 \geq 0$. Thus $m_0 = 0$ if $g(0) \neq 0$. Since $g \in \mathscr{U}_{2p}$ we may write

$$g(z) = z^{m_0} e^{S(z)} \Pi(z) \tag{24}$$

where S is a real polynomial, say

$$S(z) = -az^{2p+2} + bz^{2p+1} + cz^{2p} + dz^{2p-1} + \cdots, \qquad \text{where } a \geq 0, \tag{25}$$

and $\Pi(z)$ is the canonical product of the zeros of g (or f) other than possibly the origin, and is of genus $p_1 \leq 2p+1$. Write

$$\sigma = \text{genus}\,(g) = \max\{\deg S, p_1\} \tag{26}$$

and note that

$$2p \leq \sigma \leq 2p+2 \tag{27}$$

since $g \in \mathscr{U}_{2p}$, with

$$\sigma = 2p+2 \qquad \text{if, and only if,} \qquad a > 0, \tag{28}$$

while

$$\begin{aligned} \sigma = 2p \text{ if, and only if, } & a = b = 0 \text{ and } p_1 = 2p, \\ & \text{or } a = b = 0,\ c > 0 \text{ and } p_1 \leq 2p-1. \end{aligned} \tag{29}$$

In all other cases, $\sigma = 2p+1$.

Following Hellerstein and Williamson, we shall provide an estimate for the degree of φ. We determine this degree precisely only in the case needed for the proof of Theorem 1.1, even though a more careful analysis along the same lines would allow us to determine the degree of φ in all cases. It should be clear from the proof and from [HW1] how an interested reader can obtain more details in the general case if so desired for some application. On the other hand, the argument is mostly the same for a general Ψ as for $\Psi(z) = z^2+1$, and so we consider a general Ψ for most of the proof.

Lemma 2.2. *Let f be given by (2), where Φ is a non-constant real polynomial with no real zeros and $g \in \mathscr{U}_{2p}$, so that f has only real zeros while we do not make any a priori assumption concerning the reality of the zeros of f' or f''.*

At each zero a_k of f, we have

$$\varphi(a_k)\operatorname{sgn}\Psi(a_k) < 0. \tag{30}$$

We have

$$p_1 + \deg\Psi - 1 \le \deg\varphi \le \deg\Psi + 2p + 1. \tag{31}$$

With the above notation, and with $\Phi(z) = (z^2+1)^n$ and $\Psi(z) = z^2+1$, we have

(i) $\deg\varphi = \deg S + \deg\Psi - 1 = \deg S + 1$ *if f has no zeros;*

(ii) $\deg\varphi = \deg S + \deg\Psi = \deg S + 2$ *and f'' has at least $2p$ non-real zeros if f has at least one but only finitely many zeros, unless S is constant and the number of zeros of f with due count of multiplicity is equal $2n$;*

(iii) $2p+2 \le \deg\varphi \le 2p+3$ *if f has infinitely many zeros.*

If, in addition, f' has only real zeros, then, for a general Φ with no real zeros, we can write $f' = g_1/(\Phi\Psi)$, where $g_1 \in \mathscr{U}_{2p}$.

The proof of Lemma 2.2 will be given in Section 15. It is based on following the proof of [HW1, Lemma 8, pp. 237–243] to the extent possible.

3. The idea of the proof of Theorem 1.1

Let f be given by (2). We show that each of L and Q in (3) takes any real value at most a finite number of times in H^+. (A more careful analysis would show that there is an upper bound for this number which depends only on the order of f and on $\deg\Psi$, even if Φ has some real zeros.) We prove that Q maps each component of K (compare (6)) conformally onto H^+ and that the number of such components is bounded by a number depending on $\deg\Psi$.

Suppose for the rest of this section that $\Psi(z) = z^2+1$. Then there are at most 2 components of K. By a number of considerations from iteration theory, we show that f has only finitely many zeros. We prove that Λ is bounded (and, in fact, connected), which implies that $g \in \mathcal{U}_0$.

It is quite tedious to find a contradiction when f has only finitely many zeros and $g \in \mathcal{U}_0$. It arises as follows. We show that K has exactly 2 components, and that K^- is bounded and has a component W that separates the 2 components of K. Such a component W arises since there must be 2 consecutive zeros of f'', say a, b, that are not separated by any zeros of f or of f'. Then a, b are zeros of Q' and hence either $[a, b] \subset \partial W$ for a component W of K^-, or there is a closed segment J with $J \subset \partial W$ where the end points of J are a zero of f and one of a, b, while $f' \neq 0$ on J. But since ∂W must contain a pole of Q, that is, a zero of f'/f, there must be another segment $[c, d]$ in $\mathbf{R} \cap \partial W$. This leads to the separating property of W. We then show that such a separating component cannot exist.

It may be useful to examine what happens when g is a polynomial of degree $\leq 2n-1$, when it is possible for $ff'f''$ to have only real zeros. In that case f' and f'' collect their extraordinary zeros at the ends; for example, f' has one extraordinary zero on $(-\infty, a_\tau)$ and one on (a_ω, ∞). Hence we do not get two consecutive zeros of f'' as above, and K^- is not forced to have a separating component. In fact, K is connected. When $\deg g - 2n \in \{0, 1\}$, K is also connected even if the extraordinary zeros of f' may both be on $(-\infty, a_\tau)$ or on (a_ω, ∞).

4. Number of values taken in the upper half plane

When Sheil-Small [S] considered the number of non-real zeros of the second derivative of a real entire function f of finite order with only real zeros, it was essential for his work to get an estimate for the number of times that the function L can take a real value in the upper half plane.

One of the most crucial initial observations we make is the following. Suppose that $b \in \mathbf{R}$. Suppose that $z \in H^+$ so that $z \neq b$. Then $Q(z) = b$ if, and only if,

$$\frac{\bigl(f/(z-b)\bigr)'}{f/(z-b)} = L(z) - \frac{1}{z-b} = 0. \tag{32}$$

The function $f/(z-b) = g/\bigl((z-b)\Phi\bigr)$ has the same general properties as f, with Φ replaced by $(z-b)\Phi$. (Note that $(z-b)\Phi$ has at least one real zero.) Thus we may apply to $f/(z-b)$ some of the auxiliary arguments that we may have applied to f. This emphasizes the importance of first studying the situation on a more general level, when the polynomial Φ in (2) is reasonably arbitrary, before specializing to the case $\Phi(z) = (z^2+1)^n$, say. The above observation allows us to deduce in a more unified way that not only the function L (as noted by Sheil-Small in [S, p. 182]) but also the function Q will take any real value in H^+ at most a finite number of times (but compare [S, Lemma 5, p. 186]). This will greatly aid in the study of the behaviour of Q in H^+.

Another important tool is the study of the iterates of a branch of Q^{-1} taking H^+ onto a component U of K. Since $U \subset H^+$, we may iterate Q^{-1} in H^+ and deduce by the classical Denjoy–Wolff fixed point theorem (see, for example, [V, Ch. VI] or [Sa, p. 54]) that Q^{-1} has an attracting fixed point on $\partial H^+ = \overline{\mathbf{R}}$ or at a pole of f in H^+, which corresponds to a repelling fixed point of Q, with a proper interpretation if this fixed point is at infinity. The study of such iterates is an essential new idea that we bring to the study of the zeros of derivatives in this paper, to complement the level set methods inaugurated in this context by Sheil-Small [S]. (In Sections 8–14 we use the Fatou–Julia theory of iteration in numerous ways also.)

Lemma 4.1. *Let f be given by* (2), *where $g \in \mathcal{U}_{2p}$. Then each of the functions Q and L takes any real value only finitely many times in H^+.*

Proof of Lemma 4.1. Let f be given by (2), and suppose that $c \in \mathbf{R}$. Note that $g \in \mathcal{U}_{2p}$ if, and only if, $e^{-cz}g \in \mathcal{U}_{2p}$. By Lemma 2.1, $\varphi(c,z)$ is a polynomial. By (10), the number of times that L takes the value c in H^+ is at most $\frac{1}{2}\deg\varphi(c,z)$. This proves Lemma 4.1 for L.

Then pick $b \in \mathbf{R}$ and define $F(z) = f(z)/(z-b) = g(z)/\Psi_2(z)$ where $\Psi_2(z) = (z-b)\Phi(z)$. For $z \in H^+$, we have

$$Q(z) - b = z - \frac{1}{L(z)} - b = (z-b) - \frac{1}{L(z)} = 0$$

if, and only if, (32) holds, that is, if $(F'/F)(z) = 0$. Neither L nor $z-b$ vanishes at such a point, so that any such point has the same multiplicity as a zero of F'/F and as a zero of $Q-b$. Applying Lemma 2.1 to F instead of f we see that F'/F has only finitely many zeros in H^+. This completes the proof of Lemma 4.1.

We remark that if Φ has no real zeros, then by Lemma 2.2, the function L takes any real value at most $\frac{1}{2}\deg\Psi + p + 1$ times in H^+, with due count of multiplicity.

We shall often benefit from the following result [T, Theorem VIII.14, Corollary, p. 308], also used in [S, p. 182].

Lemma 4.2. *Let h be meromorphic in the Jordan domain D. Suppose that γ_1 and γ_2 are two disjoint open arcs of ∂D in a neighbourhood of a point $b \in \partial D$ with the point b as a common end point, and that h extends continuously to $\gamma_1 \cup \gamma_2$. Suppose that there are at least three points in $\overline{\mathbf{C}}$ that h takes only finitely many times in D.*

Suppose that $h(z) \to a_j \in \overline{\mathbf{C}}$ as $z \to b$ along γ_j, for $j = 1,2$. Then $a_1 = a_2$, and $h(z) \to a_1$ as $z \to b$ in D.

This easily extends to the following statement, by replacing, if necessary, a domain by a smaller domain which is a Jordan domain.

Lemma 4.3. *Let D be a domain whose boundary is contained in a path with at most finitely many self-intersections, none of them occurring at a point $b \in \partial D$. Let h be meromorphic in the domain D. Suppose that γ_1 and γ_2 are the two disjoint open arcs of ∂D in a neighbourhood of the point $b \in \partial D$ with the point b as a common end point, and that h extends continuously to $\gamma_1 \cup \gamma_2$. Suppose that there are at least three points in $\overline{\mathbf{C}}$ that h takes only finitely many times in D.*

Suppose that $h(z) \to a_j \in \overline{\mathbf{C}}$ as $z \to b$ along γ_j, for $j = 1, 2$. Then $a_1 = a_2$, and $h(z) \to a_1$ as $z \to b$ in D.

5. Zeros on the boundary of Λ or K

For the rest of the paper, we assume that Φ has no real zeros. For clarity, we shall occasionally repeat this assumption in the statements of some lemmas.

The following lemma explains which zeros of f or f' can be on the boundary of Λ or K. It turns out that only a finite number of points arises in this way, and their location can be described. The inequality (30) of Lemma 2.2 plays an essential role in proving the finiteness. This is useful in limiting the behaviour of the components of Λ and K, and particularly part (4) of the following lemma will be of crucial importance in extending to Q certain properties possessed by L. If x is real, we write $[x]$ for the greatest integer that does not exceed x.

Lemma 5.1. *Let f be as in (2), where g is a real entire function of finite order and Φ is a non-constant real polynomial with no real zeros, and suppose that f and f' have only real zeros.*

(1) (i) *Suppose that $x_0 \in \partial\Lambda$ and $L(x_0) = 0$. Then $f'(x_0) = 0 \neq f(x_0)$ and so $x_0 \in \mathbf{R}$.*

Next, if $x_0 \in \mathbf{R}$ and $f'(x_0) = 0 \neq f(x_0)$ then $(f''/f)(x_0) = 0$ if, and only if, x_0 is a multiple zero of L, and hence, $(f''/f)(x_0) \neq 0$ unless either $\psi(x_0) = \varphi(x_0) = 0$ or $\varphi(x_0) = 0 \neq \psi(x_0)$ and x_0 is a multiple zero of φ; and $x_0 \in \partial\Lambda$ if, and only if, $(f''/f)(x_0) \geq 0$. If $(f''/f)(x_0) > 0$ then x_0 lies on the boundary of exactly one component of Λ and x_0 has a neighbourhood that does not intersect Λ^-. If $(f''/f)(x_0) = 0$ then $x_0 \in \partial\Lambda \cap \partial\Lambda^-$. If $x_0 \in \mathbf{R}$, $\varphi(x_0) \neq 0$, and $f'(x_0) = 0 \neq f(x_0)$, then $(f''/f)(x_0) < 0$.

If $f(a) = 0$ and $(f'/f)(b) = 0$, and if $ff' \neq 0, \infty$ on the open interval with end points a and b, then $(f''/f)(b) \leq 0$.

(ii) *Let I be an interval of the form $(-\infty, a_\tau)$, (a_k, a_{k+1}), or (a_ω, ∞), containing the zeros $t_1 \leq \cdots \leq t_l$ of φ. If $I = (a_k, a_{k+1})$ then l is even and we have $(f''/f)(x_0) \geq 0$ for $\frac{1}{2}l$ of the zeros x_0 of φ among the t_j. If I is unbounded then we have $(f''/f)(x_0) \geq 0$ for $\frac{1}{2}l$ or $\frac{1}{2}(l-1)$ of the zeros x_0 of φ among the t_j according as l is even or odd. Here the points x_0 which are zeros of f' of order $m \geq 1$ have been taken into account with weight $\frac{1}{2}m$ if m is even and with weight $\frac{1}{2}(m + \varepsilon)$ if m is odd, where $\varepsilon = \operatorname{sgn}(f^{(m+1)}/f)(x_0) \in \{1, -1\}$. In partic-*

ular, those at which $(f''/f)(x_0) > 0$ *have been taken into account with weight* 1 *and those at which* $(f''/f)(x_0) < 0$ *with weight* 0.

(iii) *Thus, with these weights, the total number of points* $x_0 \in \partial\Lambda$ *with* $f'(x_0) = 0 \neq f(x_0)$ *is finite, and is at most* $1 + \left[\frac{1}{2}(\deg\varphi)\right]$. *In particular, if* $\Phi(z) = (z^2+1)^n$, *then this upper bound is* $n+1$, *which can be lowered to* n *if* $-\tau = \omega = \infty$ *(and then* $\deg\varphi$ *is even).*

(2) *Suppose that* $f(x_0) \neq \infty$ *and* $L(x_0) = \infty$. *Then* $f(x_0) = 0$ *and so* $x_0 \in \mathbf{R}$. *But if* x_0 *is a zero of* f *of order* $m \geq 1$ *then* $L(z) = m/(z-x_0) + O(1)$ *as* $z \to x_0$ *so that* $\operatorname{Im} L(z) < 0$ *if* $z \in H^+$ *and* $|z - x_0|$ *is small enough. Thus then* $x_0 \in \partial\Lambda^-$ *and* x_0 *has a neighbourhood that does not intersect* Λ.

If $x_0 \in \mathbf{R}$ *and* $f(x_0) = \infty$ *then* $x_0 \in \partial\Lambda$ *and* x_0 *has a neighbourhood that does not intersect* Λ^-.

(3) *If* $z_0 \in H^+$ *and* $f(z_0) = \infty$ *then* $z_0 \in \partial\Lambda$ *and* $z_0 \in \partial\Lambda^-$. *In fact, then* z_0 *lies on the boundary of exactly one component of* Λ *and exactly one component of* Λ^-.

(4) *Suppose that* $f'(x_0) = 0 \neq f(x_0)$ *so that* $x_0 \in \mathbf{R}$. *Then* $Q(x_0) = \infty$. *We have* $x_0 \in \partial K$ *if, and only if,* $(-1/Q)'(x_0) \geq 0$, *which is true if, and only if,* $(f''/f)(x_0) \geq 0$. *This can happen only at the finitely many points* x_0 *described in* (1). *If* $(f''/f)(x_0) > 0$ *then* x_0 *lies on the boundary of exactly one component of* K *and* x_0 *has a neighbourhood that does not intersect* K^-. *If* $(f''/f)(x_0) = 0$ *then* $x_0 \in \partial K \cap \partial K^-$. *Thus if* $(f''/f)(x_0) < 0$ *then* x_0 *lies on the boundary of exactly one component of* K^- *and* x_0 *has a neighbourhood that does not intersect* K.

(5) *Suppose that* $f(x_0) = 0$ *so that* $Q(x_0) = x_0 \in \mathbf{R}$, *and let* x_0 *be a zero of* f *or order* $m \geq 1$. *Then* $x_0 \in \partial K$ *if, and only if,* $Q'(x_0) = 0$ *or* $(f''/f)(x) > 0$ *for all real* $x \neq x_0$ *sufficiently close to* x_0.

If $m = 1$ *then* $f'(x_0) \neq 0$ *and* $Q'(x_0) = 0$. *Thus* $x_0 \in \partial K \cap \partial K^-$.

If $m \geq 2$ *then* $Q'(x_0) \neq 0$ *and* $(f''/f)(z) \sim m(m-1)/(z-x_0)^2$ *as* $z \to x_0$ *so that* $x_0 \in \partial K$ *and* $x_0 \notin \partial K^-$.

(6) *Suppose that* $x_0 \in \mathbf{R} \cap \partial K \cap \partial K^-$. *Then* $Q'(x_0) = 0$ *or* $f'(x_0) = f''(x_0) = 0 \neq f(x_0)$. *Thus*

(i) $f(x_0) = 0 \neq f'(x_0)$ *or*

(ii) $f''(x_0) = 0 \neq f(x_0)f'(x_0)$ *or*

(iii) $f'(x_0) = f''(x_0) = 0 \neq f(x_0)$.

In each case, there are m *sector-like domains in* H^+ *with vertex at* x_0, *every second sector contained in* K *and every second in* K^-. *In case* (i) *we have* $m = 2 + \operatorname{ord}(f'', x_0) \geq 2$. *In case* (ii) *we have* $m = 1 + \operatorname{ord}(f'', x_0) \geq 2$. *In case* (iii) *we have* $m = 1 + \operatorname{ord}(f'', x_0) = \operatorname{ord}(f', x_0) \geq 2$.

If m *is even then* $\frac{1}{2}m$ *sectors are contained in each of* K *and* K^-.

If $m = 2l+1$ *is odd then* $l \geq 1$, *and* l *sectors are contained in one of* K *and* K^- *while* $l+1$ *sectors are contained in the other one of* K *and* K^-. *There are exactly* $l+1$ *sectors contained in* K *if, and only if,* $(f''f)(x) > 0$ *for all real* $x > x_0$

sufficiently close to x_0, *which, in case* (iii), *is equivalent to* $(f^{(2l+2)}/f)(x_0) > 0$. *If* $f'(x_0) = 0 \neq f(x_0)$, *then the number of sectors in* K *is equal to the weight associated to the point* x_0 *as in part* (1)(ii) *of this lemma.*

(7) *Suppose that* $x_0 \in \mathbf{R} \cap \partial\Lambda \cap \partial\Lambda^-$. *Then* $L'(x_0) = 0$. *Thus* $f(x_0) \neq 0$ *and* $f(x_0)f''(x_0) = \bigl(f'(x_0)\bigr)^2$. *If also* $f'(x_0) = 0$ *then furthermore* $f''(x_0) = 0$.

In each case, there are m *sector-like domains in* H^+ *with vertex at* x_0, *every second contained in* Λ *and every second in* Λ^-. *We have* $m = 1 + \operatorname{ord}(f''f - (f')^2, x_0) \geq 2$.

If m *is even then* $\frac{1}{2}m$ *sectors are contained in each of* Λ *and* Λ^-.

If $m = 2l+1$ *is odd then* $l \geq 1$, *and* l *sectors are contained in one of* Λ *and* Λ^- *while* $l+1$ *sectors are contained in the other one of* Λ *and* Λ^-. *There are exactly* $l+1$ *sectors contained in* Λ *if, and only if,* $(f''f)(x) > \bigl(f'(x)\bigr)^2$ *for all real* $x > x_0$ *sufficiently close to* x_0 (*that is, if* $(f^{(2l+2)}/f)(x_0) > 0$ *when* $f'(x_0) = 0$).

Proof of Lemma 5.1. (1) (i) Suppose that $x_0 \in \partial\Lambda$ and $0 = L(x_0) = (f'/f)(x_0)$. Then $f(x_0) \notin \{0, \infty\}$ and $f'(x_0) = 0$. Since $f'(x_0) = 0$, we have $x_0 \in \mathbf{R}$.

Suppose that $f'(x_0) = 0 \neq f(x_0)$ so that $x_0 \in \mathbf{R}$. By (18), we have $\psi(x_0) = 0$ or $\varphi(x_0) = 0$. We have $f''/f = (f'/f)^2 + (f'/f)'$ so that $(f''/f)(x_0) = (f'/f)'(x_0)$. Hence $(f''/f)(x_0) = 0$ if, and only if, x_0 is a multiple zero of L. Since all the zeros of ψ are simple, x_0 is a multiple zero of L if, and only if, we have $\psi(x_0) = \varphi(x_0) = 0$, or $\varphi(x_0) = 0 \neq \psi(x_0)$ and x_0 is a multiple zero of φ. If $L(x_0) = 0$ and $L'(x_0) = (f''/f)(x_0) < 0$ then x_0 has a disk neighbourhood D such that $\operatorname{Im} L(z) < 0$ for all $z \in D \cap H^+$. If $L(x_0) = 0$ and $L'(x_0) = (f''/f)(x_0) > 0$ then x_0 has a neighbourhood D such that $\operatorname{Im} L(z) > 0$ for all $z \in D \cap H^+$. If $L(x_0) = 0$ and $L'(x_0) = (f''/f)(x_0) = 0$ then L has a multiple zero at x_0, and it is clear that $x_0 \in \partial\Lambda \cap \partial\Lambda^-$. We conclude that $x_0 \in \partial\Lambda$ if, and only if, $(f''/f)(x_0) \geq 0$. Also this shows that if $(f''/f)(x_0) > 0$ then x_0 lies on the boundary of exactly one component of Λ and x_0 has a neighbourhood that does not intersect Λ^-.

Next we show that $(f''/f)(x_0) < 0$ if $L(x_0) = 0$ and $\varphi(x_0) \neq 0$. So suppose that $L(x_0) = 0$ and $\varphi(x_0) \neq 0$. As we saw above, we then have $L'(x_0) = (f''/f)(x_0) \neq 0$. We have $\psi(x_0) = 0$ and so by (18), $L'(x_0) = \psi'(x_0)\varphi(x_0)/\Psi(x_0)$. This can happen only when f has at least two distinct zeros (so that ψ has at least one zero), and then it follows from (16) and (17) that $\psi'(x) > 0$ for all real x such that $\psi(x)$ is finite (this follows also directly from (34) in Section 15. In particular, $\psi'(x_0) > 0$. Further, we must have $x_0 = b_k \in (a_k, a_{k+1})$ for some k, where f has no poles on (a_k, a_{k+1}), and we have specified that b_k is the largest or smallest zero of f' on (a_k, a_{k+1}). Since $\varphi(x_0) \neq 0$, it follows that b_k is a simple zero of f'. The proof of Lemma 2.2 is independent of the rest of the paper. We apply the inequality (43), which is obtained in the course of that proof, and deduce that $\varphi/\Psi < 0$ at each of a_k and a_{k+1}. Now Ψ retains its sign on (a_k, a_{k+1}), while φ can change its sign only at a zero of f' which is also a zero of φ. Since

b_k is the largest or smallest zero of f' on (a_k, a_{k+1}), it follows that φ/Ψ has the same sign at $x_0 = b_k$ as it has at a_k or at a_{k+1}, as the case may be. So at any event, we have $(f''/f)(x_0) = L'(x_0) = (\psi'\varphi/\Psi)(x_0) < 0$, as claimed. A similar argument proves the last statement of part (1)(i).

(1) (ii) Let I be an interval of the form $(-\infty, a_\tau)$, (a_k, a_{k+1}), or (a_ω, ∞), containing the zeros $t_1 \le \cdots \le t_l$ of φ.

If $I = (a_k, a_{k+1})$ then f' has an odd number of zeros on I. One of them is b_k and the others are zeros of φ. Thus l is even. Without loss of generality, suppose that $b_k \le t_1$. As we have seen, if b_k is a simple zero of f' then $(f''/f)(b_k) < 0$ while otherwise $(f''/f)(b_k) = 0$. At the points t_j, the equalities $f''/f = L' = \psi\varphi'/\Psi$ hold, and the sign of $\psi\varphi'/\Psi$ alternates in the same way as the sign of φ' since ψ and Ψ retain their sign on (b_k, a_{k+1}). If the distinct zeros of φ on (b_k, a_{k+1}) are $u_1 < \cdots < u_\mu$ with multiplicities $\nu_1, \ldots, \nu_\mu$, then $\varphi'(u_j) = 0$ if, and only if, $\nu_j \ge 2$. We have $\nu_1 + \cdots + \nu_\mu = l - l_0$, where $l_0 + 1 = \operatorname{ord}(f', b_k)$.

If $x_0 \in (b_k, a_{k+1})$ is a zero of L of order $m_1 \ge 2$ then, since $L' = (f''/f) - L^2$, x_0 is a zero of f''/f of order $m_1 - 1$, and a zero of φ of order m_1. (If $x_0 = u_j$ then $m_1 = \nu_j + 1$.) Hence $f^{(j)}(x_0) = 0$ for $1 \le j \le m_1$ while $(f'')^{(m_1-1)}(x_0) = f^{(m_1+1)}(x_0) \ne 0$. We have

$$\operatorname{sgn} L(x) = \operatorname{sgn}(f''/f)(x) = \operatorname{sgn}(x - x_0)^{m_1-1} \operatorname{sgn}(f^{(m_1+1)}/f)(x_0)$$

for all $x \ne x_0$ that are sufficiently close to x_0. Applying this to $x_0 = u_j$, taking into account that L retains its sign on (u_j, u_{j+1}), and writing $\varepsilon_j = \operatorname{sgn}(f^{(m_1+1)}/f)(u_j)$, where m_1 depends on j, we see that $(-1)^{\nu_{j+1}}\varepsilon_j\varepsilon_{j+1} > 0$ for $1 \le j < \mu$. Further, if $t_1 = b_k$, set $u_0 = b_k$. If b_k is a zero of f' of order $m_0 \ge 2$ then t_1 is a zero of φ of order $m_0 - 1$, and $(f''/f)(t_1) = 0$. With $\varepsilon_0 = \operatorname{sgn}(f^{(m_0+1)}/f)(u_0)$, we have $(-1)^{\nu_1}\varepsilon_0\varepsilon_1 > 0$.

So if ν_{j+l} is odd while $\nu_{j+1}, \nu_{j+2}, \ldots, \nu_{j+l-1}$ are even, then $\varepsilon_j\varepsilon_{j+l} < 0$. If $f''(u_j) \ne 0$ then $\varepsilon_j = \operatorname{sgn}(f''/f)(u_j)$. Thus the weight associated with u_j is 1 if $(f''/f)(u_j) > 0$ and is 0 if $(f''/f)(u_j) < 0$. Thus these weights indeed count only those zeros of f' at which $f''/f \ge 0$, even if we take into account weights for *all* the u_j. Note that if $f''(b_k) \ne 0$ then $(f''/f)(b_k) < 0$ as shown at the end of the proof of part (1)(i). Let us now assign the weights explained in (1)(ii), to the distinct zeros of φ on (a_k, a_{k+1}), these zeros being $u_1, \ldots, u_\mu$, and possibly $u_0 = b_k$.

If $l - l_0$ is even, then an even number of the integers $\nu_1, \ldots, \nu_\mu$ is odd, and the odd numbers ν_j can be paired (the first two, the next two, and so on). We have $\varepsilon_j\varepsilon_{j+l} < 0$ if ν_j and ν_{j+l} are odd while $\nu_{j+1}, \ldots, \nu_{j+l-1}$ are even. If ν_{j+1} is even then $\varepsilon_j\varepsilon_{j+1} > 0$. Thus, if the odd numbers among the ν_j are those with $j = k_q$ where $1 \le k_1 < k_2 < \cdots < k_{2r} \le \mu$, we find that

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{k_1-1} = -\varepsilon_{k_1} = -\varepsilon_{k_1+1} = -\varepsilon_{k_2-1} = \varepsilon_{k_2} = \cdots = \varepsilon_{k_{2r}} = \cdots = \varepsilon_\mu,$$

and in particular, the signs of ε_{k_q} alternate. Thus $\sum_{q=1}^{r}\varepsilon_{k_q}=0$. Hence the total weight corresponding to the points $u_1,\ldots,u_\mu$ is $\frac{1}{2}(l-l_0)$. If $l_0=0$ then u_0 does not exist and the total weight is $\frac{1}{2}l$. If $l_0\neq 0$ the l_0 is even since l is even. The point u_0 has the weight $\frac{1}{2}(l_0+1+\varepsilon_0)$. The argument at the end of the proof of part (1)(i) shows that $L'(x)<0$ for all real $x<b_k=u_0$ that are sufficiently close to u_0. Since f' has a zero of the odd order l_0+1 at b_k, it follows that $L'(x)<0$ for $u_0<x<u_0+\delta$ as well, for some small $\delta>0$. This implies that $\varepsilon_0=-1$ so that u_0 has the weight $\frac{1}{2}l_0$. Thus the total weight is $\frac{1}{2}l$ in all cases when l is even. This argument also shows that when the total number of zeros of f' on I (including b_k) is odd (here $l+1$ where l is even) then the total weight is equal to $\frac{1}{2}\big((l+1)-1\big)$.

If one of the end points of I is infinite, essentially the same argument can be used. The conclusions of part (1)(iii) are now easily obtained by taking into account the above results on all intervals I determined by successive zeros of f, and adding up.

(2), (4), (5) These statements are clear.

(3) This follows from the fact that all the poles of $L=f'/f$ are simple.

(6) Suppose that $x_0\in\mathbf{R}\cap\partial K\cap\partial K^-$. Then $Q'(x_0)=0$ or x_0 is a multiple pole of Q, in which case $f'(x_0)=f''(x_0)=0\neq f(x_0)$. Thus one of the (obviously mutually exclusive) cases (i)–(iii) must occur.

In each case, it follows from the local behaviour of the meromorphic function Q that if x_0 is a zero of Q' of order $m-1\geq 1$ or a pole of Q of order $m\geq 2$, then there are m sector-like domains in H^+ with vertex at x_0, every second sector contained in K and every second in K^-. Since $Q'=ff''/(f')^2$, it is easy to verify the given formulas for m, and we leave this to the reader. The remaining statements of part (6) are now easy to verify and are left to the reader.

(7) Suppose that $x_0\in\mathbf{R}\cap\partial\Lambda\cap\partial\Lambda^-$. Then $L'(x_0)=0$ or x_0 is a multiple pole of L. Since all the poles of L are simple, it follows that $L'(x_0)=0$ and $L(x_0)$ is finite. Thus $f(x_0)\neq 0$ and $f(x_0)f''(x_0)=\big(f'(x_0)\big)^2$. If $f'(x_0)=0$ then clearly also $f''(x_0)=0$.

As in the proof of part (6), we see that there are m sector-like domains in H^+ with vertex at x_0, every second contained in Λ and every second in Λ^-, where $m=1+\operatorname{ord}\big(f''f-(f')^2,x_0\big)\geq 2$. The remaining statements of part (7) are now easy to verify and are left to the reader. This completes the proof of Lemma 5.1.

6. Structure of components

The next lemmas provide further information concerning the components of Λ and K. Among other things, we want to show that Λ and K have only finitely many components. It may not be quite clear what the strategy of proof ought to be, for example since L' may have infinitely many zeros in H^+. We have chosen to show first that $\partial\Lambda$ contains only finitely many bounded Jordan curves

(Lemma 6.1(3) below), and then have followed the same idea for K (Lemma 6.1(8) below).

Definition. Let γ be a Jordan arc, directed for convenience (such as an asymptotic path, which has a definite direction), and let h be a non-constant function, meromorphic in a neighbourhood of γ, with $h(\gamma) \subset \overline{\mathbf{R}}$. We say that h is monotonic on γ if h is strictly increasing on γ or strictly decreasing on γ, in the following sense. We say that h is strictly increasing on γ provided that h is strictly increasing in the usual sense on each open arc of γ from a to b (directed in accordance with the direction on γ) where a and b are successive points on γ which h maps onto infinity, and provided that at each point a with $h(a) = \infty$, one of the one-sided limits of $h(z)$ as $z \to a$ along γ is $+\infty$ and the other one is $-\infty$. The property of being strictly decreasing on γ is defined in a similar fashion.

The following is an alternative characterization of monotonicity: if M is a Möbius transformation of $\overline{\mathbf{R}}$ onto the unit circle S^1 then, as z traces γ, the point $M(h(z))$ traces S^1, moving all the time in the positive direction, or all the time in the negative direction, even if S^1 were traced several (perhaps infinitely many) times in this way.

Lemma 6.1. *Let f be given by (2), where $g \in \mathscr{U}_{2p}$ and Φ is a real polynomial with no real zeros, suppose that f' has only real zeros, and let L and Q be as defined in (3). Let Λ, Λ^-, K, K^- be as defined in (5) and (6). Let Ψ be as defined below (2). Write $l = 2p + \deg \Psi + 2$. Then the following statements are true.*

(1) *Each of the functions Q and L takes any real value only finitely many times in H^+: and L takes any real value at most $\frac{1}{2}l$ times in H^+, with due count of multiplicity.*

(2) *We have $\Lambda \subset K$. If V is a component of Λ then there is a component U of K containing V. If V is unbounded then so is U.*

(3) *There are only finitely many bounded Jordan curves in $\mathbf{C}$ contained in $\partial\Lambda$. The set Λ has only finitely many bounded components. Suppose, in addition, that Ψ has exactly one zero $\alpha \in H^+ \cup \mathbf{R}$.*

(i) *If $\alpha \in \mathbf{R}$, and if $\partial\Lambda$ contains a bounded Jordan curve γ, then we must have $\gamma = \partial G$ where G is the unique component of Λ with $\alpha \in \partial G$, and G must be bounded.*

(ii) *If $\alpha \in H^+$, and if T^+, T^- are the unique components of Λ, Λ^- with $\alpha \in \partial T^+ \cap \partial T^-$, and if $\gamma \subset \partial\Lambda$ is a bounded Jordan curve with interior D, then $D = T^+$ or $D = T^-$, or $T^+ \cup T^- \subset D \subset \overline{T^+ \cup T^-}$ with $\gamma \subset \partial T^+ \cup \partial T^-$. If some such D contains T^- then ∂T^- contains no segment of $\mathbf{R}$.*

(iii) *The set Λ has at most one bounded component, and if there is one, it contains α on its boundary.*

(4) *Let V be an unbounded component of Λ and let C be an unbounded*

component of $\mathbf{C}\cap\partial V$. *Then* C *is an arc with only finitely many self-intersections, and* $C\cup\{\infty\}$ *is a closed curve in* $\overline{\mathbf{C}}$. *The function* $L=\operatorname{Re}L$ *is monotonic on* C *and tends to the same limit* $a\in\overline{\mathbf{R}}$ *as* $z\to\infty$ *along the two branches of* C. *The limit* a *is the same for all* V. *If* $D\subset H^+$ *is a domain with* $\partial D\subset C$ *for any such component* C *then* $L(z)\to a$ *as* $z\to\infty$ *in* $\overline{D}$. *Furthermore,* $L(z)\to a$ *as* $z\to\infty$ *in* $\overline{V}$.

If C_1 *and* C_2 *are components of* $\mathbf{C}\cap\partial V_1$ *and* $\mathbf{C}\cap\partial V_2$, *where* V_1 *and* V_2 *are unbounded components of* Λ *(possibly* $V_1=V_2$*) and if* $C_1\neq C_2$, *then* $C_1\cap C_2$ *(if not empty) consists of isolated points in* $\mathbf{C}$. *If a set* C_1 *that arises in this way is given, then there is a unique component* V_1 *of* Λ *such that* C_1 *is a component of* $\mathbf{C}\cap\partial V_1$. *The set* Λ *has only finitely many unbounded components.*

(5) *If* W *is an unbounded component of* Λ^- *then any unbounded component of* $\mathbf{C}\cap\partial W$ *has at most finitely many points of self-intersection, and apart from at most* $\deg\Psi+2$ *exceptional components, we have* $L(z)\to a\in\overline{\mathbf{R}}$ *uniformly as* $z\to\infty$ *in* $\overline{W}$, *for some* a *depending on* f *only. The set* Λ^- *has only finitely many unbounded components.*

(6) *Suppose that also* f'' *has only real zeros. Let* U *be an unbounded component of* K *and let* C *be an unbounded component of* $\mathbf{C}\cap\partial U$. *Then* C *is a Jordan arc and* $C\cup\{\infty\}$ *is a Jordan curve in* $\overline{\mathbf{C}}$. *The function* $Q=\operatorname{Re}Q$ *is monotonic on* C. *The function* Q *tends to the same limit* $a\in\overline{\mathbf{R}}$ *as* $z\to\infty$ *along the two branches of* C. *The limit* a *depends on* f *only and is therefore the same for all unbounded components* C *of* $\mathbf{C}\cap\partial U$ *and all* U. *In fact, we have* $Q(z)\to a$ *as* $z\to\infty$ *in* $\overline{U}$.

If C_1 *and* C_2 *are unbounded components of* $\mathbf{C}\cap\partial U_1$ *and* $\mathbf{C}\cap\partial U_2$, *where* U_1 *and* U_2 *are components of* K *(possibly* $U_1=U_2$*) and if* $C_1\neq C_2$, *then* $C_1\cap C_2$ *(if not empty) is a subset of* $\mathbf{R}$ *and consists of isolated points on* $\mathbf{R}$. *In particular, if* C_1 *is given, then there is a unique component* U_1 *of* K *such that* C_1 *is a component of* $\mathbf{C}\cap\partial U_1$.

(7) *Let* U *be an unbounded component of* K, *and let* γ *be a Jordan arc lying in* U *such that* $Q(z)\to\alpha=\delta+i\varepsilon\in H^+$ *as* $z\to\infty$ *along* γ. *Then* $zL(z)\to 1$ *and* $L(z)\to 0$ *as* $z\to\infty$ *along* γ.

(8) *Suppose that also* f'' *has only real zeros. The sets* K *and* Λ *have only finitely many components. The set* ∂K *contains only finitely many bounded Jordan curves.*

(9) *If* Ψ *has exactly one zero* α *in* H^+, *let* V_0 *be the unique component of* Λ *with* $\alpha\in\partial V_0$. *If* $\Lambda\neq V_0$ *then* V_0 *is bounded. In particular, this holds if* $\Psi(z)=z^2+1$ *with* $\alpha=i\in\partial V_0$.

Remark. The conclusions other than (6), (8) remain valid even if the assumption that f' has only real zeros is dropped, since then f' and f'' have only finitely many zeros in H^+.

Proof of Lemma 6.1. Let the general assumptions of Lemma 6.1 be satisfied.

(1) This follows from Lemmas 2.2 and 4.1.

(2) If $z \in H^+$ and $\operatorname{Im} L(z) > 0$ then $\operatorname{Im}(1/L(z)) < 0$ so that $\operatorname{Im} Q(z) = \operatorname{Im} z - \operatorname{Im}(1/L(z)) > 0$. Hence $\Lambda \subset K$. Thus it is clear that if V is a component of Λ then there is a component U of K containing V. Hence, if V is unbounded then so is U.

(3) If V is a bounded component of Λ then ∂V contains a bounded Jordan curve γ contained in $\partial\Lambda$. There can be at most one component V of Λ with $\gamma \subset \partial V$ since any compact subset of $\partial\Lambda$ contains only finitely many points that can lie on the boundary of two or more components of Λ. Hence to prove that Λ has only finitely many bounded components, it suffices to show that there are only finitely many bounded Jordan curves γ with $\gamma \subset \partial\Lambda$.

Let W be a bounded component of Λ or of Λ^-. Then ∂W has only finitely many components, each consisting of finitely many closed Jordan arcs or curves which are disjoint apart from finitely many points where at least two such arcs or curves intersect, since any such point of intersection is a zero of L' (recall that all the poles of L are simple).

If ∂W contains no pole of L then $\operatorname{Im} L$ is harmonic and bounded in $\overline{W}$ and vanishes on ∂W. Thus $\operatorname{Im} L \equiv 0$ in W, which is impossible. Thus ∂W contains a pole of L, which must be a zero or pole of f.

Any pole of f is a zero of Ψ. There are only finitely many of them, and by Lemma 5.1(2) and (3), each zero of Ψ lies on the boundary of at most one component of Λ and at most one component of Λ^-. By Lemma 5.1(2), any zero of f lies on $\partial\Lambda^-$ but not on $\partial\Lambda$.

Now, given any bounded Jordan curve $\gamma \subset \partial\Lambda$, consider any bounded component W of Λ or of Λ^- such that $\gamma \cap \partial W$ contains an open arc, and such that if D is the bounded Jordan domain with $\partial D = \gamma$ then $W \subset D$. For each subarc of γ not containing zeros of L' but whose end points are zeros of L', there is such a domain W. Suppose that there exists $\beta \in \partial W$ with $f(\beta) = 0$, so that $\beta \in \mathbf{R}$ and $W \subset \Lambda^-$ by Lemma 5.1(2). Then there is $\varepsilon > 0$ such that $[\beta - 2\varepsilon, \beta + 2\varepsilon] \subset \partial W$. There is a Jordan domain $G \subset W \subset H^+$, bounded by $[\beta - \varepsilon, \beta + \varepsilon]$ and an open Jordan arc joining $\beta - \varepsilon$ and $\beta + \varepsilon$ in W. Then $\gamma \subset \overline{H^+} \setminus \overline{G}$ so that $D \cap G = \emptyset$. Note that $\gamma \subset \partial\Lambda$ while $[\beta - \varepsilon, \beta + \varepsilon] \cap \partial\Lambda = \emptyset$. Thus W is not contained in D, which is a contradiction. Since W is bounded, we conclude that there is a point $\alpha \in \partial W$ with $\Psi(\alpha) = 0$ and $L(\alpha) = \infty$. This allows us to associate α with γ and W.

There are only finitely many zeros α of Ψ and by Lemma 5.1(2) and (3), each α lies on the boundary of at most one component W of each of Λ and Λ^-. Hence there are only finitely many components W that can arise as above for all bounded Jordan curves $\gamma \subset \partial\Lambda$. The boundaries of these W contain only finitely many zeros of L' and hence consist of finitely many Jordan arcs and curves joining such zeros of L'. Any γ consists of some such arcs or coincides with such a curve since there is such a component W corresponding to each subarc of γ not containing zeros of L' but whose end points are zeros of L'. Thus there are only finitely many bounded Jordan curves $\gamma \subset \partial\Lambda$.

Suppose that Ψ has exactly one zero $\alpha \in H^+ \cup \mathbf{R}$. Then Lemma 5.1(2)–(3) implies the existence of the unique components G, T^+, T^- as stated. If $\gamma \subset \partial\Lambda$ is a bounded Jordan curve and W is as above, then the above proof shows that we must have $\alpha \in \partial W$ so that W coincides with G or T^+ or T^-. The remaining statements in (i) are now clear.

In (ii), it is clear that $D = T^+$ or $D = T^-$ or $T^+ \cup T^- \subset D \subset \overline{T^+ \cup T^-}$. If this actually happens, then T^+ or T^- or both must be bounded. If $D = T^-$ then $\partial T^- = \partial D = \gamma \subset \partial\Lambda$ so that ∂T^- contains no segment of $\mathbf{R}$. If $D = T^+ \cup T^-$ then any open segment in $\mathbf{R} \cap \partial T^-$ must be a subset of $\partial D = \gamma$. Again the fact that $\gamma \subset \partial\Lambda$ gives a contradiction. This proves (ii).

To prove (iii), suppose that V is a bounded component of Λ. Then the outermost boundary component of V contains a bounded Jordan curve $\gamma \subset \partial\Lambda$, and V is one of the components of Λ such that $\gamma \cap \partial V$ contains an open arc between zeros of L', or a closed curve. By (i) and (ii), we conclude that $V = G$ or $V = T^+$, as the case may be. This proves (iii), and the proof of part (3) is now complete.

(4) Let V be an unbounded component of Λ and let C be an unbounded component of $\mathbf{C} \cap \partial V$. If C is not a Jordan arc or Jordan curve, then C must return to the same point z_0 at least twice via different routes. Thus $L'(z_0) = 0$ since all the poles of L are simple. Clearly $L = \operatorname{Re} L$ is monotonic on C if C is traced in the natural way, possibly going several times through certain points. Then there exists a bounded Jordan curve $\gamma \subset C$ through z_0, on which $\operatorname{Im} L = 0$ except at the poles of L. By part (3), there are only finitely many such γ, and they contain only finitely many zeros of L'. Thus C contains only finitely many points z_0 of self-intersection. Thus $C \cup \{\infty\}$ is a closed curve in $\overline{\mathbf{C}}$ with only finitely many self-intersections.

If we trace C in the natural way, we clearly pass only finitely many times through any point of self-intersection.

Since L is monotonic on C, with values in $\overline{\mathbf{R}}$, the function $L(z)$ can fail to have a limit as $z \to \infty$ along a branch of C only if L takes each value in $\overline{\mathbf{R}}$ infinitely often on C. In particular, C would have to contain infinitely many poles of L. But the poles of L are zeros of f, which do not lie on $\partial\Lambda$ and hence not on C by Lemma 5.1(2), and the finitely many zeros of Ψ. We conclude that $L(z)$ has a limit as $z \to \infty$ along each of the two branches of C.

There is a domain $D \subset H^+$ with $\partial D = C \cup \{\infty\}$. By Lemma 4.2, there is $a \in \overline{\mathbf{R}}$ such that $L(z) \to a$ as $z \to \infty$ in $\overline{D}$ and in particular along C. Hence $L(C)$ contains $\overline{\mathbf{R}} \setminus \{a\}$.

Let C_j be an unbounded component of $\mathbf{C} \cap \partial V_j$ of a component V_j of Λ, for $j = 1, 2$. Let $\Gamma_j \subset C_j$ be a Jordan arc from a finite point to infinity. We may assume that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and join the finite end points of Γ_1 and Γ_2 by a

Jordan arc Γ_3 in H^+. We need not have $\operatorname{Im} L = 0$ on Γ_3. There is a Jordan domain $D_1 \subset H^+$ with $\mathbf{C} \cap \partial D_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Suppose that $L(z) \to a_j \in \overline{\mathbf{R}}$ as $z \to \infty$ along Γ_j for $j = 1, 2$. Applying Lemma 4.2, in view of part (1), we find that $a_1 = a_2$. This shows that all such limits are equal to some $a \in \overline{\mathbf{R}}$.

Clearly $H^+ \setminus C$ has at most three unbounded components, and V is contained in exactly one of them. If $V \subset D$ then $L(z) \to a$ as $z \to \infty$ in $\overline{V}$. Suppose that $V \cap D = \emptyset$. This is possible only if $\mathbf{R} \cap \partial V$ contains $(-\infty, x_1]$ or $[x_1, \infty)$ for some real x_1. Without loss of generality, suppose that $(-\infty, x_1] \subset \mathbf{R} \cap \partial V$. Then f has no zeros on $(-\infty, x_1]$ by Lemma 5.1(2). Thus f' and hence L has only finitely many zeros on $(-\infty, x_1]$ as they must be zeros of φ. Therefore, L being monotonic on $(-\infty, x_1]$, $L(x)$ must have a limit as $x \to -\infty$. Applying our previous arguments to the Jordan domain G in H^+ contained in V whose boundary consists of $(-\infty, x_1 - 1]$, a closed Jordan arc in $H^+ \cup \{x_1 - 1\}$, and a branch of C from a finite point to infinity (completely contained in H^+), we see that $L(z) \to a$ as $z \to \infty$ in $\overline{G}$. Hence $L(z) \to a$ as $z \to \infty$ in $\overline{V}$.

Suppose that V_1 and V_2 are unbounded components of Λ (possibly $V_1 = V_2$) and that C_j is an unbounded component of $\mathbf{C} \cap \partial V_j$, for $j = 1, 2$, with $C_1 \neq C_2$. Suppose that $C_1 \cap C_2 \neq \emptyset$ and pick $\beta \in C_1 \cap C_2$. Then $L'(\beta) = 0$ so that $C_1 \cap C_2$ consists of isolated points in $\mathbf{C}$. If C_1 is given then, since all points on C_1 that are not zeros of L', lie on the boundary of only one component of Λ (which must be V_1), it follows that C_1 determines V_1.

By Lemma 5.1(1)(i), Lemma 5.1(1)(iii), there are only finitely many zeros of L on $\partial\Lambda$, and they lie on the boundaries of finitely many components of Λ. Let V be an unbounded component of Λ such that $L \neq 0$ on $\mathbf{C} \cap \partial V$. Let C be an unbounded component of $\mathbf{C} \cap \partial V$. By what we have proved, it must be the case that $L(z) \to 0$ as $z \to \infty$ in $\overline{V}$ since necessarily $\{0\} = \overline{\mathbf{R}} \setminus L(C)$ for all such C. Thus C must contain a pole of L, which, by Lemma 5.1(2), must be a pole of f. Since f has only finitely many poles and each lies on the boundary of exactly one component of Λ, there are only finitely many components of Λ like this. It follows that Λ has only finitely many unbounded components.

(5) Let W be an unbounded component of Λ^-. Since $W \subset H^+$, there is at least one unbounded component C of $\mathbf{C} \cap \partial W$. Suppose that C has infinitely many points of self-intersection, and let $\gamma \subset C$ be a bounded Jordan curve. If γ contains no open segment of $\mathbf{R}$ then $\gamma \subset \partial\Lambda$ so that by part (3), there are only finitely many such curves γ and finitely many points of self-intersection of C lying on such curves γ. If γ contains an open segment of $\mathbf{R}$, then, since $W \subset H^+$ and $\gamma \subset C \subset \partial W$, it is easily seen that $W \subset D$ where D is the bounded Jordan domain with $\partial D = \gamma$. This is a contradiction since W is unbounded.

Hence C has only finitely many points of self-intersection, and C has two branches along which $z \to \infty$. The function $L = \operatorname{Re} L$ is monotonic along C. If $L(z)$ does not tend to a limit as $z \to \infty$ along such branch, then this branch contains infinitely many poles of L, hence infinitely many zeros of f. So it must contain $(-\infty, x_1)$ or (x_1, ∞) for some real x_1, or it must intersect $\mathbf{R}$ along infinitely many segments containing zeros of f, returning to H^+ in between. In the latter case, there are infinitely many bounded components of Λ, which is impossible by part (3). There are at most two components C taking up sets of the form $(-\infty, x_1)$ and (x_1, ∞), and at most two components W having such boundary components C.

For all but at most two components W, we may apply Lemma 4.2 in the obvious way and conclude that there is $a \in \overline{\mathbf{R}}$ such that $L(z) \to a$ as $z \to \infty$ in $\overline{W}$.

Each zero of Ψ lies on the boundary of at most one component of Λ^-. Hence, excluding these at most $\deg \Psi$ components and the above mentioned at most two unbounded components of Λ^-, we find that any other unbounded component W of Λ^- contains no zero of Ψ on its boundary, and has a point $a \in \overline{\mathbf{R}}$ associated with it such that $L(z) \to a$ as $z \to \infty$ in $\overline{W}$. By the argument used in the proof of part (4), the limit a is the same for all Jordan arcs on $\partial\Lambda^-$ going from a finite point to infinity provided that at least two such arcs exist (and two such arcs exist as soon as Λ^- has an unbounded component other than the at most two exceptional components discussed above). In fact, the limit a is obviously the same for the unbounded components of Λ and of Λ^-. Any bounded Jordan curve contained in ∂W and not containing any open segment of $\mathbf{R}$ is contained in $\partial\Lambda$. Since $\partial\Lambda$ contains only finitely many such curves by part (3), there are only finitely many W like this. If ∂W contains a Jordan curve containing an open segment of $\mathbf{R}$, it is easily seen that W is bounded.

Let Ω be the set of all unbounded components of Λ^- that have no zeros of Ψ on the boundary and are not among the at most two exceptional components, such that (a) ∂W contains no Jordan curves (hence each component of $\mathbf{C} \cap \partial W$ is a Jordan arc going to infinity at both ends), (b) ∂W does not intersect $\mathbf{R}$ at any point belonging to any interval of the form $[-M, a_\tau]$, $[a_k, a_{k+1}]$, or $[a_\omega, M]$ that contains a zero of φ, where $M > 0$ is so large that $f' \neq 0$ on $(-\infty, -M)$ or (M, ∞), as appropriate, (c) $\mathbf{R} \cap \partial W$ is connected. Thus Ω contains all but finitely many bounded components of Λ^-. For if e.g. $\mathbf{R} \cap \partial W$ is not connected then ∂W determines a bounded component D of Λ which is not similarly determined by any other component of Λ^- (if $x_1, x_2 \in \mathbf{R} \cap \partial W$, $x_1 < x_2$, and $(x_1, x_2) \cap \partial W \neq \emptyset$, then $H^+ \cap \partial W$ contains an open Jordan arc Γ joining x_1 to x_2, and D is a subset of the bounded Jordan domain inside the Jordan curve $\Gamma \cup [x_1, x_2]$). Since Λ has only finitely many bounded components by part (3), there are only finitely many such W. If $\Omega \neq \emptyset$, there is $a \in \overline{\mathbf{R}}$ such that $L(z) \to a$ as $z \to \infty$ in $\overline{W}$ for each $W \in \Omega$. If $W \in \Omega$ and C is a component of $\mathbf{C} \cap \partial W$ then L is monotonic on C. If L is one-to-one on C then $\{a\} = \overline{\mathbf{R}} \setminus L(C)$.

Combining parts (3) and (4), we see that Λ has only finitely many components, say $V_1, \dots, V_p$. Suppose Ω has at least $p+1$ elements, say $W_1, \dots, W_{p+2}$, whose boundary intersects $\mathbf{R}$. Set $\mathbf{R} \cap \partial W_j = [x_j, x_j']$, with a simple modification if $\mathbf{R} \cap \partial W_j$ is unbounded. We may assume that $x'_j \leq x_{j+1}$ for $1 \leq j \leq p+1$. There are arcs Γ_j and Γ'_j of ∂W_j from x_j and x'_j, respectively, to infinity, such that $\Gamma_j \setminus \{x_j\} \subset H^+$ and $\Gamma'_j \setminus \{x'_j\} \subset H^+$. There is a well-defined open set in H^+ between Γ'_j and Γ_{j+1}, each component of which contains at least one component of Λ. Such components of Λ are disjoint for distinct j, which leads to the existence of at least $p+1$ components of Λ. This a contradiction, and it follows that Ω contains only finitely many W with $\mathbf{R} \cap \partial W \neq \emptyset$.

Suppose that $W \in \Omega$ and $\mathbf{R} \cap \partial W = \emptyset$. Then ∂W contains no pole of L so that L is one-to-one on each component C of ∂W and $\{a\} = \{\infty\} = \overline{\mathbf{R}} \setminus L(C)$. Then C must contain exactly one zero of L, hence a zero of f', which is impossible since all the zeros of f' are real while $C \cap \mathbf{R} = \emptyset$. This completes the proof of (5).

(6) Suppose that also f'' has only real zeros. Let U be an unbounded component of K and let C be an unbounded component of $\mathbf{C} \cap \partial U$. Let $z_0 \in C$ be a point of self-intersection of C. Then $Q'(z_0) = 0$ or z_0 is a multiple pole of Q, so that $z_0 \in \mathbf{R}$. Then ∂U contains a bounded Jordan curve $\gamma \subset H^+ \cup \{z_0\}$. Let D be the bounded Jordan domain with $\partial D = \gamma$. Now D contains a bounded component G of K^-, whose boundary must contain at least one pole of Q (otherwise, $\operatorname{Im} Q$ is harmonic in G, bounded in $\overline{G}$, and vanishes on ∂G, so that $\operatorname{Im} Q \equiv 0$, which is a contradiction). But a pole z_1 of Q satisfies $f'(z_1) = 0 \neq f(z_1)$ so that $z_1 \in \mathbf{R}$. Hence $z_1 = z_0 \in \partial G$ and z_0 is a multiple pole of Q, hence a multiple zero of f' (in fact, a zero of f' of order at least 3, since ∂U returns to z_0 from H^+). In particular, $\varphi(z_0) = 0$ so that there are only finitely many points z_0 of self-intersection, involving only finitely many components U.

Thus, apart from finitely many self-intersections, C is a Jordan arc tending to infinity at both ends. Clearly $Q = \operatorname{Re} Q$ is monotonic on C. If $Q(z)$ has no limit as $z \to \infty$ along a branch of C then Q has infinitely many poles on C. But by Lemma 5.1(4), the set ∂K contains only finitely many poles of Q. Thus Q has a limit as $z \to \infty$ along each of the two branches of C. By the argument used for L in the proof of part (4), this time applying Lemma 4.1 to Q, we find that there is a fixed $a \in \overline{\mathbf{R}}$ such that $Q(z) \to a$ as $z \to \infty$ along C, for any C like this, and that $Q(z) \to a$ as $z \to \infty$ in $\overline{U}$, for every unbounded component U of K.

Again, if $\mathbf{R} \cap \partial U$ contains $(-\infty, x_1]$ or $[x_1, \infty)$ for some real x_1, say $(-\infty, x_1] \subset \mathbf{R} \cap \partial U$, then, since $\partial U \subset \partial K$ can contain only finitely many zeros of f'/f by Lemma 5.1(4), it follows that f and f' have only finitely many zeros on $(-\infty, x_1]$. Hence Q has finitely many poles on $(-\infty, x_1]$, and being monotonic on ∂U, $Q(x)$ tends to a limit as $x \to -\infty$. Now we can argue as in the proof of part (4).

Similarly, it is seen that if W is an unbounded component of K^-, if C is an unbounded component of $\mathbf{C} \cap \partial W$, and if Q has only finitely many poles on a branch of C going to infinity, then $Q(z)$ tends to this same point a as $z \to \infty$ along such a branch. If such a branch contains infinitely many poles of Q, it must intersect $\mathbf{R}$ infinitely often. There are at most two such branches (corresponding to the positive and negative real axis; this does not exclude the possibility of a single branch intersecting both the positive and negative real axis at sequences tending to infinity) and hence at most two corresponding components of K^-. Excluding such exceptional components, we see in the same way as before that $Q(z) \to a$ as $z \to \infty$ in $\overline{W}$ for any unbounded (non-exceptional) component W of K^-.

The remaining statements of (6) are verified in the same way as in part (4). Here we also note that the points in $C_1 \cap C_2$ are multiple zeros of f' or zeros of Q' and hence form a discrete subset of $\mathbf{R}$. This completes the proof of (6).

(7) Let the assumptions of (7) be satisfied. Then $zL(z) \sim z/(z-\alpha) \to 1$ and hence $L(z) \sim 1/z \to 0$ as $z \to \infty$ along γ. This proves (7).

(8) Combining parts (3) and (4) we see that Λ has only finitely many components. If $\gamma \subset \partial K$ is a bounded Jordan curve and if D is the bounded Jordan domain with $\partial D = \gamma$, then, as in the proof of (6), we see that there is $z_0 \in \gamma \cap \mathbf{R}$ with $f'(z_0) = 0 \neq f(z_0)$ and $Q(z_0) = \infty$. Also each of the two arcs of γ emanating from z_0 must be among the finitely many arcs in ∂K emanating from z_0. Since by Lemma 5.1(4) there are only finitely many points z_0 like this on ∂K, and since $Q'(z) \neq 0$ and $Q(z) \neq \infty$ in H^+ so that no further branching out of γ is possible there, it follows that there are only finitely many γ like this. As in the proof of part (3), we now see that K has only finitely many bounded components.

Let U be an unbounded component of K. By part (6), we have $Q(z) \to a \in \overline{\mathbf{R}}$ as $z \to \infty$ in $\overline{U}$. Since $Q' \neq 0$ and $Q \neq \infty$ in U and there is no path $\Gamma \subset U$ tending to ∞ such that $Q(z) \to \beta \in H^+$ as $z \to \infty$ along Γ, it follows that any branch of $(Q \mid U)^{-1}$ can be continued analytically without restriction in H^+, which defines, by the monodromy theorem, a single-valued inverse of Q mapping H^+ into U. Thus Q is one-to-one in U.

Iterating $(Q \mid U)^{-1}$ in H^+, as we may since $(Q \mid U)^{-1}(H^+) = U \subset H^+$, we see from the Denjoy–Wolff fixed point theorem (see [V, Ch. VI] or [Sa, p. 54]) that either ∂U contains a finite fixed point z_0 of Q with $|Q'(z_0)| \geq 1$, or we have $(Q \mid U)^{-1}(z) \to \infty$ as $z \to \infty$ in H^+ and $(Q \mid U)^{-1}(z)/z \to c \geq 1$ as $z \to \infty$ in any Stolz angle in H^+. In the latter case, U contains for each $\varepsilon \in \left(0, \tfrac{1}{2}\pi\right)$ the set $\{re^{i\theta} : r > r_\varepsilon,\ \varepsilon < \theta < \pi - \varepsilon\}$ for some $r_\varepsilon > 0$, so that there can be only one such U. In the former case, $f(z_0) \in \{0, \infty\}$ so that excluding finitely many U for which $f(z_0) = \infty$, we may assume that $f(z_0) = 0$. But then $|Q'(z_0)| < 1$, which is impossible. (If z_0 is a zero of f of order $m \geq 1$, then $Q'(z_0) = 1 - m^{-1} \in [0, 1)$.) We conclude that K has only finitely many unbounded components, hence only finitely many components, as asserted.

(9) Suppose that Ψ has a unique zero $\alpha \in H^+$, and let V_0 be the unique component of Λ with $\alpha \in \partial V_0$. Suppose that $\Lambda \neq V_0$. To get a contradiction, suppose that V_0 is unbounded. By part (2), K has an unbounded component U_0 with $V_0 \subset U_0$. By assumption, Λ has a component $V \neq V_0$. Now ∂V_0 contains no pole of L (by Lemma 5.1(2) and the fact that $\alpha \in \partial V_0$, $\alpha \notin \partial V$). If V is bounded, we get the same contradiction as before since then $\operatorname{Im} L = 0$ on ∂V

and $\operatorname{Im} L$ is harmonic and bounded in V. Thus V is unbounded, and we have $L(z) \to a_0 \in \overline{\mathbf{R}}$ as $z \to \infty$ in V. Since $L \neq \infty$ on ∂V, it must be the case that $a_0 = \infty$. Thus, by what we saw in the proof of part (4), $L(z) \to \infty$ as $z \to \infty$ in $\overline{V_0}$. By (6), we have $Q(z) \to a \in \overline{\mathbf{R}}$ as $z \to \infty$ in $\overline{U_0}$ and in particular, as $z \to \infty$ in V_0. Since $L(z) \to \infty$ in V_0 and $Q(z) = z - 1/L(z)$, we have $a = \infty$.

Now $Q'(z) \neq 0$ and $Q(z) \neq \infty$ in U_0, and Q maps U_0 into H^+. If any branch of Q^{-1} taking a small disk in H^+ into U_0 cannot be analytically continued throughout U_0, then there must be a path $\gamma \subset U_0$ going to infinity such that $Q(z) \to z_1 \in H^+$ as $z \to \infty$ along γ. This is impossible since $Q(z) \to \infty$ as $z \to \infty$ in $\overline{U_0}$. Thus the analytic continuation is possible, and since H^+ is simply connected, it follows from the monodromy theorem that Q^{-1} has a single-valued branch defined in H^+. Hence Q is one-to-one in U_0. Since $Q(z) \to \infty$ as $z \to \infty$ in $\overline{U_0}$, it then follows that ∂U_0 cannot contain any pole of Q, that is, any zero of L. But since $V_0 \subset U_0$ and all zeros of L are real, it follows that ∂V_0 contains no zero of L. Now we see from part (4), considering any unbounded component C of $\mathbf{C} \cap \partial V_0$, that $L(z) \to 0$ as $z \to \infty$ along C. This is impossible since $L(z) \to \infty$ as $z \to \infty$ in $\overline{V_0}$. This contradiction shows that if $\Lambda \neq V_0$ then V_0 is bounded, and the proof of part (9) is complete.

7. Number of components of K

7.1. Conformality of Q in the components of K. Let f be as in (1), where $g \in \mathscr{U}_{2p}$. Suppose that f, f', and f'' have only real zeros. If g is a polynomial of degree at most $2n+1$, then the conclusions of Theorem 1.1 follow from [H2, Theorem 8(ii)]. Hence we assume from now on that g is either transcendental or a polynomial of degree at least $2n+2$.

The following general elements enter into the proof of Theorem 1.1: the number of components of K (which we already know to be finite), the conformality of Q in each of them, the possible number (which may be infinite) and structure of the components of K^-, particularly as regards the unbounded ones (there can be only finitely many of them), the conformality of Q in each of them apart from a few possible exceptional situations, the degree of the polynomial φ, and the existence and influence of a zero x_0 of f' which is not a zero of f and which lies on ∂K (we shall call x_0 the *special* zero of f' if it exists). Different situations arise depending on if one or both of τ and ω is finite. It seems to me that there are several observations that can be made concerning these quantities in various situations. It is not clear to me what would be the absolutely shortest way to arrange the arguments so that one could make maximal use of the interdependencies that occur. I have experimented with a number of ways without finding any that would be absolutely short and uncomplicated. I will present the best choice I could find, but cannot guarantee that it could not be further simplified.

Convention. There always exists a unique component U_0 of K with $i \in U_0$ (see (8), (9)), and from now on, we reserve the notation U_0 for this component.

Lemma 7.1. *If U is a component of K then Q is one-to-one in U and maps U conformally onto H^+.*

Remark. The conclusion of Lemma 7.1 remains valid for f given by (2) instead of (1).

Proof of Lemma 7.1. Let U be a component of K. Since $Q' = (f''f)/(f')^2 \neq 0$ in H^+, any branch of Q^{-1} initially defined in a small disk in H^+, mapping the disk conformally into U, can be analytically continued without restriction in H^+ unless such continuation leads to a path going to infinity in U. This would mean that Q has an asymptotic value $\alpha = \delta + i\varepsilon \in H^+$, with $Q(z) \to \alpha$ as $z \to \infty$ along the path $\gamma \subset U$. If there is no such asymptotic value, then, since H^+ is simply connected, it follows by the monodromy theorem that we obtain a single-valued analytic function Q^{-1} in H^+, mapping H^+ onto U. Then Q is one-to-one in U and maps U conformally onto H^+. This is always the case when U is bounded for then U cannot contain a path going to infinity. Next, if Q is not one-to-one in U, then such a single-valued function Q^{-1} cannot exist in H^+, and we conclude that then Q must have an asymptotic value $\alpha \in H^+$ and in particular U must be unbounded.

By part (6) of Lemma 6.1, there is $a \in \overline{\mathbf{R}}$ such that $Q(z) \to a$ as $z \to \infty$ in $\overline{U}$ and hence, in particular, on γ. Since $Q(z) \to \alpha \in H^+$ as $z \to \infty$ along γ, we obtain a contradiction. We conclude that for every component U of K, the function Q maps U one-to-one conformally onto H^+. This proves Lemma 7.1.

Components and Denjoy–Wolff fixed points. We are able to limit the number of components of K.

Definition. If W is a domain in H^+, we say that W eventually contains every Stolz angle at infinity if for every $\varepsilon \in \left(0, \frac{1}{2}\pi\right)$ there exists $r_\varepsilon > 0$ such that

$$\{re^{i\theta} : r > r_\varepsilon,\ 0 < \varepsilon < \theta < \pi - \varepsilon\} \subset W.$$

Lemma 7.2. *The set K has at most two components U, and Q maps each one-to-one conformally onto H^+. If there are two such components then one of them, namely U_0, is bounded and contains the point i, and the other one is unbounded and eventually contains every Stolz angle at infinity.*

If a component U is bounded, then the point i is the unique Denjoy–Wolff attracting fixed point of $(Q \mid U)^{-1}$, and is a repelling fixed point of Q. If U is unbounded and does not contain i, then there is a number $c \geq 1$ such that $Q(z)/z \to 1/c$ as $z \to \infty$ in any Stolz angle at infinity. In this case, the point $\infty \in \partial U \cap \partial H^+$ is the unique Denjoy–Wolff fixed point of $(Q \mid U)^{-1}$.

The function $Q(z)$ has the same limit $a \in \overline{\mathbf{R}}$ as $z \to \infty$ in any unbounded component U of K, and the limit is $a = \infty$ if there exists $U \neq U_0$. In particular, if U_0 is unbounded then $K = U_0$ and $Q(z) \to a \in \mathbf{R}$ as $z \to \infty$ in $\overline{U_0}$.

In particular, K has at most one bounded component U, and if such a component U exists then $U = U_0$.

If U is any unbounded component of K (we may or may not have $U = U_0$) then $C = \mathbf{C} \cap \partial U$ is unbounded and connected, and is a Jordan arc tending to infinity at both ends. The domain U coincides with the Jordan domain D with $\partial D = C$ and $D \subset H^+$, and hence U lies "above" C.

The set $\mathbf{C} \cap \partial U_0$ always contains a pole of Q.

Remark. If f is given by (2) instead of (1), then the conclusion of Lemma 7.2 is to be replaced by the following: K has at most $\mu + 1$ components U, and Q maps each one-to-one conformally onto H^+, where μ is the number of distinct zeros of Φ (equivalently, the number of zeros of Ψ) in $H^+ \cup \mathbf{R}$. Namely, for each such zero z_0 of Ψ there could be a component U of K containing z_0 in its closure, and in addition, there might be one unbounded component U which eventually contains every Stolz angle at infinity.

Proof of Lemma 7.2. Let U be a component of K so that by Lemma 7.1, Q is one-to-one in U and maps U conformally onto H^+. Let h denote the branch of Q^{-1} which maps H^+ one-to-one conformally onto U. Then h maps H^+ into itself so that we may apply the Denjoy–Wolff fixed point theorem to h (see [V, p. 134] or [Sa, p. 54]). It follows that there exists a point $\zeta \in \overline{H^+}$ with $h(\zeta) = \zeta$ (so $Q(\zeta) = \zeta$) and $|h'(\zeta)| < 1$ or $h'(\zeta) = 1$, with an appropriate interpretation if $\zeta = \infty$. If $\zeta \in H^+$ then $(f/f')(\zeta) = 0$, so $f(\zeta) \in \{0, \infty\}$, which implies that $\zeta = i$. Then $i \in U$ so that $U = U_0$.

Otherwise, $\zeta \in \mathbf{R}$ or $\zeta = \infty$. If $\zeta \in \mathbf{R}$ then $f(\zeta) = 0$ and $|Q'(\zeta)| > 1$ or $Q'(\zeta) = 1$. Since $Q'(z) = \{(f''f)/(f')^2\}(z)$, we get a contradiction if $f'(\zeta) \neq 0$ for then $Q'(\zeta) = 0$. Suppose that $f'(\zeta) = 0$, and let ζ be a zero of f of order $m \geq 2$. Then $Q'(\zeta) = 1 - (1/m)$, which is also a contradiction.

Suppose that $\zeta = \infty$. In this case it follows from a theorem of Valiron (see [V, Ch. VI] or [Sa, pp. 53–54]) that $h'(z)$ has a non-tangential limit c as $z \to \infty$ in H^+, which satisfies $c \in \mathbf{R}$ and $c \geq 1$. Furthermore, $h(z)/z \to c$ as $z \to \infty$ in any Stolz angle $\{re^{i\theta} : r > 0,\ 0 < \varepsilon < \theta < \pi - \varepsilon\}$ in H^+. In particular, there can be at most one such component U. The last statement of Lemma 7.2 concerning Stolz angles is now immediate. The statement concerning limits of Q in each individual unbounded component of K follows from part (6) of Lemma 6.1, and the limit is clearly equal to infinity if $U \neq U_0$ by what we just proved. Suppose that both U and U_0 are unbounded and have unbounded boundary components C and C_0. We define a Jordan arc γ by taking an arc of C from a finite point in H^+ to ∞, and arc of C_0 from a finite point in H^+ to ∞, and an arc joining the two finite end points in H^+. Applying Lemma 4.2 to the Jordan domain D with $D \subset H^+$ and $\partial D = \gamma$ we see that $Q(z)$ tends to the same limit as $z \to \infty$ along C and C_0. By part (6) of Lemma 6.1, $Q(z)$ tends to this same limit as $z \to \infty$ in $U \cup U_0$. In fact, this argument was already given in the proof of Lemma 6.1(4).

If U is a bounded component of K then $Q \mid U$ is a conformal mapping onto

H^+ as observed before the statement of Lemma 7.1. So it follows that there is at most one such component U which, if it exists, must contain i, and hence coincide with U_0.

Let U be an unbounded component of K. Since U is simply connected, being conformally equivalent to H^+, the set ∂U has no bounded components. Hence $\mathbf{C}\cap\partial U$ has at least one unbounded component C. Now C is a Jordan arc tending to infinity at both ends. By part (6) of Lemma 6.1, Q is real and monotonic on C, and Q tends to the same limit $a\in\overline{\mathbf{R}}$ as $z\to\infty$ along either end of C. Thus for each $x\in\overline{\mathbf{R}}\setminus\{a\}$, there is $\varepsilon_x>0$ such that if $B_x=H^+\cap D_x$ where D is a disk centred at x with radius ε_x in the chordal metric, then Q takes all values in B_x in U in a small neighbourhood of a suitable point of C. Since Q is one-to-one in U, it follows that there cannot exist two distinct components C like this. Thus U has only one unbounded boundary component. It follows that $\mathbf{C}\cap\partial U=C$, which is unbounded and connected, as claimed. Now it is obvious that the domain U coincides with the Jordan domain D with $\partial D=C$ and $D\subset H^+$, so that we may say that U lies "above" C. If $U\neq U_0$ exists and if U and U_0 are both unbounded then $Q(z)\to\infty$ as $z\to\infty$ in $\overline{U}\cup\overline{U_0}$. But Q is one-to-one in U_0 and (as we shall show soon) ∂U_0 must contain a pole of Q, which leads to a contradiction. Hence U_0 is bounded if $K\neq U_0$.

Finally, we show that $\mathbf{C}\cap\partial U_0$ always contains a pole of Q. If f has at least one zero, then by Lemma 2.2(iii), we have $\deg\varphi\geq 2$ if at least one of τ,ω is infinite, and by Lemma 2.2(ii), we have $\deg\varphi\geq 3$ if τ,ω are both finite and g is transcendental. If f has no zeros (so that g must be transcendental by our assumption), Lemma 2.2(i) shows that $\deg\varphi\geq 2$. Next, there is an interval I of the form $(-\infty,a_\tau)$ or (a_k,a_{k+1}) or (a_ω,∞), or $\mathbf{R}$ if f has no zeros, such that I contains at least 2 zeros of f' with due count of multiplicity. These zeros of f' are not zeros of f, and at least one such zero z_0 of f' satisfies $(f''/f)(z_0)\geq 0$ so that $z_0\in\partial K$ by Lemma 5.1(4). Also $Q(z_0)=\infty$. If K has a component $U\neq U_0$ then, by what we have proved above, U is unbounded, Q is one-to-one in U, and $Q(z)\to\infty$ as $z\to\infty$ in $\overline{U}$. This implies that $Q\neq\infty$ on $\mathbf{C}\cap\partial U$, and so $z_0\in\mathbf{C}\cap\partial U_0$, as required.

The case remains when g is a polynomial of degree $\geq 2n+2$. Then $\deg\varphi=2$ by Lemma 2.2(ii). If there is to be no zero z_0 of f'/f with $(f''/f)(z_0)\geq 0$, then we clearly must have $x_1<a_\tau\leq a_\omega<x_2$ for the two zeros x_1,x_2 of φ. To consider this, we may assume that g has leading coefficient 1. Now $\lim_{x\to a_\omega+}(f'/f)(x)=+\infty$ while $(f'/f)(x)\sim(\deg g-2n)/x>0$ as $x\to\infty$. This shows that f'/f, or equivalently, φ, has an even number of zeros on (a_ω,∞), which gives a contradiction. This proves Lemma 7.2.

Lemma 7.3. *Let W be a component of K^-. If W is bounded then Q is one-to-one in W and maps W conformally onto H^-. If W is unbounded, let C be an unbounded component of $\mathbf{C}\cap\partial W$. Then C is a Jordan arc which can be written as $C=C^+\cup C^-$, where C^+ and C^- are Jordan arcs starting at the* *same point, going to infinity, and intersecting only their common finite end point. Concerning C^+, we have either*

(i) $Q(z) \to a^+ \in \overline{\mathbf{R}}$ *as* $z \to \infty$ *along* C^+; *or*

(ii) C^+ *intersects exactly one of* $\mathbf{R}^+$ *and* $\mathbf{R}^-$ *infinitely often at points where* $f' = 0 \neq f$, *and then* $\tau = -\infty$ *or* $\omega = \infty$.

The same choice applies independently to C^-, *except that only one of* C^+ *and* C^- *can intersect* $\mathbf{R}^+$ *infinitely often, and only one of* C^+ *and* C^- *can intersect* $\mathbf{R}^-$ *infinitely often. If both limits* a^+ *and* a^- *exist, then* $a^+ = a^-$. *In this case, if* D *is the Jordan domain with* $\mathbf{C} \cap \partial D = C$ *and* $D \subset H^+$ *then* $W \subset D$ *and* $Q(z) \to a^+$ *as* $z \to \infty$ *in* $\overline{D}$.

In particular, if C *does not intersect* $\mathbf{R}$ *outside a compact subset of* $\mathbf{R}$ *then the limits* a^+ *and* a^- *exist and are equal.*

Consequently, there can exist at most 2 *components* W *which have a boundary component* C *for which* Q *does not tend to a limit along* C^+ *or* C^-.

Remark. The conclusion of Lemma 7.3 remains valid for f given by (2) for a general Φ.

Proof of Lemma 7.3. The proof is similar to that of Lemmas 6.1, 7.1, and 7.2, and is therefore omitted.

Lemma 7.4. *Suppose that* U *is a component of* K *with* $i \notin U$, *hence unbounded. Then* $\mathbf{C} \cap \partial U$ *cannot contain any pole of* Q. *Thus any pole of* Q *on* $\mathbf{C} \cap \partial K$ *must lie on* ∂U_0. *There can be only one such pole, say* x_0. *If* x_0 *is not a simple pole of* Q, *then there are at least two sectors emanating from* x_0 *of the kind considered in part* (6) *of Lemma* 5.1. *Each sector which is contained in* K, *is contained in a different component of* K. *Hence the total number of such sectors (some of which are contained in* K^-) *can be at most* 3. *Thus* x_0 *is a pole of* Q *of order* m, *and* x_0 *is a zero of* f' *of order* m, *where* $1 \leq m \leq 3$. *If* x_0 *is a simple zero of* f' *then* $(f''/f)(x_0) > 0$. *If* x_0 *is a triple zero of* f' *then exactly one of the three sectors in* H^+ *emanating from* x_0 *lies in* K, *and* $(f^{(4)}/f)(x_0) < 0$.

Furthermore, x_0 *lies on the boundary of exactly one component* V *of* Λ, *and if* C *is the component of* ∂V *containing* x_0, *then* C *passes through* x_0 *exactly once. Also* x_0 *is an extraordinary zero of* f', *and* $\varphi(x_0) = 0$.

Convention. From now on, we reserve the notation x_0 for this (extraordinary) zero of f', which we shall call the *special zero* of f'.

Proof of Lemma 7.4. Suppose that U is a component of K with $i \notin U$, hence unbounded. Since Q is one-to-one in U and takes values with large modulus in Stolz angles at infinity, it follows that ∂U cannot contain any pole of Q. Thus any pole of Q on ∂K, if any, must lie on ∂U_0. There can be only one such pole, say x_0, since Q is one-to-one in U_0. If x_0 is not a simple pole of Q, then there are at least two sectors emanating from x_0 of the kind considered in part (6) of Lemma 5.1. Each sector which is contained in K, is contained in a different component of K since Q is one-to-one in each such component so that when z traces the boundary of the component, it can hit a pole of Q only once. If at least 2 sectors are contained in K, then x_0 must lie on the boundary of at least two distinct components of K, which is impossible as only U_0 is available for this purpose. This shows that exactly one of the sectors in H^+ emanating from x_0 and discussed in Lemma 5.1(6), lies in K. Hence the total number of such sectors can be at most 3. Thus x_0 is a pole of Q of order m, and x_0 is a zero of f' of order m, where $1 \le m \le 3$. If x_0 is a simple zero of f' then $(f''/f)(x_0) > 0$ by part (4) of Lemma 5.1. If x_0 is a triple zero of f' then exactly one of the three sectors in H^+ emanating from x_0 lies in K, which implies, considering part (6) of Lemma 5.1 with $l = 1$, that $(f^{(4)}/f)(x_0) < 0$. The statements concerning Λ now follow from Lemma 5.1(1)(i) and (7).

Since $(f''/f)(x_0) \ge 0$, it follows from Lemma 5.1(1)(i) that x_0 is an extraordinary zero of f', which by definition means that $\varphi(x_0) = 0$. This completes the proof of Lemma 7.4.

Convention. From now on, we denote by V_0 the unique component of Λ with $i \in \partial V_0$. Note that $V_0 \subset U_0$ by the definitions and by part (2) of Lemma 6.1.

Lemma 7.5. (i) *We have* $\Lambda = V_0$.

(ii) *The special zero x_0 always exists, and V_0 is always bounded. Also, ∂V_0 is a Jordan curve that contains a zero of L, which must be the special zero x_0. Furthermore, L maps V_0 one-to-one conformally onto H^+, and Q maps U_0 one-to-one conformally onto H^+. Also $x_0 \in \partial U_0$.*

Proof of Lemma 7.5. (i) Suppose that V is a component of Λ other than V_0. Now ∂V contains no pole of L so that V is unbounded and $L(z) \to \infty$ as $z \to \infty$ along any unbounded component C of ∂V. Thus $L = 0$ at some point $z_0 \in C$. So $f' = 0 \ne f$ at z_0 and hence $z_0 \in \mathbf{R}$. By part (1)(i) of Lemma 5.1, we then have $(f''/f)(z_0) \ge 0$, so that by Lemma 7.4, we must have $z_0 = x_0$. By Lemma 7.4, x_0 is a zero of f' of order $m \le 3$, and if $m = 3$, then $(f^{(4)}/f)(x_0) < 0$. Thus x_0 lies on the boundary of at most one component of Λ. This implies that there can exist at most one such C, and at most one such V. Now $V \subset U$ for some component U of K, and U is unbounded since V is. Thus $x_0 \in \partial U$ so that Q has a pole on ∂U. By Lemma 6.1(6), we have $Q(z) \to a \in \overline{\mathbf{R}}$ as $z \to \infty$ in $\overline{U}$. In particular, $Q(z) \to a$ as $z \to \infty$ along C. Now $L \to \infty$ and so $Q = z - (1/L) \to \infty$ as $z \to \infty$ along C. Thus $a = \infty$. So Q takes values in H^+ close to infinity twice in U, close to x_0 and close to infinity. Since $Q \mid U$ is one-to-one by Lemma 7.2, we get a contradiction. Thus $\Lambda = V_0$. This proves (i).

(ii) Suppose that V_0 is bounded. It follows that L maps V_0 onto H^+ (this does not imply that L must be one-to-one in V_0). Hence ∂V_0 contains at least one zero z_0 of L. Thus $f'(z_0) = 0 \ne f(z_0)$. Hence z_0 is real. Now part (1)(i) of Lemma 5.1 implies that $(f''/f)(z_0) \ge 0$, and so, by what has been proved before, we have $z_0 = x_0$, the special zero of f'. In particular, it follows that the special zero x_0 exists. The function L maps V_0 onto H^+ a certain finite number of times, say m times, where $m \geq 1$. Since V_0 is a bounded component of Λ, the function L maps each component of ∂V_0 onto $\overline{\mathbf{R}}$. Now $L(z)$ must trace $\overline{\mathbf{R}}$ also a certain number μ of times as z traces any component of ∂V_0; the sum of all such μ is m. Since L has only one zero on ∂V_0, it follows that $m = 1$ and L is one-to-one in V_0 and on ∂V_0, and ∂V_0 is connected. If ∂V_0 has a point of self-intersection, then L is not one-to-one on ∂V_0. Thus ∂V_0 is a Jordan curve. Since $V_0 \subset U_0$, we have $x_0 \in \partial U_0$.

The special zero x_0 always exists (under our standing assumption that g is transcendental or g is a polynomial with $\deg g \geq 2n+2$) by Lemma 7.2. More precisely, any finite pole of Q on ∂K must coincide with x_0. If U_0 is bounded then Q has a pole on ∂U_0. If U_0 is unbounded then $Q(z) \to a \in \mathbf{R}$ as $z \to \infty$ in $\overline{U_0}$, and since Q maps U_0 conformally onto H^+, there must then be a finite pole of Q on ∂U_0. This completes the proof that x_0 always exists.

If V_0 is unbounded then U_0 is unbounded since $V_0 \subset U_0$.

Suppose that V_0 is unbounded, and let C be an unbounded component of $\mathbf{C} \cap \partial V_0$. We first claim that $\partial V_0 = C$. Suppose that ∂V_0 also has a bounded component C_1. It follows from the structure of analytic curves (see [MO]) that C_1 is the union of finitely many Jordan curves, any two of which can only intersect at no more than one point (in particular, C_1 might be a Jordan curve). Furthermore, then $L(C_1) = \overline{\mathbf{R}}$ so that C_1 contains a pole of L, that is, the point i or a zero of f. But the zeros of f are bounded away from Λ by part (2) of Lemma 5.1, so that $i \in C_1$. Now C_1 must also contain a zero z_0 of L. Thus $f'(z_0) = 0 \neq f(z_0)$. Hence z_0 is real. Now part (1)(i) of Lemma 5.1 implies that $(f''/f)(z_0) \geq 0$, and so, by what has been proved before, we have $z_0 = x_0$, the special zero of f'. Next, $L(z)$ must trace $\overline{\mathbf{R}}$ completely a certain integral number of times as z traces C_1 once. Since by the last statement of Lemma 7.4, L has only one zero (namely, x_0) on C_1, it follows that L is one-to-one on C_1. This now shows that C_1 is a Jordan curve (for otherwise, when z traces C_1 once, at least one point would be encountered at least twice, so that L would take some value at least twice). Let D_1 be the Jordan domain bounded by C_1. Since V_0 is unbounded, we have $V_0 \cap D_1 = \emptyset$. Since C_1 intersects $\mathbf{R}$, we obtain a contradiction, since a bounded boundary component of a domain can be enclosed by a bounded Jordan curve that lies entirely inside the domain. That is impossible here since $C_1 \cap \mathbf{R} \neq \emptyset$ and $V_0 \subset H^+$. Thus ∂V_0 cannot have any bounded components.

Thus every component of $\mathbf{C} \cap \partial V_0$ is unbounded. If there are (at least) two such components, say the above C and another C_2, then we may choose C_2 so that exactly one of C and C_2 contains the point i. But $L(z)$ must tend to the same limit $a \in \overline{\mathbf{R}}$ as $z \to \infty$ along C or C_2. If $i \in C$ then $a \neq \infty$ (considering the limit on C), and then, since $i \notin C_2$, we have $a = \infty$ (considering the limit on C_2). The same contradiction arises if $i \in C_2$. Hence $\partial V_0 = C$ is connected. This also shows that if D is the Jordan domain with $\partial D = C$ and $D \subset H^+$

then $V_0 \subset D$, and V_0 lies "above" C. Indeed we now see that $D = V_0$ since $\mathbf{C} \cap \partial V_0 = C$.

Further, $L(z) \to b \in \overline{\mathbf{R}}$ as $z \to \infty$ along C. Suppose that $b \neq 0$. Since $V_0 \subset U_0$, then in view of part (6) of Lemma 6.1, $Q(z) = z - \bigl(1/L(z)\bigr) \to \infty$ as $z \to \infty$ in $\overline{U_0}$ and in particular on $C_0 = \mathbf{C} \cap \partial U_0$. Then, since Q is one-to-one in U_0, it follows that Q has no pole on C_0, and since poles of Q are at the zeros of L, we deduce that L has no zero on C_0 and hence not in $\overline{V_0}$. But then $b = 0$ since $\{b\} \subset \overline{\mathbf{R}} \setminus L(C)$, $0 \in \overline{\mathbf{R}} \setminus L(C)$, and $\overline{\mathbf{R}} \setminus L(C)$ contains at most one point. This contradicts our assumption that $b \neq 0$. Thus in any case $b = 0$. If the special zero x_0 of f' exists then $L(x_0) = 0$ and $x_0 \in \partial\Lambda$ so that $x_0 \in C$. This is impossible (since $\{b\} = \overline{\mathbf{R}} \setminus L(C)$, due to the fact that L can have at most one pole (the point i) on C so that L is one-to-one on C), and so x_0 does not exist. This is a contradiction, and it follows that V_0 must be bounded. This proves (ii), and the proof of Lemma 7.5 is complete.

8. The proof that f has only finitely many zeros

8.1. Finding a special component of K^-. Let f be as in (1), where $g \in \mathscr{U}_{2p}$. If g is a polynomial, suppose that $\deg g \geq 2n+2$. Suppose that f, f', and f'' have only real zeros. Our next major goal is to prove that f has only finitely many zeros. To do this, we need to present a number of lemmas on the dynamical properties of Q, which will repeatedly be used also later on. However, one case is easy to handle.

Lemma 8.1. *If $K = U_0$ then f has only finitely many zeros.*

Proof of Lemma 8.1. Suppose that $K = U_0$ and that f has infinitely many zeros. If $f(x_1) = 0$ then $Q(x_1) = x_1$, and by Lemma 5.1(5), we have $x_1 \in \partial K = \partial U_0$. We may choose x_1 to be any point in a sequence of zeros of f tending to ∞ or $-\infty$. This shows that U_0 is unbounded. By Lemma 7.2, we have $Q(z) \to a \in \mathbf{R}$ as $z \to \infty$ in $\overline{U_0}$. But $x_1 \in \overline{U_0}$ and $Q(x_1) = x_1 \to \pm\infty$, which is a contradiction. This proves Lemma 8.1.

For the rest of this section, we assume that K has exactly 2 components, U and U_0. By Lemma 7.2, U_0 is bounded and U is unbounded. We define $x_1 = \min\{x : x \in \mathbf{R} \cap \partial U_0\}$ and $x_2 = \max\{x : x \in \mathbf{R} \cap \partial U_0\}$. There is an open arc γ of $H^+ \cap \partial U_0$ joining x_1 to x_2. For we can start γ by moving along ∂U_0 to H^+ from x_1 so that U_0 stays on the right hand side of the oriented arc γ close to x_1. When γ hits $\mathbf{R}$ again, this must happen at x_2 (even if $x_1 = x_2$, and even if the bounded component Y of $H^+ \setminus \gamma$ were to contain one or more components of K^-). Since $Q' \neq 0$ in H^+, there are no points of $\partial U_0 \cap \partial U$ in H^+. Since $\gamma \subset H^+$, there is a single, unique component W of K^- such that $\gamma \subset \partial W$. If Y is as above then $Y \cap U = \emptyset$ since U is unbounded. Thus $U \subset H^+ \setminus \overline{Y}$ and $W \subset H^+ \setminus \overline{Y}$. Each point of $\partial W \setminus (\mathbf{R} \cup \overline{Y})$ must lie on ∂U, and there is a continuum of such points. Thus W separates U from U_0 in H^+. We reserve the notation W for this particular component for the rest of this section.

As we seek to prove that f has only finitely many zeros, we may assume that f has infinitely many zeros and derive a contradiction. So suppose that f has infinitely many zeros. Suppose that they cluster to $-\infty$. Then f' and hence f'' has infinitely many zeros, clustering to $-\infty$, such that the zeros of f'' are not zeros of either f or f'. Thus Q' has infinitely many zeros, clustering to $-\infty$. At the zeros of Q', a component of K meets a component of K^-. Apart from finitely many zeros, the component of K involved must be U (since U_0 is bounded) so that U is unbounded. Note that $i \notin U$. By Lemma 7.2, U contains eventually every Stolz angle at infinity. Also U must contain half open arcs going from any preassigned point of U to all but finitely many (and hence to infinitely many) zeros of Q'. This shows that $\operatorname{Re} z$ must be bounded below for $z \in W$. Similarly, if f has infinitely many zeros, clustering to ∞, then $\operatorname{Re} z$ is bounded above for $z \in W$.

Suppose that $\omega < +\infty$ so that the zeros of f are bounded above. If $a_\omega \in \partial U_0$, it is conceivable that $\operatorname{Re} z$ might be unbounded above for $z \in W$. But suppose that $a_\omega > \max\{x : x \in \mathbf{R} \cap \partial U_0\}$ so that $a_\omega \in \partial U$. Then U contains a half open arc γ_2 going from a point z' of U to a_ω. Let γ_3 be a half open arc in U from z' to a zero x_1 of f'' with $x_1 < \min\{x : x \in \mathbf{R} \cap \partial U_0\}$. Set $\gamma_4 = \gamma_2 \cup \gamma_3$. Then $H^+ \setminus \gamma_4$ has a bounded component Δ containing $\overline{U_0} \setminus \mathbf{R}$. We have $W \subset \Delta$ or $W \subset H^+ \setminus \Delta$. Since $\overline{U_0} \setminus \mathbf{R}$ and $\partial W \cap \partial U_0 \neq \emptyset$, we have $W \subset \Delta$. Hence W is bounded.

Thus, when $\tau = -\infty$ and $\omega < +\infty$, the domain W can be unbounded at most if $a_\omega \in \partial U_0$. If W is unbounded, we define $x_2 = \max\{x : x \in \mathbf{R} \cap \partial U_0\}$, and then $(x_2, \infty) \subset \partial W$. For otherwise, we have $x \in \partial U$ for some $x > x_2$, and using joining x to $z' \in U$ by an arc in U, we find, as above, that W is bounded.

Similarly if $\tau > -\infty$ and $\omega = +\infty$, the domain W can be unbounded at most if $a_\tau \in \partial U_0$. It is convenient to have a formal statement that summarizes what we have just proved.

Lemma 8.2. *Suppose that $K = U_0 \cup U$ and that f has infinitely many zeros. Then there is a unique component W of K^- such that ∂W intersects both $H^+ \cap \partial U$ and $H^+ \cap \partial U_0$. If W is bounded then W is a Jordan domain. If W is unbounded, then $\mathbf{C} \cap \partial W$ is connected and is a Jordan arc, and furthermore, one of the following must hold*:

(i) *$\tau = -\infty$, $\omega < +\infty$, and $a_\omega \in \partial U_0$. Even then, $\{\operatorname{Re} z : z \in W\}$ is bounded below. We have $(x_2, \infty) \subset \partial W$ for $x_2 = \max\{x : x \in \mathbf{R} \cap \partial U_0\}$.*

(ii) *$\tau > -\infty$, $\omega = +\infty$, and $a_\tau \in \partial U_0$. Even then, $\{\operatorname{Re} z : z \in W\}$ is bounded above. We have $(-\infty, x_3) \subset \partial W$ for $x_3 = \min\{x : x \in \mathbf{R} \cap \partial U_0\}$.*

The function Q maps W one-to-one conformally onto H^-.

Proof of Lemma 8.2. It is clear that if W is bounded then W is a Jordan domain. If W is unbounded, then it is seen that $\mathbf{C} \cap \partial W$ is connected and is a Jordan arc since K has no components other than U and U_0.

If W is bounded, it follows from Lemma 7.3 that Q maps W one-to-one conformally onto H^-. Suppose that W is unbounded. Now Q' has no zeros in W. If we can continue analytically without restriction in H^- any branch of $(Q \mid W)^{-1}$, then Q has a single-valued inverse in H^- by the monodromy theorem, and so Q is one-to-one in W and maps W conformally onto H^-. If not, then there is a path $\gamma \subset W$ going to infinity such that $Q(z) \to \alpha \in H^-$ as $z \to \infty$ along γ. But by what has been proved already in this lemma, f'/f has only finitely many zeros and so Q has only finitely many poles on ∂W, so that $Q(z) \to a \in \overline{\mathbf{R}}$ as $z \to \infty$ in $\overline{W}$ (see Lemma 7.3). Thus there is no such path γ.

This completes the proof of Lemma 8.2.

Thus we have the situation where U_0 is bounded and K^- has a bounded component W as above. If W is bounded then $\mathbf{R} \cap \partial W$ has at least two (and hence exactly two) components, say $[a,b]$ and $[c,d]$, where $a < b < x_0 < c < d$ and such that ∂U_0 intersects ∂W (necessarily in a closed Jordan arc which lies in H^+ apart from its end points). This follows, since K has the bounded component U_0 and the unbounded component U and no other components, and W necessarily separates U from U_0 in H^+. If W is unbounded and $\tau = -\infty$, $\omega < +\infty$, then $\mathbf{R} \cap \partial W = [a,b] \cup [x_2, \infty)$, where $x_2 = \max\{x : x \in \mathbf{R} \cap \partial U_0\}$. In this case we write $c = x_2$, $d = \infty$, so that $\mathbf{R} \cap \partial W = [a,b] \cup [c,d)$. If W is unbounded and $\tau > -\infty$, $\omega = +\infty$, then $\mathbf{R} \cap \partial W = (-\infty, x_3] \cup [c,d]$, where $x_3 = \min\{x : x \in \mathbf{R} \cap \partial U_0\}$. In this case we write $a = -\infty$, $b = x_3$, so that $\mathbf{R} \cap \partial W = (a,b] \cup [c,d]$. At this point, we could conceivably have $a = b$ or $c = d$. This could be the case only if a (or c) is a zero of Q' of order at least 2. Note that since $K = U \cup U_0$ and W separates U from U_0 in H^+, it follows that f can have no zeros in $(a,b) \cup (c,d)$, since any such zero t would satisfy $Q'(t) \geq 0$ and would place t on ∂K. Further, since $a, d \in \partial U$ whenever they are finite, and since Q has no poles at any finite point of ∂U by Lemma 7.4, it follows that $(f'/f)(a) \neq 0$ if $a > -\infty$ and $(f'/f)(d) \neq 0$ if $d < +\infty$.

8.2. Iteration theory. Next we consider some iteration theoretic properties of Q. If F is meromorphic in the plane, then the set of normality $N(F)$ consists of those $z \in \mathbf{C}$ that have a neighbourhood T such that all the iterates F^m of F are defined in T and form a normal family there. The Julia set is defined by $J(F) = \mathbf{C} \setminus N(F)$, and is a non-empty perfect set if F is not a rational function of degree at most 1. If T is a component of $N(F)$ then $F^m(T) \subset T_m$ for some component T_m of F^m. If all the T_m are distinct then T is called a wandering domain; a rational function has no wandering domains by a theorem of Sullivan. Otherwise, there are minimal $k, l \geq 1$ such that $T_k = T_{k+l}$, and then T_k is called a periodic domain for F and invariant domain for F^l. Any invariant domain T is one of six types: superattracting, attracting, parabolic, a Siegel disk, a Herman ring, or a Baker domain. In the first three cases there is a fixed point $z_0 = F(z_0)$ of F in $\overline{T}$ such that $\lim_{m\to\infty} F^m(z) = z_0$ locally uniformly for $z \in T$. We have $F'(z_0) = 0$, $0 < |F'(z_0)| < 1$, and $F'(z_0) = 1$, in the superattracting, attracting, and parabolic case, respectively. When T is a Baker domain, we have $\lim_{m\to\infty} F^m(z) = \infty \in \partial T$ locally uniformly for $z \in T$. For further details, we refer to [B].

First note that all the finite fixed points of Q occur at the zeros and poles of f so that they consist of the points $\pm i$ and all the zeros of f, if any, all the zeros of f being real. The fixed points $\pm i$ are repelling and all the other finite fixed points are attracting or superattracting. Namely, at a zero z_0 of f of order $m \geq 1$, we have $Q'(z_0) = 1 - m^{-1} \in [0, 1)$. If $m = 1$ so that $f(z_0) = 0 \neq f'(z_0)$ then Q has a superattracting fixed point at z_0, and $Q' = (ff'')/(f')^2$ has a simple zero at z_0 unless also $f''(z_0) = 0$.

We note that Q is not a rational function of degree at most 1. Since f/f' does not vanish identically, Q is not the identity mapping. Since Q fixes each of $\pm i$, it cannot be constant. If Q is a Möbius transformation, then f'/f is rational. If $(f'/f)(\infty) \neq 0$ then $Q(\infty) = \infty$ so that Q fixes three points and is the identity, a contradiction. Thus $(f'/f)(\infty) = 0$. Hence f is rational. Now f cannot have any finite zeros as such a zero would give rise to a third fixed point of Q. Hence f is a constant multiple of $(z^2+1)^{-n}$, and so $Q(z) = z - (z^2+1)/(2nz) = z\big(1 - (2n)^{-1}\big) - 1/(2nz)$, which is not a Möbius transformation. This completes the proof that Q is transcendental or Q is rational of degree at least 2.

All critical points of Q (zeros of Q' and multiple poles of Q) are real, and their images under the iterates of Q are real. If D is a Siegel disk or a Herman ring for some iterate of Q, then the images of the critical points of Q must cluster to each boundary point of D, as is classically known. Thus $\partial D \subset \overline{\mathbf{R}}$, and D contains H^+ or H^- (or both). This is impossible since the points $\pm i$ are repelling fixed points of Q and hence lie in $J(Q)$. Thus Q has no Siegel disk or Herman ring cycles.

Definition. Suppose that $f(t) = 0$ so that $t \in \mathbf{R}$. We write $D = D(t)$ for the component of the set of normality $N(Q) \subset \mathbf{C}$ containing t.

Thus $\lim_{j\to\infty} Q^j(z) = t$, locally uniformly for $z \in D(t)$, where Q^j denotes the j^{th} iterate of Q.

Lemma 8.3. *If $f(t) = 0$ and $D = D(t)$, then the domain D is symmetric with respect to the real axis. Let J be the component of $D \cap \mathbf{R}$ containing t. Then t is the only zero of f on $\mathbf{R} \cap \bar{J}$. If D is simply connected, then $J = D \cap \mathbf{R}$.*

If J is bounded, then $J = (x_1, x_2)$ with $-\infty < x_1 < t < x_2 < \infty$ and $\{x_1, x_2\} \subset J(Q)$. We have $Q(J) = J$, and Q interchanges x_1 and x_2. Further, Q is strictly decreasing on $[x_1, x_1 + \varepsilon]$ and on $[x_2 - \varepsilon, x_2]$, for some $\varepsilon > 0$.

If J is unbounded, then either $J = (-\infty, x_2)$, where x_2 is a finite pole of Q, or $J = (x_1, +\infty)$, where x_1 is a finite pole of Q. These finite poles of Q are zeros *of f'/f but cannot be equal to the special zero x_0 of f'. Also, we cannot have $J = \mathbf{R}$.*

Suppose that there are points $x_3, x_4 \in \mathbf{R}$ with $x_3 < t < x_4$ such that $f'(x_j) = 0$ for $j = 3, 4$, and such that f has no zeros on $(x_3, t) \cup (t, x_4)$. Then $x_3 < x_1 < t < x_2 < x_4$.

Proof of Lemma 8.3. Suppose that $z \in H^+ \cap D$. Then $Q^j(z) \in D$ for all $j \geq 1$ and $\lim_{j\to\infty} Q^j(z) = t \in \mathbf{R}$. Recall that Q is real so that $Q(\overline{z}) = \overline{Q(z)}$ for all $z \in \mathbf{C}$. Thus $Q^j(\overline{z}) = \overline{Q^j(z)} \in D$ for all large j. Join z to $Q(z)$ by an arc $\gamma \subset D$. Then $Q^j(\gamma)$ joins $Q^j(z)$ to $Q^{j+1}(z)$ in D, and $\overline{Q^j(\gamma)}$ joins $Q^j(\overline{z})$ to $Q^{j+1}(\overline{z})$. Now $\overline{Q^j(\gamma)} \subset D$ for all large j so that $\overline{Q^j(\gamma)} \subset N(Q)$ for all j. Since the arcs $\overline{Q^j(\gamma)}$ form a continuous path from $\overline{z}$ to t, we deduce that they all lie in D so that $\overline{z} \in D$. Thus D is symmetric with respect to $\mathbf{R}$, as claimed.

It is obvious that there is an interval $J = (x_1, x_2) \subset D \cap \mathbf{R}$ with $x_1, x_2 \in \mathbf{R} \cup \{-\infty, \infty\}$, $x_1 < t < x_2$ and $\{x_1, x_2\} \cap \mathbf{R} \subset J(Q)$, this J being the component of $D \cap \mathbf{R}$ containing t. Since $J \subset D$ and $Q(D) \subset D$, we have $Q(J) \subset D$. Since Q is real, we have $Q(J) \subset \mathbf{R} \cap D$, and since $Q(t) = t \in J$ and $Q(J)$ is an interval, we have $Q(J) \subset J$.

Any zero of f other than t belongs to a component of $N(Q)$ other than D, and hence cannot lie in J.

Since D is symmetric with respect to $\mathbf{R}$, it follows that if $J \neq D \cap \mathbf{R}$ then D contains a Jordan curve enclosing some point in $J(Q) \cap \mathbf{R}$, and so D is multiply connected. Thus if D is simply connected, then $J = D \cap \mathbf{R}$.

Suppose that J is bounded. Since for $j = 1, 2$, we have $Q(x_j) \in J(Q)$, we have $\{Q(x_1), Q(x_2)\} \subset \{x_1, x_2\}$. If $Q(x_1) = x_1$ then $f(x_1) = 0$. But then $x_1 \in N(Q)$, so $x_1 \notin J(Q)$, which is a contradiction. Thus $Q(x_1) \neq x_1$, and similarly $Q(x_2) \neq x_2$. Hence Q interchanges x_1 and x_2. Since $Q(J)$ is an interval and $\{Q(x_1), Q(x_2)\} = \{x_1, x_2\}$, we have $J \subset Q(J)$. Since also $Q(J) \subset J$, as we have seen above, we have $Q(J) = J$. Since $Q(J) = J$ and Q interchanges x_1 and x_2, it is clear that Q is strictly decreasing on $[x_1, x_1 + \varepsilon]$ and on $[x_2 - \varepsilon, x_2]$, for some $\varepsilon > 0$.

Suppose that $x_1 = -\infty$. Then f has no zeros smaller than t since each zero of f belongs to a distinct component of $N(Q)$. Also Q has no fixed points on $(-\infty, t)$.

Further, Q has no poles on $(-\infty, t)$ since any such pole would lie in D and hence in $N(Q)$. This would make sense only if Q is rational as it would require that $\infty \in N(Q)$ when $N(Q)$ is viewed as a subset of $\overline{\mathbf{C}}$. But then ∞ is a parabolic fixed point of Q and hence lies in $J(Q)$ unless $(f'/f)(\infty) = 0$. Then f is rational. If $(f'/f)(z) = O(z^{-2})$ as $z \to \infty$ then ∞ is a superattracting fixed point of Q and so belongs to a component of $N(Q)$ other than D. If $(f'/f)(z) \sim A/z$ where $A \in \mathbf{R} \setminus \{0\}$ as $z \to \infty$ then $Q(z) \sim (1 - A^{-1})z$ if $A \neq 1$, and then either $\infty \in J(Q)$ or ∞ belongs to a component of $N(Q)$ other than D. If $A = 1$ then

$Q(\infty)$ is finite while $f = P/(z^2+1)^n$ where P is a real polynomial of degree $2n+1$. Since we have excluded this situation by our standing assumptions, we may ignore it. We conclude that Q has no poles on $(-\infty, t)$.

It follows that either $Q(x) > x$ for all $x \in (-\infty, t)$, or $Q(x) < x$ for all $x \in (-\infty, t)$. Since $Q^j(x) \to t$ as $j \to \infty$, for all $x \in (-\infty, t)$, we must have $Q(x) > x$ for all $x \in (-\infty, t)$. If now x_2 is finite, then since $Q^j(x) \to t$ as $j \to \infty$, for all $x \in (t, x_2)$, we have $Q(x) < x$ for all $x \in (t, x_2)$. Now $x_2 \in J(Q)$ so that $Q(x_2) \in J(Q)$ and thus $Q(x_2) \geq x_2$ (since $(-\infty, x_2) \subset D \subset N(Q)$) or $Q(x_2) = \infty$. But $Q(x) < x$ for all $x \in (t, x_2)$, so that if $Q(x_2)$ is finite then $Q(x_2) = x_2$, and so $f(x_2) = 0$. But then $x_2 \in N(Q)$ and indeed x_2 lies in a component of $N(Q)$ other than $D = D(t)$ (namely, $x_2 \in D(x_2)$). This is a contradiction, and it follows that $Q(x_2) = \infty$. Since $Q(x) < x$ for all $x \in (t, x_2)$, we have $Q(x) \to -\infty$ as $x \to x_2-$. Thus x_2 cannot be the special zero of f'.

The case when $J = (x_1, +\infty)$ for some finite x_1 is dealt with in the same way.

Suppose that $x_1 = -\infty$ and $x_2 = \infty$. Thus $\mathbf{R} \subset D$, and then $Q(x) > x$ on $(-\infty, t)$ and $Q(x) < x$ on (t, ∞). Now t is the only zero of f (even though the multiplicity of t as a zero of f is not known). Also f' cannot have any zeros other than t since they would be finite (real) poles of Q and hence in $J(Q)$. Thus $g(z) = (z-t)^d e^{P(z)}$ for some positive integer d and real polynomial P_2. A calculation shows that f' must have at least one zero other than t, which gives a contradiction. It follows that at least one of x_1 and x_2 is always finite.

Suppose that there are points $x_3, x_4 \in \mathbf{R}$ with $x_3 < t < x_4$ such that $f'(x_j) = 0 \neq f(x_j)$ for $j = 3, 4$, and such that f has no zeros on $(x_3, t) \cup (t, x_4)$. Note that $Q(x_j) = \infty$ for $j = 3, 4$. Then $Q(x_3) = \infty$ implies that $x_3 \in J(Q)$ so that $x_3 \notin D$. Thus $x_3 \leq x_1$. Since $Q(x_1) = x_2 \neq \infty$, we have $x_3 < x_1$. In the same way we see that $x_2 < x_4$. Thus $x_3 < x_1 < t < x_2 < x_4$.

This completes the proof of Lemma 8.3.

Next we consider the question of whether $D(t)$ is simply connected. Recall that a is a repelling fixed point of F if $F(a) = a$ and $|F'(a)| > 1$, and a parabolic fixed point of F if $F(a) = a$ and $F'(a) = 1$. The next lemma, which is based on the results and ideas of Shishikura and of Przytycki, is remarkable in that it says something about the location of repelling or parabolic fixed points of a function rather than the fixed points of some iterate of the function.

Lemma 8.4. *Suppose that $f(t) = 0$ and that $D = D(t)$ is multiply connected. Then there is a Jordan curve $\gamma \subset D \subset \mathbf{C}$ that encloses a repelling or parabolic fixed point of Q. Thus γ encloses i or $-i$ or both. The complex conjugate curve $\overline{\gamma}$ is also a subset of D and encloses $-i$ or i or both, respectively. Both i and $-i$ are boundary points of D.*

Proof of Lemma 8.4. Suppose that D is multiply connected. We can deduce in the same way as in Shishikura's paper [Sh] that there is a Jordan curve $\gamma \subset$

$D \subset \mathbf{C}$ that encloses a repelling or parabolic fixed point of Q. Shishikura's main conclusions are formulated for rational functions, but the main lemma to be used in this case (when considering the attracting or superattracting domain D), [Sh, Theorem 2.1], only makes local assumptions and is equally valid for meromorphic functions such as the function Q here. (The difference is that if Q is not rational, we are not able to conclude that Q has a repelling or parabolic fixed point also in the unbounded component of $\overline{\mathbf{C}} \setminus D$. If Q is rational, we may conclude that such a fixed point exists, and it will then necessarily be the point at infinity.) Since the only non-attracting finite fixed points of Q are $\pm i$, we see that γ encloses i or $-i$. The statements made about $\overline{\gamma}$ are clear since D is symmetric with respect to $\mathbf{R}$, by Lemma 8.3.

It now suffices to prove that one of $\pm i$ lies on ∂D, for then so does the other one, by symmetry. We may argue as in a paper of Przytycki [Pr, Proof of Theorem A, p. 230], and conclude that there exist two points on distinct components of ∂D, each of which is fixed by Q provided that Q is defined at that point (that is, provided that the point is finite or that Q is rational). At least one point must be finite so that it is i or $-i$. This argument gives the extra information that if D is bounded then i and $-i$ lie on distinct components of ∂D. This completes the proof of Lemma 8.4.

Lemmas 8.3 and 8.4 are valid regardless of any assumption concerning the existence of W. For Lemmas 8.5 and 8.6 below, we assume that there is a component W of K^- separating the components U and U_0 of K, as explained in the last paragraph of subsection 8.1. Thus it makes no difference for these two lemmas whether or not f has infinitely many zeros, as long as such a component W exists. Thus Lemmas 8.5 and 8.6 can be applied later also when f has only finitely many zeros.

Lemma 8.5. *At most one component $D(t)$, where $f(t) = 0$, is multiply connected, while all the others are simply connected. In any case, every component $D(t)$ contains the points $\pm i$ on its boundary. Any simply connected component $D(t)$ is bounded.*

Proof of Lemma 8.5. Suppose that there are two multiply connected components of the form $D(t)$, say D_1 and D_2. Then by Lemma 8.4, we have $\pm i \in \partial D_1 \cap \partial D_2$, and for $j = 1, 2$, the domain D_j contains a Jordan curve γ_j that encloses the point i. Since $D_1 \cap D_2 = \emptyset$ and $i \in \partial D_2$, the domain D_2 must lie in the interior of γ_1 (that is, in the bounded component of the complement of γ_1). Thus γ_2 lies in the interior of γ_1. Similarly, γ_1 lies in the interior of γ_2. This is impossible, and so there is at most one multiply connected component $D(t)$.

In view of Lemma 8.4, it remains to be proved that if $D = D(t)$ is simply connected then at least one of i and $-i$ (and hence both) lie on ∂D. Again the argument of Przytycki [Pr] provides a point on ∂D which is fixed by Q if the point is finite. Thus $\pm i \in \partial D$ if D is bounded.

To get a contradiction, suppose now that $D = D(t)$ is simply connected and unbounded. By Lemma 8.3, we have $D \cap \mathbf{R} = (v_1, v_2)$. If $-\infty < v_1 < t < v_2 < \infty$ then Q interchanges v_1 and v_2.

Suppose first that t is neither the largest nor the smallest zero of f (this is certainly the case if f has no largest or smallest zero), and choose the nearest zeros of f to t, say t_1 and t_2, and zeros u_1 and u_2 of f', so labelled that $t_1 < u_1 < t < u_2 < t_2$. We may choose u_1 and u_2 so that f' has no zeros on $(u_1, t) \cup (t, u_2)$. In this case v_1 and v_2 are both finite.

By Lemma 8.3, we have $u_1 < v_1 < t < v_2 < u_2$. Suppose that $Q'(t) = 0$, which is true if, and only if, t is a simple zero of f. Then there is a component W' of K^- such that for some small $\varepsilon > 0$, we have $(t-\varepsilon, t) \subset \partial W'$ or $(t, t+\varepsilon) \subset \partial W'$. If W' is unbounded then either $W' = W$ and W is unbounded, or at least one of τ and ω is finite and $\partial W'$ contains $(-\infty, t]$ or $[t, \infty)$. This is deduced by considering the fact that $K = U_0 \cup U$ and W separates U from U_0. But then f cannot have any zeros on $(-\infty, t]$ or $[t, \infty)$, as the case may be, which contradicts our assumption that t is neither the largest nor the smallest zero of f. (For if $(-\infty, t] \subset \partial W'$, $f(s) = 0$, and $s < t$, then $f'(s) = 0 \neq Q'(s)$, and so $s \in \partial K \setminus \partial K^-$ by Lemma 5.1(5), which is a contradiction.) Thus W' is bounded.

Without loss of generality, suppose that $(t - \varepsilon, t) \subset \partial W'$ if $W' \neq W$. Note that then $v_1 \in \partial W'$. Now ∂D must leave W' since W' is bounded while D is unbounded. Thus there is a connected subset Γ of ∂D in H^+ joining v_1 to a point $v_3 \in \partial W' \cap \partial D \cap H^+$, such that v_3 has a neighbourhood U' such that $U' \cap \partial D$ intersects both W' and K. (The set $\Gamma \cup \{v_1, v_3\}$ is a continuum but need not be an arc.) Thus there is a sequence of points z_k in D tending to v_3 with $Q(z_k) \in \mathbf{R}$. Note that $Q(v_3) \in \mathbf{R}$ since $v_3 \in \partial W'$ and since $Q(v_3)$ is finite (since $v_3 \in H^+$). We have $Q(\partial D \cap \mathbf{C}) \subset \partial D$, and so $Q(\Gamma)$ is a connected set that lies in $\overline{H^-}$ (since $Q(W') \subset H^-$) and joins $v_2 = Q(v_1)$ to a point $Q(v_3) \in \partial(D \cap \mathbf{R})$ (since $Q(z_k) \in D \cap \mathbf{R}$). Thus $Q(v_3)$ is equal to v_1 or v_2. If $Q(v_3) = v_2$, then, roughly speaking, even though Γ may be a complicated continuum and not an arc, as z traces Γ from v_1 to v_3, if $Q(z)$ traces $\partial D \cap H^-$ from v_2 to itself (remaining finite since Q has no poles in H^+), it follows that there must be a zero of Q' on $\Gamma \subset H^+$. But this is impossible. Hence $Q(v_3) = v_1$. But then it must be the case that $Q(z)$ traces all of $\partial D \cap H^-$ from v_2 to v_1, and we conclude that $\partial D \cap H^-$ is bounded. By symmetry, $\partial D \cap H^+$ is bounded, and so D is bounded, as required. At the beginning of this paragraph we said that $v_1 \in \partial W'$ if $W' \neq W$. If $W' = W$ and $v_1 \notin \partial W'$ then $t > c$ and $v_1 \in (b, c)$. If $v_1 \in \partial W''$ for some component W'' of K^-, we can argue as above. If $v_1 \in \partial U_0$ then Q is strictly increasing on $(v_1 - \varepsilon, v_1 + \varepsilon)$ for some $\varepsilon > 0$, which contradicts Lemma 8.3. Thus the argument used in this paragraph works whether $W' = W$ or $W' \neq W$.

If $Q'(t) \neq 0$ then $Q'(t) > 0$ so that t has a neighbourhood whose intersection with H^+ is contained in K, and hence is contained in U_0 or in U. To be able to argue as above, when $Q'(t) = 0$, it suffices to show that v_1 or v_2 lies on the boundary of a bounded component W' of K^- and does not lie on ∂K. Then we can find a set Γ as above. Suppose that neither v_1 not v_2 has this property, for some component W' of K^-, whether bounded or unbounded. Since there is no pole of Q, hence no zero of f'/f, on ∂D, and since each zero of f'/f, other than the special zero x_0, lies on $\partial K^- \setminus \partial K$, this means that $[v_1, v_2] \subset \partial U_0$ or $[v_1, v_2] \subset \partial U$ according as $t \in \partial U_0$ or $t \in \partial U$. But then Q is strictly increasing on $[v_1, v_2]$, so that Q cannot interchange v_1 and v_2, which contradicts Lemma 8.3. It follows that, in fact, both v_1 and v_2 must lie on ∂K^-, corresponding to components W_1 and W_2 of K^-, say (since $Q([v_1, v_2]) = [v_1, v_2]$ and Q interchanges v_1 and v_2). We shall now see that at least one of W_1 and W_2 must be bounded.

For if $t \in \partial U_0$, move from t along $\mathbf{R}$ in the direction of the special zero x_0 of f', which lies on ∂U_0 by Lemmas 7.4 and 7.5. Since Q is strictly increasing on $(x_0 - \varepsilon_1, x_0)$ and on $(x_0, x_0 + \varepsilon_1)$ for some $\varepsilon_1 > 0$, and hence on each interval from x_0 to the nearest zero of Q' both to the left and to the right, it follows that neither v_1 nor v_2 can lie on such an interval since the property that Q interchanges v_1 and v_2 and maps $[v_1, v_2]$ onto itself, implies that Q is strictly decreasing on $(v_1, v_1 + \varepsilon_2)$ and on $(v_2 - \varepsilon_2, v_2)$, for some $\varepsilon_2 > 0$. Hence there must be a zero of Q' (thus a zero of f or of f'') between t and x_0, and such a zero of Q' must give rise to a bounded component of K^-, which will then serve as W'.

If $t \in \partial U$, suppose, for example, that $t < x$ for all $x \in \mathbf{R} \cap \partial U_0$. If W_2 is unbounded then $W_2 = W$. Since f has a zero t_1 and f' has a zero u_1 with $t_1 < u_1 < t$, we find that W_1 is bounded on the basis that U must contain an arc from some point of U to a point v of $\mathbf{R}$ with $v \le t_1$ (v might be a zero of Q'). This could only fail if Q' has no zeros $< u_1$, which would mean that t_1 is a multiple zero of f. But then $Q'(t_1) > 0$ so that $t_1 \in \partial U$ and we may take $v = t_1$. Hence W_1 is bounded.

Now the proof can be completed in the case $Q'(t) \ne 0$ in the same way as we did in the case $Q'(t) = 0$.

Suppose then that t is the largest or smallest zero of f. To be definite, suppose that t is the largest zero of f. Set $\mathbf{R} \cap D = (v_1, v_2)$ where $v_1 \in \mathbf{R}$ and $v_2 \in \mathbf{R} \cup \{+\infty\}$. If we can find zeros u_1 and u_2 of f' with $u_1 < v_1 < t < v_2 < u_2$ (we pick the nearest zeros of f' to t), we start by following the argument given when t was neither the largest nor the smallest zero of f but when $Q'(t) = 0$. Right now it does not matter whether $Q'(t) = 0$ or not. By the monotonicity of Q, we see that neither u_1 nor u_2 can be the special zero x_0 of f'. If W' is unbounded, where $u_1 \in \partial W'$, then let W'' be the component of K^- with $u_2 \in \partial W''$. If also W'' is unbounded then the one of W' and W'' whose boundary has no points in common with ∂U_0, contains no zeros of Q' on its boundary other than the point of $\mathbf{R} \cap \partial W'$ (or $\mathbf{R} \cap \partial W''$, as the case may be) closest to t.

Suppose that $\partial W'' \cap \partial U_0 = \emptyset$. By Rolle's theorem, we see that u_2 is the only zero of f' on $\mathbf{R} \cap \partial W''$ (since $\mathbf{R} \cap \partial W''$ would otherwise contain a zero of f'' which would also be a zero of Q'). Then $t \in \partial U$, and $W' \ne W''$. The fact that

W' is unbounded implies that $W' = W$. If now f has infinitely many zeros then, since $\omega < +\infty$, it follows from Lemma 8.2 that $(x_2, \infty) \subset \partial W$ for some x_2, which implies that $W' = W = W''$, a contradiction. If f has only finitely many zeros then $g = P_1 e^{P_2}$ for some real polynomials P_1 and P_2 (where $\deg P_1 \geq 2n+2$ if P_2 is constant), and $Q(z) = z - 1/\big(P_2' + O(1/z)\big)$ as $z \to \infty$. But then f'/f is rational, and $Q'(x) = 1 + (f'/f)'/(f'/f) = 1 + o(1) > 0$ as $x \to +\infty$, so that $Q'(x) > 0$ for all $x > u_2$ since Q' has no zeros or poles $> u_2$. Since u_2 is the largest finite pole of Q, and since $u_2 \neq x_0$, the residue of Q at u_2 is positive. Hence $\lim_{x\to u_2+} Q'(x) = -\infty$, which gives a contradiction.

Suppose that $\partial W' \cap \partial U_0 = \emptyset$. Then $W' \neq W'' = W$. If f has infinitely many zeros, Lemma 8.2 again gives a contradiction. If f has only finitely many zeros, we argue as above and note that $Q'(x) > 0$ for all $x < u_1$ while we should have $Q'(x) < 0$ for all $x < u_1$ since $(-\infty, u_1) \subset \partial W' \subset \partial K^-$. So this again gives a contradiction.

Suppose then that t is the largest zero of f and $\mathbf{R} \cap D = (v_1, v_2)$, but that we cannot find zeros u_1 and u_2 of f' with $u_1 < v_1 < t < v_2 < u_2$ (for example, if $v_2 = +\infty$). As long as components W' and W'' of K^- to the left and right of t can be found, the above argument goes through even if at least one of the points u_1 and u_2 cannot be found. Therefore suppose that at least one of $(-\infty, t)$ and (t, ∞) is contained in ∂K and hence in ∂U since U_0 is bounded. If $(-\infty, t) \subset \partial U$ then $Q(v_1) \neq \infty \neq v_2$, and $Q'(x) > 0$ for all $x < t$ so that $Q(v_1) < Q(t) = t < v_2$ since $v_1 < t$, contradicting the fact that $Q(v_1) = v_2$, by Lemma 8.3. If $(t, \infty) \subset \partial U$ then $Q'(x) > 0$ for all $x > t$ so that $Q(v_2) > Q(t) = t > v_1$, a contradiction, if $v_2 < +\infty$. So we must have $Q(v_1) = \infty = v_2$ and $(t, +\infty) \subset \partial U$. Now $Q'(x) < 0$ for all $x \in (v_1, t)$, and $x_0 \neq v_1 \in \partial W'$ for some component W' of K^-. If W' is bounded, we argue as before. If W' is unbounded then $W' = W$. By Lemma 8.2(i), f has only finitely many zeros. Thus, as above, $Q'(x) = 1 + o(1)$ as $x \to \pm\infty$ so that K^- and, hence, W, is bounded, which is a contradiction.

All cases have now been covered. This completes the proof of Lemma 8.5.

Remark. Lemma 8.5 may seem surprising in view of the theorem of Przytycki [Pr] that if F is a polynomial then $\infty \in \partial D$ for every attracting component D of $N(z - F/F')$. Note that Lemma 8.5 has been proved only under the assumption that f, f', and f'' have only real zeros while $\deg g \geq 2n+2$ if g is a polynomial. In view of the final conclusion of Theorem 1.1, these assumptions are never satisfied. Indeed, if f has a largest or smallest zero t, it is easily seen that $D(t)$ is unbounded, even though it is not immediately obvious if, in this case, $D(t)$ is also simply connected.

Recall the notation in the last paragraph of subsection 8.1.

Lemma 8.6. *Suppose that t is a zero of f such that $D(t)$ is simply connected. Then there is at most one such $t \in (-\infty, a] \cup [d, \infty)$, ignoring multiplicity. If $t \leq a$ then f has no zero on $(t, b]$, f' has no zero on $(t, b]$ and hence not on*

$(a,b]$, and f'/f has no zero on $[t,b]$. If $t \geq d$ then f has no zero on $[c,t)$, f' has no zero on $[c,t)$ and f'/f has no zero on $[c,t]$.

Proof of Lemma 8.6. To get a contradiction, suppose that $f(t_1) = f(t_2) = 0$ and $t_1 < t_2 \leq a$. Then a is finite. Suppose that $D = D(t_1)$ and $D' = D(t_2)$ are simply connected. By Lemma 8.3, we have $D \cap \mathbf{R} = [v_1, v_2]$, say. The function Q interchanges v_1 and v_2, and maps $[v_1, v_2]$ onto itself, so that Q is strictly decreasing on $(v_1, v_1+\varepsilon)$ and on $(v_2-\varepsilon, v_2)$, for some $\varepsilon > 0$. Hence $v_1, v_2 \in \partial K^-$. We have $Q'(t) \geq 0$ so that $t \in \partial U$. We have $v_2 < t_2 < b$. Similarly, let us write $D' \cap \mathbf{R} = [w_1, w_2]$. Then $v_2 < w_1 < t_2 < w_2$.

Let W' be the component of K^- with $v_2 \in \partial W'$. Then $\partial W' \subset \mathbf{R} \cup \partial U_0$. There exists $u \in \mathbf{R} \cap \partial W'$ with $(f'/f)(u) = 0$.

We note that Q is one-to-one in W. If f has infinitely many zeros, this follows from Lemma 8.2. If f has only finitely many zeros, the argument given in the proof of Lemma 8.2 still works and gives the same conclusion.

Pick a point $z \in D \cap H^+$ close to v_1, and a point $z' \in D' \cap H^+$ close to w_1. Since $i \in \partial D$ and since D is symmetric about $\mathbf{R}$, we can join z by a polygonal path γ in $D \cap H^+$ to any preassigned point $w \in D \cap H^+$ which we may choose as close to i as we like. There is an analogous path γ' in $D' \cap H^+$ from z' to any preassigned point $w' \in D' \cap H^+$ close to i. Now γ will have to intersect $H^+ \cap \partial W$ when moving from U into W, and again when moving from W into U_0. Let Γ_1 and Γ_2 be the components of $D \cap \partial W$ that γ intersects in this way. Note that Γ_1 and Γ_2 may be identical. Since ∂W is a Jordan curve, each of Γ_1 and Γ_2 is an open Jordan arc. Define similarly Γ_1' and Γ_2' as the components of $D' \cap \partial W$ that γ' intersects in the same way. Now Q maps each of Γ_1 and Γ_2 into $D \cap \mathbf{R} = (v_1, v_2)$. If Q does not map an end point of Γ_1 onto v_1 or v_2, then Q maps that end point into $D \cap \mathbf{R}$, which contradicts the fact that D is a component of $Q^{-1}(D)$. Hence Q maps each of Γ_1 and Γ_2 onto (v_1, v_2). Since Q is one-to-one in the closure of W, it follows that $\Gamma_1 = \Gamma_2$. Since Γ_1 must contain a point of γ obtained when γ enters W from U, and another point when γ enters U_0 from W, it follows that the Jordan arc Γ_1 must contain one of the intervals $[a,b]$ and $[c,d]$. A similar argument shows that $\Gamma_1' = \Gamma_2'$, and that Γ_1' must contain one of $[a,b]$ and $[c,d]$. Since Γ_1 and Γ_1' are disjoint (being mapped by Q onto disjoint sets), it follows that one of Γ_1 and Γ_1' contains $[a,b]$ while the other one contains $[c,d]$. Hence Q maps $[a,b] \cup [c,d]$ into $[v_1, v_2] \cup [w_1, w_2]$.

If W is bounded, then Q has a unique pole on $[a,b] \cup [c,d]$, which gives a contradiction.

If W is unbounded, then it is easily seen that f has infinitely many zeros, and by Lemma 8.2, exactly one of τ and ω is finite. If $\omega = +\infty$ then by Lemma 8.2, we have $(-\infty, x] \subset \partial W$ for some real number x, which is impossible since it means that $a = -\infty$. Thus it must be the case that $\tau = -\infty$ and $\omega < +\infty$, $[x, +\infty) \subset \partial W$ for some real x, and $\operatorname{Re} w$ is bounded below for $w \in W$. Since Q maps W one-to-one onto H^-, and since Q maps $[a,b] \cup [c,d)$ (here $d = +\infty$) into

$[v_1, v_2] \cup [w_1, w_2]$, it follows that if Q has no pole on $[a, b] \cup [c, d)$, then $Q(z) \to \infty$ as $z \to \infty$ in $\mathbf{C} \cap \overline{W}$. In particular, one of Γ_1 and Γ_1' must go through infinity and hence its image under Q must contain infinity. Thus $v_1 = -\infty$, and so $D = D(t)$ is unbounded, which contradicts Lemma 8.5.

We conclude that f has at most one zero $t \in (-\infty, a]$ such that $D(t)$ is simply connected.

Similarly, it is seen that f cannot have two distinct zeros t_1, t_2 on $[d, \infty)$ such that $D(t_1)$ and $D(t_2)$ are both simply connected.

Suppose that $f(t) = 0$, $t \le a$, and $D = D(t)$ is simply connected. Define v_1, v_2, and Γ_1 as above. Let I' be that interval of $[a, b]$ and $[c, d]$ that is contained in Γ_1 (hence $I' = [a, b]$ if W is unbounded). Then $I' \subset D \cap \mathbf{R} = (v_1, v_2)$. Thus we must have $I' = [a, b]$, and also $b < v_2$. Since Q has no pole, and no fixed point other than t, on (v_1, v_2), we then see that f has no zero on (t, v_2) and thus on $(t, b]$, and f'/f has no zero on $[t, v_2)$ and thus on $[t, b]$. Consequently, f' has no zero on (t, v_2) and thus on $(t, b]$. The same argument applies if $f(t) = 0$, $t \ge d$, and shows that then f has no zero on $[c, t)$, f' has no zero on $[c, t)$ and f'/f has no zero on $[c, t]$.

Suppose that $f(t_1) = f(t_2) = 0$ and $t_1 \le a < d \le t_2$. Then a and d are both finite. Suppose that $D(t_1)$ and $D(t_2)$ are simply connected. By what we have proved already, f'/f has no zero on $[a, b] \cup [c, d]$. But now W is bounded and Q maps W conformally onto H^- so that ∂W must contain exactly one pole u of Q, that is, a zero of f'/f. All the zeros of f'/f are real, so that $u \in \mathbf{R} \cap \partial W = [a, b] \cup [c, d]$. This gives a contradiction, showing that not both t_1 and t_2 can exist.

This completes the proof of Lemma 8.6.

8.3. Conclusion of proof that f has only finitely many zeros. We are now in a position to prove that f has only finitely many zeros.

Lemma 8.7. *Let f be as in* (1), *where $g \in \mathscr{U}_{2p}$. If g is a polynomial, suppose that $\deg g \ge 2n + 2$. Suppose that f, f', and f'' have only real zeros. Then f has only finitely many zeros.*

Proof of Lemma 8.7. Lemma 8.1 gives the conclusion when $K = U_0$. Thus we may assume that $K = U \cup U_0$, so that there is a component W of K^- separating U and U_0 in H^+. We use the notation given in the last paragraph of subsection 8.1. By Lemma 8.4, f has at most one zero t (of some multiplicity) for which $D(t)$ is multiply connected. Thus it suffices to consider from now on only zeros t of f for which $D(t)$ is simply connected. On the bounded interval $[b, c]$, f has only finitely many zeros. On $(a, b) \cup (c, d)$, f has no zeros. If f is to have at least one zero $t \le a$ then a must be finite, so that f has at most one such zero by Lemma 8.6. Similarly, by Lemma 8.6, f has at most one zero on t with $t \ge d$. This proves that f has altogether only finitely many zeros, and Lemma 8.7 is proved.

8.4. Proof that $g \in \mathscr{U}_0$**.** Knowing that f has only finitely many zeros greatly simplifies the analysis in the sense that now f'/f and Q are rational. We can now prove that $g \in \mathscr{U}_0$.

Lemma 8.8. *Let f be as in (1), where $g \in \mathscr{U}_{2p}$. If g is a polynomial, suppose that $\deg g \geq 2n+2$. Suppose that f, f', and f'' have only real zeros. Then f has only finitely many zeros and $g \in \mathscr{U}_0$. Furthermore, the set K has exactly 2 components, U and U_0, and the component W of K^- separating U from U_0 in H^+ is bounded.*

The set K^- has only finitely many components, and each is bounded and is mapped one-to-one conformally by Q onto H^-. If W' is a component of K^- other than W, then $\mathbf{R} \cap \partial W'$ is connected and consists of a closed interval or a single point. In the latter case, the single point is also a zero of f'/f.

Proof of Lemma 8.8. We have $f = P_1 e^{P_2}/\Phi$ for some real polynomials P_1 and P_2, so that

$$L = f'/f = P_1'/P_1 - \Phi'/\Phi + P_2' = P_2' + O(1/z)$$

as $z \to \infty$. By Lemma 7.5, we have $\Lambda = V_0$, and V_0 is bounded. Thus $\operatorname{Im}(f'/f)(z) < 0$ for all $z \in H^+$ outside the bounded set V_0. If $\deg P_2' \geq 2$, this is impossible. Thus $\deg P_2' \leq 1$, and we can write $P_2'(z) = -2\alpha z + \beta$ for some real constants α and β. If $\alpha < 0$, we see again that we cannot have $\operatorname{Im}(f'/f)(z) < 0$ for all $z \in H^+$ outside a bounded set. Thus $\alpha \geq 0$. Since $P_2(z) = -\alpha z^2 + \beta z + \gamma$ for some constant γ, this shows that $g = P_1 e^{P_2} \in \mathscr{U}_0$.

Suppose that $K = U_0$. If $a > 0$ then $Q(z) = z + O(1/z)$ as $z \to \infty$. If $a = 0 \neq b$ then $Q(z) = z - (1/b) + O(1/z)$. If $a = b = 0$ then g is a polynomial and $L(z) = (\deg g - \deg \Phi)/z + O(1/z^2)$ and $Q(z) = \gamma z + O(1)$, where $\gamma = 1 - 1/(\deg g - \deg \Phi) > 0$, and $\deg g - \deg \Phi \geq 2$. A more careful analysis of $\operatorname{Im} Q(z)/y$ (where $y = \operatorname{Im} z$) shows that in each of these three cases, there exists $R > 0$ such that if $z \in H^+$ and $|z| > R$ then $\operatorname{Im} Q(z) > 0$. Thus U_0 is unbounded since $K = U_0$ by our assumption. But in all these cases, we have $Q(z) \to \infty$ as $z \to \infty$. Hence $Q(z) \to \infty$ as $z \to \infty$ in U_0. But by Lemma 7.2, we have $Q(z) \to a \in \mathbf{R}$ as $z \to \infty$ in U_0. This is a contradiction. It follows that $K = U \cup U_0$, so that W exists as before. Since we still have $\{z : z \in H^+,\ |z| > R\} \subset K$, it follows that K^- and hence W is bounded.

Since K^- is bounded, it has only finitely many components, and each is bounded and is mapped one-to-one conformally by Q onto H^- by Lemma 7.3. If W' is a component of K^- other than W, then $\mathbf{R} \cap \partial W'$ is connected and consists of a closed interval or a single point. For otherwise W' separates K and hence separates U and U_0 in H^+, while only W has this property. Suppose that $\mathbf{R} \cap \partial W' = \{x_1\}$. Then $Q(x_1) = \infty$ and so $(f'/f)(x_1) = 0$, since $\mathbf{R} \cap \partial W'$ must contain a pole of Q.

This completes the proof of Lemma 8.8.

We have now proved Theorem 1.1 except in the case when $g \in \mathscr{U}_0$ and g has only finitely many zeros. The rest of the paper is devoted to this case.

9. Sign analysis of f'/f

From now on, we may assume that f is as in (1), where $g \in \mathscr{U}_0$ and g has only finitely many zeros. If g is a polynomial, we assume that $\deg g \geq 2n+2$. We further assume that f, f', and f'' have only real zeros. Furthermore, the set K has exactly 2 components, U and U_0, and the component W of K^- separating U from U_0 in H^+ is bounded. We have $\mathbf{R} \cap \partial W = [a,b] \cup [c,d]$, where $a \leq b < c \leq d$. By Lemmas 8.4 and 8.6, the function f has at most one zero t for which the component $D(t)$ of $N(Q)$ containing t is multiply connected, and at most one zero t outside (a,d) for which $D(t)$ is simply connected.

By Lemmas 7.4 and 7.5, the special zero x_0 of f', at which $f \neq 0$ and $f''f \geq 0$, exists and lies on ∂U_0, hence on $[b,c]$. At the other zeros of f'/f, we have $f''f < 0$.

Since Q is conformal in W, the set ∂W contains exactly one zero of f' which is not a zero of f, that is, exactly one pole of Q. Hence, by Lemma 8.6, at most one of the points a and d can be a zero of f, unless the other one is the only possible exceptional zero for which D is multiply connected.

The following sign analysis of $L = f'/f$ will be used repeatedly.

Lemma 9.1. *$f(z) = e^{-\xi z^2+\beta z}P(z)/(z^2+1)^n$ where $\xi \geq 0$, $\beta \in \mathbf{R}$, and P is a real polynomial with only real zeros such that if $\xi = \beta = 0$ then $\deg P \geq 2n+2$. Then*

$$L = \frac{f'}{f} = -2\xi z + \beta + \frac{P'}{P} - \frac{2nz}{z^2+1}.$$

If $\xi > 0$ then

$$\lim_{x\to\infty} L(x) = -\infty, \qquad \lim_{x\to a_\omega+} L(x) = \infty,$$
$$\lim_{x\to-\infty} L(x) = \infty, \qquad \lim_{x\to a_\tau-} L(x) = -\infty,$$

so that f' has a zero on $(-\infty, a_\tau)$ and also on (a_ω, ∞), provided that f has at least one zero. In this case f' has 4 extraordinary zeros. If $\xi > 0$ but f has no zeros then f' has 3 extraordinary zeros.

If $\xi = 0 \neq \beta$ then

$$\lim_{x\to\infty} L(x) = \beta, \qquad \lim_{x\to a_\omega+} L(x) = \infty,$$
$$\lim_{x\to-\infty} L(x) = \beta, \qquad \lim_{x\to a_\tau-} L(x) = -\infty,$$

so that f' has a zero on $(-\infty, a_\tau)$ if $\beta > 0$, and on (a_ω, ∞) if $\beta < 0$, provided that f has at least one zero. In this case f' has 3 extraordinary zeros. If $\xi = 0 \neq \beta$ but f has no zeros then f' has 2 extraordinary zeros.

If $\xi = \beta = 0$ then, as $x \to \infty$, we have $L(x) \sim (\deg P - \deg \Phi)/x > 0$, and as $x \to -\infty$, we have $L(x) \sim (\deg P - \deg \Phi)/x < 0$. In this case f' has 2 extraordinary zeros, and they are both on $(-\infty, a_\tau)$, or both on (a_ω, ∞), or both on some interval (a_k, a_{k+1}).

We can write $f' = e^{-\xi z^2 + \beta z} P_1(z)/(z^2+1)^{n+1}$, where the numbers ξ and β are the same as for f, and where P_1 is a real polynomial with only real zeros. Thus, under the assumption that f, f', and f'' have only real zeros, the above conclusions hold also for the extraordinary zeros of f''/f'.

Proof of Lemma 9.1. In each of the above cases, the number of extraordinary zeros of f', that is, $\deg \varphi$, is obtained from Lemma 2.2. The remaining conclusions on the location of the extraordinary zeros of f' are clear. In particular, it follows that any interval of the form $(-\infty, a_\tau)$, (a_ω, ∞), or (a_k, a_{k+1}) contains at most 3 zeros of f', counting multiplicities.

We may apply the conclusions to f' instead of f since we may write $f' = e^{-\xi z^2 + \beta z} P_1(z)/(z^2+1)^{n+1}$ where ξ and β are the same as for f, and P_1 is a real polynomial with only real zeros such that if $\xi = \beta = 0$ then $\deg P_1 \geq 2(n+1)+1$, which is all that is needed here, as an inspection of the proof shows. This completes the proof of Lemma 9.1.

10. Using the Euler characteristic

We now record a consequence of results of Shishikura [Sh], which will be used several times. If G is a domain in the sphere bounded by k mutually disjoint Jordan curves, we define the *Euler characteristic* $\chi(G)$ by $\chi(G) = 2 - k$. Thus $\chi(G) = 1$ if G is simply connected, and $\chi(G) = 0$ if G is doubly connected. If F is meromorphic in G, we write $\delta = \delta(F, G)$ for the sum of branching indices of F in G. Thus, if we set $v(F, z) = m - 1$ when F maps any sufficiently small neighbourhood T of z m-to-1 onto $F(T)$, then

$$\delta(F, G) = \sum_{z \in G} v(F, z).$$

Lemma 10.1. *Suppose that $g \in \mathscr{U}_0$ and that g has only finitely many zeros. Suppose that f, f', and f'' have only real zeros.*

(1) Suppose that $f(t) = 0$ and that $D = D(t)$ is multiply connected.

Then there exist domains X_0 and X_1 bounded by finitely many analytic mutually disjoint Jordan curves, with the following properties. Each iterate of Q is locally homeomorphic at each point of ∂X_0 and ∂X_1. We have $\overline{X_0} \subset D$, $\overline{X_1} \subset X_0$, $Q(X_0) = X_1$. The set X_0 is the component of $Q^{-1}(X_1)$ containing X_1. We have $t \in X_1$, so $t \in X_0$ also. The sets X_0 and X_1 are symmetric about the real axis. The domain X_1 is simply connected while X_0 is multiply connected. Let X_0 be bounded by k Jordan curves. The map $Q \mid X_0$ is a covering map that *takes X_0 onto X_1 exactly m times for some $m \geq 2$, and*

$$2 - k + \delta(Q, X_0) = \chi(X_0) + \delta(Q, X_0) = m\chi(X_1) = m. \tag{33}$$

We have $k = 2$ and $\delta(Q, X_0) = m$. The inner boundary contour of X_0 contains the points $\pm i$ in its interior.

(2) Suppose that ∞ is a parabolic fixed point of Q, and let D be a component of $N(Q)$ with $\infty \in \partial D$, such that D contains $(-\infty, x)$ or (x, ∞) for some real x, and such that $Q^j(z) \to \infty$ as $j \to \infty$, locally uniformly for $z \in D$. Suppose that D is multiply connected.

Then there exist domains X_0 and X_1 bounded by finitely many mutually disjoint Jordan curves, which are smooth except possibly at infinity, with the following properties. Each iterate of Q is locally homeomorphic at each point of ∂X_0 and ∂X_1. We have $\overline{X_0} \subset D \cup \{\infty\}$, $\overline{X_1} \subset X_0 \cup \{\infty\}$, $Q(X_0) = X_1$. The set X_0 is the component of $Q^{-1}(X_1)$ containing X_1. We have $\infty \in \partial X_1 \cap \partial X_0$. The sets X_0 and X_1 are symmetric about the real axis. The domain X_1 is simply connected while X_0 is multiply connected. Let X_0 be bounded by k Jordan curves. The map $Q \mid X_0$ is a covering map that takes X_0 onto X_1 exactly m times for some $m \geq 2$, and (33) holds. We have $k = 2$ and $\delta(Q, X_0) = m$. The inner boundary contour of X_0 contains the points $\pm i$ in its interior.

Proof of Lemma 10.1. Let the assumptions of Lemma 10.1 be satisfied. Then Q is a rational function.

Proof of (1). Suppose that $f(t) = 0$ and that $D = D(t)$ is multiply connected.

As in [Pr], let γ be a Jordan curve in D surrounding t such that γ does not intersect the set $\bigcup_{j=0}^{\infty} Q^j(E)$ where the finite set E consists of the critical points of Q (critical points are the points at which Q is not a local homeomorphism). The points in $D \cap \{\bigcup_{j=0}^{\infty} Q^j(E)\}$ cluster to t only. Thus it is easy to take γ to be an analytic curve and still satisfying these conditions. We further take γ to be symmetric about $\mathbf{R}$. Let G be the Jordan domain with $t \in G$ and $\partial G = \gamma$. We take γ to be such that $\overline{Q(G)} \subset G$. Let G_j be the component of $Q^{-j}(G)$ containing G. Then each G_j is symmetric about $\mathbf{R}$. Since $\bigcup_{j=0}^{\infty} G_j = D$, there is a smallest value of j such that $X_0 = G_j$ is multiply connected while $X_1 = Q(G_j) = G_{j-1}$ is simply connected. Thus X_1 is a Jordan domain, X_0 is bounded by finitely many, say k, analytic Jordan curves, and $\overline{X_1} \subset X_0$. The domains X_0 and X_1 are bounded and their closures are compact subsets of D, since $\infty \in J(Q)$, being a parabolic or repelling fixed point of Q.

It is clear from the construction that each iterate of Q is locally homeomorphic at each point of ∂X_0 and ∂X_1, that $\overline{X_0} \subset D$, $\overline{X_1} \subset X_0$, $Q(X_0) = X_1$, that the set X_0 is the component of $Q^{-1}(X_1)$ containing X_1, that $t \in X_1$ and $t \in X_0$, and that the sets X_0 and X_1 are symmetric about the real axis.

The Euler characteristics of these domains are given by $\chi(X_0) = 2 - k$ and $\chi(X_1) = 1$. Now $Q \mid X_0$ is a proper map onto X_1 that covers X_1 m times, say.

By the Riemann–Hurwitz formula [B, Theorem 5.4.1, p. 87], we obtain (33), that is, $\delta = k - 2 + m$, where $\delta = \delta(Q, X_0)$. By a result of Shishikura [Sh, Proposition 4.1(β) with $p = 1$, p. 15], each component of $\overline{\mathbf{C}} \setminus X_0$ contains at least one repelling or parabolic fixed point of Q. The only such fixed points for Q are $\pm i$ and ∞, which implies that $k \le 3$. Also $k \ge 2$ as X_0 is multiply connected. Hence either $k = 2$ and $\delta = m$, or $k = 3$ and $\delta = m + 1$.

The case $k = 3$ remains to be ruled out. To get a contradiction, suppose that $k = 3$. Let γ_j for $1 \le j \le 3$ be the components of ∂X_0, numbered so that γ_1 contains i and γ_2 contains $-i$ in its interior while γ_3 is the outer contour. Then, by symmetry, γ_2 is the reflection of γ_1 in the real axis. We have $\gamma_1 \subset H^+$, for if γ_1 intersects $\mathbf{R}$ then, since X_0 is symmetric about $\mathbf{R}$, we would have $\gamma_2 = \gamma_1$ (since γ_1 contains i and γ_2 contains $-i$ in its interior). This is impossible, and so $\gamma_1 \subset H^+$. Since γ_3 is symmetric about $\mathbf{R}$ also, it is now seen that $X_0 \cap \mathbf{R}$ is a single open interval. Since $X_1 = Q(X_0) \subset X_0$ is also symmetric about $\mathbf{R}$ and is a Jordan domain, we see that $X_1 \cap \mathbf{R}$ is also a single open interval.

Now Q maps each γ_j for $1 \le j \le 3$ onto the Jordan curve ∂X_1, say m_j times, and Q is a local homeomorphism at each point of ∂X_0. It follows from the argument principle that Q maps the interior of γ_j onto the interior of ∂X_1, m_j times, for $j = 1, 2$, since Q has no poles in the interior of γ_j. This is impossible since Q fixes the point i, which lies in the interior of γ_1 but not in X_1, which is equal to the interior of ∂X_1. This contradiction shows that the case $k = 3$ cannot occur.

Since $k = 2$, it follows that the inner boundary contour of X_0 contains the points $\pm i$ in its interior.

This completes the proof of part (1) of Lemma 10.1.

Proof of (2). Let D be a multiply connected component of $N(Q)$ as in the assumptions of part (2).

Let γ be a Jordan curve in $D \cup \{\infty\}$ such that γ is symmetric about $\mathbf{R}$, γ is infinitely differentiable at each of its finite points, and γ does not intersect the set $\bigcup_{j=0}^{\infty} Q^j(E)$ where E consists of the critical points of Q. The points in $D \cap \bigl\{\bigcup_{j=0}^{\infty} Q^j(E)\bigr\}$ cluster to ∞ only, so that we can satisfy these conditions. Let G be the Jordan domain with $\partial G = \gamma$, containing $(-\infty, x)$ or $(x, +\infty)$ for some real x, depending on which of these rays is contained in D (exactly one of them is contained in D). We take γ to be such that $\overline{Q(G)} \subset G \cup \{\infty\}$. Let G_j be the component of $Q^{-j}(G)$ containing G. Then each G_j is symmetric about $\mathbf{R}$. Since $\bigcup_{j=0}^{\infty} G_j = D$, there is a smallest value of j such that $X_0 = G_j$ is multiply connected while $X_1 = Q(G_j) = G_{j-1}$ is simply connected. Thus X_1 is a Jordan domain, X_0 is bounded by finitely many, say k, disjoint Jordan curves, that are smooth outside infinity, and $\overline{X_1} \subset X_0 \cup \{\infty\}$. Exactly one component of ∂X_0 is unbounded.

It is clear from the construction that each iterate of Q is locally homeomorphic at each point of ∂X_0 and ∂X_1, that $\overline{X_0} \subset D \cup \{\infty\}$, $\overline{X_1} \subset X_0 \cup \{\infty\}$, $Q(X_0) =$

X_1, that the set X_0 is the component of $Q^{-1}(X_1)$ containing X_1, that $\infty \in \partial X_1 \cap \partial X_0$, and that the sets X_0 and X_1 are symmetric about the real axis.

The Euler characteristics of these domains are given by $\chi(X_0) = 2 - k$ and $\chi(X_1) = 1$. Again $Q \mid X_0$ is a proper map onto X_1 that covers X_1 m times, say. By the Riemann–Hurwitz formula, we obtain (33), that is, $\delta = k - 2 + m$, where $\delta = \delta(Q, X_0)$. By a result of Shishikura [Sh, Proposition 4.1(β) with $p = 1$, p. 15], each bounded component (more precisely, any component not containing the parabolic fixed point ∞) of $\overline{\mathbf{C}} \setminus X_0$ contains at least one repelling or parabolic fixed point of Q. The only such fixed points for Q are $\pm i$, which implies that $k \leq 3$. Also $k \geq 2$ as X_0 is multiply connected. Hence either $k = 2$ and $\delta = m$, or $k = 3$ and $\delta = m + 1$.

The case $k = 3$ remains to be ruled out. To get a contradiction, suppose that $k = 3$. Let γ_j for $1 \leq j \leq 3$ be the components of ∂X_0, numbered so that γ_1 contains i and γ_2 contains $-i$ in its interior while γ_3 is the unbounded outer contour. Then, by symmetry, γ_2 is the reflection of γ_1 in the real axis. We have $\gamma_1 \subset H^+$, for if γ_1 intersects $\mathbf{R}$ then, since X_0 is symmetric about $\mathbf{R}$, we would have $\gamma_2 = \gamma_1$ (since γ_1 contains i and γ_2 contains $-i$ in its interior). This is impossible, and so $\gamma_1 \subset H^+$. Since γ_3 is symmetric about $\mathbf{R}$ also, it is now seen that $X_0 \cap \mathbf{R}$ is a single unbounded open interval. Since $X_1 = Q(X_0) \subset X_0$ is also symmetric about $\mathbf{R}$ and is a Jordan domain, we see that $X_1 \cap \mathbf{R}$ is also a single unbounded open interval.

Now Q maps each γ_j for $1 \leq j \leq 3$ onto the Jordan curve ∂X_1, say m_j times, and Q is a local homeomorphism at each point of ∂X_0. It follows from the argument principle that Q maps the interior of γ_j onto the interior of ∂X_1, m_j times, for $j = 1, 2$, since Q has no poles in the interior of γ_j. This is impossible since Q fixes the point i, which lies in the interior of γ_1 but not in X_1, which is equal to the interior of ∂X_1. This contradiction shows that the case $k = 3$ cannot occur.

Since $k = 2$, it follows that the inner boundary contour of X_0 contains the points $\pm i$ in its interior.

This completes the proof of part (2) of Lemma 10.1. The proof of Lemma 10.1 is therefore complete.

11. When g is transcendental

11.1. Connectivity of parabolic domains. Recall that $f(z) = g(z)/(z^2 + 1)^n$, where $g \in \mathscr{U}_0$ and g has only finitely many zeros, and that f, f', and f'' are assumed to have only real zeros. If g is transcendental then $g(z) = e^{-\xi z^2 + \beta z} P(z)$ where $\xi \geq 0$, $\beta \in \mathbf{R}$, and P is a real polynomial with only real zeros. Furthermore, at least one of ξ and β is non-zero. Thus

$$L = \frac{f'}{f} = -2\xi z + \beta + \frac{P'}{P} - \frac{2nz}{z^2 + 1}$$

and

$$Q(z) = z - \frac{1}{-2\xi z + \beta + O(1/z)} \qquad \text{as } z \to \infty.$$

Hence Q is rational and has a parabolic fixed point at infinity. For the definition and discussion of petals at a parabolic fixed point, we refer to [B, pp. 110–132]. If $\xi > 0$ then there are exactly 2 petals at infinity, one containing $(-\infty, x_1)$ and the other containing (x_2, ∞) for some real $x_1 \le x_2$. If $\xi = 0 \ne \beta$ then there is exactly 1 petal at infinity, and it contains $(-\infty, x_1)$ for some real x_1 if $\beta > 0$, while it contains (x_2, ∞) for some real x_2 if $\beta < 0$. (If g is a polynomial of degree at least $2n+2$ then Q has a repelling fixed point at infinity.)

Let D denote any component of $N(Q)$ with $\infty \in \partial D$, such that D contains $(-\infty, x)$ or (x, ∞) for some real x, and such that $Q^j(z) \to \infty$ as $j \to \infty$, locally uniformly for $z \in D$. Thus D contains one petal. Note that all components D arising as above are symmetric about $\mathbf{R}$.

Lemma 11.1. *The domain D is multiply connected.*

Proof of Lemma 11.1. Suppose that D is simply connected. We may assume that $(x, \infty) \subset D$ for some real x. Let x_1 be the smallest real number such that $(x_1, \infty) \subset D$. Then $x_1 \in \partial D \subset J(Q)$. Thus $Q(x) > x$ for all $x > x_1$, and either $Q(x_1) = x_1$ or $Q(x_1) = \infty$, for otherwise, there is $\varepsilon > 0$ such that $(x_1 - \varepsilon, \infty) \subset D$. If $Q(x_1) = x_1$ then $f(x_1) = 0$ so that x_1 is an attracting or superattracting fixed point of Q and hence lies in the interior of a component of $N(Q)$ other than D. This is impossible. Hence $Q(x_1) = \infty$, and so $(f'/f)(x_1) = 0$. Since $Q(x) > x$ for all $x > x_1$, we have $\lim_{x \to x_1+} Q(x) = +\infty$, so that $x_1 \ne x_0$, and also $\lim_{x \to +\infty} Q(x) = +\infty$. Hence there must be at least one zero x_2 of Q' on (x_1, ∞). There are no zeros of f or of f' on (x_1, ∞). Thus $f''(x_2) = 0$. So the values in $[Q(x_2), \infty)$ are taken by Q at least twice on (x_1, ∞).

Since D is simply connected, there is a conformal mapping λ of D onto the unit disk $B(0,1)$. Then $Q_1 = \lambda \circ Q \circ \lambda^{-1}$ maps $B(0,1)$ onto itself, and it is known that Q_1 is a finite Blaschke product (see [Po, p. 118]). Let Q_1 have degree m. Then Q_1 has no fixed points in $B(0,1)$ but Q_1 has exactly $m+1$ fixed points on $\partial B(0,1) = S(1)$, with due count of multiplicity. Exactly one of these is the Denjoy–Wolff fixed point of Q_1 and corresponds to the point ∞ for Q. The other m points are repelling fixed points of Q_1 [Po, p. 118]. They correspond to repelling fixed points of Q on ∂D (see [Po, Theorem 1, p. 118]) in the sense that if ζ is a repelling fixed point of Q_1 then $\lambda^{-1}(\zeta)$ exists as an angular limit and is equal to a repelling fixed point of Q. Since Q has only two repelling fixed points, namely $\pm i$, each of multiplicity 1, it follows that $m \le 2$. Since Q cannot be one-to-one in D, we have $m = 2$. This also shows that $\pm i \in \partial D$ and that the points $\pm i$ are accessible boundary points of D (images of certain radii in $B(0,1)$ under λ^{-1} are paths in D tending to i and $-i$).

More precisely, in our case, for each of $\pm i$ there can be only one $\zeta \in S(1)$ corresponding to this point. For if the distinct repelling points $\zeta_1, \zeta_2 \in S(1)$ of Q_1

satisfy $\lambda^{-1}(\zeta_1) = \lambda^{-1}(\zeta_2) = i$, let γ_j be the image under λ^{-1} of a short radial segment going to ζ_j, for $j = 1, 2$. We can join a point in $H^+ \cap D$ to the starting points of γ_1 and γ_2 to obtain a set which together with the point i is a Jordan curve γ in $D \cup \{i\}$ (it is a Jordan curve since λ^{-1} is one-to-one). By reflecting in $\mathbf{R}$ those parts of γ that are in H^- and modifying γ slightly at its points of intersection with $\mathbf{R}$, we may assume that $\gamma \setminus \{i\} \subset H^+ \cap D$. Let Y be the interior of γ. The set $Q(\gamma)$ is a Jordan curve in $D \cup \{i\}$, $Q(i) = i$, and $\partial Q(Y) \subset Q(\gamma)$. The function Q has no poles in Y.

Let Γ be any connected component of $Y \cap (\mathbf{C} \setminus D)$. Thus $\partial\Gamma \subset J(Q)$. Since $Q(i) = i$ and $Q(D) = D$, Q maps Γ into Γ_1 where Γ_1 is also such a component. More precisely, since each of ζ_1 and ζ_2 is a repelling fixed point of Q_1, the arcs $Q_1\big(\lambda(\gamma_j)\big)$, for $j = 1, 2$, are very close to the arcs $\lambda(\gamma_j)$ and also start from ζ_1 and ζ_2. Hence $Q(\gamma_j)$ determines the same prime end of D (whose impression contains i) as γ_j. Now $Q(Y)$ can be larger than Y and might contain points of $\mathbf{R}$. Since $Q(\Gamma)$ is connected, it follows that $Q(\Gamma)$ is contained in some component Γ_1 as soon as it is clear that $Q(\Gamma) \cap D = \emptyset$. If $Q(\Gamma) \cap D \neq \emptyset$, then Γ contains a component D_1 of $Q^{-1}(D)$ that Q maps onto D. Hence $\overline{D_1}$ and thus $\overline{\Gamma}$ must contain a pole of Q. But this is not the case since $\overline{\Gamma} \subset \overline{Y} \subset H^+$, which contains no poles of Q.

Now consider the iterates of Q in the open set Y. If $z \in Y \setminus D$ then $Q^j(z) \in Y$ for all $j \geq 1$ by the above. If $z \in Y \cap D$ then $Q^j(z) \in D$ for all $j \geq 1$. Thus the Q^j omit at least 3 points in Y and therefore form a normal family in Y. Since D is a component of $N(Q)$, it follows that $Y \subset D$. Now the conformality of λ (via Lindelöf's theorem) implies that $\zeta_1 = \zeta_2$. This proves our assertion that only one point on $S(1)$ corresponds to i, and the same holds for $-i$.

Since $m = 2$, x_2 is the unique zero of f'' on (x_1, ∞). Since D is simply connected and symmetric about $\mathbf{R}$, we have $\mathbf{R} \cap D = (x_1, \infty)$. Hence the only real values that Q takes in D are in (x_1, ∞), and each such value is taken twice. There is a component W' of K^- such that $x_2 \in \partial W'$. Then $x_1 \in \partial W'$ also. If $W' \neq W$, we have $\mathbf{R} \cap \partial W' = [x_3, x_2]$ where $d \leq x_3 \leq x_1$. If $W' = W$ then $x_2 = d$ and $c \leq x_1$.

Now D contains an arc γ_1 of $\partial W' \cap H^+$ starting from x_2, as well as the complex conjugate arc γ_2 of γ_1. The map Q takes the same real values on γ_1 and γ_2, and these values lie in (x_1, x_2). If we follow $\partial W'$ starting from x_2 and entering H^+ from x_2 but then continuing as long as necessary along $\partial W'$, we will stay in D until we come to a point ζ at which $Q(\zeta) = x_1$. Then $\zeta \in \partial D$. The values of Q decrease as we move along γ_1 from x_2 towards ζ.

If $W' \neq W$ and $x_3 < x_1$ then $Q(x_3) \leq x_3 < x_1$. For if $Q(x_3) \geq x_3$, then, since Q is decreasing on (x_3, x_1), there is a fixed point x_4 of Q, hence a zero of f, on (x_3, x_1). If $f'(x_4) \neq 0$, then $Q'(x_4) = 0$, hence $\mathbf{R} \cap \partial W' \subset [x_4, x_2]$, which contradicts the definition of x_3. If $f'(x_4) = 0$ then $Q'(x_4) > 0$ so that $x_4 \in \partial K \setminus \partial K^-$, which contradicts the fact that $x_4 \in \partial W'$. We conclude that

$Q(x_3) \le x_3 < x_1$ or $x_3 = x_1$, and in each case we encounter the point ζ before coming to the point x_3. So the arc γ_1 stays in H^+ and goes from x_2 to ζ. We see that Q cannot take real values in D outside $\gamma_1 \cap \gamma_2 \cup (x_1, +\infty)$. But there is a path in D from a point in γ_1 to i, and this path must intersect ∂U_0, where Q takes real values. This is a contradiction.

It follows that $W' = W$. When z moves along γ_1, z would first reach a before any other real number. If $a \in D$ then D is multiply connected, which is against our assumption. Hence we come to ζ before reaching a. Thus all the real values that Q ever takes in D are attained in the outer contour of W, its complex conjugate, and in $(x_1, +\infty)$. In the same way as above, we get a contradiction with the fact that i is an accessible boundary point of D.

This completes the proof of Lemma 11.1.

11.2. Some special cases. The cases dealt with in the following lemma do not fit into our general scheme so that they need to be handled separately.

Lemma 11.2. *Suppose that $\xi = 0 \ne \beta$ and that f has no zeros, or that $\xi = \beta = 0$ and f has only one zero, ignoring multiplicities. Then if f and f' have only real zeros, f'' must have at least one non-real zero.*

Proof of Lemma 11.2. To get a contradiction, suppose that f, f', and f'' have only real zeros. Throughout the proof, W' denotes a component of K^- with $W' \ne W$, and we write $\mathbf{R} \cap \partial W' = [w_1, w_2]$. There must exist a unique zero w_3 of f'/f on $[w_1, w_2]$.

In the first case, $g(z) = e^{\beta z}$ where $\beta \ne 0$. Now f' has exactly 3 zeros, counting multiplicities, all of them extraordinary, and f'' has exactly 3 extraordinary zeros and altogether 5 zeros, by Lemma 9.1. Thus a, b, c, d are all zeros of f''. We may assume that the unique zero of f' on ∂W lies in $[a, b]$. In addition, f' has the special zero $x_0 \in [b, c]$ and one further zero v, which must be a simple zero unless it coincides with x_0 or u.

If there exists W' whose boundary intersects (d, ∞), then $v \in [w_1, w_2]$. Hence there is only one such W'. Thus also $f'f'' \ne 0$ on $(-\infty, a)$ (a zero of f'' there would imply the existence of a bounded component of K^- there and hence the existence of a zero of f' there). So if $a \ne u$ then a is an extraordinary zero of f'', and the two other extraordinary zeros of f'' are on the same interval determined by the zeros of f'. But $v \ne w_2$ as v is a simple zero of f', and so w_2 is the only zero of f'' on (v, ∞), a contradiction. If $a = u$ then u is a double zero of f', so $u = v$ and W' cannot exist. We conclude that $(d, \infty) \subset \partial K$. A similar argument shows that K^- has no component W' whose boundary intersects $(-\infty, a)$.

If $a = u$ then $u = v$. Now $K^- = W$, for otherwise f' would have to have a zero other than x_0 on $[b, c]$. Hence the zeros of f'' exactly at a, b, c, d, and c, d are the only extraordinary zeros of f''. Since f'' must have exactly 3 extraordinary zeros, this is impossible.

So $a \neq u$, hence a is an extraordinary zero of f'', and the two other extraordinary zeros of f'' are on the same interval determined by the zeros of f'. Thus also $v \in [b, c)$, and there exists W' with $v \in \partial W'$. If $x_0 \leq v < w_2$ then f'' has the 3 zeros w_2, c, d greater than the largest zero of f', which is a contradiction. If $x_0 \leq v = w_2$ then f' has too many zeros unless $x_0 = v$, and then $w_1 = w_2 = v = x_0$. But now z passes through x_0 twice when z traces ∂U_0 once, contradicting the fact that Q is one-to-one in U_0. This proves Lemma 11.2 in the first case.

In the second case, by our standing assumption, $g(z) = (z-t)^\mu$ for some real t and for some integer $\mu = \deg g \geq 2n+2$. Now by Lemma 9.1, each of f' and f'' has 2 extraordinary zeros, and they lie on the same interval determined by the zeros of f or f', respectively. Now $Q'(t) > 0$ so that $t \in \partial K \setminus \partial K^-$. The only zeros of f' other than t are x_0 and a zero $u \in \partial W$. Possibly $u = x_0$ is a double zero, and otherwise u is a simple zero of f' and hence distinct from all of a, b, c, d. If $u = x_0$ then $u = b$ or $u = c$, so we always have $a \neq u \neq d$. Now there can exist no W', and so $K = W$.

Suppose that $t \in \partial U$. We may assume that $t > d$.

If $c \leq u < d$ then a, b are the extraordinary zeros of f''. Since $u \neq d$, also d is an extraordinary zero of f'', which is impossible.

If $a < u \leq b$ then a is the only zero of f'' less than the smallest zero u of f', which contradicts Lemma 9.1.

This proves Lemma 11.2 in the second case, and completes the proof of Lemma 11.2.

11.3. On multiply connected domains.

Lemma 11.3. *Suppose that f has only finitely many zeros and $g \in \mathscr{U}_0$, so that Q is rational. Suppose that f, f', and f'' have only real zeros.*

(i) *Let ∞ be a parabolic fixed point of Q with only one petal, and let D be the associated component of $N(Q)$. If f has at least one zero, then D is simply connected.*

(ii) *Let ∞ be a parabolic fixed point of Q, and suppose that there are two distinct components D_1 and D_2 of $N(Q)$ associated with infinity. Then at least one of D_1 and D_2 is simply connected.*

(iii) *Suppose that g is a polynomial. Let f have at least two distinct zeros. Then $D(t)$ is simply connected whenever $f(t) = 0$.*

Therefore the function g cannot be transcendental if f, f', and f'' are to have only real zeros.

Proof of Lemma 11.3. Suppose that parts (i), (ii), and (iii) have been proved. Suppose that f has only finitely many zeros and $g \in \mathscr{U}_0$. Write $g = e^{-\xi z^2 + \beta z} P$ as before. Suppose that f, f', and f'' have only real zeros.

If $\xi > 0$ then there are two components of $N(Q)$, say D_1 and D_2, at the parabolic fixed point at infinity, and by Lemma 11.1, each is multiply connected. This contradicts (ii). So $\xi = 0$.

If $\beta \neq 0$ then the case when f has no zeros is ruled out by Lemma 11.2. Suppose that f has at least one zero. Now ∞ is a parabolic fixed point of Q with an associated component D of $N(Q)$. By (i), D is simply connected. Now Lemma 11.1 gives a contradiction.

So g cannot be transcendental, and we have proved the last paragraph of Lemma 11.3.

We proceed to prove (ii) of Lemma 11.3.

Let D_1, D_2 be as in (ii). There are real numbers x_1, x_2 such that $(-\infty, x_1) \subset D_1$ and $(x_2, +\infty) \subset D_2$, say. Suppose that D_1 and D_2 are multiply connected.

The points $\pm i \in \partial D_1 \cap \partial D_2$ and both are accessible boundary points of D_1 and of D_2. It follows from [Sh, Proposition 4.1(β) with $p = 1$, p. 15, and Remark 2, p. 16] that at least one of $\pm i$, and hence by symmetry, each of them, lies on ∂D_1 and similarly on ∂D_2. To prove that the points $\pm i$ are accessible boundary points, let D be either one of D_1 and D_2, and consider D. For convenience, suppose that $(x_2, +\infty) \subset D$. We use the argument of Przytycki [Pr]. Pick $w \in D \cap H^+$ with a large positive real part such that $Q(w) \in D \cap H^+$. Choose w so that w and $Q(w)$ can be joined by a path Γ in $D \cap H^+$ such that Γ (including its end points) does not intersect $\bigcup_{j=0}^{\infty} Q^j(E)$, where E consists of the critical points of Q. We may assume that the only points z_1, z_2 of Γ satisfying $z_2 = Q(z_1)$ are $z_1 = w$, $z_2 = Q(w)$. Set $\Gamma_0 = \Gamma$. Define inductively branches of Q^{-1} taking Γ_j onto Γ_{j+1} so that Γ_j and Γ_{j+1} share exactly one point. The set $\Gamma' = \bigcup_{j=0}^{\infty} \Gamma_j$ is a continuous path. The various branches of Q^{-j} defined in this way form a normal family in a neighbourhood of Γ. Each limit function of a convergent subsequence of such branches of Q^{-j} is a constant function and its constant value belongs to ∂D. The spherical diameters of the Γ_j therefore tend to zero as $j \to \infty$. If the Γ_j cluster to more than one point, then the cluster points form a continuum, and each such cluster point z satisfies $Q(z) = z$. But there is no such continuum. Hence the sequence Γ_j tends to a single point, which must be a fixed point of Q in $J(Q)$, and hence can only be $\pm i$ or ∞. If we can choose Γ so that the Γ_j remain bounded, then the limit point is i or $-i$. This shows also that the points $\pm i$ are accessible boundary points of D since a path of the form Γ' can tend to one of them, and the complex conjugate path tends to the other.

Now we present the details of the construction of Γ'. Applying Lemma 10.1 we find domains $X_1 \subset X_0 \subset D$ with the properties given there. Let $\gamma_0 \subset D$ be the inner boundary contour of X_0. We can choose $w \in \gamma_0 \cap H^+$, and assume that $\Gamma \setminus \{w, Q(w)\} \subset X_0 \setminus X_1$. Then the points $\pm i$ lie in the interior Ω of γ_0. Since $\Gamma \cap \bigcup_{j=0}^{\infty} Q^j(E) = \emptyset$, there is a unique lift Γ_1 of Γ by Q^{-1} starting from w. That is, $Q(\Gamma_1) = \Gamma$ and Γ_1 starts at w, corresponding to tracing Γ starting at $Q(w)$.

We may assume that Γ is orthogonal to ∂X_0 at w and to ∂X_1 at $Q(w)$. Then Γ_1 enters into $D \setminus \overline{X_0}$ at w. We claim that Γ_1 remains in $D \setminus X_0$. If not, then Γ_1 meets γ_0 so that $\Gamma = Q(\Gamma_1)$ meets $Q(\gamma_0) = \partial X_1$ at a point other than $Q(w)$, which is against the construction of Γ. Also Γ_1 continues until it hits a point $w_1 \in D \setminus X_0$ with $Q(w_1) = w$. Otherwise, Γ_1 clusters to $\partial D \subset J(Q)$, contradicting the fact that Q is rational and Γ goes (from $Q(w)$) to $w \in D$. We obtain Γ_j for $j \geq 2$ by continuing in this way. Suppose that $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$ stay in $D \setminus \overline{X_0}$ while Γ_{j+1} hits γ_0. Then $\Gamma_j = Q(\Gamma_{j+1})$ hits $Q(\gamma_0) = \partial X_1$, which is against our assumption. This produces a bounded path Γ', as required.

Consider both D_1 and D_2 again. Applying Lemma 10.1 we find domains $X_1 \subset X_0 \subset D_1$ and $X_1' \subset X_0' \subset D_2$ with the properties given there. Let $\gamma_0 \subset D_1$ be the inner boundary contour of X_0, and define $\gamma_0' \subset D_2$ in the same way in terms of X_0'. Let D_0 be the interior of the Jordan curve γ_0, and let D_0' be the interior of γ_0'. Then by Lemma 10.1, the points $\pm i$ lie in $D_0 \cap D_0'$. Since $\gamma_0 \subset D_1$ and $D_1 \cap D_2 = \emptyset$, and since D_2 is unbounded, it follows that $D_2 \cap D_0 = \emptyset$. Thus $i \notin \partial D_2$, which is a contradiction. This proves (ii).

Consider D as in (i), and assume that D is multiply connected while f has at least one zero. Note that $\xi = 0 \neq \beta$. Again we may assume that $(x, \infty) \subset D$ for some real x. This is equivalent to $\beta < 0$, which can be achieved by replacing $f(z)$ by $f(-z)$, if necessary. Let x_1 be the smallest real number such that $(x_1, \infty) \subset D$. As in the proof of Lemma 11.1, we find that $Q(x_1) = \infty$, and so $(f'/f)(x_1) = 0$. Again $Q(x) > x$ for all $x > x_1$, and $\lim_{x \to x_1+} Q(x) = \lim_{x \to +\infty} Q(x) = +\infty$, and $x_1 \neq x_0$. There is a zero x_2 of Q' but no zeros of f or of f' on (x_1, ∞). Thus $f''(x_2) = 0$. So the values in $\big[Q(x_2), \infty\big)$ are taken by Q at least twice on (x_1, ∞).

We apply Lemma 10.1 and find the domains X_0 and X_1 with the properties given there.

As in the proof of part (ii), we see that $\pm i \in \partial D$ and that both are accessible boundary points of D.

The function f cannot have any zero t for which $D(t)$ is multiply connected. For if there is such a t, then, since $\pm i \in \partial D(t)$ by Lemma 8.4, it follows that $D(t)$ is contained in the interior of the inner boundary contour of X_0. But by Lemma 8.4, $D(t)$ contains a Jordan curve γ such that i lies in the interior of γ. This prevents i from lying on ∂D. This contradiction shows that $D(t)$ is simply connected for each zero t of f. (This is the same argument that we used to show that there cannot exist two multiply connected components D_1, D_2 in part (ii).)

So we assume that f has at least one zero t. Now $D(t)$ is simply connected, and by Lemma 8.5, we have $\pm i \in \partial D(t)$. Thus $\overline{D(t)}$ is a continuum that lies outside D and contains $\pm i$.

For any $j \geq 1$, let Y_j be the component of $Q^{-j}(X_0)$ containing X_0. Then each Y_j is multiply connected and is bounded by finitely many disjoint Jordan curves, each smooth outside infinity, and $\overline{Y_j} \subset Y_{j+1} \cup \bigcup_{l=0}^{j} Q^{-l}(\infty)$. Also each Y_j

is symmetric about $\mathbf{R}$.

Let γ_j and γ_j' be the components of ∂Y_j such that the Jordan curve γ_j contains the point i in its interior, and γ_j' contains the point $-i$ in its interior. Since $\pm i$ belong to the compact connected set $\overline{D(t)}$ that lies outside D, we have $\gamma_j = \gamma_j'$. Clearly γ_j is symmetric about $\mathbf{R}$ and contains $\pm i$ in its interior D_j.

Since D_j is a Jordan domain symmetric about $\mathbf{R}$, we have $\mathbf{R} \cap D_j = (v_j, w_j)$, where $-\infty < v_j < w_j < +\infty$. We have $\overline{D_{j+1}} \subset D_j \cup \bigcup_{l=0}^{j} Q^{-l}(\infty)$, since $\overline{Y_j} \subset Y_{j+1} \cup \bigcup_{l=0}^{j} Q^{-l}(\infty)$. Hence $v_j \le v_{j+1} < w_{j+1} \le w_j$. As $j \to \infty$, the sequences v_j, w_j converge to limits v, w with $v \le w$. Note that $v, w \in \partial D$.

Let $t_1, \dots, t_k$ be all the distinct zeros of f. As we have seen, each $D(t_j)$ is simply connected, symmetric about $\mathbf{R}$, and connects i to $-i$. Let F be the component of $\overline{\mathbf{C}} \setminus D$ containing i. Since $k \ge 1$ and $\overline{D(t_1)} \subset \overline{\mathbf{C}} \setminus D$ contains $\pm i$, we have $\pm i \in F$. Thus $F \subset D_j$ for all j. Suppose that $t_1 < \cdots < t_k$. Then the set $\bigcup_{j=1}^{k} \overline{D(t_j)}$ and all points separated by this set from infinity, lie in F. If $k \ge 2$ then F contains a point $s \in (t_1, t_2)$ with $(f'/f)(s) = 0$, hence $Q(s) = \infty$.

Suppose first that $k \ge 2$, and pick $s \in F$ with $(f'/f)(s) = 0$.

There is a component T of $Q^{-1}(D)$, symmetric about $\mathbf{R}$, with $s \in \partial T$. There are $u_1, u_2 \in \partial T$ with $Q(u_1) = i$, $Q(u_2) = -i$. The points u_1, s lie in different components of ∂T, for otherwise they are joined by a continuum F_1 in $J(Q)$, and $Q(F_1)$ is a continuum in $J(Q)$ joining i to ∞, which makes it impossible for D to be multiply connected. Hence there is a Jordan curve $\gamma \subset T$ separating s from u_1. We may take γ to be symmetric in $\mathbf{R}$. There is a component F_1 of $Q^{-1}(F)$ with $u_1 \in \partial F_1$. The set $F_1 \cap \partial T$ contains points w_1, w_2 with $Q(w_1) = i$ and $Q(w_2) = -i$ (in fact, $w_1 = u_1$).

We may take s with $s \in \partial W'$ for a component $W' \ne W$ of K^-. For otherwise, $x_0 \in (t_1, t_2)$ so that f' has 3 zeros on (t_1, t_2). If one of these at least is distinct from x_0, we take s to be that zero. Then $(f''/f)(s) < 0$ by the last statement of Lemma 5.1(1)(i). (Otherwise, x_0 is a zero of f' of order 3. If x_0 is a multiple zero of f' then $x_0 \in \{b, c\}$ for otherwise z passes through x_0 twice when tracing ∂U_0, contradicting the fact that Q is one-to-one in U_0. But now $x_0 \in \{b, c\}$ and $x_0 \notin \partial(K^- \setminus W)$ imply that x_0 is a double zero of f', for otherwise, e.g., $x_0 = b = a$, in which case Q has the pole x_0 on ∂U, which is impossible by Lemma 7.4.)

So $\mathbf{R} \cap \partial W' = [t', t'']$ where one of t' is a zero of f and the other is a zero of f'', one belonging to the component $D(t_1)$, the other to $D(t_2)$.

Thus $\mathbf{R} \cap \partial T \subset [v', v'']$, where $t' < v' < v'' < t''$, and where $v' \in \partial D(t_1)$, $v'' \in \partial D(t_2)$. The set T must lie between $D(t_1)$ and $D(t_2)$, and the only points there that Q maps to $\pm i$ are a point $w_3 \in W'$ with $Q(w_3) = -i$, the point $\overline{w_3}$, and possibly the points $\pm i$ on the boundary of the region between $D(t_1)$ and $D(t_2)$ (unless the closures of $D(t_1)$ and $D(t_2)$ meet also outside $\pm i$ so as to prevent access from s to $\pm i$). If one of w_1, w_2 lies in H^+ and the other in H^-, then F_1 intersects $[v', v'']$ also. Hence Q maps some subset of $[v', v'']$

onto $[t_1,t_k]$. But now $Q(t')<t'$ and $Q(t'')>t''$, for otherwise f has a zero on (t',t'') (compare the proof of Lemma 11.1). Hence, on the set (t',t'') and thus on $[v',v'']$, Q omits the values in $[Q(t'),Q(t'')]\supset[t',t'']$ while $[t',t'']\subset[t_1,t_k]$. This is a contradiction. Hence w_1,w_2 lie both in H^+ or both in H^-. Suppose that $w_1,w_2\in H^+$. Then $w_1=i$ and $w_2=w_3$. But Q is locally homeomorphic at i and maps each of $D(t_1)$ and $D(t_2)$ onto itself. Hence it cannot map any set close to i and between $D(t_1)$ and $D(t_2)$ onto $F\cap N$ for some neighbourhood N of i. The same contradiction is obtained at $-i$ if $w_1,w_2\in H^-$.

For (i), it remains to assume that f has only one zero t. Note that there is a component T of $Q^{-1}(D)$ with $x_0\in\partial T$. Now $T\cap U_0\neq\emptyset$ (or, if $f''(x_0)=0$, we may choose T with $T\cap U_0\neq\emptyset$). There is a path $\lambda\subset D\cap H^+$ from infinity to i, and a path $\lambda_1\in T\cap U_0$ with $Q(\lambda_1)=\lambda$. Since Q is one-to-one in U_0, λ_1 goes from x_0 to i. Let λ_2 be the closure of the union of λ_1 and its complex conjugate. Now $\overline{D(t)}\cup\lambda_2\subset F$, and if $f''(x_0)\neq 0$, then by the last statement of Lemma 5.1(1)(i), there is a zero s of f'/f strictly between t and x_0. Now we can proceed as above after we found $s\in(t_1,t_2)\cap F$ with $(f'/f)(s)=0$, to arrive at a contradiction.

Suppose that $f''(x_0)=0$ and that we cannot find a pole s of Q strictly between t and x_0. Then it must be the case that if $t>x_0$ then $x_0=b$, and if $t<x_0$ then $x_0=c$. Suppose that $t>x_0=b$ (a similar argument works if $t<x_0=c$). If x_0 is a triple zero and hence the only zero of f'/f, then consideration of $\mathbf{R}\cap D(t)$ in view of Lemma 8.3 shows that D contains no zero of Q' other than the point a. This contradicts (33) of Lemma 10.1(1).

Suppose that $x_0=b$ is a double zero of f'/f. Then $(x_0,x_0+\varepsilon)\subset\partial U_0\subset\partial K$. If t is a multiple zero of f, then $Q'(t)>0$ and $t\in\partial K\setminus\partial K^-$. But then also $f'(t)=0$ so that $f''(x_1)=0$ for some $x_1\in(x_0,t)$. Thus $(x_1,x_1+\varepsilon)\subset\partial W'$ for some component $W'\neq W$ of K^-. Now $\mathbf{R}\cap\partial W'=[x_1,x_2]$, say. If $x_2\le t$ then there is $x_3\in[x_1,x_2)\subset(x_0,t)$ with $Q(x_3)=\infty$ and hence $f'(x_3)=0$, which is against our assumption. If $x_2>t$ then $t\in\partial K^-$, which is also impossible.

Hence t is a simple zero of f, and $Q'(t)=0$. If there is $x_1\in(x_0,t)$ with $f''(x_1)=0=Q'(x_1)$, then, as above, there is a component $W'\neq W$ of K^- such that $(x_1,x_1+\varepsilon)\subset\partial W'$ so that $\mathbf{R}\cap\partial W'=[x_1,x_2]$ for some $x_2\le t$. Thus there is $x_3\in[x_1,x_2]$ with $f'(x_3)=0$, a contradiction. Hence $f''\neq 0$ on (x_0,t), and so $[x_0,t]\subset\partial U_0$ while $[t,t+\varepsilon)\subset\partial W'$ for some component $W'\neq W$ of K^-. There must be $x_3\in[t,x_2]=\mathbf{R}\cap\partial W'$ with $Q(x_3)=\infty$, hence $(f'/f)(x_3)=0$. Thus there is $x_4\in(x_0,x_3)$ with $f''(x_4)=0$. But then $t<x_4<x_3\le x_2$ so that $\mathbf{R}\cap\partial W'$ must end at x_4 and be equal to $[t,x_4]$. This is impossible. This contradiction completes the proof of (i).

To prove (iii), we proceed as in the proof of (i), replacing the component D in (i) by the component $D(t)$, which we assume to be multiply connected, noting that f has a zero $s\neq t$ for which $D(s)$ is simply connected by Lemma 8.5. We note that if x_0 is a double zero of f'/f then f'/f has exactly 2 distinct zeros, and now $\deg Q = 3$. We omit further details. This completes the proof of Lemma 11.3.

12. When g is a polynomial

For the rest of the paper, we assume that g is a polynomial. By our standing assumption, $\deg g \geq 2n+2$. Recall that $f(z) = g(z)/(z^2+1)^n$, and that f, f', and f'' are assumed to have only real zeros. Now ∞ is a repelling fixed point of Q since $Q(z) = \bigl(1-(\deg g - 2n)^{-1}\bigr)z + O(1)$ as $z \to \infty$, and the multiplier of Q at infinity is $\bigl(1-(\deg g-2n)^{-1}\bigr)^{-1} > 1$. By Lemmas 11.2 and 11.3, we may and will assume that f has at least 2 distinct zeros. Hence by Lemma 11.3(iii), the domain $D(t)$ is simply connected whenever $f(t)=0$. By Lemma 9.1, each of f' and f'' has exactly 2 extraordinary zeros.

We next obtain a number of lemmas concerning what happens on the interval $[b,c]$.

Lemma 12.1. *At least one of the intervals $[b,x_0)$ and $(x_0,c]$ contains no zeros of f, and the corresponding one of the intervals (b,x_0) and (x_0,c) contains no zeros of any one of the functions f, f', and f''.*

Proof of Lemma 12.1. Suppose that there are zeros x,y of f with $b \leq x < x_0 < y \leq c$. We have $Q(x_0)=\infty$, and Q maps U_0 one-to-one onto H^+. We have $x,y \in \partial U_0$. As z traces ∂U_0 in the positive direction, starting from and ending at x_0, $Q(z)$ traces $\mathbf{R}$ once in the increasing direction. Thus $y = Q(y) < Q(x) = x$, which is a contradiction.

Hence we may assume, for example, that $[b,x_0)$ contains no zeros of f. Since $Q'(b)=0$, we have $f''(b)=0$. If neither f' nor f'' vanishes anywhere on (b,x_0), there is nothing more to prove. So suppose that there is a zero of f' or of f'' on (b,x_0). Any such point is on the boundary of K^-. Similarly, if there is a zero of f or of f' or of f'' on (x_0,c) then such a point is on the boundary of K^-. Let G be the Jordan domain bounded by $[b,c]$ and that arc of ∂U_0 that joins b and c in H^+. Let the components of $G \setminus \overline{U_0}$ be W_j with $\mathbf{R} \cap \partial W_j = [c_j,d_j]$ for $1 \leq j \leq N$. Of course, the W_j are components of K^-. Label the components so that $b < c_j < d_j < c_{j+1} \leq c$ for all j. Recall that x_0 is bounded away from K^- so that x_0 does not lie in $[c_j,d_j]$ for any j. We now need to show that if there is a component W_j with $c_j < x_0$ then we must have $c_j < x_0$ for all j with $1 \leq j \leq N$. Therefore, to get a contradiction, we assume that this is not true, which means that $c_1 < x_0 < c_N$.

Suppose that also $c_2 < x_0$. For each j, there is a unique zero u_j of f'/f on $[c_j,d_j]$. So if, for example, $f''(c_j) = 0 \neq f(c_j)$, we might have $c_j = u_j$ and $f'(c_j)=0$, while if $f(c_j)=0$, we cannot have $c_j = u_j$. Between any two zeros of f, there is exactly one zero of f', apart from the interval (a_k,a_{k+1}) between successive zeros of f that contains x_0. On the interval (a_k,a_{k+1}), the point x_0 is the middle zero of f', with appropriate interpretation if x_0 is a multiple zero of f'. Since $f \neq 0$ on $[b, x_0]$ while $[b, x_0]$ contains the zeros $u_1 < u_2 < x_0$ of f', we get a contradiction. This shows that if $c_1 < x_0 < c_N$ then $c_2 > x_0$.

Suppose that (a, b) contains a zero u of f'/f. Then (u, u_1) must contain a zero of f which must be b or c_1. This is impossible, and it follows that $f' \neq 0$ on $[a, b]$. Hence $f'(u) = 0$ for exactly one point $u \in (c, d)$.

We now deduce a number of inequalities involving x and $Q(x)$, for various values of $x \in \partial U_0 \cup \partial W$. All these inequalities can be obtained by checking what happens to $Q(z)$ as z traces ∂U_0 or ∂W or ∂W_j once in the positive direction, starting from and ending at x_0 or at u or at u_j, as the case may be. Note that $Q(x_0) = Q(u) = \infty$.

Suppose that $f(t) = 0$ for some $t \in (x_0, c]$. Since $Q(t) = t > x_0$, we have $Q(x) > t$ for all $x \in [b, x_0)$. In particular, we have $c_1 < d_1 < x_0 < t < Q(c_1) < Q(d_1)$. Now Q is decreasing on each of $[c_1, u_1)$ and $(u_1, d_1]$, and $Q(u_1) = \infty$. Thus $Q(x)$ decreases from $Q(c_1)$ to $-\infty$ as x increases from c_1 to u_1. It follows that Q has a fixed point and so f has a zero on $(c_1, u_1) \subset [b, x_0)$, which is impossible. So if $c_1 < x_0$ (and if $f \neq 0$ on $[b, x_0)$) then f has no zero on $(x_0, c]$, either. Now, since f has no zero on $(x_0, c]$, we may use the above argument which proved that $c_2 > x_0$, to show that $c_{N-1} < x_0$. It follows that $N = 2$.

Now all of the points b, c_1, d_1, c_2, d_2 are zeros of Q' and hence zeros of f'' and not zeros of f. Recall that $f'(u) = 0$ for exactly one point $u \in (c, d)$. The interval (u_2, u) between successive zeros of f' now contains exactly two zeros of f'', namely, the points d_2 and c, which is impossible. (The point c or d_2 could be a multiple zero of f'' only if $d_2 = c$, which would then have to be a zero of f'' or order 2 for otherwise there would exist more components W_j than allowed, leading to the same result.) This contradiction shows that if $f \neq 0$ on $[b, x_0)$ and if f' or f'' has a zero on (b, x_0), then $f(c) \neq 0$ and none of f, f', and f'' can vanish on (x_0, c). This proves Lemma 12.1.

Lemma 12.2. (i) *Suppose that $ff'f'' \neq 0$ on (b, x_0). If c_1 exists, we have $f''(c_1) = 0 \neq f(c_1)$.*

(ii) *Suppose that $ff'f'' \neq 0$ on (x_0, c). If c_N exists, we have $f''(c_N) = 0 \neq f(c_N)$.*

Proof of Lemma 12.2. It suffices to consider the case when $ff'f'' \neq 0$ on (b, x_0). Suppose that c_1 exists so that c_1 is a zero of Q' and hence of f or of f''. It therefore suffices to prove that $f(c_1) \neq 0$. To get a contradiction, suppose that $f(c_1) = 0$. Write $D = D(c_1)$. By Lemma 8.3, we have $(v_1, v_2) = D \cap \mathbf{R}$ where $x_0 < v_1 < c_1 < v_2 < u_1$, and $v_1, v_2 \in J(Q)$ with $Q(v_1) = v_2$ and $Q(v_2) = v_1$. On the other hand, Q is increasing on (x_0, c_1) and decreasing on (c_1, u_1) as can be seen by letting z trace ∂U_0 and ∂W_1. Hence $Q(v_1) < Q(c_1) = c_1 < v_2$. This is a contradiction, and Lemma 12.2 is proved.

Lemma 12.3. *If $N \geq 2$ then for $1 \leq j \leq N - 1$, exactly one of the points d_j and c_{j+1} is a zero of f but not of f'' and the other one is a zero of f'' but* *not of f. If (c,d) contains a zero of f' then also exactly one of d_N and c is a zero of f but not of f'' and the other one is a zero of f'' but not of f.*

Proof of Lemma 12.3. Suppose that $N \geq 1$. Suppose that for some j with $1 \leq j \leq N-1$, both d_j and c_{j+1} are zeros of f so that neither one is a zero of f''. Then there is no zero of f' between d_j and c_{j+1}, which contradicts Rolle's theorem. Suppose then that both d_j and c_{j+1} are zeros of f'' and hence not zeros of f. Then there are exactly two zeros of f'' (namely, d_j and c_{j+1}) between the successive zeros u_j and u_{j+1} of f', which is also impossible. If (c,d) contains a zero of f', we argue in the same way. Lemma 12.3 is proved.

13. There are no zeros of $ff'f''$ outside $[a,d]$

Recall that $f(z) = g(z)/(z^2+1)^n$, where g is a polynomial of degree $\geq 2n+2$, and that f, f', and f'' are assumed to have only real zeros.

Lemma 13.1. *Suppose that g is a polynomial with at least 2 distinct zeros, and that $f(t) = 0$. Then $a < t < d$.*

Proof of Lemma 13.1. Let the assumptions of Lemma 13.1 be satisfied. Then Q is a rational function. By Lemma 11.3(iii), the domain $D(t)$ is simply connected whenever $f(t) = 0$. To get a contradiction, suppose that $f(t) = 0$, that $D = D(t)$ is simply connected, and that $t \notin (a,d)$. Then $t \leq a$ or $t \geq d$. We consider the case $t \geq d$ as the case $t \leq a$ is similar. So suppose that $t \geq d$. By Lemma 8.6, all the other zeros t' of f (for which $D(t')$ is simply connected, hence in this case all other zeros t' of f), satisfy $a < t' < d$. Also, f'/f has no zero on (c,t) and hence not on (c,d). But f'/f has a zero on $[a,b]$.

Suppose that all zeros of f are $\geq b$. Suppose that $f''(a) = 0 \neq f'(a)$. If f'' has no zeros $< a$ then f'' has exactly one extraordinary zero $< u$, which contradicts Lemma 9.1, unless f' has a zero $< a$. If f' has a zero $v < a$, take v to be maximal. Then $v \in \partial W'$ where W' is a component of K^-, and $v \in \mathbf{R} \cap \partial W' = [w_1, w_2]$, say. Since f has no zeros $\leq b$, we have $f''(w_1) = f''(w_2) = 0$. Whether some of v, w_2 and a coincide or not, we see that there are exactly 2 zeros of f'' between consecutive zeros of f' (such as v, u or other pairs to the left of them in case K^- has components other then W' further left), or f'' has exactly one extraordinary zero to the left of the smallest zero of f'. In any case we have a contradiction. So suppose that $f''(a) = 0 = f'(a)$. Then $a = u$, and f' has a double zero at a. If K^- has a component W' to the left of a, we get the same contradiction as above for the smallest zero of f'', or two zeros of f'' between zeros of f' in case there are two or more such components W'. Otherwise $ff'f'' \neq 0$ on $(-\infty, a)$. Consideration of possible components $W_j \neq W$ of K^- with boundaries intersecting $[b,c]$, for $1 \leq j \leq N$, as in the proof of Lemma 12.1 (whether $ff'f'' \neq 0$ on (b, x_0) or on (x_0, c)), shows that the extraordinary zeros of f'' must occur somewhere on (d_N, ∞). In the same way as above, we see that

K^- cannot have a component $W' \neq W$ with $t \notin \partial W'$ and $\mathbf{R} \cap \partial W' = [w_1, w_2]$ where $w_1 \geq d$.

Since f' has a double zero at a, f must have a zero on (b, x_0) for otherwise f' has at least the 3 zeros $a = u, x_0$ less than the smallest zero of f, violating Lemma 9.1. Thus by Lemma 12.1, $ff'f'' \neq 0$ on (x_0, c). Also, $N \geq 1$ in the notation of the proof of Lemma 12.1.

If K^- has a component W' with $\mathbf{R} \cap \partial W' = [w_1, w_2]$, where $w_1 \geq d$, then $t = w_1$ or $t = w_2$, and there is $w_3 \in [w_1, w_2]$ with $f'(w_3) = 0$. If $t = w_1$ then $f''(w_2) = 0$, so that if $f'(w_2) \neq 0$, then f'' has exactly one extraordinary zero to the right of the largest zero of f', which is impossible. If $f''(w_2) = f'(w_1) = 0$ then f' has exactly one extraordinary zero to the right of the largest zero of f, which is impossible. If $t = w_2$ and $f''(d_N) = 0 \neq f(d_N)$, then there are exactly 4 zeros (which are w_1, d, c, d_N) of f'' between the consecutive zeros w_3, u_N of f' (true even if $w_2 = w_3$), which is a contradiction. If $t = w_2$ then $w_2 \neq w_3$ for otherwise w_2 is a multiple zero of f and hence not on ∂K^-. If $t = w_2$ and $f(d_N) = 0 \neq f''(d_N)$ and $x_0 > d_N$, then there are exactly 2 zeros of f' (which are x_0 and w_3) between the consecutive zeros w_2, d_N of f. So there is no such component $W' \neq W$ at all. Thus $f'' \neq 0$ on (d, ∞) and $f' \neq 0$ on $[c, \infty)$.

Since K^- has no component W' with $\partial W' \cap (d, \infty) \neq \emptyset$, then either $t = d$, or $t > d$ and t is a multiple zero of f so that $Q'(t) > 0$ and $t \in \partial K$, hence $t \in \partial U$.

Suppose that $t = d$. Since $ff'f'' \neq 0$ on (x_0, c) then f'' has exactly one zero, c, greater than the largest zero x_0 of f', which is impossible.

Suppose that $t > d$ and t is a multiple zero of f. Now the extraordinary zeros of f'' are c, d which are greater than the largest zero x_0 of f', and so f'' has no other extraordinary zeros, by Lemma 9.1.

We use notation as in the proof of Lemma 12.1. If $f''(c_1) = 0 \neq f(c_1)$ then, unless $c_1 = u_1$, f'' has at least 2 zeros between the consecutive zeros a, u_1 of f', which is impossible. If $c_1 = u_1$ then f' has the 4 zeros a, u_1 less than the smallest zero of f (counting multiplicities), which contradicts Lemma 9.1.

If $f(c_1) = 0 \neq f''(c_1)$ and $N = 1$, then $c_1 \neq u_1$, and f' has exactly the 2 zeros u_1, x_0 between the consecutive zeros c_1, t of f, which is impossible.

If $f(c_1) = 0 \neq f''(c_1)$ and $N \geq 2$, then one of d_{N-1}, c_N is a zero s of f, and f' has exactly the two zeros u_N, x_0 between the consecutive zeros s, t of f, which is impossible.

Suppose then that f has a zero $s < b$. Since f'/f has a zero on $[a, b]$, it follows from Lemma 8.6 that $D(s)$ is multiply connected, and then from Lemma 11.3(iii) that there is no such zero s at all.

This completes the proof of Lemma 13.1.

Lemma 13.2. *Suppose that g is a polynomial with at least 2 distinct zeros. Then $(-\infty, x_0)$ or (x_0, ∞) contains at most one zero of f', not counting multiplicities.*

Proof of Lemma 13.2. Suppose that g is a polynomial with at least 2 distinct zeros. We know from Lemma 13.1 that f has no zeros outside (a,d). Thus $f''(a)=f''(d)=0$. All zeros of f' on $(-\infty,b)$ and (c,∞) are therefore extraordinary zeros of f', and by Lemma 9.1, there are only 2 of them, counting multiplicities (but they do not have to lie on $(-\infty,b)\cup(c,\infty)$). In view of Lemma 12.1, we may assume that (b,x_0) contains no zeros of $ff'f''$. If f' has no zeros on $(-\infty,a)$, then $(-\infty,x_0)$ contains at most one zero of f', namely, one on $[a,b]$, if any. If f' has a zero w on $(-\infty,a)$ then there is at most one such zero, for otherwise there will be exactly 2 zeros of f'' (counting multiplicities) between successive such zeros of f' since each such zero of f' lies on the boundary of a component of K^-. If now there is also a zero u of f' on $[a,b]$ then both zeros of f' must be simple (in view of Lemma 9.1). But now there are exactly 2 zeros of f'' (counting multiplicities) between the successive zeros u and w of f', again a contradiction. This shows that f' has at most one zero on $(-\infty,x_0)$. This completes the proof of Lemma 13.2.

Lemma 13.3. *Suppose that g is a polynomial with at least 2 distinct zeros. Then the unbounded component of $H^+\setminus\overline{W}$ must be contained in K and must therefore coincide with U. Thus f' and f'' have no zeros outside $[a,d]$.*

Proof of Lemma 13.3. Suppose that g is a polynomial with at least 2 distinct zeros. We know from the previous lemmas that f has no zeros outside (a,d). To get a contradiction, suppose that the unbounded component of $H^+\setminus\overline{W}$ intersects K^-. Then there is a zero u' of f' outside $[a,d]$, lying on the boundary of a bounded component W' of K^-, and both end points w_1 and w_2 of $\mathbf{R}\cap\partial W'$ must be zeros of f'' and not zeros of f. We may assume that $w_1<u'<w_2$, and take u' so that there are no zeros of f' between u' and a if $u'<a$ and between u' and d if $u'>d$. There cannot be two such zeros of f' on the same side of a as there would be exactly two zeros of f'' between a pair of successive zeros of f' like that, which is impossible. Thus there might be one such zero $<a$ and one $>d$. Let us denote such a zero $<a$ by u' if one exists, and denote such a zero $>d$ by u'' if one exists. Then choose the nearest zeros w_j of f'' for $1\le j\le 4$ so that $w_1\le u'\le w_2$ and $w_3\le u''\le w_4$, where either 2 or 4 of these zeros of f'' exist. Thus $[w_1,w_2]=\mathbf{R}\cap\partial W'$ for the component W' of K^- whose boundary contains u', and similarly for $[w_3,w_4]$.

Suppose, for example, that there exists $u'<a$. Suppose that f' has no zero on $[a,b]$. Then a,b must be zeros of f'' and not of f. (For $f(a)\ne 0$ by Lemma 13.1, and if $f(b)=0$, then by Lemma 9.1, f' must have exactly two extraordinary zeros on $(-\infty,b)$, and they must be u' and one other zero, which will have to lie on $[a,b]$.) Now there are three zeros w_2,a,b of f'' on (u',x_0) (two of which are therefore extraordinary) and hence exactly one zero of f'' between any other pair of successive zeros of f'. By Lemma 9.1, f'' cannot have any other extraordinary zeros, which contradicts the fact that w_1, which is smaller than the

smallest zero u' of f', is an extraordinary zero of f''.

So if u' exists, it must be the case that f' has a zero u on $[a, b]$ and no zeros on $[c, d]$. Now if $a < u$ and $u' < w_2$, then (u', u) contains exactly 2 zeros of f'', namely w_2 and a. Since u' and u are consecutive zeros of f', this is impossible. If $a = u$ or $u' = w_2$, then f' has at least 3 zeros smaller than the smallest zero of f (which is $\geq b$), which contradicts Lemma 9.1.

Thus u' cannot exist, and by a similar argument, u'' cannot exist.

We have now proved that K contains the unbounded component of $H^- \setminus W$. If f' or f'' had a zero x outside $[a, d]$, then by what we have already proved and by Lemma 5.1(4), we cannot have $f'(x) = 0$, and so $f''(x) = 0 = Q'(x)$. But since $Q'(x) = 0$, it follows that $x \in \partial K^-$, which is impossible. This completes the proof of Lemma 13.3.

14. Finishing the proof of Theorem 1.1

We assume that $f(z) = g(z)/(z^2 + 1)^n$, where g is a polynomial of degree $\geq 2n + 2$ with at least 2 distinct zeros, and that f, f', and f'' have only real zeros. Thus $K = U_0 \cup U$ and there is a bounded component W of K^- separating U_0 and U in H^+. We assume, as we may in view of the preceding lemmas, that all the zeros of f lie in (a, d) and that $D(t)$ is simply connected for every zero t of f.

We consider two cases. In view of Lemma 12.1, we may and will assume without loss of generality that none of f, f', and f'' vanishes on (b, x_0) and that $f(b) \neq 0$. Then f' has a unique zero u on $[a, b] \cup [c, d]$.

We know by Lemma 13.3 that $f'f''$ has no zeros outside $[a, d]$, either. We assume that $ff'f''$ has only real zeros and derive a contradiction.

Case I. We assume that $u \in [c, d]$. Thus $f'(b) \neq 0$, and so $b < x_0$. Hence $(b, x_0) \subset \partial U_0$. Now each of a, b is a zero of f'' and is not a zero of f. Then by Lemma 9.1, f'' has exactly 2 extraordinary zeros, which then must be a and b. Since $f(d) \neq 0$, we must have $f'(d) = 0 = f''(d)$, for otherwise d is another extraordinary zero of f''. Thus also $u = d$. Now f' must also have exactly 2 extraordinary zeros, by Lemma 9.1, and x_0 is also an extraordinary zero by Lemma 7.4. But d counts as 2 extraordinary zeros of f' (since the largest zero of f is $< d$), so there are no others by Lemma 9.1. This is a contradiction.

Case II. We assume that $u \in [a, b]$, so that $f' \neq 0$ on $[c, d]$. Suppose first that each of c, d is a zero of f'' and is not a zero of f. Thus c, d are extraordinary zeros of f'', so by Lemma 9.1, f'' has no other extraordinary zeros. Since a cannot be an extraordinary zero of f'', we have $f'(a) = f''(a) = 0$ (otherwise, a is smaller than the smallest zero of f'). So $u = a$. Since there needs to be a zero of f'' between the consecutive zeros a, x_0 of f', we have $f''(b) = 0$. Now a is a double extraordinary zero of f' while x_0 is also an extraordinary zero of f'. This contradicts Lemma 9.1.

Thus $f''(d)=0$ and $f(c)=0\neq f(b)f(a)f(d)$. If $a<u$ then f'' has exactly one zero (which is extraordinary) on $(-\infty,u)$, which contradicts Lemma 9.1. For if a is a multiple zero of f'' while $f'(a)\neq 0$, then a is a multiple zero of Q' so that there is a component W' of K^- with $(a-\varepsilon,a)\subset\partial W'$ for some $\varepsilon>0$. This contradicts Lemma 13.3. Thus $a=u$ and $f'(a)=0$. Now a is a double extraordinary zero of f', and by Lemma 9.1, f' has no other extraordinary zeros. Hence f must have a zero on (a,x_0). Since $f\neq 0$ on $(a,b)\cup(b,x_0)$, we must then have $f(b)=0$, which is a contradiction.

This completes the proof of Theorem 1.1.

15. Proof of Lemma 2.2

Proof of Lemma 2.2. Let f be given by (2), where Φ is a non-constant real polynomial with no real zeros and $g\in\mathscr{U}_{2p}$, so that f has only real zeros while we do not make any a priori assumption concerning the reality of the zeros of f' or f''. We have the representation (compare [HW1, Lemma 3, p. 231], [L, Theorem 2, p. 310])

$$\psi(z)=\gamma z+\delta+\sum_{k=\tau}^{\omega}{}' A_k\left(\frac{1}{a_k-z}-\frac{1}{a_k}\right)-\frac{A_{k_0}}{z}, \tag{34}$$

where $\gamma\geq 0$, $\delta\in\mathbf{R}$, $A_k>0$, and where

$$\sum_{k=\tau}^{\omega}{}'\frac{A_k}{a_k^2}<\infty. \tag{35}$$

The dash in the summation in the above (34) and (35) indicates that if there is k_0 with $a_{k_0}=0$ (it would be $k_0=0$ by our convention) then $k=k_0$ is omitted in the summation and then the extra term $-A_{k_0}/z$ is included in (34); if there is no such k_0 then the term $-A_{k_0}/z$ is omitted in (34).

We have, by (24),

$$\begin{aligned}\frac{f'}{f}\equiv L&=\frac{m_0}{z}-\frac{\Phi'}{\Phi}(z)+S'(z)+\frac{\Pi'}{\Pi}(z)\\ &=\frac{m_0}{z}-\frac{\Phi'}{\Phi}(z)-a(2p+2)z^{2p+1}+b(2p+1)z^{2p}+c(2p)z^{2p-1}\\ &\quad+d(2p-1)z^{2p-2}+\cdots+z^{p_1}\sum_{\substack{k=\tau\\ a_k\neq 0}}^{\omega}\frac{m_k}{a_k^{p_1}(z-a_k)}\end{aligned} \tag{36}$$

where m_k is the multiplicity of a_k as a zero of g.

Suppose that f has no zeros so that $\psi \equiv 1$. Then by (36) and (18) we have

$$\frac{f'}{f}(z) \sim C_1 z^{\deg S - 1} \qquad \text{and} \qquad \frac{f'}{f}(z) \sim C_2 z^{\deg \varphi - \deg \Psi},$$

as $z \to \infty$, for some $C_1, C_2 \in \mathbf{C} \setminus \{0\}$. Thus

$$\deg \varphi = \deg S + \deg \Psi - 1 \qquad \text{if } f \text{ has no zeros.} \tag{37}$$

This proves (i).

If f has at least one but only finitely many zeros then $\psi(z) \sim -1/z$ as $z \to \infty$ by (17), so by (36) and (18),

$$\frac{f'}{f}(z) \sim C_1 z^{\deg S - 1} \sim C_2 z^{\deg \varphi - 1 - \deg \Psi} \qquad \text{if } S' \not\equiv 0 \tag{38}$$

so that

$$\deg \varphi = \deg S + \deg \Psi. \tag{39}$$

We remark that by a result of the author [H1], if f has only finitely many zeros and $g \in \mathscr{U}_{2p}$ then f'' has at least $2p$ non-real zeros.

Suppose that f has at least one but only finitely many zeros and $S' \equiv 0$. So $\deg S = 0$. Then g is a polynomial and by (18),

$$\frac{f'}{f}(z) \sim C_1 z^{\deg \varphi - 1 - \deg \Psi} \qquad \text{as } z \to \infty$$

while by (36),

$$\frac{f'}{f}(z) = -\frac{\Phi'}{\Phi}(z) + \sum_k \frac{m_k}{z - a_k}. \tag{40}$$

If the number of zeros f, counting multiplicities, is not equal to $\deg \Phi$, then

$$\frac{f'}{f}(z) \sim C_2 z^{-1} \qquad \text{as } z \to \infty$$

so that $\deg \varphi = \deg \Psi$ which is the same as (39). This proves (ii).

Suppose then that f has infinitely many zeros. Combining (36), (18), and (34), we get

(41)

$$\begin{aligned}
&\frac{m_0}{z} - \frac{\Phi'}{\Phi}(z) - a(2p+2)z^{2p+1} + b(2p+1)z^{2p} + c(2p)z^{2p-1} + \cdots \\
&+ z^{p_1} {\sum_{k=\tau}^{\omega}}' \frac{m_k}{a_k^{p_1}(z - a_k)} = \frac{\varphi(z)}{\Psi(z)} \left\{ \gamma z + \delta - \frac{A_{k_0}}{z} + {\sum_{k=\tau}^{\omega}}' A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right) \right\}.
\end{aligned}$$

Equating residues at $z = a_k$, we get

$$m_k = -A_k \frac{\varphi(a_k)}{\Psi(a_k)} \tag{42}$$

which is valid for all k, also for $k = k_0$. Thus

$$\varphi(a_k) \operatorname{sgn} \Psi(a_k) = -m_k \frac{\Psi(a_k) \operatorname{sgn} \Psi(a_k)}{A_k} < 0, \tag{43}$$

which is valid by the same argument even if f has only finitely many zeros. This proves (30). Write $\rho = \deg \varphi$ and choose $M > 1$ so that for some positive integer l, we have

$$0 < -\varphi(a_k) \operatorname{sgn} \Psi(a_k) \le M |a_k|^\rho \tag{44}$$

and $k \neq k_0$ whenever $|k| \ge l$. Then by (44), (43), and (35)

$$\frac{1}{M} \sum_{|k| \ge l} \frac{m_k}{|a_k|^{\rho + 2 - \deg \Psi}} \le \sum_{|k| \ge l} \frac{A_k}{|\Psi(a_k)| |a_k|^{2 - \deg \Psi}} < \infty.$$

Hence by the definition of p_1 we have

$$p_1 + 1 \le \rho + 2 - \deg \Psi = \deg \varphi - \deg \Psi + 2, \tag{45}$$

which proves the lower bound for $\deg \varphi$ in (31).

Note that by (43),

$$\text{if } \tau = -\infty \text{ and } \omega = \infty \text{ then } \rho + \deg \Psi \text{ is even.} \tag{46}$$

Next, since $p_1 \le 2p + 1$, we get from (36) and [HW1, Lemma 4, p. 231] that

$$\left| \frac{f'}{f}(iy) \right| \le a(2p+2)|y|^{2p+1} + o(|y|^{2p+1}) \qquad \text{as } |y| \to \infty, \tag{47}$$

and

$$\operatorname{Im} \frac{f'}{f}(iy) = -a(-1)^p (2p+2) y^{2p+1} + o(|y|^{2p+1}) \qquad \text{as } |y| \to \infty. \tag{48}$$

Next using (18) with $\Psi_1 \equiv 1$ and (22) we get

$$\left| \frac{f'}{f}(iy) \right| \ge C_1 |y|^{\rho - \deg \Psi - 1} \tag{49}$$

and

$$\left|\frac{f'}{f}(iy)\right| \le C_1^{-1}|y|^{\rho-\deg\Psi+1}$$

for $|y| \ge 2$, for some $C_1 \in (0,1)$. By (49) and (47) we get

$$\rho = \deg\varphi \le 2p+2+\deg\Psi.$$

If $\rho = 2p+2+\deg\Psi$ and $a=0$ then (49) and (47) yield $\rho \le 2p+1+\deg\Psi$. So if $\rho = 2p+2+\deg\Psi$, then $a>0$. But then (43) shows that $B_1<0$ where B_1 is the leading coefficient of φ while (18) implies that

$$\text{(50)} \qquad \operatorname{Im}\frac{f'}{f}(iy) \sim (-1)^{p+1}B_1 y^{2p+2}\operatorname{Im}\psi(iy) \qquad \text{as } |y|\to\infty.$$

Now (50) and (48) combine to give

$$\text{(51)} \qquad \operatorname{Im}\psi(iy) \sim \frac{a}{B_1}(2p+2)y^{-1} \qquad \text{as } |y|\to\infty.$$

Since $a>0$ and $B_1<0$, (51) contradicts the fact that $\operatorname{Im}\psi(iy)>0$ when $y>0$. Thus $\rho = 2p+2+\deg\Psi$ is impossible, and so we always have

$$\text{(52)} \qquad \rho = \deg\varphi \le 2p+1+\deg\Psi.$$

This proves the general upper bound for $\deg\varphi$ in (31).

From now on, we assume that $\Phi(z) = (z^2+1)^n$, so $\Psi(z) = z^2+1$. We have $\deg\varphi \ge \deg\Psi - 1 = 1$. Next we multiply both sides of (41) by $\Psi(z)$ and then differentiate q times where

$$\text{(53)} \qquad q \ge \max\{\sigma+\deg\Psi-1, \deg\varphi+\deg\Psi-1, p_1+1\} \ge \max\{1, 2p+1\}$$

since $\sigma \ge 2p$. Then if $1 \le q = \deg\Psi - 1$, we have $\sigma = \rho = 0 = p_1$, so that by (45), $\deg\Psi - 1 \le 0$, which is impossible. Hence $q \ge \deg\Psi$. This yields

(54)

$$\begin{aligned}
&\frac{\Psi(0)m_0(-1)^q q!}{z^{q+1}} + \frac{d^q}{dz^q}\left\{\Psi(z)z^{p_1}\sum_{k\ne k_0}\frac{m_k}{a_k^{p_1}(z-a_k)}\right\} + q!B \\
&= \frac{d^q}{dz^q}\left\{\varphi(z)(\gamma z+\delta)\right\} - A_{k_0}\frac{d^q}{dz^q}\frac{\varphi(z)}{z} + \sum_{k\ne k_0}\left\{\frac{d^q}{dz^q}\frac{A_k\varphi(z)}{a_k - z} - \frac{A_k}{a_k}\frac{d^q}{dz^q}\varphi(z)\right\}
\end{aligned}$$

where, in view of (26)–(29), (45), and (53),

$$\begin{aligned}
B &= -(2p+2)a \le 0 && \text{if } q = 2p+\deg\Psi+1,\\
B &= (2p+1)b && \text{if } q = 2p+\deg\Psi,\\
B &= 2pc > 0 && \text{if } q = 2p+\deg\Psi-1,\\
B &= 0 && \text{if } q \ge 2p+\deg\Psi+2.
\end{aligned}$$

Recall that Ψ has leading coefficient 1. Consider the left hand side of (54). As in [HW1, p. 240], we see that

$$\Psi(z)z^{p_1}\frac{1}{a_k^{p_1}(z-a_k)}=\frac{\Psi(z)}{z-a_k}+\frac{\Psi(z)}{a_k}T\left(\frac{z}{a_k}\right)$$

where

$$T(z)=z^{p_1-1}+z^{p_1-2}+\cdots+z+1.$$

Also

$$\frac{\Psi(z)}{z-a_k}=\Psi_2(z;a_k)+\frac{\Psi(a_k)}{z-a_k},$$

where $\Psi_2(z;a_k)$ is a polynomial of degree $\deg\Psi-1$ in z. Note that $\deg\Psi\geq 2$. By (45), $q\geq\rho+\deg\Psi-1\geq p_1+2$. So

$$\frac{d^q}{dz^q}\Psi(z)z^{p_1}\sum_{a_k\neq 0}\frac{m_k}{a_k^{p_1}(z-a_k)}=\sum_{a_k\neq 0}m_k\Psi(a_k)\frac{d^q}{dz^q}\frac{1}{z-a_k}.$$

Further, since $q\geq\rho+1$, we have, denoting the leading coefficient of φ by B_1,

$$\frac{d^q}{dz^q}\big(\varphi(z)(\gamma z+\delta)\big)=\gamma\frac{d^q}{dz^q}\big(z\varphi(z)\big)=\gamma B_1(\rho+1)!\left(\frac{d}{dz}\right)^{q-\rho-1}1,$$

$$-A_{k_0}\frac{d^q}{dz^q}\frac{\varphi(z)}{z}=-A_{k_0}\frac{\varphi(0)(-1)^q q!}{z^{q+1}}=\frac{\Psi(0)m_0(-1)^q q!}{z^{q+1}}$$

by (42) for $k=k_0$, and $(d/dz)^q\varphi=0$, so that (54) becomes (using also (42))

$$q!B+\sum_{a_k\neq 0}m_k\Psi(a_k)\frac{d^q}{dz^q}\frac{1}{z-a_k}=\gamma B_1(\rho+1)!\left(\frac{d}{dz}\right)^{q-\rho-1}1+\sum_{a_k\neq 0}m_k\frac{\Psi(a_k)}{\varphi(a_k)}\frac{d^q}{dz^q}\frac{\varphi(z)}{z-a_k}. \tag{55}$$

But since $q\geq\rho+1$ we have

$$\frac{1}{\varphi(a_k)}\frac{d^q}{dz^q}\frac{\varphi(z)}{z-a_k}=\frac{d^q}{dz^q}\frac{1}{z-a_k},$$

using $\varphi(z)=\varphi(a_k)+\varphi'(a_k)(z-a_k)+\cdots+\varphi^{(\rho)}(a_k)(z-a_k)^\rho/\rho!$. Thus

$$q!B=\gamma B_1(\rho+1)!\left(\frac{d}{dz}\right)^{q-\rho-1}1. \tag{56}$$

Claim. If $\sigma = 2p$ then ρ is even. We assume that $\sigma = 2p$ so that by (29), $a = b = 0$ and $p_1 \leq 2p$. Also either $p_1 = 2p$, or $c > 0$ and $p_1 \leq 2p - 1$. If $\tau = -\infty$ and $\omega = \infty$, then ρ is even by (46). Suppose that $\omega < \infty$ so that $\tau = -\infty$. Then (36) gives

$$\frac{f'}{f}(z) = \frac{m_0}{z} - \frac{\Phi'}{\Phi}(z) + 2pcz^{2p-1} + \cdots + z^{p_1} \sum_{k \neq k_0} \frac{m_k}{a_k^{p_1}(z - a_k)}. \tag{57}$$

Thus with $z = x \in \mathbf{R}$,

$$\lim_{x \to a_\omega +} \frac{f'}{f}(x) = +\infty \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{f'}{f}(x) = +\infty. \tag{58}$$

A similar analysis shows that φ has an even number of zeros on each interval of the form (a_k, a_{k+1}) or $(a_\omega, +\infty)$. Thus ρ is even. A similar argument shows that if $\tau > -\infty$ then ρ is even.

Thus

$$\text{if } \sigma = 2p \text{ then } \rho = \deg \varphi \text{ is even.} \tag{59}$$

Claim. If ρ is even then $\rho = 2p+2$. Suppose that ρ is even so that $\rho \leq 2p+2$ by (52). If $p_1 \geq 2p$ then $\rho \geq 2p+1$ by (45), hence $\rho = 2p+2$. Suppose then that $p_1 \leq 2p - 1$. If $\rho \geq 2p + 1$ then $\rho = 2p + 2$, so suppose also that $\rho \leq 2p$. Then, since $\sigma \geq 2p$ by (27), we may, in view of this and (53), choose $q = \sigma + 1 \geq 2p + 1$ in (54) to get (56). Moreover, since

$$p_1 \leq 2p - 1 < 2p \leq \sigma = \max\{p_1, \deg S\}$$

by (26), we have $\sigma = \deg S \geq 2p$. Hence $B \neq 0$ by the formulas for B after (54) and (25). Thus by (56), $\gamma \neq 0$ and $q = \rho + 1$, so since $q = \sigma + 1$, we have (since also $\sigma \geq 2p$ and $\rho \leq 2p$)

$$\deg \varphi = \rho = \sigma = \deg S = 2p \qquad \text{and} \qquad B = 2pc > 0. \tag{60}$$

But then, since by (43), $\varphi(a_k) < 0$ for all k and since $\deg \varphi$ is even, the leading coefficient of φ must be negative. This contradicts (56) since $B = c > 0$ by (60) and $\gamma \geq 0$ by (34) (and now $\gamma > 0$ since $\gamma \neq 0$). Thus this case is impossible and we conclude that

$$\text{if } \rho \text{ is even then } \rho = 2p + 2. \tag{61}$$

Claim. If ρ is odd then $\rho = 2p+3$. Suppose that ρ is odd so that $\rho \leq 2p+3$ by (52). To prove that $\rho = 2p + 3$, it suffices to show that $\rho \geq 2p + 3$. To get a contradiction, suppose that $\rho \leq 2p + 1$. Suppose that $p_1 \geq 2p + 1$ so that then

$p_1 = 2p+1$. Then by (45), $\rho \geq p_1 + 1 = 2p+2$, a contradiction with $\rho \leq 2p+1$. So we may assume that $p_1 \leq 2p$ and $\rho \leq 2p+1$. By (27) and (59), we must have $\sigma \geq 2p+1$. Thus we may choose $q = \sigma + 1$ in (53) and (54) and get (56). Now

$$p_1 \leq 2p < 2p+1 \leq \sigma = \max\{\deg S, p_1\} \tag{62}$$

so that

$$\sigma = \deg S > p_1, \qquad \deg S \geq 2p+1. \tag{63}$$

Thus $a \neq 0$ or $b \neq 0$, and $q = \deg S + 1$. By the formulas for B after (54), and by (25), $B \neq 0$ in the formulas for B after (54) and in (56) so that $\gamma \neq 0$ (hence $\gamma > 0$), and $q = \rho + 1$. Hence

$$\sigma = \rho = \deg \varphi = \deg S = 2p+1 \qquad \text{or} \qquad 2p+2 \tag{64}$$

and therefore

$$\deg S = \sigma = \rho = 2p+1 \qquad \text{since } \rho \text{ is odd.} \tag{65}$$

Thus by the formulas for B after (54),

$$B = (2p+1)b \neq 0, \qquad a = 0. \tag{66}$$

By (46), and since ρ is odd, we have $\tau > -\infty$ or $\omega < \infty$. Suppose that $\omega < \infty$. Since $\varphi(a_k) < 0$ for all k and $\deg \varphi$ is odd, the leading coefficient B_1 of φ must be positive. Hence, since $\gamma > 0$, (56) gives $B = (2p+1)b > 0$. By (36), (26), (66), and (65) we have

$$\frac{f'}{f}(z) = \frac{m_0}{z} - \frac{2nz}{z^2+1} + b(2p+1)z^{2p} + \cdots + z^{p_1} \sum_{k \neq k_0} \frac{m_k}{a_k^{p_1}(z - a_k)} \tag{67}$$

where now $b > 0$, $p_1 \leq 2p$ and $\omega < \infty$. Thus,

$$\lim_{x \to a_\omega +} \frac{f'}{f}(x) = +\infty \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{f'}{f}(x) = +\infty. \tag{68}$$

Thus f' and hence φ has an even number of zeros on $(a_\omega, +\infty)$ and so $\deg \varphi$ is even, a contradiction. A similar argument works if $\tau > -\infty$. We conclude that

$$\text{if } \rho \text{ is odd then } \rho = \deg \varphi = 2p+3. \tag{69}$$

We have proved that if f has infinitely many zeros and $\deg \Psi = 2$ then

$$2p+2 \leq \rho = \deg \varphi \leq 2p+3, \tag{70}$$

and this proves (iii).

For any $c \in \mathbf{R}$, we have $g \in \mathscr{U}_{2p}$ if, and only if, $ge^{-cz} \in \mathscr{U}_{2p}$. Thus, what has been proved for $\psi(0,z)$ and $\varphi(0,z)$ above, is true for $\psi(c,z)$ and $\varphi(c,z)$ for any real c.

We can write (compare (24))

$$f' = \frac{z^{m'_0} e^{S_1} \Pi_1}{\Phi(z)\Psi(z)} = \frac{g_1}{\Phi(z)\Psi(z)}, \tag{71}$$

where g_1 is a real entire function of finite order and of the same order as g.

Since f' has only finitely many extraordinary zeros and otherwise has zeros at the multiple zeros of f and one on each interval (a_k, a_{k+1}), we deduce that

$$\text{genus } (\Pi_1) = \text{ genus } (\Pi) = p_1. \tag{72}$$

As in [HW1, p. 232],

$$\deg (S_1 - S) \leq p_1 \leq 2p + 1. \tag{73}$$

Note that the development above up to this point after (9) only needs the assumption that $g \in \mathscr{U}_{2p}$, not any assumption on the reality of the zeros of f' or of f''. If we assume that all the zeros of f' are real then by (71)–(73), we obtain also that $g_1 \in \mathscr{U}_{2p}$.

This completes the proof of Lemma 2.2.

References

[B] Beardon, A.F.: Iteration of Rational Functions. - Springer, New York, 1991.

[E] Eremenko, A.E.: Distribution of zeros of some real polynomials and iteration theory. - Preprint, Purdue University, 1989.

[H] Hayman, W.K.: Meromorphic Functions. - Clarendon Press, Oxford, 1964.

[HSW] Hellerstein, S., L.C. Shen and J. Williamson: Reality of the zeros of an entire function and its derivatives. - Trans. Amer. Math. Soc. 275, 1983, 319–331.

[HW1] Hellerstein, S., and J. Williamson: Derivatives of entire functions and a question of Pólya. - Trans. Amer. Math. Soc. 227, 1977, 227–249.

[HW2] Hellerstein, S., and J. Williamson: Derivatives of entire functions and a question of Pólya II. - Trans. Amer. Math. Soc. 234, 1977, 497–503.

[H1] Hinkkanen, A.: Iteration and the zeros of the second derivative of a meromorphic function. - Proc. London Math. Soc. (3) 65, 1992, 629–650.

[H2] Hinkkanen, A.: Reality of zeros of derivatives of meromorphic functions. - Ann. Acad. Sci. Fenn. Math. 22, 1997, 21–38.

[H3] Hinkkanen, A.: Zeros of derivatives of strictly non-real meromorphic functions. - Ann. Acad. Sci. Fenn. Math. 22, 1997, 39–74.

[L] Levin, B.Ja.: Distribution of Zeros of Entire Functions, 2nd ed., vol. 5. - Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1980.

[LO] LEVIN, B.JA., and I.V. OSTROVSKIĬ: On the dependence of the growth of an entire function on the distribution of the zeros of its derivatives. - Trans. Amer. Math. Soc. (2) 32, 1963, 323–357.

[MO] MILLER, T.L., and R. OLIN: Analytic curves. - Amer. Math. Monthly 91, 1984, 127–130.

[P] PÓLYA, G.: Collected Papers, Vol. II. - The MIT Press, Cambridge–London, 1974.

[Po] POMMERENKE, CH.: On conformal mapping and iteration of rational functions. - Complex Variables Theory Appl. 5, 1986, 117–126.

[Pr] PRZYTYCKI, F.: Remarks on the simple connectedness of basins of sinks for iterations of rational maps. - In: Dynamical Systems and Ergodic Theory, edited by K. Krzyzewski. Banach Center Publ., Vol. 23, PWN – Polish Scientific Publishers, Warsaw, 1989, 229–235.

[R] ROSSI, J.: The reciprocal of an entire function of infinite order and the distribution of the zeros of its second derivative. - Trans. Amer. Math. Soc. 270, 1982, 667–683.

[Sa] SARASON, D.: Sub-Hardy Hilbert Spaces in the Unit Disk. - Wiley, New York, 1994.

[S] SHEIL–SMALL, T.: On the zeros of the derivatives of real entire functions and Wiman's conjecture. - Ann. Math. 129, 1989, 179–193.

[Sh] SHISHIKURA, M.: The connectivity of the Julia set and fixed point. - IHES preprint IHES/M/90/37, 1990.

[T] TSUJI, M.: Potential Theory in Modern Function Theory. - Maruzen, Tokyo, 1959.

[V] VALIRON, G.: Fonctions analytiques. - Presses Universitaires de France, Paris, 1954.

Received 22 November 1996