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CONJUGATIONS ON ROTATION DOMAINS AS LIMIT FUNCTIONS OF THE GEOMETRIC MEANS OF THE ITERATES

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Abstract. Let E be a region in $\mathbf{C}$ and $h\colon E \to E$ be a holomorphic function such that the family $\{h^n : n \in \mathbf{N}\}$ of iterates is normal, does not contain id_E, and only possesses non-constant limit functions. It is proved by elementary function theoretic arguments that the geometric means of the iterates converge to an injective holomorphic function ϕ which conjugates h to an irrational rotation. Subsequently it is outlined how to use this theorem to get a more elementary access to the classification theorem of periodic components of the Fatou set.

1. The main result and its applications to iteration theory

Recall that, for each closed rectifiable curve c in $\mathbf{C}$, its interior $\mathrm{int}(c)$ is defined to be the set of all $a \in \mathbf{C}$ which do not lie on the trace $\mathrm{tr}(c)$ of c and satisfy

$$n(c;a) := \frac{1}{2\pi i}\int_c \frac{1}{z-a}\,dz \neq 0.$$

For each region $D \subset \mathbf{C}$, let

$$\widehat{D} := \bigcup\{\mathrm{int}(c) : c \text{ is a closed rectifiable curve in } D\}.$$

The aim of this paper is to discuss and to prove the following result:

Theorem 1. *Let $E \subset \mathbf{C}$ be a region and $h\colon E \to E$ be holomorphic such that*
(1) *the set $\{h^n : n \in \mathbf{N}\}$ of iterates of h forms a normal family;*
(2) *for each $n \in \mathbf{N}$, $h^n \neq \mathrm{id}_E$;*
(3) *each function in the accumulation set of $(h^n)_{n\in\mathbf{N}}$ is not constant.*
Let $a \in \mathbf{C}$ such that
(4) *$a \in E$ and $h(a) = a$ if E is simply connected, and $a \in \widehat{E} \setminus E$ otherwise.*
Then there exists a sequence $(\phi_n)_{n\in\mathbf{N}}$ of holomorphic functions on E such that
(5) *for each $n \in \mathbf{N}$ and $z \in E$, $\phi_n(z)^n = \prod_{j=0}^{n-1}\bigl(h^j(z) - a\bigr)$;*
(6) *$(\phi_n)_{n\in\mathbf{N}}$ converges locally uniformly, $\phi := \lim_{n\to\infty}\phi_n$ is an injective holomorphic function, and there exists $\lambda \in \exp\bigl(2\pi i(\mathbf{R}\setminus\mathbf{Q})\bigr)$ such that $\phi\circ h = \lambda\phi$.*

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Roughly speaking, Theorem 1 says that, if conditions (1), (2), and (3) hold, then the holomorphic geometric means of the (centralized) iterates of h converge to an injective holomorphic function ϕ which conjugates h to an irrational rotation. Note that by Montel–Carathéodory's theorem condition (1) is satisfied if $\mathbf{C} \setminus E$ consists of at least two points.

Theorem 1 is of interest for iteration theory from two points of view. First of all, it includes the fact that, given conditions (1), (2), and (3), the function h is conjugated to an irrational rotation at all. To state this as a precise theorem (which we call the weak rotation theorem), we need a few more notations. Denote the extended complex plane by $\overline{\mathbf{C}}$. For each $a \in \mathbf{C}$ and $r \in]0,\infty]$, denote the open disk centered at a with radius r by $B(a;r)$. For each $a \in \mathbf{C}$, $r \in]0,\infty]$, $s \in]r,\infty]$, denote the open annulus centered at a with inner radius r and outer radius s by $A(a;r;s)$.

Theorem 2 (Weak Rotation Theorem). *Let $E \subset \overline{\mathbf{C}}$ be a region and $h\colon E \to E$ be holomorphic such that conditions* (1), (2), *and* (3) *hold. Then there exist $\lambda \in \exp\bigl(2\pi i(\mathbf{R} \setminus \mathbf{Q})\bigr)$ and*

$$G \in \{\overline{\mathbf{C}}\} \cup \{B(0;r) : r \in]0,\infty]\} \cup \{A(0;r;s) : 0 \le r < s \le \infty\}$$

such that h and $\lambda\,\mathrm{id}_G$ are conjugated. If E is simply connected then condition (3) *may be replaced by the condition*

(7) *there is a non-constant function in the accumulation set of $(h^n)_{n\in\mathbf{N}}$.*

Proof. For the case when E is multiply connected, it is evident that there exists an $a \in \mathbf{C}$ such that (4) holds. Hence in this case the weak rotation theorem is an easy consequence of Theorem 1. This is not so trivial for the case when E is simply connected. In this case one uses the Riemann mapping theorem, Proposition 1 (see Section 2), and statements about iteration of automorphisms of $\overline{\mathbf{C}}$, $\mathbf{C}$, and $B(0;1)$ to see that the conclusion holds even under the weaker condition (7) instead of (3). □

It is well known that the conclusion of the weak rotation theorem remains true even in the multiply connected case if condition (3) is replaced by (7). This result, which in the following is referred to as the strong rotation theorem, is due to Cremer [5]. Proofs can also be found in [1], [3], [8], and [9]. But to prove this stronger version, some fundamental knowledge of Riemann surfaces like the Riemann uniformization theorem is necessary.

The proof of Theorem 1 (and hence of the weak rotation theorem) given in Section 2 does not require such fundamental theorems and is much more elementary, as far as the choice of tools is concerned. As outlined in Section 3, the weak form of the rotation theorem suffices to deduce the classification theorem of periodic components of the Fatou set. In this way Theorem 1 yields a more elementary access to this fundamental theorem in iteration theory.

The second aspect of Theorem 1 is that the conjugating function on Siegel disks and Herman rings can always be constructed as the limit of the holomorphic geometric means of the iterates. Before, it was only known that the conjugating function on Siegel disks can be constructed as the limit of the arithmetic means of the iterates, namely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{h^j - a}{\lambda^j},$$

where λ denotes the eigenvalue of the fixed point a of h (see for instance [9, p. 81]). It is unknown whether the same is true for Herman rings.

Note that the construction by the geometric means does not require any knowledge of the rotation number λ.

2. Proof of Theorem 1

2.1. Structure of the proof and notations. The proof of Theorem 1 falls into five parts. The first step is to transform condition (3) into a sharper condition on the accumulation set of $(h^n)_{n\in\mathbf{N}}$. This is done in Section 2.2. In Section 2.3 we use this result to reduce the proof of Theorem 1 to the case when E is multiply connected. These are the preliminary parts of the proof.

The main problem in the proof of Theorem 1 is to show that the holomorphic geometric means of the iterates exist. This is solved by transition to the logarithmic derivatives. The logarithmic derivatives of the geometric means of the iterates are the arithmetic means of the logarithmic derivatives of the iterates, which naturally exist. Following this concept we shall prove the following lemma, which represents the main step in the proof of Theorem 1, in Section 2.4:

Main Lemma 1. *Let E, h, and a be as in Theorem 1. Suppose that E is multiply connected. For each $n \in \mathbf{N}$, let*

$$s_n\colon E \to \mathbf{C}, \qquad z \mapsto \frac{1}{n} \sum_{j=0}^{n-1} \frac{(h^j)'(z)}{h^j(z) - a}.$$

Then $(s_n)_{n\in\mathbf{N}}$ converges locally uniformly and $s := \lim_{n\to\infty} s_n$ is not constant and holomorphic such that

(8) $(s \circ h)h' = s$;

(9) *for each closed rectifiable curve d in E, $\int_d s(z)\,dz = 2\pi i n(d;a)$.*

As explained in Section 2.6, one easily concludes from Main Lemma 1 that there is a normal sequence $(\phi_n)_{n\in\mathbf{N}}$ such that (5) holds and, for each limit function ϕ of $(\phi_n)_{n\in\mathbf{N}}$, there exists a $\lambda \in \exp\big(2\pi i(\mathbf{R}\setminus\mathbf{Q})\big)$ satisfying $\phi\circ h = \lambda\phi$.

The second problem in the proof of Theorem 1 is to show that each function ϕ in the accumulation set of $(\phi_n)_{n\in\mathbf{N}}$ is injective. To this end we shall prove in Section 2.5:

Main Lemma 2. *Let $E \subset \mathbf{C}$ be a multiply connected region and $h\colon E \to E$ be a holomorphic function such that* (1) *and* (3) *hold. Let $\phi\colon E \to \mathbf{C}\backslash\{0\}$ be holomorphic and $\lambda \in \exp\bigl(2\pi i(\mathbf{R} \setminus \mathbf{Q})\bigr)$ such that $\phi \circ h = \lambda\phi$. Then ϕ is locally injective and there exists $k \in \mathbf{N}$ such that, for each $w \in \phi(E)$, $\operatorname{card}\bigl(\phi^{-1}(w)\bigr) = k$.*

In our situation Main Lemma 2 means that, for each function ϕ in the accumulation set of $(\phi_n)_{n\in\mathbf{N}}$, (E, ϕ) is a covering space of $\phi(E)$. This enables us to apply the abstract monodromy theorem and, as we shall see in Section 2.6, to prove that ϕ is injective.

Finally, we have to show that the sequence $(\phi_n)_{n\in\mathbf{N}}$ of geometric means of the iterates, which is uniquely determined up to roots of unity, can be chosen in such a way that $(\phi_n)_{n\in\mathbf{N}}$ itself is convergent. As explained in Section 2.6, this is settled by the following theorem, which is known as Weyl's equidistribution theorem:

Theorem 3 (Weyl's Equidistribution Theorem). *Let $f\colon \exp(i\mathbf{R}) \to \mathbf{R}$ be a continuous function and $\lambda \in \exp\bigl(2\pi i(\mathbf{R} \setminus \mathbf{Q})\bigr)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\lambda^j) = \int_0^1 f\bigl(\exp(2\pi i t)\bigr)\, dt.$$

The proof of Theorem 3 is elementary (see for instance Koerner [7, p. 11]). We complete the proof of Theorem 1 in Section 2.6 by combining the various lemmas as indicated.

We shall make use of the following notations. For all regions $E, G \subset \overline{\mathbf{C}}$, denote the set of holomorphic functions between the Riemann surfaces E and G by $\mathscr{H}(E;G)$. Note that, for each region $E \subset \mathbf{C}$, the meromorphic functions on E belong to $\mathscr{H}(E;\overline{\mathbf{C}})$. Moreover, for each region $E \subset \overline{\mathbf{C}}$ and each $h \in \mathscr{H}(E;E)$, denote the family $\{h^n : n \in \mathbf{N}\}$ of iterates of h by $\Phi(h)$ and the accumulation set of $(h^n)_{n\in\mathbf{N}}$ in $\mathscr{H}(E;\overline{\mathbf{C}})$ by $\mathscr{G}(h)$. Furthermore $\mathscr{N}(h)$ is defined to be the set of all $g \in \mathscr{G}(h)$ which are not constant. Finally, for each region $E \subset \overline{\mathbf{C}}$, denote the group of biholomorphic self-maps of E by $\operatorname{Aut}(E)$.

2.2. Transformation of condition (3). The first step is to transform condition (3) into a sharper condition on the accumulation set of $(h^n)_{n\in\mathbf{N}}$:

Proposition 1. *Let $E \subset \overline{\mathbf{C}}$ be a region and $h \in \mathscr{H}(E;E)$ such that conditions* (1) *and* (7) *hold. Then $\mathscr{N}(h)$ is a commutative subgroup of $\operatorname{Aut}(E)$ and $\Phi(h) \subset \mathscr{N}(h)$.*

In principle this result is due to Fatou [6]. The proof is elementary and for instance also given in [1, p. 163]. The following consequences of this proposition will be used later:

Lemma 3. *Let $E \subset \overline{\mathbf{C}}$ be a region and $h \in \mathscr{H}(E;E)$ such that conditions* (1), (2), *and* (3) *hold. Let $\phi \in \mathscr{H}(E;\overline{\mathbf{C}})$ such that $\phi \circ h^k = \phi$ for some $k \in \mathbf{N}$. Then ϕ is constant.*

Proof per contraposition. Suppose that conditions (1) and (3) hold and ϕ is not constant. Then there exist $x \in E$ and a region $U \subset E$ such that $x \in U$ and $\phi|_U$ is injective. By Proposition 1 we have $\mathrm{id}_E \in \mathscr{G}(h^k)$. Hence we find an open neighbourhood W of x in U and $n \in \mathbf{N}$ such that $h^{nk}(W) \subset U$. Now the functional equation $\phi \circ h^k = \phi$ in particular implies that $\phi \circ h^{nk}|_W = \phi|_W$. Since $\phi|_U$ is injective, we obtain that $h^{nk}|_W = \mathrm{id}_W$, which implies that $h^{nk} = \mathrm{id}_E$ by the identity theorem. □

Lemma 4. *Let $E \subset \mathbf{C}$ be a region and $h \in \mathscr{H}(E;E)$ such that conditions (1) and (3) hold. Let $a \in \mathbf{C} \setminus E$. Then*

(i) *for each compact subset $K \subset E$, the set $\bigcup_{g \in \mathscr{G}(h)} g(K)$ is compact;*

(ii) *the family*

$$\left\{ g - a, \frac{1}{g-a}, g', \frac{1}{g'} : g \in \mathscr{G}(h) \right\}$$

is locally bounded.

Proof. Part (i) follows from the compactness of $\mathscr{G}(h)$ and the continuity of the map $\mathscr{H}(E;\mathbf{C}) \times E \to \mathbf{C}, (\phi, z) \mapsto \phi(z)$. Part (ii) also uses Proposition 1 and the continuity of the map $\mathscr{H}(E;\mathbf{C}) \to \mathscr{H}(E;\mathbf{C})$, $\phi \mapsto \phi'$ as well as of the map $\mathscr{H}(E;\mathbf{C}) \times \mathscr{H}(E;\mathbf{C}\setminus\{0\}) \to \mathscr{H}(E;\mathbf{C})$, $(\phi, \psi) \mapsto \phi/\psi$. □

Lemma 5. *Let $E \subset \mathbf{C}$ be a region and $h \in \mathscr{H}(E;E)$ such that conditions (1) and (3) hold. Let $a \in \mathbf{C} \setminus E$. Let M be a finite set of closed rectifiable curves in E. Then the set*

$$Q(M;a) := \{ j \in \mathbf{N}; \textit{for all } n \in \mathbf{N} \textit{ for all } d \in M : n(h^{jn} \circ d; a) = n(d;a) \}$$

is not empty.

Proof. Denote the length of a closed rectifiable curve d in E by $L(d)$. Since M is finite, Lemma 4 implies that $K := \bigcup_{g \in \mathscr{G}(h)} \bigcup_{d \in M} g\big(\mathrm{tr}(d)\big)$ is compact. So by Lemma 4:

$$s := \sup\{|g'(z)| : z \in K,\ g \in \mathscr{G}(h)\} \max\{L(c) : c \in M\} < \infty.$$

By Proposition 1 we find a strictly increasing $\alpha \in \mathbf{N}^{\mathbf{N}}$ such that $\mathrm{id}_E = \lim_{n \to \infty} h^{\alpha(n)}$. For each $n \in \mathbf{N}$, let $\psi_n := (h^{\alpha(n)})'/(h^{\alpha(n)} - a)$. Then there exists $n \in \mathbf{N}$ such that

$$\sup\left\{ \left| \psi_n(z) - \frac{1}{z-a} \right| : z \in K \right\} < \frac{2\pi}{s}.$$

Now let $d \in M$. For each $g \in \mathscr{G}(h)$,

$$\begin{aligned}
2\pi\left|n(h^{\alpha(n)} \circ g \circ d; a) - n(g \circ d; a)\right| &= \left|\int_{h^{\alpha(n)}\circ g\circ d} \frac{1}{z-a}\, dz - \int_{g\circ d} \frac{1}{z-a}\, dz\right| \\
&= \left|\int_{g\circ d} \psi_n(z) - \frac{1}{z-a}\, dz\right| \\
&\le \sup\left\{\left|\psi_n(z) - \frac{1}{z-a}\right| : z \in \operatorname{tr}(g\circ d)\right\} L(g \circ d) \\
&\le \sup\left\{\left|\psi_n(z) - \frac{1}{z-a}\right| : z \in K\right\} \sup\{|g'(z)| : z \in \operatorname{tr}(d)\} L(d) \\
&\le \sup\left\{\left|\psi_n(z) - \frac{1}{z-a}\right| : z \in K\right\} s < 2\pi.
\end{aligned}$$

Since $n(h^{\alpha(n)} \circ g \circ d; a) - n(g \circ d; a) \in \mathbf{Z}$, we conclude that $n(h^{\alpha(n)} \circ g \circ d; a) = n(g \circ d; a)$, for each $g \in \mathscr{G}(h)$. Since we know from Proposition 1 that $h^{\alpha(n)} \in \mathscr{G}(h)$, it follows by induction that $\alpha(n) \in Q(M; a)$. □

2.3. Reduction to the multiply connected case. As a further consequence of Proposition 1 we obtain the

Reduction of the proof of Theorem 1 *to the multiply connected case.* Suppose that E is simply connected. Then condition (4) implies that $h(a) = a$. The region $F := E \setminus \{a\}$ is multiply connected and since Proposition 1 implies that $h \in \operatorname{Aut}(E)$, we conclude that $f := h|_F \in \operatorname{Aut}(F)$. In particular, conditions (1), (2), and (3) hold for (f, F) instead of (h, E). Now using Riemann's removability theorem and Hurwitz's theorem it is easy to prove that the statement of Theorem 1 for (f, F) implies the statement for (h, E). □

2.4. Proof of Main Lemma 1. Let E, h, and a be as in Theorem 1. Suppose that E is multiply connected. For each $j, n \in \mathbf{N}$, let

$$s_{j,n} := \frac{1}{n} \sum_{i=0}^{n-1} \frac{(h^{ji})'}{h^{ji} - a}.$$

We have to prove that $s := \lim_{n\to\infty} s_{1,n}$ exists and satisfies (8) and (9). As a first step to prove convergence and verify condition (8) we shall prove that

(10) $\Psi := \{s_{j,n} : j, n \in \mathbf{N}\}$ is a normal family and $(\sigma \circ h^{jk})(h^{jk})' = \sigma$ for each $j \in \mathbf{N}$, $k \in \mathbf{N}$, and σ in the accumulation set Σ_j of $(s_{j,n})_{n\in\mathbf{N}}$.

Proof of (10). By Proposition 1 we have

$$\left\{\frac{(h^k)'}{h^k - a} : k \in \mathbf{N}_0\right\} \subset \left\{\frac{g'}{g-a} : g \in \mathscr{G}(h)\right\},$$

which by Lemma 4 implies that $\{(h^k)'/(h^k - a) : k \in \mathbf{N}_0\}$ is locally bounded. Using the triangle inequality we see that Ψ is locally bounded, so that by Montel's theorem Ψ is normal and $\overline{\Psi} \subset \mathscr{H}(E;\mathbf{C})$. Now, for each $j \in \mathbf{N}$, define Σ_j to be the accumulation set of $(s_{j,n})_{n\in\mathbf{N}}$ in $\mathscr{H}(E;\mathbf{C})$. Applying Lemma 4 again it is easy to prove that, for each $j \in \mathbf{N}$ and $z \in E$,

$$\lim_{n\to\infty} \left| s_{j,n}\bigl(h^j(z)\bigr)(h^j)'(z) - s_{j,n}(z) \right| = 0.$$

For each $j \in \mathbf{N}$ and $\sigma \in \Sigma_j$, this implies $(\sigma \circ h^j)(h^j)' = \sigma$, which by induction leads to $(\sigma \circ h^{jn})(h^{jn})' = \sigma$, for each $n \in \mathbf{N}$. □

We see from Lemma 5 that, for each closed rectifiable curve d in E, the set

$$Q_d := \{j \in \mathbf{N}; \text{for all } n \in \mathbf{N} : n(h^{jn} \circ d; a) = n(d;a)\}$$

is not empty. As a first step to verify condition (9) we prove

(11) for each closed rectifiable curve d in E and each $j \in Q_d$ and $\sigma \in \Sigma_j$,

$$\int_d \sigma(z)\, dz = 2\pi i n(d;a).$$

Proof of (11). Let d be a closed rectifiable curve in E. Let $j \in Q_d$ and $\sigma \in \Sigma_j$. For each $n \in \mathbf{N}$,

$$\begin{aligned}\int_d s_{j,n}(z)\, dz &= \frac{1}{n}\sum_{k=0}^{n-1} \int_d \frac{(h^{jk})'(z)}{h^{jk}(z) - a}\, dz = \frac{1}{n}\sum_{k=0}^{n-1} \int_{h^{jk}\circ d} \frac{1}{z-a}\, dz \\ &= \frac{1}{n}\sum_{k=0}^{n-1} 2\pi i n(h^{jk} \circ d; a) = \frac{1}{n}\sum_{k=0}^{n-1} 2\pi i n(d;a) = 2\pi i n(d;a).\end{aligned}$$

Since the map $\mathscr{H}(E;\mathbf{C}) \to \mathbf{C}$, $\phi \mapsto \int_d \phi(z)\, dz$ is continuous, we conclude that $\int_d \sigma\, dz = 2\pi i n(d;a)$. □

Because $a \in \widehat{E} \setminus E$, we find a closed rectifiable curve c in E such that $n(c;a) \neq 0$. Now (11) implies that

(12) for each $j \in Q_c$, each function in Σ_j is not constant.

Using Lemma 3 we can show that

(13) for each $j, m \in Q_c$ and $\sigma \in \Sigma_j$ and $\tau \in \Sigma_m$, $\sigma = \tau$.

Proof of (13). Let $j, m \in Q_c$ and $\sigma \in \Sigma_j$ and $\tau \in \Sigma_m$. Applying (12) we see that $g := \sigma/\tau$ is a meromorphic function and hence belongs to $\mathscr{H}(E;\overline{\mathbf{C}})$. Let $n := jm$. Applying (10) we have $(\sigma \circ h^n)(h^n)' = \sigma$ as well as $(\tau \circ h^n)(h^n)' = \tau$ so that $g \circ h^n = g$ holds. Using Lemma 3 we find $u \in \mathbf{C}$ such that $\sigma = u\tau$. Applying (11) we see that

$$u 2\pi i n(c;a) = u \int_c \tau(z)\, dz = \int_c \sigma(z)\, dz = 2\pi i n(c;a)$$

so that $n(c;a) \neq 0$ implies that $u = 1$. □

Now fix $j_0 \in Q_c$ and $\sigma_0 \in \Sigma_{j_0}$ and define $s := \sigma_0$. From (12) we know that s is not constant. From (13) we conclude that

(14) for each $j \in Q_c$, $s = \lim_{n\to\infty} s_{j,n}$.

Now it remains to show (8) and (9). To see this, note that, for each $k \in \mathbf{N}$, conditions (8) and (9) imply that

$$2\pi i n(h^k \circ c; a) = \int_{h^k\circ c} s(z)\,dz = \int_c s(h^k(z))(h^k)'(z)\,dz = \int_c s(z)\,dz = 2\pi i n(c; a),$$

which means that $1 \in Q_c$ and because of (14) implies that $s = \lim_{n\to\infty} s_{1,n}$.

To prove (9), let d be a closed rectifiable curve in E. By Lemma 5 there exists $m \in Q_c \cap Q_d$. Because $s \in \Sigma_m$, from (11) we obtain

$$\int_d s(z)\,dz = 2\pi i n(d; a).$$

To prove (8), let $u := (s \circ h)h'$. Choose an element $j \in Q_c$. Because $s \in \Sigma_j$, from (10) we obtain, for each $z \in E$, that

$$\begin{aligned} u\bigl(h^j(z)\bigr)(h^j)'(z) &= s\bigl(h^{j+1}(z)\bigr)h'\bigl(h^j(z)\bigr)(h^j)'(z) = s\bigl(h^j\bigl(h(z)\bigr)\bigr)(h^{j+1})'(z) \\ &= s\bigl(h^j\bigl(h(z)\bigr)\bigr)(h^j)'(h(z))h'(z) = s\bigl(h(z)\bigr)h'(z) = u(z). \end{aligned}$$

Since s is not constant, $g := u/s$ is a meromorphic function and hence belongs to $\mathscr{H}(E;\overline{\mathbf{C}})$. Since $(s \circ h^j)(h^j)' = s$, we have that $g \circ h^j = g$. By Lemma 3 we find an element $\nu \in \mathbf{C}$ such that $(s \circ h)h' = u = \nu s$. By induction this implies that $(s \circ h^n)(h^n)' = \nu^n s$, for each $n \in \mathbf{N}$. In particular, we have that $\nu^j s = (s \circ h^j)(h^j)' = s$ and obtain that $\nu^j = 1$. On the other hand, applying (9) we see that

$$\begin{aligned} 2\pi i n(h \circ c; a) &= \int_{h\circ c} s(z)\,dz = \int_c s\bigl(h(z)\bigr)h'(z)\,dz = \int_c \nu s(z)\,dz \\ &= \nu \int_c s(z)\,dz = \nu 2\pi i n(c; a), \end{aligned}$$

which leads to $\nu = \bigl(n(h \circ c; a)/n(c; a)\bigr) \in \mathbf{Q}$. Thus we conclude that $\nu \in \{-1, 1\}$, and it remains to show that $\nu \neq -1$. To this end note that, for each closed rectifiable curve d in E, we know from (9) that

$$\int_d s(z)\,dz \in 2\pi i \mathbf{Z}.$$

It is a well-known fact that this implies the existence of a holomorphic function $\phi\colon E \to \mathbf{C}\setminus\{0\}$ such that $s = \phi'/\phi$. Since s is not constant, ϕ is not constant either. Hence by Lemma 3 we obtain that $\phi \circ h^2 \neq \phi$. In terms of the function $v := \phi(\phi \circ h)$ this means that $(v \circ h/v) = (\phi \circ h^2/\phi) \neq 1$. This implies that v is not constant and there exists $z \in E$ such that $v'(z) \neq 0$. Thus we see that

$$\begin{aligned} 0 \neq v'(z) &= \phi'(z)\phi\bigl(h(z)\bigr) + \phi(z)\phi'\bigl(h(z)\bigr)h'(z) = s(z)v(z) + s\bigl(h(z)\bigr)h'(z)v(z) \\ &= v(z)\bigl(s(z) + s\bigl(h(z)\bigr)h'(z)\bigr) = v(z)s(z)(1 + \nu) \end{aligned}$$

and conclude that $\nu \neq -1$. □

2.5. Proof of Main Lemma 2. Let $E \subset \mathbf{C}$ be a multiply connected region and $h \in \mathscr{H}(E;E)$ such that (1) and (3) hold. Let $\phi\colon E \to \mathbf{C}\setminus\{0\}$ be a holomorphic function and $\lambda \in \exp\bigl(2\pi i(\mathbf{R}\setminus\mathbf{Q})\bigr)$ such that $\phi \circ h = \lambda\phi$.

We first prove that ϕ is locally injective:

Proof of local injectivity of ϕ. Because $\phi(E) \subset \mathbf{C}\setminus\{0\}$ and $\lambda \neq 1$ we know that ϕ is not constant. Fix $x \in E$. By induction and differentiation we get, for each $n \in \mathbf{N}$,

$$\phi'\bigl(h^n(x)\bigr)(h^n)'(x) = \lambda^n\phi'(x).$$

By Proposition 1 $\{h^n : n \in \mathbf{N}\} \subset \operatorname{Aut}(E)$ and $x \in \overline{\{h^n(x) : n \in \mathbf{N}\} \setminus \{x\}}$. Now assume that $\phi'(x) = 0$. Then the identity theorem yields that $\phi' = 0$, which implies that ϕ is constant. This is a contradiction. □

However, it is much more difficult to prove that ϕ is a proper mapping of finite degree. The proof is based on the following abstract theorem, which will be proved first:

Theorem 4. *Let X be a connected Hausdorff space. Let $g\colon X \to \mathbf{R}$ be a continuous, open, and locally injective mapping. Then g is injective.*

Proof. The first step is to show that

(15) for all regions $U, D \subset X$ satisfying $U \cap D \neq \emptyset$,

$$g|_U \text{ and } g|_D \text{ are injective } \implies g|_{U\cup D} \text{ is injective.}$$

Proof of (15) *by contraposition.* Let $U, D \subset X$ be regions such that $U\cap D \neq \emptyset$ and $g|_{U\cup D}$ is not injective and $g|_U$ is injective. By assumption $g(U)$ and $g(D)$ and so $Z := g(U) \cap g(D)$ are open intervals in $\mathbf{R}$. Since $I := (g|_U)^{-1}$ is open and continuous, $F := I(Z)$ is a region, too. Because $g(U \cap D) \subset Z$, we see that $U \cap D \subset F$ and so $F \cap D \neq \emptyset$. Let $W := U \cup D$.

Case 1. $F \subset D$. Since $g|_W$ is not injective and $g|_U$ is injective, we find $w \in W$ and $v \in D$ such that $v \neq w$ and $g(v) = g(w)$. Since $w \in U$ implies that $w = I\bigl(g(w)\bigr) \in I(Z) = F \subset D$, we have that $w \in D$ and conclude that $g|_D$ is not injective.

Case 2. $F \setminus D \neq \emptyset$. Since F is connected and $F \cap D \neq \emptyset$, there exists $s \in F \cap \partial D$. Because $g(F) = Z \subset g(D)$ there exists $t \in D$ such that $g(t) = g(s)$. Since X is a Hausdorff space and g is open and continuous, there are neighbourhoods S of s in $F \subset U$ and T of t in D such that $S \cap T = \emptyset$ and $g(T) = g(T) \cap g(S) \subset g(D) \cap g(U) = Z$. Hence $I\bigl(g(T)\bigr)$ is a neighbourhood of s in S. Since $s \in \partial D$, we find an element $w \in D \cap I\bigl(g(T)\bigr)$. We choose $v \in T$ such that $w = I\bigl(g(v)\bigr)$ and note that $v, w \in D$ and $v \neq w$ and $g(v) = g(w)$. □

Now fix $z, w \in X$ such that $z \neq w$. Let

$$\mathscr{Q} := \{D \subset X : D \text{ is a region such that } z \in D \text{ and } g|_D \text{ is injective}\}.$$

Since g is locally injective, applying (15) it is easy to show that $G := \bigcup_{D\in\mathscr{Q}} D$ is a non-empty, open, and closed subset of X. Since X is connected, we conclude that $G = X$. Hence we can find a region $D \in \mathscr{Q}$ such that $w \in D$. Since $g|_D$ is injective, we conclude that $g(w) \neq g(z)$. □

In the situation of Main Lemma 2 this abstract theorem leads to the following consequence.

Lemma 6. *In the situation of Main Lemma 2, for all $x, y \in E$,*

$$|\phi(x)| = |\phi(y)| \Longrightarrow y \in \{g(x) : g \in \mathscr{G}(h)\}.$$

Proof. First of all using Proposition 1 one can easily check that

$$x \sim y :\Longleftrightarrow \exists g \in \mathscr{G}(h) : g(x) = y$$

defines an equivalence relation on E such that, for each $x \in E$,

(16) $[x] := \{y \in E : x \sim y\}$ is compact and $\phi([x]) = |\phi(x)| \exp(i\mathbf{R})$.

Hence the map

$$H\colon E/{\sim} \to \mathbf{R}, \qquad [x] \mapsto |\phi(x)|$$

is well defined, and we have to show that H is injective. To this end let

$$\iota\colon E \to E/{\sim}, \qquad x \mapsto [x]$$

and denote the final topology on $E/{\sim}$ with respect to ι by τ. It is clear that ι is continuous and hence $E/{\sim}$ is connected. Since ϕ is open and continuous without zeros, we conclude that $H \circ \iota = |\phi|$ is open and continuous. Hence we obtain that H is open and continuous. In view of Theorem 4 it remains to show that $E/{\sim}$ is a Hausdorff space and H is locally injective.

First of all one can easily prove that ι is open. To realize the Hausdorff property of $E/{\sim}$, fix $x, y \in E$ such that $[x] \neq [y]$. Since $[y]$ is compact, we find an open and relatively compact neighbourhood K of x in E such that $\overline{K} \cap [y] = \emptyset$. Now $[y]$ and the set $M := \bigcup_{g\in\mathscr{G}(h)} g(K)$, which is compact by Lemma 4, are disjoint, too. Hence we find an open and relatively compact neighbourhood V of y in E such that $M \cap \overline{V} = \emptyset$. Since ι is open, we conclude that $X := \iota(K)$ and $Y := \iota(V)$ are disjoint neighbourhoods of $[x]$ and $[y]$ in $E/{\sim}$.

To prove the local injectivity of H, let $x \in E$. Since ϕ is locally injective, we find a neighbourhood U of x in E and $\varepsilon \in]0, |\phi(x)|[$ such that

$$\phi_U\colon U \to B\big(\phi(x); \varepsilon\big), \qquad z \mapsto \phi(z)$$

is biholomorphic. Define $W := \iota(U)$ and fix $z, y \in U$ such that $H([z]) = H([y])$. Then we have that

$$\{y, z\} \subset Z := \phi_U^{-1}\Big(B\big(\phi(x); \varepsilon\big) \cap \partial B\big(0; |\phi(z)|\big)\Big).$$

Since Z is connected, it remains to show that the non-empty set $A := \{w \in Z : [w] = [z]\}$, which is closed with respect to Z by continuity of ι, is open with respect to Z. So let $v \in \overline{Z \setminus A} \cap Z$. Then we find a sequence $(v_n)_{n\in\mathbf{N}}$ in $Z \setminus A$ such that $v = \lim_{n\to\infty} v_n$.

Assumption. For each $n \in \mathbf{N}$, $[v] \neq [v_n]$. We may assume that $[v_i] \neq [v_j]$ if $i \neq j$. Now, for each $n \in \mathbf{N}$, we know that $|\phi(v)| = |\phi(v_n)|$, which by (16) implies that

$$\phi(v) \in |\phi(v)| \exp(i\mathbf{R}) = \phi([v_n]).$$

Hence, for each $n \in \mathbf{N}$, we find $w_n \in [v_n]$ such that $\phi(w_n) = \phi(v)$. We note that

$$S := \{w_n : n \in \mathbf{N}\} \subset M := \bigcup_{g\in\mathscr{G}(h)} g\bigl(\{v_n : n \in \mathbf{N}\} \cup \{v\}\bigr).$$

Since M is compact by Lemma 4, we conclude that S has an accumulation point in $M \subset E$. Since $\phi|_S$ is constant, we conclude by the identity theorem that ϕ is constant and obtain a contradiction.

Thus there exists $n \in \mathbf{N}$ such that $[v] = [v_n] \neq [z]$. Hence $v \in Z \setminus A$ and $Z \setminus A$ is closed with respect to Z. □

Completion of the proof of Main Lemma 2. Define M to be the set of all $g \in \mathscr{G}(h)$ for which $\phi \circ g = \phi$ holds. Fix $x \in E$. Let $y \in E$ such that $\phi(x) = \phi(y)$. By Lemma 6 we find $g \in \mathscr{G}(h)$ such that $y = g(x)$. Since $\mathscr{G}(h)$ is commutative, we conclude, for each $n \in \mathbf{N}$, that

$$\phi\bigl(g\bigl(h^n(x)\bigr)\bigr) = \phi\bigl(h^n\bigl(g(x)\bigr)\bigr) = \lambda^n \phi\bigl(g(x)\bigr) = \lambda^n \phi(y) = \lambda^n \phi(x) = \phi\bigl(h^n(x)\bigr).$$

Now the identity theorem implies that $g \in M$. Hence the map

$$\alpha\colon M \to \phi^{-1}\bigl(\phi(x)\bigr), \qquad g \mapsto g(x)$$

is surjective. Using the commutativity of $\mathscr{G}(h)$ and the identity theorem again we get that α is injective. Since $\phi^{-1}\bigl(\phi(x)\bigr) = \{g(x) : g \in M\}$ is a closed subset of the compact set $\bigcup_{g\in\mathscr{G}(h)}\{g(x)\}$, we conclude that $\phi^{-1}\bigl(\phi(x)\bigr)$ is compact, too. By the identity theorem this means that $\phi^{-1}\bigl(\phi(x)\bigr)$ and hence M is finite. Now the conclusion follows with $k := \operatorname{card}(M)$. □

2.6. Completion of the proof of Theorem 1. In view of Section 2.3 we may assume that E is multiply connected. First we shall show that there exists a sequence $(\phi_n)_{n\in\mathbf{N}}$ in $\mathscr{H}(E;\mathbf{C})$ such that (5) holds. To this end let, for each $n \in \mathbf{N}$,

$$u_n := \prod_{j=0}^{n-1}(h^j - a) \qquad \text{and} \qquad s_n := \frac{1}{n}\frac{u_n'}{u_n} = \frac{1}{n}\sum_{j=0}^{n-1}\frac{(h^j)'}{h^j - a}.$$

By Main Lemma 1 there is a non-constant function $s \in \mathscr{H}(E;\mathbf{C})$ such that $s = \lim_{n\to\infty} s_n$ and conditions (8) and (9) are satisfied. Hence, for each closed rectifiable curve d in E and each $n \in \mathbf{N}$, we obtain that

$$\begin{aligned}
2\pi i n(u_n \circ d; 0) &= \int_d \frac{u_n'(z)}{u_n(z)}\,dz = \sum_{j=0}^{n-1} \int_d \frac{(h^j)'(z)}{h^j(z)-a}\,dz \\
&= 2\pi i \sum_{j=0}^{n-1} n(h^j \circ d; a) = \sum_{j=0}^{n-1} \int_{h^j\circ d} s(z)\,dz \\
&= \sum_{j=0}^{n-1} \int_d s\big(h^j(z)\big)(h^j)'(z)\,dz = \sum_{j=0}^{n-1} \int_d s(z)\,dz \\
&= 2\pi i n n(d;a) \in 2\pi i n\mathbf{Z}.
\end{aligned}$$

Now this implies that there exists a sequence $(\phi_n)_{n\in\mathbf{N}}$ such that condition (5) is satisfied. By Lemma 4 we see that $\{\phi_n : n \in \mathbf{N}\}$ is locally bounded and hence normal by Montel's theorem. Define Γ to be the accumulation set of $(\phi_n)_{n\in\mathbf{N}}$ in $\mathscr{H}(E;\mathbf{C})$. Let $\phi \in \Gamma$.

Now we shall prove that there exists $\lambda \in \exp\big(2\pi i(\mathbf{R}\setminus\mathbf{Q})\big)$ such that $\phi \circ h = \lambda\phi$. Lemma 4 implies that $\phi^{-1}(0) = \emptyset$. Since $\phi_n'/\phi_n = s_n$, for each $n \in \mathbf{N}$, we conclude that $s = \phi'/\phi$. The function $w := \phi\circ h/\phi$ is well defined and holomorphic and from the functional equation $(s\circ h)h' = s$ we conclude that, for each $z \in E$,

$$\begin{aligned}
w'(z) &= \frac{\phi'\big(h(z)\big)h'(z)\phi(z) - \phi'(z)\phi\big(h(z)\big)}{\phi(z)^2} \\
&= \frac{s\big(h(z)\big)h'(z)\phi\big(h(z)\big)\phi(z) - s(z)\phi(z)\phi\big(h(z)\big)}{\phi(z)^2} \\
&= \frac{\big(s(z)-s(z)\big)\phi\big(h(z)\big)}{\phi(z)} = 0.
\end{aligned}$$

Hence we find $\lambda \in \mathbf{C}$ such that $w \equiv \lambda$. Therefore $\phi\circ h = \lambda\phi$ and induction leads to $\phi\circ h^n = \lambda^n\phi$, for each $n \in \mathbf{N}$. By Proposition 1 we know that $\mathrm{id}_E \in \mathscr{G}(h)$ which means that ϕ is in the accumulation set of $(\lambda^n\phi)_{n\in\mathbf{N}}$. Since $\phi^{-1}(0) = \emptyset$, we conclude that $|\lambda| = 1$. Moreover, $\lambda \in \exp\big(2\pi i(\mathbf{R}\setminus\mathbf{Q})\big)$, for otherwise ϕ would be constant by Lemma 3 and hence $s = \phi'/\phi = 0$. From the equation $\lambda\phi(E) = \phi\big(h(E)\big) = \phi(E)$ we conclude that $\phi(E)$ is an open annulus centered at 0.

Next, using Main Lemma 2 and the abstract monodromy theorem, we shall prove that ϕ is injective. Let $z, w \in E$ such that $\phi(z) = \phi(w)$. Choose a rectifiable curve γ in E connecting z and w. Hence $\phi\circ\gamma$ is a closed rectifiable curve in $\phi(E)$. Since E is multiply connected, condition (4) implies that there exists a closed rectifiable curve c in E such that $n(c;a)=1$. We may assume that z is the end-point of c. Applying (9) we see that

$$2\pi i n(\phi\circ c;0)=\int_c \frac{\phi'(z)}{\phi(z)}\,dz=\int_c s(z)\,dz=2\pi i n(c;a)=2\pi i.$$

Thus $\delta := n(\phi\circ\gamma;0)c$ is a closed rectifiable curve in E such that $n(\phi\circ\delta;0)=n(\phi\circ\gamma;0)$. Since $\phi(E)$ is an annulus it follows from $n(\phi\circ\gamma;0)=n(\phi\circ\delta;0)$ that $\phi\circ\gamma$ and $\phi\circ\delta$ are fixed-end-point homotopic in $\phi(E)$. By Main Lemma 2 we see that (E,ϕ) is a covering space of $\phi(E)$. Hence the abstract monodromy theorem ([4, p. 247]) implies that γ and δ are fixed-end-point homotopic, too. Since δ is closed, we conclude that γ is closed and obtain $z=w$. Hence ϕ is injective.

Finally, we have to prove that the sequence $(\phi_n)_{n\in\mathbf{N}}$, which by (5) is uniquely determined up to roots of unity, can be chosen in such a way that $(\phi_n)_{n\in\mathbf{N}}$ is convergent. To this end, fix $x\in E$. Since $\phi(x)\neq 0$ and $\phi(E)$ is an annulus, the function

$$f\colon \exp(i\mathbf{R})\to\mathbf{R},\qquad \mu\mapsto \log\bigl|\phi^{-1}\bigl(\mu\bigl(\phi(x)\bigr)\bigr)-a\bigr|$$

is well defined and continuous. For each $n\in\mathbf{N}$,

$$\begin{aligned}\log|\phi_n(x)| &= \log\left(\sqrt[n]{\prod_{j=0}^{n-1}|h^j(x)-a|}\right)=\frac{1}{n}\sum_{j=0}^{n-1}\log|h^j(x)-a|\\ &=\frac{1}{n}\sum_{j=0}^{n-1}\log\bigl|\phi^{-1}\bigl(\lambda^j\phi(x)\bigr)-a\bigr|=\frac{1}{n}\sum_{j=0}^{n-1}f(\lambda^j).\end{aligned}$$

By Theorem 3 (Weyl's equidistribution theorem) we conclude that

$$\lim_{n\to\infty}\log|\phi_n(x)|=\int_0^1 f\bigl(\exp(2\pi i t)\bigr)\,dt.$$

Hence $\bigl(|\phi_n(x)|\bigr)_{n\in\mathbf{N}}$ is convergent. This implies that there exists a sequence $(\theta_n)_{n\in\mathbf{N}}$ such that $\theta_n^n=1$, for each $n\in\mathbf{N}$, and $\phi(x)=\lim_{n\to\infty}\theta_n\phi_n(x)$. Passing over to $(\theta_n\phi_n)_{n\in\mathbf{N}}$ instead of $(\phi_n)_{n\in\mathbf{N}}$ if necessary we may assume that $\theta_n=1$, for each $n\in\mathbf{N}$. Now let $\psi\in\Gamma$. Then we know that $\psi^{-1}(0)=\emptyset$ and $\psi'/\psi=s=\phi'/\phi$. Hence $\psi'\phi=\phi'\psi$ and thus ψ/ϕ is constant by the quotient rule. Since $\psi(x)=\phi(x)$, we see that $\phi=\psi$, which means that ϕ is the only function in Γ. Hence we conclude that $\phi=\lim_{n\to\infty}\phi_n$. □

3. A modified proof of the classification theorem of periodic components of the Fatou set

Our aim in this section is to show how to use our weaker version of the rotation theorem (compared to the stronger version which holds under condition (7) instead of (3) but whose proof requires the Riemann uniformization theorem) to prove the classification theorem of periodic components of the Fatou set. This famous theorem is already well documented in the literature (see for instance [1], [3], [8], [9], and especially [2]) so that we confine ourselves to stating this result without discussing its significance and history.

General assumptions and further notations. Let $\Delta \in \{\overline{\mathbf{C}}, \mathbf{C}, \mathbf{C}\setminus\{0\}\}$ and $f\colon \Delta \to \Delta$ be holomorphic and neither constant nor injective. Let D be a component of the Fatou set $\mathscr{F}$ of f such that $f(D) \subset D$. Denote the accumulation set of $(f^n|_D)_{n\in\mathbf{N}}$ by $\mathscr{G}$. Denote the Julia set of f by $\mathscr{J}$.

Theorem 5 (Classification Theorem). *Exactly one of the following statements holds*:

(A) *D is an attracting domain of f, i.e., there exists $p \in D$ such that $f(p) = p$, $|f'(p)| < 1$, and $\lim_{n\to\infty} f^n|_D \equiv p$.*
(P) *D is a parabolic domain of f, i.e., there exists $p \in \partial D \cap \Delta$ such that $f(p) = p$, $f'(p) = 1$, and $\lim_{n\to\infty} f^n|_D \equiv p$.*
(B) *D is a Baker domain of f, i.e., there exists $p \in \overline{\mathbf{C}}\setminus\Delta$ such that $\lim_{n\to\infty} f^n|_D \equiv p$.*
(S) *D is a Siegel disk, i.e., there exist $\lambda \in \exp\big(2\pi i(\mathbf{R}\setminus\mathbf{Q})\big)$ and a biholomorphic map $\phi\colon D \to B(0;1)$ such that $\phi \circ f|_D = \lambda\phi$.*
(H) *D is a Herman ring of f, i.e., there exist $\lambda \in \exp\big(2\pi i(\mathbf{R}\setminus\mathbf{Q})\big)$, $r \in]0,1[$, and a biholomorphic map $\phi\colon D \to A(0;r;1)$ such that $\phi \circ f|_D = \lambda\phi$.*

If $\Delta = \overline{\mathbf{C}}$ then (B) *does not hold. If $\Delta = \mathbf{C}$ then* (H) *does not hold.*

The traditional proof of the classification theorem is determined by the following distinction (see for instance [1], [3], [8], and [9]):

Case 1. Each function g in the accumulation set $\mathscr{G}$ of $(f^n|_D)_{n\in\mathbf{N}}$ is constant. Then it is easy to show that $\operatorname{card}(\mathscr{G}) = 1$. Let $g \in \mathscr{G}$ and $p \in \overline{D}$ such that $g \equiv p$. Then it depends on $p \in D$, $p \in \partial D \cap \Delta$, and $p \in \overline{\mathbf{C}} \setminus \Delta$ to verify statement (A), (P), or (B).

Case 2. There exists a non-constant function $g \in \mathscr{G}$. Then the strong rotation theorem is used to verify statement (S) or (H).

If we replace the strong rotation theorem by Theorem 2 the above scheme can still be taken over for the case when D is simply connected because in this case Theorem 2 works under condition (7) instead of (3). As the following lemma shows, this may be done for the case when $\Delta = \mathbf{C}$ and for the case when $\Delta = \mathbf{C}\setminus\{0\}$ and $\widehat{D}$ (which in Section 1 was defined to be the union of the interiors of the closed rectifiable curves in D) does not contain 0:

Lemma 7. *Suppose that $\Delta \in \{\mathbf{C}, \mathbf{C}\backslash\{0\}\}$, $\widehat{D} \subset \Delta$, and there is a non-constant function $g \in \mathscr{G}$. Then D is simply connected.*

Lemma 7 is an immediate consequence of the following lemma, which can easily be proved using Cauchy's formula.

Lemma 8. *Suppose that $D \subset \mathbf{C}$ and there is a non-constant function $g \in \mathscr{G}$. Let c be a closed rectifiable curve in D such that $\operatorname{int}(c) \subset \Delta$ and, for each $n \in \mathbf{N}$, $(f^n)^{-1}(\infty) \cap \operatorname{int}(c) = \emptyset$. Then $\operatorname{int}(c) \subset D$.*

Proof. By Proposition 1 there is a strictly increasing $\alpha \in \mathbf{N}^{\mathbf{N}}$ such that $\lim_{n\to\infty} f^{\alpha(n)}|_D = \mathrm{id}_D$. By Cauchy's formula we obtain that $\lim_{n\to\infty} f^{\alpha(n)}|_{\operatorname{int}(c)} = \mathrm{id}_{\operatorname{int}(c)}$. From this one concludes that $\operatorname{int}(c) \subset \mathscr{F}$ and hence $\operatorname{int}(c) \subset D$. ▫

However, if $\Delta = \overline{\mathbf{C}}$ or $\Delta = \mathbf{C}\backslash\{0\}$ and $0 \in \widehat{D}$, then D can be multiply connected and, if we want to use the weak rotation theorem instead of the strong one, we have to modify the traditional structure of the proof of the classification theorem. Now statements (A), (P), and (B) have to be verified under the weaker condition that there exists a constant function $g \in \mathscr{G}$ instead of that every function $g \in \mathscr{G}$ is constant.

This modification does not cause any trouble if the value p of the existing constant function $g \in \mathscr{G}$ belongs to D. In this case we can use the following proposition, which itself is easy to prove, to show that D is an attracting domain of f at p.

Proposition 2. *Let $E \subset \overline{\mathbf{C}}$ be a region and $h\colon E \to E$ be holomorphic such that $\{h^n : n \in \mathbf{N}\}$ forms a normal family and there exist a strictly increasing $\alpha \in \mathbf{N}^{\mathbf{N}}$ and $p \in E$ such that $\lim_{n\to\infty} h^{\alpha(n)} \equiv p$. Then $h(p) = p$, $|h'(p)| < 1$ and $\lim_{n\to\infty} h^n \equiv p$.*

Proof. See for instance [1, p. 163]. ▫

But some more effort has to be done if $p \in \partial D \cap \Delta$ or $p \in \overline{\mathbf{C}} \setminus \Delta$. We first prove:

Lemma 9. *Let $p \in \partial D \cap \Delta$ such that the constant function $D \to \overline{\mathbf{C}}$, $z \mapsto p$ belongs to $\mathscr{G}$. Then D is a parabolic domain of f.*

The proof of the weaker form of this lemma (where the additional assumption that every function $g \in \mathscr{G}$ is constant is at disposal) is due to Fatou [6] and well documented in the literature (see for instance [1, p. 165]). We adopt the structure of this proof and just have to show that the single steps in this proof carry over to the case when a non-constant function $g \in \mathscr{G}$ exists.

Proof. By conjugating if necessary we may assume that $p \in \mathbf{C}$. Let $\alpha \in \mathbf{N}^{\mathbf{N}}$ be strictly increasing such that $\lim_{n\to\infty} f^{\alpha(n)}|_D \equiv p$. Since $f(D) \subset D$,

$$f(p) \equiv f \circ (\lim_{n\to\infty} f^{\alpha(n)}|_D) = \lim_{n\to\infty} f^{\alpha(n)}|_D \circ f|_D \equiv p$$

so that p is a fixed point of f. Let $\lambda := f'(p)$. Since $p \in \partial D \cap \Delta \subset \mathscr{J}$, we conclude that $|\lambda| \geq 1$.

The first step is to prove that

(17) there is a region $W \subset D$ such that $f(W) \subset W$ and $f|_W$ is injective.

Proof of (17). For the case when each function $g \in \mathscr{G}$ is constant, this is proved in [1, p. 126]. For the case when there is a non-constant function $g \in \mathscr{G}$, Proposition 1 yields that one can choose $W := D$. □

Now fix W according to (17). By conjugating if necessary we may assume that $W \subset \mathbf{C}$. Fix $\zeta \in W$ and, for each $n \in \mathbf{N}$, let $\phi_n := (f^n|_W - p)/\big(f^n(\zeta) - p\big)$. Since $f|_W$ is injective, one easily proves by Montel–Carathéodory's theorem that $(\phi_n)_{n\in\mathbf{N}}$ is normal. By transition to a subsequence of α if necessary we may assume that $(\phi_{\alpha(n)})_{n\in\mathbf{N}}$ converges to a holomorphic function $\phi\colon W \to \mathbf{C}$. An easy calculation shows that $\phi\big(f(z)\big) = \lambda\phi(z)$, for each $z \in W$. Induction leads to $\phi\big(f^n(z)\big) = \lambda^n\phi(z)$, for each $z \in W$ and $n \in \mathbf{N}$. The next step is to prove that

(18) $|\lambda| = 1$.

Proof of (18). For the case when each function $g \in \mathscr{G}$ is constant, see [1, p. 165]. For the case when there is a non-constant function $g \in \mathscr{G}$, Proposition 1 implies that there exists a strictly increasing $\gamma \in \mathbf{N}^{\mathbf{N}}$ such that $\lim_{n\to\infty} f^{\gamma(n)}(\zeta) = \zeta$. Now

$$1 = \phi(\zeta) = \lim_{n\to\infty} \phi\big(f^{\gamma(n)}(\zeta)\big) = \lim_{n\to\infty} \lambda^{\gamma(n)}\phi(\zeta) = \lim_{n\to\infty} \lambda^{\gamma(n)},$$

which implies that $|\lambda| = 1$. □

It follows easily from (18) that ϕ is constant (see [1, p. 166]). Hence we obtain that

$$1 = \phi(\zeta) = \phi\big(f(\zeta)\big) = \lambda\phi(\zeta) = \lambda.$$

It remains to show that $\lim_{n\to\infty} f^n|_D \equiv p$. To this purpose we shall make use of some facts concerning the behaviour of the iterates near the parabolic fixed point p of f (see e.g. [1, §6.5], [8, §7]).

Choose a region $U \subset \Delta$ such that $p \in U$, $\zeta \in \mathbf{C} \setminus \big(U \cup f(U)\big)$, and $f|_U$ is injective. Let $I := (f|_U)^{-1}$ and $k := \inf\{m \in \mathbf{N} : f^{(m+1)}(p) \neq 0\}$. By the flower theorem (see e.g. [8, p. 45]) we then find simply connected regions $G_1, \dots, G_{2k}$ in $U \cup f(U)$ such that $(G_1, \dots, G_{2k})$ forms a flower of f at p, which in particular implies that

(F1) for each $i \in \{1, \dots, k\}$, G_{2i-1} is an attracting petal of f at p and G_{2i} is a repelling petal of f at p,

(F2) for each $j, m \in \{1, \dots, 2k\}$, one has $G_j \cap G_m \neq \emptyset$ if and only if there exists $\sigma \in \{-1, 0, 1\}$ such that $(j - m) \equiv \sigma (\operatorname{mod} 2k)$,

(F3) $\bigcup_{j=1}^{2k} G_j \cup \{p\}$ is open and connected.

Now, for each $i \in \{1,\dots,k\}$, the attracting petal G_{2i-1} is contained in a component D_i of the Fatou set $\mathscr{F}$, which of course is a parabolic domain of f at p. Thus it suffices to show that there exists $i \in \{1,\dots,k\}$ such that $D \cap D_i \neq \emptyset$ because then we have $D = D_i$ and conclude that D is a parabolic domain of f at p.

To this end choose a region $K \subset W$ such that $\{\zeta, f(\zeta)\} \subset K$ and $\overline{K} \subset W$. Since $\lim_{n\to\infty} f^{\alpha(n)}|_D \equiv p$, condition (F3) implies that there exists $n \in \mathbf{N}$ such that $f^n(K) \subset \bigcup_{j=0}^{2k} G_j \cup \{p\}$. Now we prove that

(19) $f^n(K) \not\subset \bigcup_{i=1}^{k} G_{2i}$.

Proof of (19). Assume that $f^n(K) \subset \bigcup_{i=1}^{k} G_{2i}$. Since $f^n(K)$ is connected, condition (F2) implies that there exists $i \in \{1,\dots,k\}$ such that $f^n(K) \subset G_{2i}$. Let $m := \min\{j \in \mathbf{N} : f^j(K) \subset G_{2i}\}$. Choose a neighbourhood X of ζ in K such that $Y := f(X) \subset K$. Let $y \in Y$. Then $f^{m-1}(y) \in f^m(X) \subset G_{2i}$. On the other hand, $y \in Y \subset K$ and hence $f^m(y) \in G_{2i}$. Since G_{2i} is a repelling petal of f at p, we know that $I\bigl(f^m(y)\bigr) \in G_{2i}$. Since $f\bigl(f^{m-1}(y)\bigr) = f^m(y) = f\bigl(I\bigl(f^m(y)\bigr)\bigr)$ and $f|_{G_{2i}}$ is injective, we conclude that $f^{m-1}(y) = I\bigl(f^m(y)\bigr)$. Thus we have proved that $f^{m-1}|_Y = I \circ f^m|_Y$. By the identity theorem we conclude that $f^{m-1}|_K = I \circ f^m|_K$ and hence $f^{m-1}(K) \subset G_{2i}$. This implies that $m = 1$ and $K \subset G_{2i}$. Since $G_{2i} \subset U \cup f(U)$, this is a contradiction to $\zeta \in \mathbf{C} \setminus \bigl(U \cup f(U)\bigr)$. □

Since $p \in \mathscr{J}$ and $f^n(K) \subset \mathscr{F}$, we then conclude that there exists $i \in \{1,\dots,k\}$ such that $f^n(K) \cap G_{2i-1} \neq \emptyset$. Hence we obtain that $D \cap D_i \neq \emptyset$, which implies that $D = D_i$. Thus D is a parabolic domain of f at p. □

If $\Delta = \overline{\mathbf{C}}$ then the classification theorem follows by combining the weak rotation theorem, Proposition 2, and Lemma 9. To settle the remaining case when $\Delta = \mathbf{C}\setminus\{0\}$ and $0 \in \widehat{D}$, one also uses the following lemma.

Lemma 10. *Suppose that $\Delta = \mathbf{C}\setminus\{0\}$ and $0 \in \widehat{D}$. If one of the constant functions 0 or ∞ belongs to $\mathscr{G}$ then each function $g \in \mathscr{G}$ is constant.*

Proof by contraposition. Suppose that $\mathscr{G}$ contains a non-constant function. It suffices to show that the constant function 0 does not lie in $\mathscr{G}$, for then we can conjugate with $z \mapsto 1/z$ and obtain the same result for the constant function ∞.

Choose a closed rectifiable curve c in D such that $0 \in \operatorname{int}(c)$. Denote the connected component of 0 in $\operatorname{int}(c)$ by Z. There are two cases to consider:

Case 1. $Z \setminus \{0\} \subset D$. Since Proposition 1 shows that $f|_D$ is injective, one can easily verify that 0 is not an essential singularity of f and, if 0 is a pole of f, then ∞ is not an essential singularity of f, either. Hence there is a region $G \in \{\overline{\mathbf{C}}, \mathbf{C}\}$ and a holomorphic map $F\colon G \to G$ such that $F|_{\mathbf{C}\setminus\{0\}} = f$. Denote the component of the Fatou set of F which contains D by E. Since we have already verified the classification theorem for holomorphic self-maps of $\overline{\mathbf{C}}$ and $\mathbf{C}$, we see that E is a Siegel disk or a Herman ring of F. Hence each function in $\mathscr{G}$ is not constant.

Case 2. *There exists* $x \in Z \setminus (D \cup \{0\})$. In this case we shall prove that $n(f^n \circ c; x) \neq 0$, for each $n \in \mathbf{N}$. This implies that $\operatorname{tr}(f^n \circ c) \not\subset B(0; |x|)$, for each $n \in \mathbf{N}$, which in particular means that no subsequence of $(f^n|_D)_{n \in \mathbf{N}}$ converges locally uniformly to zero.

Let $n \in \mathbf{N}$. We first prove:

(20) $n(f^n \circ c; 0) \neq 0$.

Proof of (20). From Proposition 1 we know that $\mathrm{id}_D \in \mathscr{G}$. In particular, this implies that there exists $k \in \mathbf{N}$ such that $k > n$ and $n(f^k \circ c; 0) = n(\mathrm{id}_E \circ c; 0) = n(c; 0) \neq 0$. Assume that $n(f^n \circ c; 0) = 0$. Then $f^n \circ c$ is homotopic to zero in $\mathbf{C} \setminus \{0\}$. Since f^{k-n} is a self-map of $\mathbf{C} \setminus \{0\}$ and hence omits 0, Cauchy's theorem implies that

$$2\pi i n(f^k \circ c; 0) = \int_{f^{k-n+n} \circ c} \frac{1}{z}\, dz = \int_{f^n \circ c} \frac{(f^{k-n})'(z)}{f^{k-n}(z)}\, dz = 0.$$

This is a contradiction to $n(f^k \circ c; 0) \neq 0$. □

Now choose a rectifiable curve γ in D connecting the end-points of c and $f^n \circ c$. Then

$$d := n(c; 0) f^n \circ c - \gamma - n(f^n \circ c; 0) c + \gamma$$

is a closed rectifiable curve in D such that $\operatorname{int}(d) \subset \Delta$. By Lemma 8 we obtain that $\operatorname{int}(d) \subset D$ and hence $n(d; x) = 0$. Thus we conclude that

$$n(c; 0) n(f^n \circ c; x) = n(f^n \circ c; 0) n(c; x) \neq 0,$$

which implies that $n(f^n \circ c; x) \neq 0$. □

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