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SPACES OF GEOMETRICALLY FINITE REPRESENTATIONS

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Abstract. We explore conditions under which the property of geometrical finiteness is open among type-preserving representations of a given group into the group of isometries of hyperbolic n-space. We give general criteria under which this is the case, for example if every maximal parabolic subgroup has rank at least $n-2$. In dimension $n=3$, we deduce Marden's theorem that geometrical finiteness is always an open property. We give examples to show that, in general, additional constraints of the type we describe are necessary in dimension 4 and higher.

0. Introduction

In this paper, we consider the space of type-preserving representations of a given finitely generated group into the group, $\mathrm{Isom}\,\mathbf{H}^n$, of isometries of hyperbolic n-space, $\mathbf{H}^n$. We are particularly interested in the subset of geometrically finite representations without accidental parabolics, and consider the question of when this subset is open. This need not always be the case, and we give examples in dimension $n=4$ of sequences of non-discrete (or of discrete non geometrically finite representations) which converge on a geometrically finite representation (Section 5). We also give a positive result (Theorem 1.5) which clarifies how and when this phenomenon can occur.

Deformations of Kleinian groups play an important role in hyperbolic geometry. In dimension $n=3$, there is a well developed theory (see for example [BeP] for an exposition), which plays a crucial role in Thurston's hyperbolisation theorem for Haken 3-manifolds. In higher dimensions, spaces of deformations are much less well understood in general, but can be useful in constructing interesting examples.

Suppose that Γ is a finitely generated group with a (possibly empty) collection of virtually abelian "peripheral" subgroups. We say that a representation from Γ to $\mathrm{Isom}\,\mathbf{H}^n$ is "type preserving" if it sends every peripheral subgroup to a discrete parabolic group. By a "peripheral element", we mean an infinite order element lying in some peripheral subgroup. An "accidental parabolic" is a non-peripheral element which gets sent to a parabolic. Let $\mathscr{R}=\mathscr{R}(\Gamma,n)$ be the set of type preserving representations. Thus, $\mathscr{R}$ carries a natural "algebraic" topology

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(Section 1). Let $\mathscr{R}_D \subseteq \mathscr{R}$ be the subset of discrete representations, and let $\mathscr{R}_F \subseteq \mathscr{R}_D$ be the subset of geometrically finite representations without accidental parabolics. (We do not necessarily assume that such representations are faithful, though the kernel of such a representation is, by hypothesis, finite.) It turns out that $\mathscr{R}_D$ is always closed in $\mathscr{R}$, provided that Γ is not virtually abelian [Wi]. (See the end of Section 1 for a discussion of this). The properties of $\mathscr{R}_F$, depend on the dimension, n, and the group Γ.

In dimension $n \leq 3$, $\mathscr{R}_F$ is always open in $\mathscr{R}$. For $n = 3$, this is a result of Marden [Mard]. These cases are discussed more fully in Section 1. The set $\mathscr{R}$ is also open in the "convex cocompact" case—where Γ has no peripheral elements. This is a consequence of the Holonomy Theorem (see [L], [G2]), and will also be proven directly in Section 4 (Proposition 4.1). Also, in the finite covolume case, Mostow rigidity tells us that $\mathscr{R}_F$ is a point, and in particular open. In fact, Proposition 1.8 tells us that $\mathscr{R}_F$ is open provided that the rank of every peripheral subgroup is at least $n-2$. All the above examples are special cases of this result.

Note that our definition of "type preserving" is more restrictive than that sometimes used, in that we are supposing that the parabolic groups are discrete. Elsewhere it has been taken to mean that every peripheral element gets sent to a parabolic. In low dimension, this makes no real difference. Note that a rank 1 parabolic group is necessarily discrete, and a discrete rank $n-1$ parabolic group cannot be deformed to a non-discrete parabolic group. We thus recover the usual results in dimension $n \leq 3$. In higher dimension, however, some assumption of discreteness is essential. Consider, for example, a rank 2 free abelian group acting properly discontinuously by translation on euclidean 3-space. We can perturb this action so that it becomes non-discrete fixing setwise a 1-dimensional subspace. We can extend this to parabolic action on hyperbolic 4-space. This gives a fairly trivial example. In fact, Misha Kapovich has observed that this phenomenon can occur in more interesting situations, for example, by deforming a finite volume hyperbolic 3-manifold into 4-space by bending along a totally geodesic finite-area surface.

Even assuming discreteness of parabolic groups, however, $\mathscr{R}_F$ need not be open in $\mathscr{R}$ or even in $\mathscr{R}_D$. Again, the simplest examples occur in dimension $n = 4$. In Section 5, we describe a sequence of representations in $\mathscr{R} \setminus \mathscr{R}_D$ which converge on a point of $\mathscr{R}_F$. We also give a sequence of representations in $\mathscr{R}_D \setminus \mathscr{R}_F$ which converge to a point of $\mathscr{R}_F$. In both these examples, there is a cyclic parabolic group $G \leq \Gamma$ which acts by euclidean isometry on a horosphere. For each representation in the converging sequence, this group acts as a "screw motion". As the sequence converges, the rotational part of the screw motion tends to 0, so that, in the limit, we are left with a euclidean translation. In other words, the parabolic group, in some sense, changes character in the limit. We shall see that this must always be the case in such examples.

The main positive result of this paper (Theorem 1.5) gives a "stratification" of the space $\mathscr{R}$, such that the intersection of $\mathscr{R}_F$ with each stratum is open in that stratum. This is achieved by associating to each parabolic subgroup a canonical euclidean subspace in the corresponding horosphere. On a stratum, the dimension of these subspaces is constant. In the case of a screw motion on euclidean 3-space, the canonical subspace is just the axis of the screw motion. For a translation, it is the whole of euclidean 3-space. Thus, in the examples described earlier, we see that the dimension jumps up in the limit. In contrast, there are cases (for example as described by Proposition 1.8) where the whole of $\mathscr{R}$ consists of a single stratum, or at least where each stratum is open and closed in $\mathscr{R}$. Thus, in such cases, $\mathscr{R}_F$ is open in $\mathscr{R}$.

As remarked earlier, representations in $\mathscr{R}_F$ or $\mathscr{R}_D$ are not assumed to be faithful. However the above results would remain valid if "discrete" were everywhere replaced by "discrete and faithful", and "geometrically finite" by "geometrically finite and faithful". (See Lemma 1.9.)

In this paper, we shall concentrate on the case of constant curvature. However, the notion of geometrical finiteness can be defined for groups acting on any pinched Hadamard manifold (i.e. a complete simply connected Riemannian manifold of pinched negative curvature). The definitions are described in general in [Bo3], and have been explored explicitly in the case of complex hyperbolic space by Goldman and others, (see for example [G3]). I suspect that many of the results described here can be generalised to that situation. The problem reduces to understanding the geometry of parabolic groups. For complex hyperbolic space (and the other symmetric spaces) this may be possible along similar lines. In the general case, one could not hope to associate canonical subspaces to parabolic groups, but one might be able to rephrase the stratification in terms of rotational parts of parabolic isometries, which can be defined in the general context [Bo2]. Note that peripheral subgroups in this case are allowed to be virtually nilpotent.

In the special case of convex cocompact groups (no peripheral elements), none of these problems arise. Indeed the proof given here of Proposition 4.1 generalises, essentially unchanged, to show that in variable curvature, the set of convex cocompact representations is open.

Returning to the constant curvature case, we remark that there are close connections between geometrical finiteness and structural stability. These were explored by Sullivan [S] in dimension $n = 3$. In [Tu], Tukia shows that geometrically finite representations are structurally stable (among geometrically finite representations) in any dimension. (In Tukia's paper, a more restrictive definition of geometrical finiteness is used, namely that the group should possess a finite-sided fundamental domain. However, all the arguments appear to go through in general.) This means that the limit sets of geometrically finite representations vary continuously in the Hausdorff topology. Moreover the action of Γ on the boundary of $\mathbf{H}^n$ of actions which are close in $\mathscr{R}_F$ are quasiconformally conjugate, by a map with quasiconformal constant close to unity. In some sense, these results are complementary to the kind of results we describe in this paper. Thus, Sullivan and Tukia consider representations which remain in the class of geometrically finite groups. In dimension 3, this restriction is superfluous by the result of Marden [Mard], but not necessarily in higher dimensions.

1. Summary of results

We use $\mathbf{S}^n$, $\mathbf{E}^n$ and $\mathbf{H}^n$ to denote, respectively, the n-dimensional spherical, euclidean and hyperbolic spaces. We write d_{sph}, d_{euc} and d_{hyp} for the metrics on these spaces. If X is any of these spaces, we shall write $\operatorname{Isom} X$ for the Lie group of isometries of X.

Suppose Γ is a finitely generated group. The space of representations $\varrho\colon \Gamma \longrightarrow \operatorname{Isom} X$ carries an *algebraic topology.* This is defined by choosing a finite generating set $\{\gamma_1, \dots, \gamma_k\}$ for Γ, and embedding the representation space in the cartesian product $(\operatorname{Isom} X)^k$ by the map $\bigl[\varrho \mapsto \bigl(\varrho(\gamma_1), \dots, \varrho(\gamma_k)\bigr)\bigr]$. Note that, for any $\gamma \in \Gamma$, the map $[\varrho \mapsto \varrho(\gamma)]$ is thus continuous. In particular, it follows that the topology is independent of the choice of finite generating set.

We say that a representation is *discrete* if it is finite-one and $\varrho(\Gamma)$ is a discrete subset of $\operatorname{Isom} X$. Thus, ϱ is discrete if and only if the induced action on X is properly discontinuous.

Let us begin our discussion with euclidean space $\mathbf{E}^m$. The Bieberbach Theorems tell us that:

Lemma 1.1. *If $\varrho\colon G \longrightarrow \operatorname{Isom} \mathbf{E}^m$ is discrete, then G is finitely generated virtually abelian.* □

We shall assume, from now on, that G is finitely generated virtually abelian. We write $r(G)$ for the rank of a finite index free abelian subgroup. (This is independent of the choice of such a subgroup.)

Suppose that $\varrho\colon G \longrightarrow \operatorname{Isom} \mathbf{E}^m$ is discrete. A *crystallographic subspace* of $\mathbf{E}^m$ is a $\varrho(G)$-invariant affine subspace, $\mu \subseteq \mathbf{E}^m$, such that $\mu/\varrho(G)$ is compact. Note that the latter condition is equivalent to saying that $\dim(\mu) = r(G)$. We say that G is a *crystallographic group* if $r(G) = m$, or equivalently if $\mathbf{E}^m/\varrho(G)$ is compact.

For any discrete representation, ϱ, we set $\Sigma = \Sigma(\varrho)$ to be the set of all crystallographic subspaces. We set $\sigma = \sigma(\varrho) = \bigcup \Sigma(\varrho) \subseteq \mathbf{E}^m$. The following is another consequence of the Bieberbach Theorems (see for example [Wo] or [Bo1]).

Theorem 1.2. *The set σ is a non-empty $\varrho(G)$-invariant subspace of $\mathbf{E}^m$, which is foliated by the elements of Σ. Any two elements of Σ are parallel, and the action of $\varrho(G)$ on two such elements commutes with orthogonal projection between them.* □

We write $s(\varrho) = \dim\bigl(\sigma(\varrho)\bigr)$. Clearly $r(G) \le s(\varrho) \le m$.

We shall write $\mathscr{S} = \mathscr{S}(G, m)$ for the space of discrete representations of G into $\operatorname{Isom} \mathbf{E}^m$ with the algebraic topology. We shall see (Section 2) that:

Lemma 1.3. *The map* $s\colon \mathscr{S} \longrightarrow \mathbf{N}$ *is upper semicontinuous.*

We now move on to hyperbolic space, $\mathbf{H}^n$. We write $\partial\mathbf{H}^n$ for the ideal sphere. Given $p \in \partial\mathbf{H}^n$, we write $\operatorname{Isom}_p \mathbf{H}^n$ for the subgroup of isometries which preserve some, and hence any, horosphere about p. Identifying some such horosphere with $\mathbf{E}^{n-1}$, we get an identification of $\operatorname{Isom}_p \mathbf{H}^n$ with $\operatorname{Isom} \mathbf{E}^{n-1}$.

If G is a finitely generated infinite virtually abelian group, we say that a representation $\varrho\colon G \longrightarrow \operatorname{Isom} \mathbf{H}^n$ is *parabolic*, with fixed point p, if $\varrho(G) \subseteq \operatorname{Isom}_p \mathbf{H}^n$ and ϱ is discrete. The fixed point, p, is uniquely determined by ϱ.

Now suppose that Γ is a finitely generated group. A *peripheral structure* on Γ consists of a set, Π, and an action of Γ on Π, together with a collection $(G_i)_{i\in\Pi}$ of subsets of Γ indexed by Π, satisfying the following conditions:

(1) If $\gamma \in \Gamma$ and $i \in \Pi$, then $G_{\gamma i} = \gamma G_i \gamma^{-1}$,

(2) Π/Γ is finite,

(3) If $i \neq j$ then $G_i \cap G_j$ is finite, and

(4) For each $i \in \Pi$, G_i is an infinite finitely generated virtually abelian group.

Note that these conditions imply that G_i is the stabiliser of i (so that we could have defined peripheral structure purely in terms of the action of Γ on a set Π).

Definition. A representation $\varrho\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^n$ is *type-preserving* (relative to the given peripheral structure) if for each $i \in \Pi$, $\varrho \mid G_i$ is discrete parabolic.

We shall write $\mathscr{R} = \mathscr{R}(\Gamma, n)$ for the space of type-preserving representations in the algebraic topology.

We can think of $\mathbf{N}^{\Pi/\Gamma}$ as the set of maps $\omega\colon \Pi \longrightarrow \mathbf{N}$ such that $\omega(\gamma i) = \omega(i)$ for all $i \in \Pi$ and $\gamma \in \Gamma$. We define a partial order, $\leq$, on $\mathbf{N}^{\Pi/\Gamma}$ by $\omega \leq \omega'$ if and only if $\omega(i) \leq \omega'(i)$ for all $i \in \Pi$. Let $W(\Gamma)$ be the finite subset of $\mathbf{N}^{\Pi/\Gamma}$ defined by $\omega \in W(\Gamma)$ if and only if $\omega(i) \leq n-1$ for all $i \in \Pi$.

Now, given any $\varrho \in \mathscr{R}(\Gamma)$, and $i \in \Pi$, we get a parabolic representation $\varrho \mid G_i \longrightarrow \operatorname{Isom}_{p_i} \mathbf{H}^n$, where p_i is the fixed point of $\varrho(G_i)$. We set $\Delta(\varrho)(i) = s(\varrho \mid G_i)$. This defines a map $\Delta\colon \mathscr{R} \longrightarrow W$. The following is a corollary of Lemma 1.3:

Lemma 1.4. *The map* $\Delta\colon \mathscr{R}(\Gamma, n) \longrightarrow W(\Gamma)$ *is upper semicontinuous.*

This gives us our stratification of $\mathscr{R}$, where each "stratum" has the form $\mathscr{R}_\omega = \Delta^{-1}(\omega)$, for $\omega \in W(\Gamma)$.

By a *peripheral element* of Γ, we mean an infinite order element of $\bigcup_{i\in\Pi} G_i$. If ϱ is type-preserving, then an *accidental parabolic* of ϱ is a non-peripheral element whose image under ϱ is parabolic. We write $\mathscr{R}_F \subseteq \mathscr{R}$ for the subspace of geometrically finite representations without accidental parabolics.

Our main result can now be stated:

Theorem 1.5. *For each* $\omega \in W$, *the set* $\mathscr{R}_F \cap \mathscr{R}_\omega$ *is open in* $\mathscr{R}_\omega$.

This is proved in Section 4.

In our definition of a geometrically finite representation, we did not assume that the representation was faithful, though there would be no loss in doing this. (Note that the kernel of a geometrically finite representation is a finite normal subgroup. Provided that the group is not elementary, this will be the unique maximal finite normal subgroup. Moreover the trivial representation of any finite group is isolated in the algebraic topology. It follows easily that we cannot have a sequence of non-faithful geometrically finite representations of a non-elementary group converging on a faithful geometrically finite representation.)

To describe some applications of this result, we return for a moment, to euclidean space $\mathbf{E}^m$. Let G be a finitely generated virtually abelian group, and suppose that $\varrho \in \mathscr{S}(G,m)$. Now $r(G) \leq s(\varrho) \leq m$, so if $r(G) = m$ then $s(\varrho) = m$. If $\varrho(\Gamma) = m-1$, then $s \in \{m-1, m\}$. In fact:

Lemma 1.6. *If $r(G) = m-1$, then s is locally constant on $\mathscr{S}(G,m)$.*

This is fairly intuitive, since for any crystallographic subspace, either each component of the normal bundle is preserved by $\varrho(G)$, or there is some element which interchanges the two components. In the former case, $s(\varrho) = m$ and in the latter $s(\varrho) = m-1$. This situation remains stable under small perturbations. A more detailed argument is given at the end of Section 2.

Returning to hyperbolic space, with Γ as above, we obtain the following corollaries.

Corollary 1.7. *If $r(G_i) \geq n-2$ for each $i \in \Pi$, then Δ is locally constant on $\mathscr{R}$.*

This uses the fact that the parabolic fixed point of a group $\varrho(G_i)$ varies continuously with ϱ (see Section 4). Putting this together with Theorem 1.5, we deduce:

Proposition 1.8. *If $r(G_i) \geq n-2$ for each $i \in \Pi$, then $\mathscr{R}_F$ is open in $\mathscr{R}$.* □

As remarked in the introduction, Theorem 1.5 and its corollaries would remain valid if $\mathscr{R}_F$ were everywhere replaced by the set $\mathscr{R}_F^0$ of faithful geometrically finite representations. This follows from the following observation (which works equally well in variable curvature). Let $\mathscr{R}_D^0 \subseteq \mathscr{R}_D$ be the set of discrete faithful representations.

Lemma 1.9. *The set $\mathscr{R}_D^0$ is open and closed in $\mathscr{R}_D$.*

Proof. To see that $\mathscr{R}_D^0$ is open, suppose that the sequence $\varrho_i \in \mathscr{R}_D \setminus \mathscr{R}_D^0$ converges to $\varrho \in \mathscr{R}_D$. The fact that Γ admits a discrete representation tells us that it contains a maximal finite normal subgroup which contains all other finite normal subgroups. We can therefore assume that $\ker \varrho_i$ is constant—equal to $F \triangleleft \Gamma$. Since each $\varrho_i \mid F$ is trivial, we see that $\varrho \mid F$ is trivial, and so $\varrho \notin \mathscr{R}_D^0$.

To see that $\mathscr{R}_D^0$ is closed, suppose that the sequence $\varrho_i \in \mathscr{R}_D^0$ converges on $\varrho \in \mathscr{R}_D$. Now, the trivial representation of a finite group is isolated in the algebraic topology, and so $\varrho_i \mid \ker \varrho$ is trivial for all sufficiently large i. Thus $\ker \varrho$ is trivial, and so $\varrho \in \mathscr{R}_D^0$. □

As mentioned in the introduction, there are several circumstances under which the hypotheses of Proposition 1.8 are satisfied.

One obvious case is if $\Pi = \emptyset$, so the space of convex cocompact representations is always open in the algebraic topology. To see this directly one can bypass most of the proof of Theorem 1.5, as is done in the first part of Section 4 (Proposition 4.1). The technical details arise mainly in dealing with parabolics.

We also see that the space of finite covolume representations of a group are open. In fact, we know by Mostow rigidity that this space consists of an isolated point.

The hypotheses are also satisfied for $n \leq 3$. In the case $n = 2$, all discrete representations are geometrically finite, though they might contain accidental parabolics. In the finite coarea case, however, we see that $\mathscr{R}_F = \mathscr{R}_D$, and so $\mathscr{R}_F$ is both open and closed. In fact, we get two copies of Teichmüller space (one for each choice of orientation). For a compact surface, it is known that the connected components of the representation space correspond to the possible values of the Euler class of a representation [G1]. The discrete representations correspond to the extreme cases where the Euler class is plus or minus the Euler characteristic of the surface.

In the case $n = 3$, we recover Marden's result [Mard], that $\mathscr{R}_F$ is open in $\mathscr{R}$. It has been conjectured that $\mathscr{R}_F$ is dense in $\mathscr{R}_D$, though (as far as I know) the only cases for which this is known are the finite covolume groups, where both spaces reduce to a point, (or of course if $\mathscr{R}_D = \emptyset$). It can be shown, by a roundabout argument using Thurston's geometrisation theorem for Haken manifolds, that if $\mathscr{R}_D \neq \emptyset$ then $\mathscr{R}_F \neq \emptyset$. In general, the space $\mathscr{R}_D$ admits two other natural topologies, namely the "geometric topology" and the "quasiconformal topology". These are discussed in [Th]. On the space $\mathscr{R}_F$, all three topologies agree.

In dimension $n = 4$, the $\mathscr{R}_F$ need not be open in $\mathscr{R}$, or even in $\mathscr{R}_D$. In Section 5, we give examples of sequences in $\mathscr{R} \setminus \mathscr{R}_D$ which converge to a point of $\mathscr{R}_F$, and a sequence in $\mathscr{R}_D \setminus \mathscr{R}_F$ which also converges to a point of $\mathscr{R}_F$.

There are other constraints under which the space of geometrically finite representations will be open; for example if we demand that all parabolic elements should have zero rotational part. This however, seems rather unnatural.

We finish this section with a few remarks about the result that $\mathscr{R}_D(\Gamma)$ is closed if Γ is non-elementary. The term "elementary" can be interpreted to mean virtually abelian in the constant curvature case. This result is due, with varying generality to Chuckrow [C], Marden, Jørgensen and Wielenberg [Wi]. In fact it is true for pinched Hadamard manifolds, if "elementary" is interpreted to mean virtually nilpotent. Note that the type preserving assumption is irrelevant here (so we might as well take Γ to have empty peripheral structure). Our situation is slightly different to that described in [Wi], in that we are not assuming that discrete representations are faithful—just that they have finite kernels. For completeness, we give an argument below, which works in variable curvature. The result we want is:

Proposition 1.10. *Suppose that X is a pinched Hadamard manifold, and that Γ is a finitely generated group. If $\varrho\colon \Gamma \longrightarrow \mathrm{Isom}\, X$ is an algebraic limit of discrete representations, then either ϱ is discrete, or Γ is virtually nilpotent.*

Proof. The fact that Γ admits a discrete representation tells us immediately that any locally finite subgroup of Γ is finite. Suppose that $\varrho\colon \Gamma \longrightarrow \mathrm{Isom}\, X$ is not discrete. If U is a neighbourhood of the identity in $\mathrm{Isom}\, X$, then $\varrho^{-1}U$ is an infinite subset of Γ. Thus we can find $\beta_1, \dots, \beta_p \in \varrho^{-1}U$ such that $\langle \beta_1, \dots, \beta_p \rangle$ is infinite (otherwise $\langle \varrho^{-1}U \rangle$ would be an infinite locally finite subgroup of Γ).

Now let $\varepsilon > 0$ be less than the Margulis constant. Let $\{\gamma_1, \dots, \gamma_q\}$ be a finite generating set for Γ. Choose any $x \in X$, and let

$$K = \max\bigl\{d\bigl(x, \varrho(\gamma_j)x\bigr) \mid 1 \leq j \leq q\bigr\}.$$

Let U be a neighbourhood of the identity in $\mathrm{Isom}\, X$ such that if $g \in U$ and $y \in N(x, K+1)$ then $d(y, gy) \leq \varepsilon/2$. Let $\beta_1, \dots, \beta_p \in \varrho^{-1}U$ be chosen as above. Let ϱ' be a discrete representation, close to ϱ, such that $d\bigl(y, \varrho'(\beta_i)y\bigr) \leq \varepsilon$ for all $y \in N(x, K+1)$ and $i \in \{1, \dots, p\}$, and such that $d\bigl(x, \varrho'(\gamma_j)x\bigr) \leq K+1$ for all $j \in \{1, \dots, q\}$.

Let $T_\varepsilon = \{y \in X \mid \Gamma_\varepsilon(y) \text{ is infinite}\}$, where

$$\Gamma_\varepsilon(y) = \langle \gamma \in \Gamma \mid d\bigl(y, \varrho'(\gamma)y\bigr) \leq \varepsilon \rangle.$$

Let T_0 be the connected component of T_ε containing the point x (i.e. the "Margulis region" containing x). Now, $N(x, K+1) \subseteq T_\varepsilon$ and so $T_0 \cap \gamma_j T_0 \neq \emptyset$ for each $j \in \{1, \dots, q\}$. Thus, T_0 is $\varrho'(\Gamma)$-invariant. From the structure of Margulis regions, it follows that $\varrho'(\Gamma)$ is virtually nilpotent (see Section 3.5 of [Bo3]). Now $\varrho'(\Gamma) \cong \Gamma / \ker \varrho'$, and $\ker \varrho'$ is finite. A theorem of P. Hall tells us that a nilpotent extension of a finite group is virtually nilpotent. (This is not hard in the finitely generated case.) It follows that Γ is virtually nilpotent. $\square$

Proposition 1.10 remains valid if we replace "discrete" by "discrete and faithful" in both the hypothesis and conclusion. This is a corollary of the above result and Lemma 1.9. Note that, in the case of constant curvature, every nilpotent subgroup of $\mathrm{Isom}\, \mathbf{H}^n$ is virtually abelian, so we recover the statement given in [Wi].

Note that we have not made much use of the curvature bound away from 0, other than the convenience of referring to [Bo3]. We suspect that the above argument can be modified to work for any Hadamard manifold with curvature bounded away from $-\infty$.

Other proofs of Proposition 1.10 can be found in [GM] for complex hyperbolic space, and in [Mart] in the case of pinched negative curvature (where the metric is also allowed to vary).

2. Euclidean groups

In this section we prove some results relating to discrete representations into euclidean space.

For convenience of notation, we shall use $\mathbf{R}^m$ to denote euclidean space $\mathbf{E}^m$ with a preferred basepoint $\underline{0} \in T$. Thus, $\mathbf{R}^m$ is an inner-product space, and we can identify $\mathbf{S}^{m-1}$ with the unit sphere in $\mathbf{E}^m$. In this way $\operatorname{Isom}\mathbf{S}^{m-1}$ acts by isometry on $\mathbf{R}^m$. Also, there is a natural homomorphism rot: $\operatorname{Isom}\mathbf{E}^m \longrightarrow \operatorname{Isom}\mathbf{S}^{m-1}$, where $\operatorname{rot}(g)$ is the rotational part of g.

Given $p \in \{0, \dots, m\}$, we write $\mathscr{F}_p$ for the Grassmannian of p-dimensional vector subspaces of $\mathbf{R}^m$, and we write $\mathscr{F} = \bigsqcup_{p=0}^m \mathscr{F}_p$. Given $V \in \mathscr{F}_p$, we write $V^\perp \in \mathscr{F}_{m-p}$ for the orthogonal complement. Clearly, the map $[V \mapsto V^\perp]$ is continuous.

It will be convenient to introduce a "parameter space", T, which might, in specific cases, be the whole or part of the representation space we are dealing with. All we really need to assume about T is that it is a hausdorff topological space, though it is convenient to assume also that it is first countable. This will allow us to speak about continuity in terms of sequences.

In view of the fact that $\mathscr{F}$ is compact, we can define "upper semicontinuity" as follows. Suppose we have some function $V\colon T \longrightarrow \mathscr{F}$. We say that V is *upper semicontinuous* if whenever we have a sequence $(t_i)_{i\in\mathbf{N}}$ converging to some $t \in T$, and with $V(t_i)$ converging to some $V_\infty \in \mathscr{F}$, then $V_\infty \subseteq V(t)$. Note that one can also characterise continuous maps in this fashion by demanding that we always have $V_\infty = V(t)$.

The following are simple observations:

Lemma 2.1. *If $V\colon T \longrightarrow \mathscr{F}$ is upper semicontinuous, then the map $[t \mapsto \dim V(t)]\colon T \longrightarrow \mathbf{N}$ is also upper semicontinuous. If $\dim V(t)$ is constant on T, then V is continuous.* □

Lemma 2.2. *If $V, W\colon T \longrightarrow \mathscr{F}$ are upper semicontinuous, then so is the map $[t \mapsto V(t) \cap W(t)]$.* □

Given $g \in \operatorname{Isom}\mathbf{S}^{m-1}$, let $\operatorname{fix}(g) \in \mathscr{F}$ be the set of fixed points of g in $\mathbf{R}^m$. We see easily that:

Lemma 2.3. *The map* fix: $\operatorname{Isom}\mathbf{S}^{m-1} \longrightarrow \mathscr{F}$ *is upper semicontinuous.* □

More generally, given a subset $A \subseteq \operatorname{Isom}\mathbf{S}^{m-1}$, we write $\operatorname{fix}(A) = \bigcap_{g\in A}\operatorname{fix}(g)$.

Let G be any finitely generated group. Suppose to each $t \in T$ we associate a representation $\varrho_t\colon G \longrightarrow \operatorname{Isom}\mathbf{S}^{m-1}$.

Lemma 2.4. *If the map $[t \mapsto \varrho_t]$ is continuous (with respect to the algebraic topology), then the map $\bigl[t \mapsto \mathrm{fix}\bigl(\varrho_t(G)\bigr)\bigr]$ is upper semicontinuous.*

Proof. Choose a finite generating set, $\{\gamma_1, \ldots, \gamma_k\}$ for G. Then $\mathrm{fix}\bigl(\varrho_t(G)\bigr) = \bigcap_{i=1}^{k} \varrho_t(\gamma_i)$. For each i, the map $[t \mapsto \varrho_t(\gamma_i)]$ is continuous, so the result follows by Lemmas 3.2 and 3.3. □

We now consider general euclidean groups. Given $x, y \in \mathbf{E}^m$, we write $\overrightarrow{xy} \in \mathbf{R}^m$ for the vector from x to y.

Given $p \in \{0, \ldots, m\}$, let $\mathscr{E}_p$ be the set of all p-dimensional subspaces of $\mathbf{E}^m$. Let $\mathscr{E} = \bigsqcup_{p=0}^{m} \mathscr{E}_p$. Given $\tau \in \mathscr{F}_p$, set $D(\tau) = \{\overrightarrow{xy} \mid x, y \in \tau\} \in \mathscr{F}_p$. We put a topology on $\mathscr{E}_p$ by choosing as base the collection of all sets of the form $\{\tau \in \mathscr{E}_p \mid \tau \cap U \neq \emptyset,\ D(\tau) \in O\}$, where U runs over all open subsets of $\mathbf{E}^m$, and O runs over all open subsets of $\mathscr{F}_p$. We give $\mathscr{E}$ the topology as a disjoint union. In this way, the map $D\colon \mathscr{E} \longrightarrow \mathscr{F}$ is continuous.

We shall need the following observation about convergence in $\mathscr{E}$. Suppose that $(\tau_i)_{i \in \mathbf{N}}$ is a sequence of elements of $\mathscr{E}$, and $\tau \in \mathscr{E}$. Suppose that $D(\tau_i) \to D(\tau)$, and that there are points $x_i \in \tau_i$ and $x \in \tau$, with $x_i \to x$. Then $\tau_i \to \tau$.

Now, suppose G is a finitely generated virtually abelian group, and $\varrho\colon G \longrightarrow \mathrm{Isom}\,\mathbf{E}^m$ is a discrete representation. There are several natural subspaces one can associate to ϱ. Recall that $\sigma(\varrho) \in \mathscr{E}$ is foliated by the set $\Sigma(\varrho)$ of parallel crystallographic subspaces (Theorem 1.2). We write $M(\varrho) = D(\mu) \in \mathscr{F}$ for some (hence any) such subspace $\mu \in \Sigma$. Let $S(\varrho) = D\bigl(\sigma(\varrho)\bigr)$. Clearly, $M(\varrho) \subseteq S(\varrho)$. By definition, $\dim\bigl(S(\varrho)\bigr) = s(\varrho)$. Let $F(\varrho) = \mathrm{fix}\bigl(\mathrm{rot}\bigl(\varrho(G)\bigr)\bigr)$.

Lemma 2.5. $S(\varrho) = M(\varrho) + F(\varrho)$.

Proof. We know that $M(\varrho) \subseteq S(\varrho)$. We claim that $F(\varrho) \subseteq S(\varrho)$.

To see this, suppose $\zeta \in F(\varrho)$. Choose $x \in \mu \in \Sigma$, and let $y \in \mathbf{E}^n$ be the point such that $\overrightarrow{xy} = \zeta$. Let μ' be the subspace through y parallel to μ (i.e. $D(\mu') = D(\mu)$). Note that this is defined independently of the choice of $x \in \mu$. If $\gamma \in G$, then $\zeta = \mathrm{rot}\bigl(\varrho(\gamma)\bigr)(\zeta)$ is also the vector from $\varrho(\gamma)x$ to $\varrho(\gamma)y$. But $\varrho(\gamma)x \in \mu$, so $\varrho(\gamma)y \in \mu'$. Thus μ' is $\varrho(G)$-invariant. Since $\dim(\mu') = \dim(\mu) = r(G)$, we see that $\mu' \in \Sigma$, and so $y \in \mu' \subseteq \sigma(\varrho)$. Thus $\zeta = \overrightarrow{xy} \in D\bigl(\sigma(\varrho)\bigr) = S(\varrho)$ as claimed.

It remains to show that $S(\varrho) \subseteq M(\varrho) + F(\varrho)$. Suppose that $\zeta \in S(\varrho)$. Choose $x, y \in \sigma(\varrho)$ with $\zeta = \overrightarrow{xy}$. Let $x \in \mu \in \Sigma$ and $y \in \mu' \in \Sigma$. Let $z \in \mu$ be the nearest point on μ to y. Thus $\overrightarrow{xy} = \overrightarrow{xz} + \overrightarrow{zy}$, and $\overrightarrow{xz} \in M(\varrho)$ and $\overrightarrow{zy} \in \bigl(M(\varrho)\bigr)^{\perp}$. Now, μ and μ' are $\varrho(G)$-invariant, and the action of G commutes with orthogonal projection between them (Theorem 1.2). Thus, if $\gamma \in G$, then $\varrho(\gamma)z$ is the nearest point on μ to $\varrho(\gamma)y$. In other words, $\mathrm{rot}\bigl(\varrho(\gamma)\bigr)(\overrightarrow{zy}) = \overrightarrow{zy}$ and we deduce that $\overrightarrow{zy} \in F(\varrho)$. Thus $\zeta = \overrightarrow{xy} = \overrightarrow{xz} + \overrightarrow{zy} \in M(\varrho) + F(\varrho)$. □

(Note that the argument shows in fact that the subspaces $M(\varrho)$ and $F(\varrho)$ meet orthogonally along their intersection. In other words, we can write $S(\varrho)$ as an orthogonal direct sum $S(\varrho) = \big(F(\varrho)\cap M(\varrho)\big)\oplus F'(\varrho)\oplus M'(\varrho)$, where $F'(\varrho)\subseteq F(\varrho)$ and $M'(\varrho)\subseteq M(\varrho)$.)

As in Section 1, $\mathscr{S} = \mathscr{S}(G,m)$ denotes the set of discrete representations from G into $\operatorname{Isom}\mathbf{E}^m$. Using Lemma 2.3, and the fact that rotational part is continuous, we deduce:

Lemma 2.6. *The map $F\colon \mathscr{S} \longrightarrow \mathscr{F}$ is upper semicontinuous.* □

Let us restrict attention to the case of crystallographic groups for the moment. Suppose that $r(G) = m$, and that $\varrho, \varrho' \in \mathscr{S}(G,m)$ are faithful. Now one of the Bieberbach Theorems [Wo] tells us that there is an affine transformation, A, of $\mathbf{E}^m$, which conjugates ϱ to ϱ', i.e. $\varrho'(\gamma) = A\varrho(\gamma)A^{-1}$ for all $\gamma\in G$. (In fact, as observed elsewhere, this follows from Theorem 1.2.—Take the product action of Γ on $\mathbf{E}^m\times\mathbf{E}^m \equiv \mathbf{E}^{2m}$. A crystallographic subspace for this action is a graph of the desired affine transformation.) Now such an affine transformation has a "rotational part" which in this case is a linear endomorphism, B, of $\mathbf{R}^m$. Thus, $\zeta\in\operatorname{fix}\big(\operatorname{rot}\varrho(\gamma)\big)$ if and only if $B\zeta\in\operatorname{fix}\big(B\big(\operatorname{rot}\varrho(\gamma)\big)B^{-1}\big) = \operatorname{fix}\big(\operatorname{rot}\big(A\varrho(\gamma)A^{-1}\big)\big)$. We see that $F(\varrho') = B\big(F(\varrho)\big)$. In particular, we see that $f(\varrho') = f(\varrho)$.

Now suppose $\varrho\in\mathscr{S}(G,m)$, still with $r(G)=m$. We claim that $\ker(\varrho)$ is the unique maximal finite normal subgroup of G. (For if H were a finite normal subgroup of G, then $\operatorname{fix}\big(\varrho(H)\big)$ would be a non-empty $\varrho(G)$-invariant subspace of $\mathbf{E}^m$, and hence the whole of $\mathbf{E}^m$. Thus, $H\le\ker(\varrho)$.) In particular, the kernel is completely determined by the group structure of G. Thus the previous paragraph applies equally well to non-faithful discrete representations. In summary, we have shown:

Lemma 2.7. *If $r(G)=m$, then f is constant on $\mathscr{S}(G,m)$.* □

In this case, we can define $h(G) = f(\varrho)$ for some, and hence any, $\varrho\in\mathscr{S}(G,m)$.

We now drop the constraint on dimension. If $\varrho\in\mathscr{S}(G,m)$, then we have $r(G) = \dim\mu$ for any $\mu\in\Sigma$. Applying the above result to the action of G on μ, we see that $h(G) = \dim\big(F(\varrho)\cap M(\varrho)\big)$.

In summary, if $\varrho\in\mathscr{S}(G,m)$ we have $\dim M(\varrho) = r(G)$, $\dim F(\varrho) = f(\varrho)$, $\dim S(\varrho) = s(\varrho)$ and $\dim\big(F(\varrho)\cap M(\varrho)\big) = h(G)$. Moreover, by Lemma 2.5, we have $S(\varrho) = M(\varrho)+F(\varrho)$. Using the formula $\dim S(\varrho) + \dim\big(F(\varrho)\cap M(\varrho)\big) = \dim M(\varrho) + \dim F(\varrho)$, we arrive at the identity $s(\varrho) = r(G) - h(G) + f(\varrho)$. Since, by Lemma 2.6, f is upper semicontinuous, so we deduce that s is also upper semicontinuous. This proves Lemma 1.3.

It remains to consider the way in which the subspaces, $\sigma(\varrho)$ vary with σ. Given $q\in\{r(G),\dots,m\}$, we shall write $\mathscr{S}_q = s^{-1}(q)$.

Note that, by Lemma 2.6, F is upper semicontinuous on $\mathscr{S}$. Now, by the above identity, $f(\varrho) = \dim F(\varrho)$ is constant on $\mathscr{S}_q$. By Lemma 1.2, we see that $F\colon \mathscr{S}_q \longrightarrow \mathscr{F}$ is continuous.

Moreover, we shall show:

Proposition 2.8. *For each $q \in \{r(G), \dots, m\}$, the map $\sigma\colon \mathscr{S}_q \longrightarrow \mathscr{E}_q$ is continuous.*

Proof. Suppose that $(\varrho_i)_{i\in\mathbf{N}}$ is a sequence of representations in $\mathscr{S}_q$, converging on some $\varrho_\infty \in \mathscr{S}_q$. From the continuity of F on $\mathscr{S}_q$, we know that $F(\varrho_i) \to F(\varrho_\infty)$. We write $\sigma_i = \sigma(\varrho_i)$. We want to show that $\sigma_i \to \sigma(\varrho_\infty)$.

Suppose that σ_i fails to converge to $\sigma(\varrho_\infty)$. Then, passing to a subsequence, we can assume that either $\sigma_i \to \sigma_\infty \in \mathscr{E}$ with $\sigma_\infty \neq \sigma(\varrho_\infty)$, or else σ_i tends to infinity in the sense that for any compact set $K \subseteq \mathbf{E}^m$, $\{i \in \mathbf{N} \mid \sigma_i \cap K \neq \emptyset\}$ is finite.

Let us consider the first case. Again passing to a subsequence, we can suppose that $M(\varrho_i)$ converges to some $M_\infty \in \mathscr{F}_{r(G)}$. Now $D(\sigma_i) \to D(\sigma_\infty)$ and for each i, we have $M(\varrho_i) \subseteq D(\sigma_i)$. It follows that $M_\infty \subseteq D(\sigma_\infty)$.

Given $x \in \sigma_\infty$, let $\mu \in \mathscr{E}$ be the subspace through x with $D(\mu) = M_\infty$. Since $D(\mu) \subseteq D(\sigma_\infty)$, we have $\mu \subseteq \sigma_\infty$. Now, since $\sigma_i \to \sigma_\infty$, there are points $x_i \in \sigma_i$ with $x_i \to x$. Let μ_i be the subspace through x_i with $D(\mu_i) = M(\varrho_i)$. Since $D(\mu_i) \to D(\mu)$, we see that $\mu_i \to \mu$.

Now, $\mu_i \in \Sigma(\varrho_i)$. In particular, μ_i is $\varrho_i(G)$-invariant. It now follows that μ is $\varrho(G)$-invariant. To see this, suppose $y \in \mu$, and choose a sequence $y_i \in \mu_i$ with $y_i \to y$. Now given $\gamma \in G$, we have $\varrho_i(\gamma) \to \varrho_\infty(\gamma)$ and so $\varrho_i(\gamma)y_i \to \varrho_\infty(\gamma)y$. We have $\varrho_i(\gamma)y_i \in \mu_i$ and so $\varrho_\infty(\gamma)y \in \mu$. This shows that μ is $\varrho_\infty(G)$-invariant as claimed. Now since $\dim\mu = r(G)$, we see that $\mu \in \Sigma(\varrho_\infty)$, and so $\mu \subseteq \sigma(\varrho_\infty)$. In particular $x \in \sigma(\varrho_\infty)$.

We have thus shown that $\sigma_\infty \subseteq \sigma(\varrho_\infty)$. But since $\dim\sigma_\infty = \dim\sigma(\varrho_\infty)$, we get that $\sigma_\infty = \varrho(\sigma_\infty)$.

We now move on to consider the second case, namely where σ_i tends to infinity.

Given any $x \in \mathbf{E}^m$, let $y_i = y_i(x) \in \sigma_i$ be the nearest point in σ_i to x. We can suppose (for sufficiently large i) that $x \notin \sigma_i$. Let $\zeta_i(x)$ be the unit vector in the direction $\overrightarrow{xy_i}$. Now, passing to a subsequence, we can suppose that $\zeta_i(x)$ converges to some $\zeta \in \mathbf{R}^m$. Now, if $x' \in \mathbf{E}^m$ is any other point, we see that $d_{\mathrm{euc}}\big(y_i(x), y_i(x')\big) \leq d_{\mathrm{euc}}(x, x')$. Since the distance between x and σ_i is tending to infinity, we see that the angle between $\zeta_i(x)$ and $\zeta_i(x')$ tends to 0. We see that $\zeta_i(x')$ also tends to ζ as $i \to \infty$. Note that for each i, $\zeta_i(x)$ is perpendicular to σ_i. In other words, $\zeta_i(x) \in D(\sigma_i)^\perp = S(\varrho_i)^\perp \subseteq F(\varrho_i)^\perp$. Now $F(\varrho_i) \to F(\varrho_\infty)$ and so we see that $\zeta \in F(\varrho_\infty)^\perp$.

Now suppose $\gamma \in G$. We have $\varrho_i(\gamma) \to \varrho_\infty(\gamma)$ and so $\varrho_i(\gamma)x \to \varrho_\infty(\gamma)x$. Now, since σ_i is $\varrho_i(G)$-invariant, we see that $\xi_i = \operatorname{rot}\varrho_i(\gamma)\big(\zeta_i(x)\big)$ is the unit vector from $\varrho_i(\gamma)x$ to the nearest point $y_i\big(\varrho_i(\gamma)x\big) = \varrho_i(\gamma)\big(y_i(x)\big)$ in σ_i. Since $\varrho_i \to \varrho_\infty$ and $\zeta_i(x) \to \zeta$, we see that $\xi_i \to \operatorname{rot}\varrho_\infty(\gamma)(\zeta)$. Now, since $\varrho_i(\gamma)x \to \varrho_\infty(\gamma)x$, we see that the angle between ξ_i and $\zeta_i\big(\varrho_\infty(\gamma)x\big)$ tends to 0. But (as discussed earlier with $x' = \varrho_\infty(\gamma)x$) the latter vector tends to ζ. We see that $\xi_i \to \zeta$. Thus $\operatorname{rot} \varrho_\infty(\gamma)(\zeta) = \zeta$. We thus conclude that $\zeta \in F(\varrho_\infty)$ contradicting the earlier statement that $\zeta \in F(\varrho_\infty)^\perp$. □

Lemma 2.9. *The map $M\colon \mathscr{S}_q \longrightarrow \mathscr{F}$ is continuous.*

Proof. Fix some point $x_0 \in \mathbf{E}^m$. Given $\varrho \in \mathscr{S}_q$, let $x_0(\varrho) \in \sigma(\varrho)$ be the nearest point in $\sigma(\varrho)$ to x_0. Using Proposition 2.8, we see easily that $x_0(\varrho)$ depends continuously on ϱ. Let $\mu(\varrho)$ be the crystallographic subspace in $\Sigma(\varrho)$ containing the point $x_0(\varrho)$. We claim that $[\varrho \mapsto \mu(\varrho)]\colon \mathscr{S}_q \longrightarrow \mathscr{E}$ is continuous. It then follows that $M(\varrho) = D\big(\mu(\varrho)\big)$ varies continuously in ϱ as required.

To prove the claim, choose a set of generators, $\{\gamma_1, \dots, \gamma_r\}$ for a finite index free abelian subgroup of G, where $r = r(G)$. For $1 \le i \le r$, let $x_i(\varrho) = \varrho(\gamma_i)\big(x_0(\varrho)\big)$. Thus each of the points $x_i(\varrho)$ vary continuously in ϱ. But $\mu(\varrho)$ is the subspace spanned by the points $x_0(\varrho), x_1(\varrho), \dots, x_r(\varrho)$, and has constant dimension r. The claim now follows. □

Given $r, s \in \mathbf{N}$, with $r \le s$, we shall write

$$\mathscr{L}(r,s) = \{(V,\tau) \mid V \in \mathscr{F}_r,\ \tau \in \mathscr{E}_s,\ V \subseteq D(\tau)\}.$$

We give $\mathscr{L}(r,s)$ the subspace topology as a subset of $\mathscr{F}_r \times \mathscr{E}_s$. Intuitively, we can think of a point of $\mathscr{L}(r,s)$ as consisting of an s-dimensional subspace of $\mathbf{E}^m$ (namely τ) foliated by parallel r-dimensional subspaces (whose direction is given by V).

Given $\varrho \in \mathscr{S}(G,m)$, we define $\lambda(\varrho) = \big(M(\varrho), \sigma(\varrho)\big) \in \mathscr{L}(r,s)$ where $r = r(G)$ and $s = s(G)$. Thus, we get a map $\lambda\colon \mathscr{S}_s \longrightarrow \mathscr{L}(r,s)$ where $s = m - d$. Putting Proposition 2.8 and Lemma 2.9 together, we see that:

Proposition 2.10. *The map $\lambda\colon \mathscr{S}_s \longrightarrow \mathscr{L}(r,s)$ is continuous.* □

We shall also need the following observation:

Lemma 2.11. *Suppose $\varrho \in \mathscr{S}_s$. Given any $x \in \mathbf{E}^m$, and $\eta > 0$, there is a neighbourhood, U, of ϱ in $\mathscr{S}_s$ such that for all $\varrho' \in U$ and $\gamma \in G$, we have $d_{\mathrm{euc}}\big(x, \varrho'(\gamma)x\big) \ge (1-\eta) d_{\mathrm{euc}}\big(x, \varrho(\gamma)x\big)$.* □

We shall only need this result for crystallographic groups, for which it is a fairly simple exercise. We state in general, since Proposition 2.10 allows us fairly easily to reduce to that case anyway.

Finally, we give a proof of Lemma 1.6:

Proof of Lemma 1.6. Suppose that $r(G) = m-1$, and that $\varrho_i \to \varrho_\infty$. Suppose that $s(\varrho_i) = \dim \sigma(\varrho_i) = m-1$ for all i. We want to show that $s(\varrho_\infty) = m-1$. Since G is finitely generated, we can suppose (on passing to a subsequence) that there is a fixed $\gamma_0 \in G$, such that $\varrho_i(\gamma_0)$ swaps the two components of $\mathbf{E}^m \setminus \sigma(\varrho_i)$.

Now, as in the proof of Proposition 2.8, we have either that $\sigma(\varrho_i)$ tends to infinity as $i \to \infty$, or that $\sigma(\varrho_i)$ converges on some $\sigma_\infty \in \mathscr{E}_{m-1}$. But, in the former case, we see easily that the isometries $\varrho_i(\gamma_0)$ move any fixed basepoint an arbitrarily large distance, contradicting the fact that $\varrho_i(\gamma_0)$ converges to some isometry $\varrho_\infty(\gamma_0)$. We thus have that $\sigma(\varrho_i) \to \sigma_\infty$. Now it is easily seen that σ_∞ is $\varrho_\infty(G)$-invariant, and hence a crystallographic subspace, and that $\varrho_\infty(\gamma_0)$ swaps the two components of $\mathbf{E}^m \setminus \sigma_\infty$. Thus $\sigma_\infty = \sigma(\varrho_\infty)$. In particular, $\dim \sigma(\varrho_\infty) = m-1$ as required.

3. Coverings by connected sets

In this section, (X,d) can be any simply-connected metric space in which open metric balls $N(x,r) = \{y \in X \mid d(x,y) < r\}$ are connected for all $x \in X$ and $r > 0$. (In fact it would be enough to assume this for all r less than some fixed positive constant.) Our principal application will be to hyperbolic space: $X = \mathbf{H}^n$.

Suppose that (I,E) is a connected graph with vertex set I, and edge set E. We shall denote by ij the edge with endpoints $i,j \in I$. For convenience we shall assume that $ii \in E$ for all $i \in I$. (In other words, E is a reflexive symmetric relation on I, whose transitive closure has just one equivalence class.)

The main result of the this section is used in the proof Theorem 1.5.

Theorem 3.1. *Suppose that (I,E) is a non-empty connected graph, and that $(A_i)_{i\in I}$ is a collection of non-empty open connected subsets of X, indexed by the vertex set I, satisfying:*

(A1) *if $ij \in E$, then $A_i \cap A_j \neq \emptyset$,*

(A2) *if $ij, ik \in E$ and $A_i \cap A_j \cap A_k \neq \emptyset$ then $jk \in E$, and*

(A3) $(\exists \varepsilon > 0)(\forall i \in I)(\forall x \in A_i)(\exists j \in I)(ij \in E$ *and* $N(x,\varepsilon) \subseteq A_j)$.

Then, $X = \bigcup_{i\in I} A_i$ and if $A_i \cap A_j \neq \emptyset$ then $ij \in E$.

The rest of this section is devoted to proving this result. The idea is to construct abstractly a covering space for X out of the sets A_i, together with the combinatorial information from (I,E) telling us how to glue them together.

We give I the discrete topology, and $X \times I$ the product topology. Given $i \in I$, we set $\Xi_i = \{(x,i) \in X \times I \mid x \in A_i\}$, and set $\Xi = \bigcup_{i\in I} \Xi_i \subseteq X \times I$. We give Ξ the subspace topology. Thus Ξ can be thought of as a disjoint union of the sets A_i.

We define a relation $\sim$ on Ξ by $(x,i) \sim (y,j)$ if and only if $x = y$ and $ij \in E$. Using hypothesis (A2), we see easily that $\sim$ is an equivalence relation on Ξ. We let $\Sigma = \Xi/\!\sim$ with the quotient topology, and let $\pi\colon \Xi \longrightarrow \Sigma$ be the quotient map. Given $(x,i) \in \Xi$, write $[x,i] = \pi\big((x,i)\big) \in \Sigma$ for the equivalence class. If $U \subseteq A_i$, write $[U,i] = \{[x,i] \mid x \in U\} \subseteq \Xi_i$. Let $B_i = [A_i,i] = \pi(\Xi_i)$. Thus, B_i is connected. Define the map $\phi\colon \Sigma \longrightarrow X$ by $\phi([x,i]) = x$.

Lemma 3.2. *The map ϕ is continuous.*

Proof. If $U \subseteq X$ is open, then for all $i \in I$, $\Xi_i \cap \pi^{-1}\phi^{-1}U = (U \cap A_i) \times \{i\}$ is open in $\Xi_i = A_i \times \{i\}$. Thus, $\pi^{-1}\phi^{-1}U$ is open in Ξ, so $\phi^{-1}U$ is open in Σ. ◻

Lemma 3.3. *The set Σ is connected.*

Proof. Suppose $[x,i],[y,j] \in \Sigma$. By hypothesis, (I,E) is connected, so there is a sequence $i = i(0), i(1), \dots, i(n) = j$ in I, with $i(m-1)i(m) \in E$ for each $m \in \{1,\dots,n\}$. By hypothesis (A1), for each $m \in \{1,\dots,n\}$, we can choose $x(m) \in A_{i(m-1)} \cap A_{i(m)}$. Now $[x(m), i(m-1)] = [x(m), i(m)] \in B_{i(m-1)} \cap B_{i(m)}$. In particular, $B_{i(m)} \cap B_{i(m-1)} \neq \emptyset$. Now each $B_{i(m)}$ is connected, and so $B = \bigcup_{m=1}^{n} B_{i(m)}$ is connected. But now, $[x,i] \in B_{i(0)} \subseteq B$ and $[y,i] \in B_{i(n)} \subseteq B$. We have shown that every pair of points of Σ lie inside a connected subset of Σ. This shows that Σ itself is connected. ◻

We next claim that ϕ is a covering map. We begin with a couple of preliminary observations. If $U \subseteq A_i$ is open, then $[U,i] \subseteq \Sigma$ is open. This follows since $\Xi_j \cap \pi^{-1}([U,i]) = (U \cap A_i) \times \{j\}$ is open in Ξ_j for each $j \in I$. Also if $U \subseteq A_i \cap A_j$, then $[U,i] = [U,j]$ if $ij \in E$, and $[U,i] \cap [U,j] = \emptyset$ if $ij \notin E$. Also $x \in N = N(x, \varepsilon/2)$. Using hypothesis (A3), we see that if $N \cap A_i \neq \emptyset$, then $N \subseteq A_j$ for some $j \in I$ with $ij \in E$.

Lemma 3.4. *The map ϕ is a covering map.*

Proof. Suppose $x \in X$. Let $N = N(x, \varepsilon/2)$. We aim to show that $\phi^{-1}N$ is topologically a disjoint union of its connected components, each of which is mapped homeomorphically to N under ϕ.

Suppose $[y,i] \in \phi^{-1}N \subseteq \Sigma$. Now $y \in A_i$, so $N \cap A_i \neq \emptyset$, and so $N \subseteq A_j$ for some $j \in I$ with $ij \in E$. Thus $[y,i] = [y,j] \in [N,j]$. We see that, as a set, $\phi^{-1}N$ is a disjoint union of sets of the form $[N,i]$ for certain $i \in I$. Now each such set $[N,i]$ is open in Σ and hence in $\phi^{-1}N$. It follows that each $[N,i]$ is also closed in $\phi^{-1}N$. Now $[N,i] = \pi(N \times \{i\})$ is connected, and is thus a connected component of $\phi^{-1}N$.

Now, $\phi \mid [N,i]$ maps bijectively onto N. We know that ϕ is continuous. Also any subset of $[N,i]$ has the form $[U,i]$ with $U \subseteq N$. If $[U,i]$ is open in $[N,i]$, then $\Xi_i \cap \pi^{-1}([U,i]) = U \cap \{i\}$ is open in $N \times \{i\}$. It follows that U is an open subset of X, and so $\phi([U,i]) = U$ is open. This shows that ϕ maps $[N,i]$ homeomorphically onto N. ◻

Proof of Theorem 3.1. By hypothesis, Σ is non-empty. By Lemma 3.3, it is connected. By Lemma 3.4, $\phi\colon \Sigma \longrightarrow X$ is a covering map. By hypothesis, X is simply connected. It follows that ϕ is a homeomorphism.

By construction, we have $\Sigma = \bigcup_{i \in I} B_i$ and $B_i \cap B_j \neq \emptyset$ if and only if $ij \in E$. Also, ϕ maps B_i homeomorphically onto A_i. We deduce that $X = \bigcup_{i \in I} A_i$ and $A_i \cap A_j \neq \emptyset$ if and only if $ij \in E$. ◻

4. Proofs of the main results

The main object of this section is to give a proof of Theorem 1.5. We shall motivate the argument by first giving a direct proof along similar lines in the convex cocompact case, i.e. where the peripheral structure is empty. The only input we need for this is Theorem 3.1. Afterwards we shall worry about how to deal with parabolics. For this, we shall also need the results of Section 2.

Suppose that Γ is a finitely generated group (with empty peripheral structure). Thus, $\mathscr{R}(\Gamma, n)$ is the space of all representations, $\varrho\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^n$, and $\mathscr{R}(\Gamma, n)$ the subset of convex cocompact representations (where "convex cocompact" means "geometrically finite without parabolics"). In this special case we shall show:

Proposition 4.1. *The set $\mathscr{R}_F$ is open in $\mathscr{R}$.*

This result is not new (at least in constant curvature, or for symmetric spaces). For example, it can be deduced from the Holonomy Theorem, versions of which are proven in [L] and [G2]. (To do this, one may need a slight variant of the Holonomy Theorem for manifolds with boundary. The manifold in question is then viewed as locally modelled on compactified hyperbolic space. If Γ is a convex cocompact group, with discontinuity domain Ω, then the quotient of $\mathbf{H}^n \cup \Omega$ by Γ is a compact manifold having a structure of this type.)

The argument we give here can be applied equally well in the case of variable negative curvature with curvature bounded away from 0. (In case there is no curvature bound away from $-\infty$ one should reinterpret "convex" as "quasiconvex" in some sense. This adds some technical complications, but the argument should still go through.) For the sake of simplicity, we shall give the argument here only with reference to hyperbolic space, $\mathbf{H}^n$.

We begin with some general discussion of convex sets in $\mathbf{H}^n$. Given a subset $Q \subseteq \mathbf{H}^n$, we write $\overline{Q}$ for its closure in the ball $\mathbf{H}^n \cup \partial \mathbf{H}^n$. If Q_1 and Q_2 are convex, then $\overline{Q}_1 \cap \overline{Q}_2 = \emptyset$ if and only if $d_{\mathrm{hyp}}(Q_1, Q_2) > 0$. In such a case we shall say that Q_1 and Q_2 are *strictly disjoint.*

We shall phrase everything in terms of continuous families of representations. Let T be a "parameter space", i.e. a first countable Hausdorff topological space, with a preferred basepoint $0 \in T$. We shall speak about a parameter $t \in T$ being "small" if it lies in some small neighbourhood of 0.

Suppose that the map $[t \mapsto Q(t)]$ associates to each $t \in T$ a set $Q \subseteq \mathbf{H}^n$. We say that $Q(t)$ is a *continuous translation* (of some fixed set Q), if there is a continuous map $[t \mapsto g(t)]\colon T \longrightarrow \operatorname{Isom} \mathbf{H}^n$ such that $Q(t) = g(t)Q$ for all $t \in T$. Without loss of generality, we can take $Q = Q(0)$.

Suppose that $K \subseteq Q$ is closed, and that $\mathscr{Q}$ is a collection of closed convex subsets of $\mathbf{H}^n$. We say that $\mathscr{Q}$ *properly ε-covers* K if for all $x \in K$ there is some $Q \in \mathscr{Q}$ such that $N(x, \varepsilon) \subseteq Q$, and if every point of $\overline{K}$ lies in the interior of $\overline{Q}$ for some $Q \in \mathscr{Q}$. One easily verifies the following:

Lemma 4.2. *Suppose that I is some finite indexing set, and that for each $i \in I$, we have a continuous translation $[t \mapsto Q_i(t)]$ of a closed convex set $Q_i(0)$. Suppose that $K \subseteq \mathbf{H}^n$ is closed and is properly ε-covered by $\{Q_i(0) \mid i \in I\}$. Then K is properly $(\varepsilon/2)$-covered by $\{Q_i(t) \mid i \in I\}$ for all sufficiently small $t \in T$.* □

Returning to our finitely generated group, Γ, we can characterise convex cocompact representations as follows:

Lemma 4.3. *Suppose $\varrho \in \mathscr{R}(\Gamma)$, then $\varrho \in \mathscr{R}_F(\Gamma)$ if and only if there is a closed convex set $Q \subseteq \mathbf{H}^n$ such that $\{\gamma \in \Gamma \mid Q \cap \varrho(\gamma)Q \neq \emptyset\}$ is finite, and such that the sets $\{\varrho(\gamma)Q \mid \gamma \in \Gamma\}$ form a locally finite cover of $\mathbf{H}^n$.* □

If $\varrho \in \mathscr{R}_F$, we could take Q to be, for example, a Dirichlet domain. Note that the sets $\{\varrho(\gamma)Q \mid \gamma \in \Gamma\}$ are necessarily locally finite on $\mathbf{H}^n \cup \Omega(\Gamma)$ (see [Bo1]). Note that we can "thicken up" Q a bit so that ∂Q is properly ε-covered by the sets $\varrho(\gamma)Q$ as γ ranges over the finite set of non-trivial $\gamma \in \Gamma$ such that $Q \cap \varrho(\gamma)Q \neq \emptyset$. (This can be done by replacing Q by some convex neighbourhood of itself in $\mathbf{H}^n \cup \Omega(\Gamma)$, using, for example, the Klein model. By the above remark, the finiteness properties of the cover remain valid.) We can now thicken up Q a bit further so that it has the additional properties that if $Q \cap \varrho(\gamma)Q = \emptyset$ then Q and $\varrho(\gamma)Q$ are strictly disjoint, whereas if $Q \cap \varrho(\gamma)Q \neq \emptyset$ then this intersection contains a non-empty open set. (This can be done, without affecting the ε-covering property, by replacing Q by a small uniform neighbourhood of itself in the hyperbolic metric.) The point of imposing these additional conditions is that they are stable with respect to small perturbations (cf. Lemma 4.2).

To avoid notational confusion later on, we shall imagine the the elements of Γ as indexed by a set Υ. Thus, for each $i \in \Upsilon$, we have a corresponding element $g(i) \in \Gamma$. Thus Γ acts on Υ by left multiplication, i.e. such that $g(\gamma i) = \gamma g(i)$ for all $i \in \Upsilon$ and $\gamma \in \Gamma$.

Now suppose that $\varrho_0 \in \mathscr{R}_F$, and that $Q \subseteq \mathbf{H}^n$, is as described by Lemma 4.3 and the subsequent discussion. Given $i \in \Upsilon$, set $A_i = \varrho_0\big(g(i)\big)Q$. Thus, if $\gamma \in \Gamma$, $A_{\gamma i} = \varrho_0(\gamma)A_i$. We define a graph (I, E) by setting $I = \Upsilon$, and letting $ij \in E$ if and only if $A_i \cap A_j \neq \emptyset$ (so that A_i and A_j are strictly disjoint). Note that this graph is connected, every vertex has finite degree, and that E/Γ is finite. Note that the hypotheses of Theorem 3.1 are satisfied. In particular, (A3) follows from the fact that ∂A_i is properly ε-covered by the sets $\{A_j \mid ij \in E,\ i \neq j\}$. Of course, we deduce nothing new from the conclusion in this case.

Suppose that $[t \mapsto \varrho_t]\colon T \longrightarrow \operatorname{Isom}\mathbf{H}^n$ is a continuous family of representations, with $\varrho_0 \in \mathscr{R}_F$. Given $t \in T$ and $i \in I$, let $A_i(t) = \varrho_t\big(g(i)\big)Q$. Thus, $A_i(0) = A_i$, and $A_{\gamma i}(t) = \varrho_t(\gamma)A_i(t)$ for all $i \in I$, $\gamma \in \Gamma$ and $t \in T$. Also $[t \mapsto A_i(t)]$ is a continuous translation of $A_i(0)$.

Fix any $i \in I$. We can assume (after conjugating by $\varrho_t\big(g(i)\big)$) that $A_i(t) = A_i$ is constant. (In fact, by taking i so that $g(i)$ is the identity, we need not bother with this here, but the principle will be used in the proof of the general case.) Now, applying Lemma 4.2, we see that for all sufficiently small $t \in T$, the sets $\{A_j(t) \mid ij \in E,\ i \neq j\}$ give a proper $(\varepsilon/2)$-cover of $\partial A_i = A_i(t)$. From the $\varrho_t(\Gamma)$-equivariance, we see that this is true simultaneously for all $i \in I$, and so we see that the hypothesis (A3) is satisfied for the collection $\{A_i(t) \mid i \in I\}$.

If $ij \in E$, we arranged that $A_i(0) \cap A_j(0)$ contains a non-empty open set, and so $A_i(t) \cap A_j(t) \neq \emptyset$ for all sufficiently small $t \in T$. But, since E/Γ is finite, we see, again from the $\varrho_t(\Gamma)$-equivariance, that hypothesis (A1) is satisfied for all sufficiently small t.

Finally, suppose $ij, ik \in E$ and $jk \notin E$. Then $A_j(0)$ and $A_k(0)$ are strictly disjoint, and so $A_j(t) \cap A_k(t) = \emptyset$ for all sufficiently small t. Now, for a given i, there are only finitely many such pairs j, k, and so there are only finitely many such triples, i, j, k, up to the action of Γ. We see that hypothesis (A2) is satisfied for all sufficiently small t.

In summary, we see that for all sufficiently small $t \in T$, the hypotheses of Theorem 3.1 are satisfied, and so $\mathbf{H}^n = \bigcup_{i\in I} A_i(t) = \bigcup_{\gamma\in\Gamma} \varrho_t(\gamma)Q$. Moreover, we have $A_i(t) \cap A_j(t) \neq \emptyset$ if and only if $ij \in E$. It follows that the collection $\{A_i(t) \mid i \in I\} = \{\varrho_t(\gamma)Q \mid \gamma \in \Gamma\}$ is a locally finite cover of $\mathbf{H}^n$, with Q meeting only finitely many images of itself under the action of $\varrho_t(\Gamma)$. Lemma 4.2 now tells us that $\varrho_t \in \mathscr{R}_F$.

By taking T to be the whole representation space $\mathscr{R}$, this proves Proposition 4.1.

We now do the same thing again in the general case. This time, we have to deal with parabolic groups. The idea will be to include a set of "standard parabolic regions" in our family of covers, $\big(A_i(t)\big)_{i\in I}$.

As before, let $\mathbf{R}^n_+$ be the upper half space model, and identify $\partial\mathbf{R}^n_+ \equiv \mathbf{R}^{n-1} \equiv \mathbf{E}^{n-1}$. Given any $r \in \{0, \dots, n-1\}$, we write $\tau(r) \subseteq \mathbf{E}^{n-1} \subseteq \mathbf{R}^n_+ \cup \partial\mathbf{R}^n_+$ for the "standard" r-dimensional subspace $\tau(r) = \{(\xi_1, \dots, \xi_{n-1}, 0) \mid (\forall k > r)(\xi_k = 0)\}$. We write $C(r) = \big\{x \in \mathbf{R}^n_+ \mid d_{\text{euc}}\big(x, \tau(r)\big) \geq 1\big\}$, where d_{euc} is the usual euclidean metric on $\mathbf{R}^n_+$.

Now suppose that G is a finitely generated virtually abelian group with $r(G) = r$. Let $\mathscr{S}(G, n-1)$ be the space of discrete representations into $\mathbf{E}^{n-1}$ (as described in Section 2). Suppose that $\varrho \in \mathscr{S}(G, n-1)$. By the Bieberbach Theorem (Theorem 1.3), we can conjugate ϱ by an element of $\operatorname{Isom}\mathbf{E}^{n-1}$ so that $\tau(r)$ is a crystallographic subspace. In particular, $\tau(r)$ is $\varrho(G)$-invariant. Also, if $H(r)$ is the closure of $\mathbf{R}^n_+ \setminus C(r)$ in $\mathbf{R}^n_+ \cup \partial\mathbf{R}^n_+$, then the quotient $H(r)/\varrho(G)$ is compact. We refer to $C(r)$ as a *standard parabolic region* for the group $\varrho(G)$.

More generally, suppose that $\varrho\colon G \longrightarrow \operatorname{Isom}\mathbf{H}^n$ is a parabolic representation, as discussed in Section 1, with fixed point $p \in \partial\mathbf{H}^n$. By a *parabolic region*, C, we shall mean a closed convex $\varrho(G)$-invariant subset of $\mathbf{H}^n$ such that $H/\varrho(G)$ is compact, where H is the closure of $\mathbf{H}^n \setminus C$ in $\mathbf{H}^n \cup \partial\mathbf{H}^n \setminus \{p\}$.

There is an isometry, $\beta\colon \mathbf{H}^n \longrightarrow \mathbf{R}^n_+$ with $\beta(p) = \infty$. Let $\varrho^\beta \in \mathscr{S}(G, n-1)$ be the conjugate representation, defined by $\varrho^\beta(\gamma) = \beta\varrho(\gamma)\beta^{-1}$. Now, we can also assume that β is chosen so that $\tau(r)$ is a crystallographic subspace for $\varrho^\beta(G)$. We shall say that $C \subseteq \mathbf{H}^n$ is a *standard parabolic region* for ϱ if it has the form $C = \beta^{-1}C(r)$ for some such isometry β.

Suppose Γ is a finitely generated group, with peripheral structure $(G_i)_{i\in\Pi}$. Recall that $\mathscr{R}$ is the space of type-preserving representations, and $\mathscr{R}_F$ is the subset of geometrically finite representations without accidental parabolics. We may characterise elements of $\mathscr{R}_F$ as follows:

Proposition 4.4. *Suppose that $\varrho \in \mathscr{R}$. Then $\varrho \in \mathscr{R}_F$ if and only if there is a collection $(C_i)_{i\in\Pi}$ of closed convex subsets of $\mathbf{H}^n$, together with another closed convex set $Q \subseteq \mathbf{H}^n$, satisfying the following:*

(1) $(\forall\gamma \in \Gamma)(\forall i \in \Pi)(C_{\gamma i} = \varrho(\gamma)C_i)$,

(2) *If $i \neq j$, then $C_i \cap C_j = \emptyset$,*

(3) *For all $i \in \Pi$, C_i is a cusp region corresponding to the parabolic representation $\varrho \mid G_i$,*

(4) *The collection $\{C_i \mid i \in \Pi\} \cup \{\varrho(\gamma)Q \mid \gamma \in \Gamma\}$ form a locally finite cover of $\mathbf{H}^n$.* □

This is essentially Marden's definition of geometrical finiteness [Mard], (or "GF1" as described in [Bo1]). Note that we could obtain Q, for example, by taking a Dirichlet domain, removing its intersection with the standard parabolic regions, and then taking the convex hull.

The local finiteness part of condition (4) should be interpreted to mean that given any compact set $K \subseteq \mathbf{H}^n$, the sets $\{i \in \Pi \mid K \cap C_i \neq \emptyset\}$ and $\{\gamma \in \Gamma \mid K \cap \varrho(\gamma)Q \neq \emptyset\}$ are finite. In fact, it follows that the collection is locally finite on $\mathbf{H}^n \cup \Omega(\Gamma)$ (see [Bo1]). It follows that Q, or indeed any neighbourhood of Q in $\mathbf{H}^n$ which is relatively compact in $\mathbf{H}^n \cup \Omega(\Gamma)$, meets only finitely many images of itself under Γ, and only finitely many of the sets C_i. Similarly, each C_i meets only finitely many images of Q, up to the action of $\varrho(G_i)$. There is no loss in assuming that each C_i is in fact a standard cusp region, and that if $i \neq j$ then C_i and C_j are strictly disjoint. It then follows, in fact, that there is some fixed $\varepsilon > 0$ such that if $i \neq j$, then $d_{\mathrm{hyp}}(C_i, C_j) \geq \varepsilon$.

There are further conditions we could impose on Q, similar to those in the cocompact case. Thus, by "thickening" it up a bit we can assume that ∂Q is properly ε-covered by the (finite) collection of those γQ and C_i which meet Q. Also, for each i, ∂C_i is "properly ε-covered" the set of those images of Q which meet ∂C_i. We are abusing terminology slightly here, since the parabolic fixed point p_i lies in the closure of ∂C_i, but is not contained in the interior of any set $\varrho(\gamma)\overline{Q}$. What we really mean can be expressed by saying that there is a set $P \subseteq \partial C_i$, whose images under $\varrho(G_i)$ cover ∂C_i and which is properly ε-covered by the set of those images of Q which meet P. Finally, we can assume that if Q

meets a given set $\varrho(\gamma)Q$ or C_i then the intersection contains a non-empty open set, whereas if these sets are disjoint, then they are strictly disjoint.

Suppose that $\varrho_0 \in \mathscr{R}_F$, and that $Q \subseteq \mathbf{H}^n$ and $(C_i)_{i\in\Pi}$ satisfy all the conditions described above (with $\varrho = \varrho_0$). Let $I = \Upsilon \sqcup \Pi$. Given $i \in I$, we define the convex set $A_i \subseteq \mathbf{H}^n$ by $A_i = C_i$ if $i \in \Pi$, and $A_i = \varrho_0\big(g(i)\big)Q$ if $i \in \Upsilon$. We define a graph (I, E) be letting $ij \in E$ if and only if $A_i \cap A_j \neq \emptyset$. Note that there is a natural Γ-action on this graph. Given $i \in I$, write $I(i) = \{j \in I \mid ij \in E,\ i \neq j\}$. Thus, if $i \in \Upsilon$, then $I(i)$ is finite. If $i \in \Pi$, then $I(i) \subseteq \Upsilon$, and $I(i)/G_i$ is finite. Also, I/Γ is finite, and so E/Γ is finite. Note that the sets $(A_i)_{i\in I}$ satisfy the hypotheses of Theorem 3.1.

The idea of the proof of Theorem 1.5 is to consider a continuous family $[t \mapsto \varrho_t]\colon T \longrightarrow \mathscr{R}$, of deformations of ϱ_0, such that $\Delta(\varrho_t) = \Delta(\varrho_0)$ for all t. We construct a family of continuous translations $[t \mapsto A_i(t)]$, so that $A_i(0) = A_i$, and $A_{\gamma i}(t) = \varrho_t(\gamma)A_i(t)$, for all γ, i and t. We verify that for all sufficiently small t, the collection $\big(A_i(t)\big)_{i\in I}$ satisfies the hypotheses of Theorem 3.1. It then follows by Proposition 4.4, that $\varrho_t \in \mathscr{R}_F$.

Given $p \in \partial\mathbf{H}^n$, we can find a hyperbolic isometry $\theta_p\colon \mathbf{H}^n \longrightarrow \mathbf{R}^n_+$ which extends to a map $\mathbf{H}^n \cup \partial\mathbf{H}^n \longrightarrow \mathbf{R}^n_+ \cup \partial\mathbf{R}^n_+ \cup \{\infty\}$ such that $\theta_p(p) = \infty$. For notational convenience, we shall choose a preferred point of $\mathbf{H}^n$ which we shall denote by ∞, and identify $\mathbf{H}^n$ and $\mathbf{R}^n_+$ in such a way that θ_∞ is the identity map. Moreover, we can assume that θ_p varies continuously in a neighbourhood of ∞. More precisely, there is a neighbourhood, U, of ∞ in $\partial\mathbf{H}^n$, such that the map $[(p, x) \mapsto \theta_p(x)]\colon U \times \mathbf{H}^n \longrightarrow \mathbf{H}^n$ is continuous.

Now, given points $p, q \in \partial\mathbf{H}^n$, we set $\theta_{p,q} = \theta_{\theta_p(q)} \circ \theta_p$. Thus, $\theta_{p,q}$ is an isometry of $\mathbf{H}^n$. Note that $\theta_{p,q}(q) = \theta_{\theta_p(q)}\big(\theta_p(q)\big) = \infty$. Also, $\theta_{\infty,q} = \theta_q$. Moreover, for a fixed p, $\theta_{p,q}$ varies continuously in q, as q varies over a neighbourhood of p (namely $\theta_p^{-1}U$).

There is a similar construction in euclidean space. Recall, from Section 2, that $\mathscr{L}(r, s) = \{(V, \tau) \mid V \in \mathscr{F}_r,\ \tau \in \mathscr{E}_s,\ V \subseteq D(\tau)\}$. We fix a "standard" element of $\mathscr{L}(r, s)$, namely $\lambda_0 = \lambda_0(r, s) = \big(D\big(\tau(r)\big), \tau(s)\big)$, where $\tau(k)$ is the standard k-dimensional subspace of $\mathbf{E}^{n-1}$ as defined earlier. Now, given any $\lambda \in \mathscr{L}(r, s)$, we can find $\phi_\lambda \in \operatorname{Isom}\mathbf{E}^{n-1}$ such that $\phi_\lambda(\lambda) = \lambda_0$. We can assume that ϕ_{λ_0} is the identity, and that ϕ_λ varies continuously on a neighbourhood of λ_0 in $\mathscr{L}(r, s)$. Given $\kappa, \lambda \in \mathscr{L}(r, s)$, we set $\phi_{\lambda,\kappa} = \phi_{\phi_\lambda(\kappa)} \circ \phi_\lambda$. Thus $\phi_{\lambda,\kappa}(\kappa) = \lambda_0$, $\phi_{\lambda_0,\kappa} = \phi_\kappa$, and $\phi_{\lambda,\kappa}$ varies continuously in κ in a neighbourhood of λ in $\mathscr{L}(r, s)$. We extend $\phi_{\lambda,\kappa}$ to a hyperbolic isometry fixing ∞.

Suppose that $[t \mapsto \varrho_t]\colon T \longrightarrow \mathscr{R}$ is a continuous family, with $\varrho_0 \in \mathscr{R}_F$ and $\Delta(\varrho_t) = \Delta(\varrho_0)$ for all $t \in T$. Let Q, $(C_i)_{i\in\Pi}$, (I, E) and $(A_i)_{i\in I}$ be as described earlier.

Now given $i \in \Pi$, $\varrho_t \mid G_i$ is parabolic. Let $p_i(t)$ be the fixed point of $\varrho_t(G_i)$. Note that $p_i(t)$ is the unique fixed point of $\varrho_t(\gamma)$ for some fixed $\gamma \in G_i$, and so we see that it varies continuously in t. Let $r_i = r(G_i)$ and $s_i = \Delta(\varrho_t)(i)$.

Let $\Pi_0 \subseteq \Pi$ be a transversal to the action of Γ on Π. Thus, Π_0 is finite. Given $i \in \Pi_0$, let $\theta_i(t) = \theta_{p_i(0),p_i(t)}$, and let $\varrho_t^{\theta_i(t)}$ be the representation ϱ_t conjugated by $\theta_i(t)$. Now $\varrho_t^{\theta_i(t)} \mid G_i$ is parabolic representation with fixed point $\theta_i(t)\bigl(p_i(t)\bigr) = \infty$. In other words, we can consider $\varrho_t^{\theta_i(t)}$ as an element of $\mathscr{S}(G_i, n-1)$. Let $\lambda_i(t) = \lambda(\varrho_t^{\theta_i(t)}) \in \mathscr{L}(r_i, s_i)$. By Proposition 2.10, $\lambda_i(t)$ is continuous in t. Let $\phi_i(t) = \phi_{\lambda_i(0),\lambda_i(t)}$, so that $\varrho_t^{(\phi_i(t)\circ\theta_i(t))} \mid G_i \in \mathscr{S}(G_i, n-1)$, and $\lambda\bigl(\varrho_t^{(\phi_i(t)\circ\theta_i(t))}\bigr) = \lambda(r_i, s_i)$.

There is a fixed hyperbolic isometry (or euclidean homothety), ψ_i such that $\psi_i \circ \phi_i(0) \circ \theta_i(0)(C_i)$ is the standard parabolic region $C(r_i)$ (in $\mathbf{R}^n_+$). Let $\beta_i(t) = \psi_i \circ \phi_i(t) \circ \theta_i(t)$. Thus, $\varrho_t^{\beta_i(t)} \mid G_i \in \mathscr{S}(G_i, n-1)$, and $\lambda(\varrho_t^{\beta_i(t)}) = \lambda_0$.

Each $i \in \Pi$ has the form γj for some $\gamma \in \Gamma$ and $j \in \Pi_0$. We choose one such γ for each $i \notin \Pi_0$. Now, $G_i = \gamma G_j \gamma^{-1}$, so $\varrho_t \mid G_i$ is parabolic with fixed point $\varrho_t(\gamma)\bigl(p_j(t)\bigr)$. In this case, we set $\beta_i(t) = \beta_j(t) \circ \varrho_t(\gamma^{-1})$. Again, we get $\varrho_t^{\beta_i(t)} \mid G_i \in \mathscr{S}(G_i, n-1)$.

For each $i \in \Pi$, set $C_i(t) = \beta_i(t)^{-1} C(r_i)$. Thus, $C_{\gamma i}(t) = \varrho_t(\gamma) C_i(t)$, and $C_i(t)$ is a standard cusp region for the parabolic representation $\varrho_t \mid G_i$. Also $C_i(0) = C_i$, and $C_i(t) = h_i(t) C_i(0)$, where $h_i(t) = \beta_i(t)^{-1}\beta_i(0)$. Thus, $[t \mapsto C_i(t)]$ is a continuous translation (for sufficiently small t).

Let $I = \Upsilon \sqcup \Pi$ be as above. We define the collection $\bigl(A_i(t)\bigr)_{i\in I}$ by setting $A_i(t) = C_i(t)$ for $i \in \Pi$ and $A_i(t) = \varrho_t\bigl(g(i)\bigr)Q$ for $i \in \Upsilon$. Thus, $\bigl(A_i(t)\bigr)_{i\in I}$ has all the properties outlined earlier. We want to show that it satisfies the hypotheses of Theorem 3.1 for all sufficiently small t.

Now hypothesis (A1) follows exactly as in the cocompact case, given that I/Γ and E/Γ are finite. Hypothesis (A3), in the case where $i \in \Upsilon$, also follows as in the cocompact case, given that $I(i)$ is finite. Hypothesis (A3), in the case where $i \in \Pi$, calls for some comment. Note that after conjugating the family of representations by a continuous family of elements of $\operatorname{Isom} \mathbf{H}^n$ (namely $\beta_i(t)$) we can suppose that $A_i(t) = C_i$ is constant. Now since $\varrho_t \mid G_i$ varies continuously, we can find a constant set $P \subseteq \partial C_i$ such that the images of P under each group $\varrho_t(G_i)$ cover C_i. Now P is properly ε-covered by some finite subset of the sets $A_j(0)$ for $j \in I(i) \subseteq \Upsilon$. We now apply Lemma 4.2 to show that it is properly $(\varepsilon/2)$-covered by the corresponding sets $A_j(t)$ for all sufficiently small t.

Again, property (A2) in the case where $i \in \Upsilon$ follows as in the cocompact case, given that $I(i)$ is finite. The case where $i \in \Pi$ requires a bit more work.

Again up to a continuous conjugacy, we can assume that $A_i(t)$ is constant, and equal to the standard cusp region $C(r_i)$ in $\mathbf{R}^n_+$. In particular, $\tau = \tau(r_i)$ is a crystallographic subspace. Let $\underline{0} \in \tau \subseteq \mathbf{E}^{n-1} \equiv \mathbf{R}^{n-1}$ be the origin, and let $N(\underline{0}, h)$ denote the euclidean h-neighbourhood of $\underline{0}$ in $\mathbf{R}^n_+ \cup \partial\mathbf{R}^n_+$.

Now, $I(i)/G_i$ is finite. Let $I_0 \subseteq I(i)$ be a (finite) transversal to the G_i-action. For each $j \in I_0$, the set $A_j(0)$ is relatively compact in $\mathbf{R}^n_+ \cup \partial\mathbf{R}^n_+$, and so $A_j(0) \subseteq N(\underline{0}, R_j)$ for some $R_j > 0$. Thus, for all sufficiently small t, we have $A_j(t) \subseteq N(\underline{0}, R_j + 1)$. Let $R = \max\{R_j + 1 \mid j \in I_0\}$. Thus, we can assume that $A_j(t) \subseteq N(\underline{0}, R)$ for all $j \in I_0$ and $t \in T$.

Let $H = \{\gamma \in \Gamma \mid d_{\mathrm{euc}}(\underline{0},\ \varrho_0(\gamma)\underline{0}) \leq 4R\}$. Applying Lemma 2.11, we see that for all $\gamma \in G_i$, and for all sufficiently small t, $d_{\mathrm{euc}}\big(\underline{0}, \varrho_t(\gamma)\underline{0}\big) \geq \frac{1}{2} d_{\mathrm{euc}}\big(\underline{0}, \varrho_0(\gamma)\underline{0}\big)$. In particular, for all $\gamma \in G_i \setminus H$, we have $d_{\mathrm{euc}}\big(\underline{0}, \varrho_t(\gamma)\underline{0}\big) \geq 2R$.

Suppose that $\gamma \in H$, $j, l \in I_0$, and $A_j(0) \cap A_{\gamma l}(0) = \emptyset$. For all sufficiently small t, we have $A_j(t) \cap A_{\gamma l}(t) = \emptyset$. Since there are only finitely many such j, l and γ, we can suppose this is true for all $t \in T$.

Now suppose that $j, k \in I(i)$ with $A_j(t) \cap A_k(t) \neq \emptyset$. Up to the action of G_i, we can suppose that $j \in I_0$ and that $k = \gamma l$, where $\gamma \in G_i$ and $l \in I_0$. Thus $A_k(t) = A_{\gamma l}(t) = \varrho_t(\gamma) A_l(t)$. Now, $A_j(t) \cap \varrho_t(\gamma) A_l(t) \neq \emptyset$ and $A_j(t), A_l(t) \subseteq N(\underline{0}, R)$ and so $d_{\mathrm{euc}}\big(\underline{0}, \varrho_t(\gamma)\underline{0}\big) \leq 2R$. Thus $\gamma \in H$. From the previous paragraph, we see that $A_j(0) \cap A_k(0) \neq \emptyset$, so $jk \in E$.

Since there are only finitely many such i up to the action Γ, this verifies hypothesis (A2). Thus all the hypotheses of Theorem 3.1 are satisfied for all sufficiently small t. It follows that $\big(A_i(t)\big)_{i \in I}$ forms a locally finite cover for $\mathbf{H}^n$, and that if $A_i(t) \cap A_j(t) \neq \emptyset$ then $ij \in E$. In particular, we see that the parabolic regions $C_i(t)$ and $C_j(t)$ for $i, j \in \Pi$ are disjoint unless $i = j$. Proposition 4.4 now tells us that $\varrho_t \in \mathscr{R}_F$ as required. This proves Theorem 1.5.

5. Examples

In this section we give examples of geometrically finite representations into $\operatorname{Isom} \mathbf{H}^4$ which are limits of non-geometrically finite representations. By Theorem 1.5, we know that this has to be the result of the canonical subspaces associated to the cusps jumping up in dimension in the limit.

In the first example, we describe type-preserving representations $\varrho_n \colon \mathbf{Z} * \mathbf{Z} \longrightarrow \operatorname{Isom} \mathbf{H}^4$, such that $\varrho_n \to \varrho$, where ϱ is faithful and geometrically finite without accidental parabolics, but where $\varrho_n(\mathbf{Z} * \mathbf{Z})$ is not discrete for any n.

In the second example, we describe type-preserving representations $\varrho_n \colon \mathbf{Z} * \mathbf{Z} * \mathbf{Z} \longrightarrow \operatorname{Isom} \mathbf{H}^4$, again converging to a faithful geometrically finite representation $\varrho \colon \mathbf{Z} * \mathbf{Z} * \mathbf{Z} \longrightarrow \operatorname{Isom} \mathbf{H}^4$. This time, each ϱ_n is discrete but not geometrically finite.

In each case, there is just one conjugacy class of peripheral subgroup, namely the set of conjugates of one of the free factors isomorphic to $\mathbf{Z}$.

To construct these representations, we shall need the following elementary combination lemmas. Given a discrete subgroup, $\Gamma \subseteq \operatorname{Isom} \mathbf{H}^4$, we write $\Omega(\Gamma) \subseteq \partial \mathbf{H}^4$ for the discontinuity domain. We write $M(\Gamma) = \big(\mathbf{H}^4 \cup \Omega(\Gamma)\big)/\Gamma$. If $Q \subseteq \Omega(\Gamma)$ we say that Q is *dispersed* by Γ if $Q \cap \gamma Q = \emptyset$ for all non-trivial $\gamma \in \Gamma$.

We shall say that a pair of round balls, $D_1, D_2 \in \partial \mathbf{H}^4$, are *complementary* if they meet along their common boundary $D_1 \cap D_2 = \partial D_1 = \partial D_2$.

Lemma 5.1. *Suppose that D_1 and D_2 are complementary round balls in $\partial \mathbf{H}^4$, and $\Gamma_1, \Gamma_2 \subseteq \mathrm{Isom}\,\mathbf{H}^4$ are discrete groups. Suppose that $D_2 \subseteq \Omega(\Gamma_1)$ and that $D_1 \subseteq \Omega(\Gamma_2)$, and are dispersed by Γ_1 and Γ_2 respectively. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle \cong \Gamma_1 * \Gamma_2$, and Γ is discrete. Moreover, if Γ_1 and Γ_2 are both geometrically finite, then so is Γ.*

Proof. Let H_i be the convex hull of D_i, so that H_i is a half-space in $\mathbf{H}^4 \cup \partial \mathbf{H}^4$. Now H_2 projects to a half space in $M(\Gamma_1)$, and we form the manifold M_1' by removing the interior of H_2 from $M(\Gamma_1)$. We construct M_2' similarly, and form a another manifold M by gluing together M_1' and M_2' along their common boundary (a copy of hyperbolic 3-space). If we do this right, then $M = \big(\mathbf{H}^n \cup \Omega(\Gamma)\big)/\Gamma$, so that Γ is discrete. Now, using Marden's description of geometrical finiteness [Mard] ("GF1" in [Bo1]), we see that if Γ_1 and Γ_2 are geometrically finite, then so is Γ. ◻

The following variation can be proved by a similar argument:

Lemma 5.2. *Suppose that $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathrm{Isom}\,\mathbf{H}^4$ are discrete, and that $D_2, D_3 \subseteq \Omega(\Gamma_1)$ are round balls. Suppose that D_2 and D_3 are both dispersed by Γ_1 and that their complements are dispersed by Γ_2 and Γ_3 respectively. Suppose that $D_2 \cap \gamma D_3 = \emptyset$ for all $\gamma \in \Gamma$. Then, $\Gamma = \langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle \cong \Gamma_1 * \Gamma_2 * \Gamma_3$, and Γ is discrete. Moreover, if Γ_1, Γ_2 and Γ_3 are all geometrically finite, then so is Γ.* ◻

Let $\mathbf{R}^4_+$ be the upper-half-space model for $\mathbf{H}^4$. We identify $\mathbf{E}^3 \equiv \mathbf{R}^3 \equiv \partial \mathbf{R}^4_+$. Given any $h > 0$, let X_h and Z_h be, respectively, the translations of $\mathbf{E}^3$ given by $[(x,y,z) \mapsto (x+h,y,z)]$ and $[(x,y,z) \mapsto (x,y,z+h)]$. Given $r > 0$, let $\alpha(r)$ be the line parallel to the z-axis given by $\alpha(y) = \{(0,r,z) \mid z \in \mathbf{R}\}$. If $r, h > 0$, let $T(r,\theta,h)$ be the "screw motion" on $\mathbf{E}^3$ given by a rotation through an angle of θ about the axis $\alpha(r)$ composed with the translation Z_h. Note that each these maps extend to a parabolic isometry of $\mathbf{H}^4$ fixing ∞, which we shall denote by the same symbols.

Suppose $\zeta, \xi > 0$ are constants (to be described later). For each $n \in \mathbf{N}$, let $T_n = T_n(\zeta,\xi) = T(\zeta n/2\pi, 2\pi/n, \xi/n)$. Thus $T_n^n = Z_\xi$. Also, as $n \to \infty$, the map T_n converges to X_ζ.

Let D be the ball of radius 1 about the origin, $\underline{0}$ in $\mathbf{E}^3$, and let D' be the complementary ball in $\partial \mathbf{H}^4$. We fix any $\zeta > 2$. Note that the ball D is dispersed by the cyclic parabolic group $\langle X_\zeta \rangle$.

First example. Choose $\varepsilon > 0$ to be some number less than the Margulis constant in dimension 4. Then there is some $\eta > 0$ such that if $\beta, \beta' \subseteq \mathbf{H}^4$ are two distinct loxodromic axes in a discrete subgroup of $\mathrm{Isom}\,\mathbf{H}^n$, which are each translated a distance less that ε by the corresponding loxodromic elements, then $d_{\mathrm{hyp}}(\beta, \beta') > \eta$.

Let $0 < t < 1$, and let $\beta \subseteq \mathbf{H}^4$ be the bi-infinite geodesic joining the ideal points $(-t,0,0)$ and $(t,0,0)$. Let $\gamma \in \mathrm{Isom}\,\mathbf{H}^4$ be a hyperbolic isometry which translates the axis β by a hyperbolic distance ε. (For definiteness, we can take $(t,0,0)$ to be the attracting fixed point, and assume that γ has trivial rotational part.) By choosing t small enough, we can ensure that the ball D' is dispersed by the group $\Gamma_2 = \langle \gamma \rangle$.

Let $\Gamma_1 = \langle X_\zeta \rangle$, so that D is dispersed by Γ_1. Thus, by Lemma 5.1, $\langle \Gamma_1, \Gamma_2 \rangle \cong \mathbf{Z} * \mathbf{Z}$ is geometrically finite.

Suppose $\xi > 0$. Now, $Z_\xi \beta$ is the bi-infinite geodesic joining the ideal points $(-t,0,\xi)$ and $(t,0,\xi)$. We fix ξ small enough so that $d_{\rm hyp}(\beta, Z_\xi \beta) < \eta$. Thus, the isometries $Z_\xi \gamma Z_\xi^{-1}$ generate a non-discrete group.

Let $\Gamma = \mathbf{Z} * \mathbf{Z}$ with free generators a and b. We choose a peripheral structure for Γ consisting of the cyclic subgroup $\langle a \rangle$ and all its conjugates in Γ. We define a type-preserving representation, $\varrho\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^4$, by setting $\varrho(a) = X_\zeta$ and $\varrho(b) = \gamma$. As described above, ϱ is faithful and geometrically finite.

Given $n \in \mathbf{N}$, we define the type-preserving representation $\varrho_n\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^4$ by setting $\varrho_n(a) = T_n = T_n(\zeta, \xi)$ and $\varrho_n(b) = \gamma$. Now $Z_\xi = T_n^n \in \varrho_n(\Gamma)$, and so $\gamma, Z_\xi \gamma Z_\xi^{-1} \in \varrho_n(\Gamma)$. Thus $\varrho_n(\Gamma)$ is not discrete.

We also have that $\varrho_n(a) = T_n$ converges to $X_\zeta = \varrho(b)$, and that $\varrho_n(b) = \varrho(b)$ for all n. Thus ϱ_n converges to ϱ.

Second example. For this we shall need the following fact [C]:

Lemma 5.3. *There exists a discrete subgroup of* $\operatorname{Isom} \mathbf{H}^3$ *isomorphic to* $\mathbf{Z} * \mathbf{Z}$, *which has no parabolics and which is not geometrically finite.*

Proof. (Sketch) There is a homeomorphism from $\mathscr{R} = \mathscr{R}(\mathbf{Z} * \mathbf{Z})$ onto $(\operatorname{Isom} \mathbf{H}^3)^2$ given by $\big[\varrho \mapsto \big(\varrho(a), \varrho(b)\big)\big]$, where a, b are free generators of $\mathbf{Z} * \mathbf{Z}$. (Here $\mathbf{Z} * \mathbf{Z}$ is taken to have empty peripheral structure, so that every representation is type-preserving.) Thus, $\mathscr{R}$ is a 12-manifold. If $\gamma \in \mathbf{Z} * \mathbf{Z}$, then $P(\gamma) = \{\varrho \in \mathscr{R} \mid \operatorname{tr} \varrho(\gamma) = \pm 2\}$ is a 10-dimensional subvariety of $\mathscr{R}$. Thus, $P = \bigcup_{\gamma \in \mathbf{Z}*\mathbf{Z}} P(\gamma)$ has Hausdorff dimension 10 (with respect to any Riemannian metric on $\mathscr{R}$).

Let $\mathscr{R}_F^0$ and $\mathscr{R}_D^0$ be, respectively, the sets of faithful geometrically finite and faithful discrete representations. Now, $\mathscr{R}_F^0$ and $\mathscr{R} \setminus \mathscr{R}_D^0$ are both non-empty and open in $\mathscr{R}$. It follows that the closed set $\mathscr{R}_D^0 \setminus \mathscr{R}_F^0$ has dimension at least 11, and so $\mathscr{R}_D^0 \setminus (\mathscr{R}_F^0 \cup P) \neq \emptyset$. The image of any representation in this set has the desired property. For further details, see [C]. ◻

We now embed $\mathbf{H}^3$ as a subspace of $\mathbf{H}^4$. Let $\gamma, \delta \in \operatorname{Isom} \mathbf{H}^3$ be free generators of a non geometrically finite group as described by Lemma 5.3, and extend this action to $\mathbf{H}^4$ as a 4-dimensional Fuchsian group, $\Gamma' = \langle \gamma, \delta \rangle$. Now, Γ' acts properly discontinuously on $\partial \mathbf{H}^4 \setminus \partial \mathbf{H}^3$, so we can find a round ball in $\partial \mathbf{H}^4 \setminus \partial \mathbf{H}^3$ which is dispersed by Γ'. After conjugating by a suitable element of $\operatorname{Isom} \mathbf{H}^4$, we can assume that D' is such a ball (i.e. the complementary ball to the unit ball, D, centred at the origin of $\mathbf{E}^3$).

Fix any $\zeta, \xi > 2$, and let $T_n = T_n(\zeta, \xi)$. For all sufficiently large n, we see that the ball D is dispersed by the group $\langle T_n \rangle$. It follows by Lemma 5.1, that $\langle \Gamma', T_n \rangle \cong \Gamma' * \langle T_n \rangle \cong \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$ is discrete (but not geometrically finite).

Suppose $\Gamma_1 = \langle X_\zeta \rangle$, $\Gamma_2 = \langle \gamma \rangle$, and $\Gamma_3 = \langle Z_\xi \delta Z_\xi^{-1} \rangle$. Thus, Γ_1, Γ_2 and Γ_3 are geometrically finite cyclic groups, with Γ_1 parabolic, and Γ_2, Γ_3 loxodromic. Now D is dispersed by Γ_1, and D' is dispersed by Γ_2 (since $\Gamma_2 \leq \Gamma'$). Since Z_ξ and X_ζ commute, $Z_\xi D$ is also dispersed by Γ_1. Moreover, $Z_\xi = T_n^n \in \Gamma'$, and so $\Gamma_3 \subseteq \Gamma' = Z_\xi \Gamma' Z_\xi^{-1}$. Thus, $Z_\xi D'$ is dispersed by Γ_3. We also note that $D \cap \gamma Z_\xi D = \emptyset$ for all $\gamma \in \Gamma_1$. Thus, the hypotheses of Lemma 5.2 are satisfied, so we see that $\langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle \cong \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$ is geometrically finite.

Let $\Gamma \cong \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$ with free generators a, b and c. We take a peripheral structure of the cyclic group $\langle a \rangle$ together with all its conjugates in Γ. We define a representation $\varrho\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^4$ by setting $\varrho(a) = X_\zeta$, $\varrho(b) = \gamma$ and $\varrho(c) = Z_\xi \delta Z_\xi^{-1}$. We see that ϱ is faithful, type preserving and geometrically finite without accidental parabolics.

Now, given $n \in \mathbf{N}$, we define a type-preserving representation $\varrho_n\colon \Gamma \longrightarrow \operatorname{Isom} \mathbf{H}^4$ by setting $\varrho_n(a) = T_n$, $\varrho_n(b) = \gamma$ and $\varrho_n(c) = Z_\xi \delta Z_\xi^{-1}$. Now, $Z_\xi = T_n^n$, and so $\delta = \varrho_n(a^{-n} c a^n) \in \varrho_n(\Gamma)$. Thus $\varrho_n(\Gamma) = \langle T_n, \gamma, \delta \rangle = \langle T_n, \Gamma_1 \rangle$. From the earlier discussion, we see that for all sufficiently large n, we have that ϱ_n is discrete and faithful, but not geometrically finite.

Finally note that as $n \to \infty$, $T_n \to X_\zeta$, and so ϱ_n converges to ϱ.

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