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WANDERING DOMAINS AND INVARIANT CONFORMAL STRUCTURES FOR MAPPINGS OF THE 2-TORUS

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Abstract. A C^1 circle diffeomorphism with irrational rotation number need not have any dense orbits. However, any C^2 circle diffeomorphism with irrational rotation number must in fact be topologically conjugate to an irrational rotation.

This paper addresses the analogous matter for the 2-torus. We say that a diffeomorphism f of T^2, isotopic to the identity, has *Denjoy type* if $hf = Rh$, where R is some minimal translation of the torus, and h is a continuous torus mapping homotopic to the identity such that $\{x \in T^2 :$ cardinality $(h^{-1}(x)) > 1\}$ is nonempty and countable. If f has Denjoy type, the interior of any fiber $h^{-1}(x)$, if nonempty, is a *wandering domain* for f.

It is known that there are C^2 diffeomorphisms of Denjoy type, but not known whether they can be C^3. Our main results imply the following

Theorem. *Let $f \in \mathrm{Diff}_1(T^2)$ have Denjoy type, with minimal set $\Gamma \neq T^2$.*

(i) *If f preserves a measurable, essentially bounded conformal structure on Γ, then the collection $\{f^n\}$ (considered as mappings of the ideal boundaries of the wandering domains) has unbounded quasisymmetric distortion, and*

(ii) *if f preserves a C^{1+Z} conformal structure on Γ, then f cannot be C^{2+Z}.*

A simple corollary of (ii) is that no C^3 diffeomorphism of Denjoy type exists with, for example, *circular* wandering domains.

0. Introduction

Let $f\colon S^1 \to S^1$ be a homeomorphism of the circle without periodic points. There is always a continuous monotone function h such that $hf = Rh$, where R is some irrational rotation. (In fact $h(x) = \int_c^x d\mu$, where c is some point on the circle, and μ is the (unique) invariant probability measure for f.) If h is a homeomorphism, f is said to be *topologically conjugate* to R. Otherwise f is *semiconjugate* to R, and f permutes countably many pairwise disjoint closed intervals of S^1 [8].

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Poincaré [16] essentially understood both these possibilities, and asked whether a homeomorphism of this second type (now called a "Denjoy counterexample") could be realized as an analytic function. About 100 years later, Yoccoz [19] showed that the answer is "no". Previously, Hall [5] showed that there are nevertheless C^∞ Denjoy counterexamples.

For diffeomorphisms (i.e., smooth homeomorphisms without critical points), this question was settled by A. Denjoy [4] in 1932 with the following theorem.

Denjoy's theorem. *If $f\colon S^1 \to S^1$ is a C^1 diffeomorphism without periodic points and Df has bounded variation ($f \in C^{1+b.v.}$), then f is topologically conjugate to an irrational rotation.*

Furthermore, Denjoy, and before him Bohl [2], provided examples of C^1 diffeomorphisms semiconjugate but not conjugate to an irrational rotation. (The condition $C^{1+b.v.}$ is sharp in the sense that there are $C^{1+\alpha}$ counterexamples for all $\alpha \in (0,1)$ [8]. Also $C^{1+b.v.}$ in the theorem can be replaced by the condition $C^{1+\mathrm{Zygmund}}$, [17], and Hu [10] showed recently that this condition can be replaced by one weaker than both $C^{1+b.v.}$ and $C^{1+\mathrm{Zygmund}}$.)

Denjoy's techniques make strong use of the one-dimensionality of the problem, and so do subsequent proofs. E.g. see [8], [12], [18].

It is natural to ask whether similar phenomena occur in higher dimensions. For example, one might consider the dynamics on an invariant circle in the plane. A famous argument of Ghys shows that a complex analytic homeomorphism defined on a neighborhood of an invariant closed curve in $\mathbf{C}$ with no periodic points on the curve has only dense orbits on the curve. (Use the Riemann mapping theorem and the Schwarz reflection principle, followed by Denjoy's theorem.) On the other hand, J. Harrison constructed in [7] a $C^{2+\varepsilon}$ diffeomorphism of the annulus in which there is a (non-rectifiable) invariant closed curve with no periodic points and no dense orbits, and which has a wandering domain. G.R. Hall constructed in [6] a C^∞ diffeomorphism of the annulus with an invariant rectifiable curve having also no dense orbits and no periodic points.

In this paper we generalize the setting of Denjoy's theorem more fully to two dimensions by considering diffeomorphisms of the torus. (This matter has been previously addressed in [14] and [15].) An equivalent formulation of Denjoy's theorem, more suggestive for this kind of generalization, is the following:

Denjoy's theorem revisited. *If $f\colon S^1 \to S^1$ is a $C^{1+b.v.}$ diffeomorphism, R_α is the irrational rotation $R_\alpha(x) = x + \alpha \bmod 1$, and $hf = R_\alpha h$ for some continuous degree one mapping h, then h is a homeomorphism (and so f is topologically conjugate to R_α).*

In two dimensions we take as the analog of irrational rotation the minimal translation $R_{\alpha,\beta}\colon (x,y) \mapsto (x+\alpha \bmod 1, y+\beta \bmod 1)$, where α and β are rationally independent irrationals.

Question. If $f\colon T^2 \to T^2$ is a diffeomorphism, $h\colon T^2 \to T^2$ is a continuous map homotopic to the identity, and $hf = R_{\alpha,\beta}h$, are there natural geometric conditions (e.g. smoothness) on f that force h to be a homeomorphism?

We will restrict our attention, in analogy with the S^1 theory, to pairs (f, h) for which h has only countably many nontrivial fibers. Accordingly, we say that a diffeomorphism has *Denjoy type* if it is semiconjugate to a minimal translation by a map h, as above, with the property that the fiber $h^{-1}(x)$ is a singleton for all but countably many x. This is motivated by and includes the *wandering domains problem*: can one "blow up" one or more orbits of $R_{\alpha,\beta}$ to make a smooth diffeomorphism with wandering domains?

P. McSwiggen [13] shows the answer to the wandering domains problem is "yes" if one does not demand too much smoothness: he constructs a $C^{2+\alpha}$ diffeomorphism of Denjoy type having a smooth wandering domain. There are no known C^{2+Z} examples.

In this paper, we show that Denjoy type mappings must, if they exist, possess abundant geometric distortion in the sense of quasiconformal dilatation on the (unique) minimal set Γ. There are two main theorems (see below for definitions).

Theorem 1. *Let $f \in QC(T^2)$ have Denjoy type. Then either*

(i) $\operatorname{area}(\Gamma) > 0$ *and the collection $\{f^n|_\Gamma\}$ has unbounded quasiconformal distortion, or*

(ii) *the collection $\{f^n\}$, considered as mappings of the ideal boundaries of the wandering domains, has unbounded quasisymmetric distortion.*

The meaning of part (ii) is as follows. Let B denote the unit disk in the complex plane. For every wandering domain Δ and $n \in Z$, choose any conformal isomorphism $\phi_n\colon f^n(\Delta) \to B$. The homeomorphisms $g_n\colon \phi_{n+1} \circ f \circ \phi_n^{-1}$ are quasiconformal, and so extend to the boundary ∂B and are quasisymmetric there. The statement is that for every constant $C > 0$, there exist Δ, n, and k such that $g_{n+k} \circ \ldots \circ g_n|_{\partial B}$ is not C-quasisymmetric.

Theorem 1 is proved by means of a quasiconformal extension operator that respects composition—see Section 3.

Theorem 2. *Let $f \in \mathrm{Diff}_1(T^2)$ have Denjoy type, and suppose f preserves a C^{1+Z} conformal structure on Γ.*

Then f cannot be C^{2+Z}.

This has a simple, concrete corollary:

Corollary of Theorem 2. *Let $f \in \mathrm{Diff}_{2+Z}(T^2)$ have Denjoy type, with $\Gamma \neq T^2$. Then the components of $T^2 \setminus \Gamma$ (i.e., the wandering domains) cannot all be circular disks.*

This Corollary remains true if "circular disks" is replaced, for example, by "square regions". More generally, the components of $T^2 \setminus \Gamma$ cannot all be homothetic copies of a single domain D provided that D satisfies the mild condition

$$\{A \in \mathrm{SL}(2, \mathbf{R}) : A(D) \text{ is homothetic to } D\} = \mathrm{SO}(2, \mathbf{R}).$$

See Section 5.

Theorems 1 and 2 together imply the following (slightly weaker) summary statement:

Theorem 3. *Let $f \in \mathrm{Diff}_1(T^2)$ have Denjoy type, with minimal set $\Gamma \neq T^2$.*

(i) *If f preserves a measurable, essentially bounded conformal structure on Γ, then the collection $\{f^n\}$ (considered as mappings of the ideal boundaries of the wandering domains) has unbounded quasisymmetric distortion, and*
(ii) *if f preserves a C^{1+Z} conformal structure on Γ, then f cannot be C^{2+Z}.*

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1. Definitions and topological facts

Definitions. $\mathrm{Diff}_k(T^2)$ will denote the space of C^k diffeomorphisms of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ that are homotopic to the identity. (When $k = 0$, this is the space of homeomorphisms homotopic to the identity.) $\mathrm{QC}(T^2)$ will denote the space of quasiconformal homeomorphisms of T^2 homotopic to the identity; note that $\mathrm{Diff}_1(T^2) \subset \mathrm{QC}(T^2) \subset \mathrm{Diff}_0(T^2)$. The notation $f \in C^{r+H}$ means $f \in C^r$ and $D^r f$ satisfies a Hölder condition with exponent less than one; $f \in C^{r+\mathrm{Lip}}$ means $D^r f$ is Lipschitz, and $f \in C^{r+Z}$ means $D^r f$ is Zygmund. (Recall that $f\colon R^n \to R^m$ is Zygmund if f is continuous and $|f(x+h)+f(x-h)-2f(x)| \leq C|h|$ for some constant $C > 0$ and all x, h.) Then for any nonnegative integer r,

$$C^{r+1} \subset C^{r+\mathrm{Lip}} \subset C^{r+Z} \subset C^{r+H} \subset C^r$$

and these classes are all distinct.

A *trivial value* of a map $h\colon T^2 \to T^2$ is a point $y \in T^2$ such that the fiber $h^{-1}(y)$ is a single point. The set of nontrivial values of h is denoted V_h.

A set is *cellular* if it is the monotone intersection of closed topological disks.

We say a homeomorphism $f \in \mathrm{Diff}_0(T^2)$ has *Denjoy type* if

(1) f is semiconjugate to a minimal translation R (that is, $hf = Rh$ for some continuous map h homotopic to the identity), but not conjugate to one, and
(2) the semiconjugacy h has only countably many nontrivial values.

Certain topological properties obtain for Denjoy type homeomorphisms; we summarize these below in the following proposition. Our main results are Theorem 1 in Section 4 and Theorem 2 in Section 5, which are more geometric in nature.

Proposition 1. *Let $f \in \mathrm{Diff}_0(T^2)$ have Denjoy type. Then:*

(1) *f has a unique minimal set Γ. Moreover $\Gamma = T^2 \setminus \cup\{\text{interior}\,(h^{-1}(x)) : x \in V_h\} = \mathrm{cl}(T^2 \setminus h^{-1}(V_h))$, and this set is connected.*

(2) *If $\Gamma \neq T^2$, then Γ is nowhere dense, and each component of $T^2 \setminus \Gamma$ is a wandering simply-connected domain for f.*

(3) *The map h is monotone. In fact, every h-fiber is cellular.*

(4) *f is uniquely ergodic.*

Proof. The proofs of (1), (2), and (4) appear in [15]. We give here the proof of (3).

Since h is continuous and $hf = Rh$, h is surjective. Given $y \in T^2$, we show that the fiber $h^{-1}(y)$ is cellular.

Let C be a small circle in T^2 with center y and disjoint from V. (Since V is countable, most circles centered at y miss V.)

Let D be the open disk bounded by C, and let $\widetilde{C} = h^{-1}(C)$. Since $\widetilde{C}$ is compact and h is injective on $\widetilde{C}$, $h|_{\widetilde{C}}$ is a homeomorphism of $\widetilde{C}$ to C. Hence $\widetilde{C}$ is a simple closed curve. Since C is homotopically trivial and h is homotopic to the identity, $\widetilde{C}$ must also be homotopically trivial, and so bounds a disk $\widetilde{D}$.

Claim. $\widetilde{D} = h^{-1}(D)$. Hence $\widetilde{D}$ is a disk containing the fiber $h^{-1}(y)$.

Assuming the claim, we now complete the proof. Let $\{C_i\}$ be a sequence of circles with center y, all disjoint from V, and with diameters tending to zero as $i \to \infty$. Let D_i be the disk bounded by C_i. Then $\widetilde{D}_i = h^{-1}(D_i)$ is a disk containing $h^{-1}(y)$. Since $\cap D_i = \{y\}$, we must have $\cap \widetilde{D}_i = h^{-1}(y)$, and so $h^{-1}(y)$ is cellular.

Proof of Claim. Note that $h^{-1}(D)$ is nonempty, open, and disjoint from $\widetilde{C}$. An easy argument shows that $\partial h^{-1}(D) \subset \widetilde{C}$. Now since h can send no nontrivial loops into D, the only possibility is that $h^{-1}(D) = \widetilde{D}$.

2. Invariant conformal structures

In addition to the (C^∞) smooth structure on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$, it is convenient also to consider various complex analytic structures on T^2 that are compatible with the original smooth structure. Such a complex structure is determined by an atlas of charts $\{(U, \phi)\}$ such that each homeomorphism $\phi\colon U \to \mathbf{C}$ is locally quasiconformal, and the overlaps $\phi^{-1} \circ \psi$ are complex analytic where defined. (A good reference for standard material relating to quasiconformal maps and Riemann surfaces is [11].)

A C^r complex structure is one in which the chart mappings are all C^r relative to the C^∞ smooth structure on T^2.

By the uniformization theorem, any Riemann surface defined this way is conformally equivalent to $\mathbf{C}/L$, where $L = \langle 1, \tau \rangle$ is the two-dimensional (linear) lattice generated by the complex numbers 1 and $\tau \in \mathbf{H}$. Hence we may always consider T^2 to be conformally represented by $\mathbf{C}/L$ for some such two-dimensional lattice L. (Note that any two such representations are smoothly equivalent.) This permits us to use a global complex coordinate without having to specify a particular complex structure in advance. With respect to this coordinate, a smooth (respectively conformal, quasiconformal, etc.) mapping of T^2 is represented by a smooth (resp. conformal, quasiconformal, etc.) mapping of $\mathbf{C}$ that commutes with L.

In contrast, we define a *conformal structure* on a smooth 2-manifold M as follows. A *similarity class* on a two dimensional real vector space V is an equivalence class of positive definite inner products on V, up to scale. A *conformal structure* is then a choice $c(x)$ of a similarity class on each tangent space T_xM of M. Equivalently, the class of conformal structures on M, denoted $\mathscr{C}(M)$, is the set of equivalence classes of Riemannian metrics on M, where two metrics are identified if they differ by a multiplicative scalar function on M. A smooth (or measurable, etc.) conformal structure is one that comes from a smooth (measurable, etc.) Riemannian metric.

We now claim that the set of similarity classes $\mathscr{C}(V)$ of V is naturally isometric to the hyperbolic plane with respect to a natural Poincaré metric.

To see this, first choose a basis B for V, which is then identified with $\mathbf{R}^2$. Then every positive definite inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{R}^2$ is of the form $\langle v, w\rangle = Av\cdot Aw$, where $\cdot$ refers to the Euclidean inner product and $A \in \mathrm{GL}(2,\mathbf{R})$. The collection of all positive definite inner products is in this way identified with $\mathrm{GL}(2,\mathbf{R})/O(2,\mathbf{R})$, where $O(2,\mathbf{R})$ acts on the left. Then $\mathscr{C}(V)$ is identified with $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$.

Recall $\mathrm{SL}(2,\mathbf{R})$ acts naturally on the upper half plane H with the Poincaré metric as the group of isometries. Since each element of $\mathrm{SL}(2,\mathbf{R})$ is determined up to rotation by the image of $i \in H$, $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$ is naturally identified with H, and so $\mathscr{C}(V)$ is identified with H and inherits its Poincaré metric. This metric does not depend on the choice of basis: a change of the basis B changes the identification with $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$ by conjugacy with an element of $\mathrm{GL}(2,\mathbf{R})$, which merely acts as an isometry.

Hence we have a natural hyperbolic metric ϱ_x on each fiber $\mathscr{C}(T_xM)$ of $\mathscr{C}(M)$, and with this metric we will say that the distance between two conformal structures c and c' is given by

$$\varrho(c,c') = \operatorname*{ess\,sup}_{x\in M} \varrho_x\big(c(x), c'(x)\big).$$

Given a conformal structure $c \in \mathscr{C}(M)$, a differentiable mapping $f\colon M \to M$ acts by pullback: if ξ is a metric representing c, then

$$f^*\xi_x(v,w) = \xi_{f(x)}\big(D_xf(v), D_xf(w)\big),$$

for $v, w \in T_xM$. We then say that c is f-invariant if $f^*c = c$.

Relative to any Riemannian metric ξ representing c, the dilatation of f at x is defined to be

$$\operatorname{dil}(f, x) = \frac{\max_v |D_x f(v)|}{\min_v |D_x f(v)|},$$

where v ranges over the unit circle in T_xM, and the norm is the one induced by ξ. Since this quantity is unchanged by scaling ξ, it depends only on c and not on the choice of representative.

On $\mathbf{C}$, we have the standard conformal structure c_0 induced by the Euclidean metric. Given any measurable conformal structure $c \in \mathscr{C}(\mathbf{C})$, we say that c is essentially bounded if the distance between c and c_0 is finite. When this is the case, the measurable Riemann mapping theorem of Morrey–Bojarski–Ahlfors–Bers says that there is a quasiconformal homeomorphism $\phi\colon \mathbf{C} \to \mathbf{C}$ such that $\phi^* c_0 = c$.

Hence if c is an essentially bounded f-invariant conformal structure on $\mathbf{C}$ then $\phi f \phi^{-1}$ leaves c_0 invariant—i.e., $\phi f \phi^{-1}$ is 1-quasiconformal, hence conformal, on $\mathbf{C}$. The mapping ϕ is unique up to post-composition by a Möbius transformation of $\mathbf{C}$.

The importance of the Zygmund class in this connection is that it is preserved by "integration" of conformal structures: if c is a C^{k+Z} conformal structure, $k = 0, 1, 2, \ldots$, then the induced quasiconformal mapping ϕ is C^{k+1+Z}.

Now consider $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ as a smooth manifold. If c is a measurable conformal structure on T^2, then c lifts to a conformal structure $\tilde{c}$ on $\mathbf{C}$. If $\tilde{c}$ is essentially bounded, then by the measurable Riemann mapping theorem there is $\tilde{\phi}$ such that $\tilde{\phi}^* c_0 = \tilde{c}$. Since the action of $\mathbf{Z}^2$ preserves $\tilde{c}$, $\tilde{\phi}\mathbf{Z}^2\tilde{\phi}^{-1}$ must be conformal on $\mathbf{C}$, and hence is itself a two-dimensional linear lattice L. Hence c induces the (unique) complex structure $\mathbf{C}/L$ on T^2, and $\tilde{\phi}$ projects to a global complex coordinate $\phi\colon T^2 \to \mathbf{C}/L$.

Relative to this coordinate, we can compute the Beltrami coefficient of $f\colon T^2 \to T^2$:

$$\mu_f \equiv \mu_{\phi f \phi^{-1}} \equiv \frac{(\phi f \phi^{-1})_{\bar{z}}}{(\phi f \phi^{-1})_z}.$$

Then $\mu_f = 0$ if and only if $\phi f \phi^{-1}$ is conformal, if and only if f preserves the conformal structure c. Moreover

$$\operatorname{dil}(f, x) = \frac{1 + \left|\mu_f\big(\phi(x)\big)\right|}{1 - \left|\mu_f\big(\phi(x)\big)\right|}.$$

We will use the following

Proposition 2. *Let C be a collection of transformations of a Riemannian surface S which is closed under composition and such that for some $K > 0$, every member of C is K-quasiconformal. Then there is a complex analytic structure on S (compatible with its original quasiconformal structure) such that every member of C becomes complex analytic* [18].

As an application, consider a diffeomorphism $f\colon T^2 \to T^2$ such that the iterates $\{f^n\}$ are all K-quasiconformal. Equivalently, we are assuming that the essential supremum of the dilatation $\mathrm{dil}(f^n, x)$ is uniformly bounded by K. The main point of the proof of Proposition 2 is that this is equivalent, by means of a barycenter construction in the set of similarity structures on the tangent space at each point, to the existence of a measurable essentially bounded conformal structure invariant by f.

By the measurable Riemann mapping theorem as before, f is quasiconformally conjugate to a conformal isomorphism of $\mathbf{C}/L$ homotopic to the identity, and hence a translation.

Summarizing, we have the following

Corollary of Proposition 2. *If $f\colon T^2 \to T^2$ is a diffeomorphism homotopic to the identity and*

$$\sup\{\mathrm{dil}(f^n, x) : x \in T^2\}$$

is bounded uniformly in n, then there is a quasiconformal map ϕ such that $\phi f \phi^{-1}$ is a (linear) translation.

Uniformly bounded dilatation of iterates is a sufficient but not necessary condition for *topological* conjugacy to a translation, as the following example shows.

Example. There exists a real-analytic diffeomorphism F of T^2 such that F is topologically conjugate to a minimal translation, but has no essentially bounded invariant conformal structure.

To see this, let $F = f \times g$, where f and g are two Arnol'd examples [3] of analytic circle diffeomorphisms conjugate to irrational rotations R_α and R_β via conjugacies ϕ and ψ. Here f and g are constructed so that ϕ and ψ are not absolutely continuous, and we may choose α and β so that $R = R_\alpha \times R_\beta$ is minimal.

Now $H^{-1}FH = R$, where $H = \phi \times \psi$. If F had an essentially bounded invariant conformal structure, then by the above argument $Q^{-1}FQ = R$ for some qc homeomorphism Q of T^2. This means $(H^{-1}Q)R(H^{-1}Q)^{-1} = R$, and minimality of R implies that $H^{-1}Q$ is itself a translation. But this implies that H is quasiconformal and thus absolutely continuous on lines, contrary to our construction.

3. Quasiconformal extension of boundary values

If $f\colon \mathbf{R} \to \mathbf{R}$ is an orientation preserving homeomorphism, we say that f is K-quasisymmetric (K-qs) for $K \geq 1$ if for all $x, t \in \mathbf{R}$,

$$1/K \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K.$$

We define $\mathrm{qs}(f)$ to be the smallest constant K for which this is true. It is well-known that any quasiconformal map of $\mathbf{H}$ extends to a quasisymmetric boundary homeomorphism of $\partial\mathbf{H} = \mathbf{R}$. Conversely, if f is K-qs, f extends to a $K'(K)$-qc map $\hat{f}$ of $\mathbf{H}$. There are many ways to make such an extension, for example the Ahlfors–Beurling extension.

For dynamical reasons, however, we would like to use an extension operator that respects composition, but unfortunately no such operator is known. In this connection we state two open problems and a related theorem.

Notation. For $K \geq 1$, denote by $\mathrm{QS}(K, \mathbf{R})$ the space of all K-quasisymmetric homeomorphisms of $\mathbf{R}$, and by $\mathrm{QC}(K, \mathbf{H})$ the space of K-quasiconformal homeomorphisms of $\mathbf{H}$. Write

$$\mathrm{QS}(\mathbf{R}) = \bigcup_{K>1} \mathrm{QS}(K, \mathbf{R}), \qquad \text{and} \qquad \mathrm{QC}(\mathbf{H}) = \bigcup_{K>1} \mathrm{QC}(K, \mathbf{H}).$$

Extension problem 1. Is there an extension operator $\wedge\colon \mathrm{QS}(\mathbf{R}) \to \mathrm{QC}(\mathbf{H})$ such that the dilatation of $\hat{f}$ depends only on $\mathrm{qs}(f)$, and such that for all $f, g \in \mathrm{QS}(\mathbf{R})$,

$$\widehat{f \circ g} = \hat{f} \circ \hat{g}?$$

Extension problem 2. Is there an extension operator $\wedge\colon \mathrm{QS}(\mathbf{R}) \to \mathrm{QC}(\mathbf{H})$ such that the dilatation of $\hat{f}$ depends only on $\mathrm{qs}(f)$, and such that for any $f_1, \dots, f_n \in \mathrm{QS}(\mathbf{R})$,

$$\mathrm{qs}(f_1 \circ \cdots \circ f_n) \leq K \Longrightarrow \mathrm{dil}(\hat{f}_1 \circ \cdots \hat{f}_n) \leq K'(K)?$$

Note that an affirmative answer to 1 also solves 2.

There are some partial results: for example it follows readily from a result of Hinkkanen [9] that if $f \in \mathrm{QS}(\mathbf{R})$ and f^n is K-qs for all n, then there is an extension $\hat{f} \in \mathrm{QC}(\mathbf{H})$ so that $\mathrm{dil}(\hat{f}^n) \leq K'(K)$ for all n.

We also remark that there is a simple solution of problem 1 in the restricted category of biLipschitz homeomorphisms: the “triangular extension”

$$\wedge\colon \mathrm{Lip}(\mathbf{R}) \to \mathrm{QC}(\mathbf{H})$$

defined by

$$\hat{f}(x, y) = \bigl(\tfrac{1}{2}\bigl(f(x+y) + f(x-y)\bigr), \tfrac{1}{2}\bigl(f(x+y) - f(x-y)\bigr)\bigr)$$

(see Figure 1). One can compute that the dilatation of $\hat{f}$ is controlled by the ratio K_2/K_1, where $K_1|x-y| \leq |f(x) - f(y)| \leq K_2|x-y|$; it is easy to check that $\widehat{f \circ g} = \hat{f} \circ \hat{g}$.

Figure 1.

For our purposes, it will suffice to solve a weak version of problem 1, applicable to mappings with disjoint domains, as follows.

Let $\mathrm{Hom}(\mathbf{R})$ denote the set of all homeomorphisms of $\mathbf{R}$, and $\mathrm{PL}(\mathbf{H})$ denote the set of locally piecewise affine homeomorphisms of $\mathbf{H}$.

The standard binary grid S on $\mathbf{R}$ is defined to be the collection

$$S = \{S_k\}_{k=-\infty}^{+\infty},$$

where $S_k = \{p2^{-k} : p \in Z\}$. A general binary grid $G = \{G_k\}$ is the image $h(S) = \{h(S_k)\}$ of S by a homeomorphism $h\colon \mathbf{R} \to \mathbf{R}$. The collection of all such grids is denoted $\mathscr{G}$, and the collection of K-quasistandard grids is defined by

$$Q\mathscr{G}(K) = \big\{\{h(S_k)\} : h \in \mathrm{QS}(K, \mathbf{R})\big\} \subset \mathscr{G}.$$

A given $G = \{G_k\} \in \mathscr{G}$ determines a triangulation of $\mathbf{H}$ as follows. Fix k for the moment and above each element x of G_k erect a vertical line segment in $\mathbf{H}$ whose length is the distance between x and the nearest right hand neighbor to x in G_k. Then connect the upper endpoint of each such vertical segment to its two nearest neighbors by two line segments. Repeating this for all k gives a tiling of $\mathbf{H}$ by pentagons, which are then further subdivided into triangles so the picture is homeomorphic to the standard triangulation, a finite approximation of which is indicated in Figure 2. Note that this triangulation is completely determined by the grid G.

Figure 2.

Now we can define an extension operator

$$\mathscr{E}\colon \mathrm{Hom}(\mathbf{R}) \times \mathscr{G} \to \mathrm{PL}(\mathbf{H})$$

so that $\mathscr{E}(f, G)$: $\mathbf{H} \to \mathbf{H}$ is the unique homeomorphism that (a) sends the vertices of the triangulation determined by G to the corresponding vertices of the triangulation determined by $f(G)$, and (b) in the interiors of the triangles is the unique affine map determined by the source and target vertices.

Extension theorem. (a) *The extension operator* $\mathscr{E}$ *respects composition:*

$$\mathscr{E}\big(f, g(G)\big) \circ \mathscr{E}(g, G) = \mathscr{E}(f \circ g, G).$$

(b) $\mathscr{E}$ *sends* $\mathrm{QS}(\mathbf{R}) \times Q\mathscr{G}$ *into* $\mathrm{QC}(\mathbf{H})$, *and in fact for every* $K > 1$ *there is a* $K' > 1$ *such that*

$$\mathscr{E}\big(\mathrm{QS}(K, \mathbf{R}) \times Q\mathscr{G}(K)\big) \subset \mathrm{QC}(K', \mathbf{H}).$$

Proof. Part (a) is immediate from the construction.

To see (b), first show by induction on K that $(f, G) \in \mathrm{QS}(K, \mathbf{R}) \times Q\mathscr{G}(K)$ implies $f(G) \in Q\mathscr{G}(K^K)$. Then apply the following

Lemma 1. *If* $f \in \mathrm{Hom}(\mathbf{R})$, $G_1, G_2 \in Q\mathscr{G}(K)$, *and* $f(G_1) = G_2$, *then*

$$\mathrm{dil}\big(\mathscr{E}(f, G_1)\big) \le K'$$

for some K' *depending only on* K.

Proof. One can check that all of the triangles in the triangulation of $\mathbf{H}$ induced by any $G \in Q\mathscr{G}(K)$ are K''-quasiequilateral in the sense that they are images of an equilateral triangle by an affine mapping with dilatation at most $K'' = K''(K)$.

Since $\mathscr{E}(f, G_1)$ affinely takes G_1-triangles to G_2-triangles, it must have dilatation at most $K' = (K'')^2$ a.e. □

The purpose of the "cocycle property" in part (a) of the extension theorem is to control the qc distortion of a long composition of extensions by the qs distortion of the corresponding composition of boundary mappings (see Section 4). This happens because

$$\mathscr{E}\big(f_k, f_{k-1} \circ \cdots \circ f_1(G)\big) \circ \cdots \circ \mathscr{E}(f_1, G) = \mathscr{E}(f_k \circ \cdots \circ f_1, G).$$

Remark. The virtue of the above technique is that the concrete extension operator $\mathscr{E}$ is geometrically transparent. However, R. de la Llave points out that the same result can be achieved abstractly as follows. Let E: $\mathrm{QS}(\mathbf{R}) \to \mathrm{QC}(\mathbf{H})$ be any suitable extension operator, for example the Ahlfors–Beurling extension. If $\{f_0, f_1, \dots\}$ is a sequence in $\mathrm{QS}(\mathbf{R})$, let $H_k = E(f_k \circ \cdots \circ f_0)$ and $F_k = H_k \circ (H_{k-1})^{-1}$. Then, for each k, F_k is a quasiconformal extension of f_k.

Suppose that all ordered compositions $f_j \circ \cdots \circ f_0$ are K-quasisymmetric, and for some K', E: $\mathrm{QS}(K, \mathbf{R}) \to \mathrm{QC}(K', \mathbf{C})$. Then

$$F_{n+k} \circ \cdots \circ F_n = H_{n+k} \circ (H_{n-1})^{-1}$$

is $(K')^2$-quasiconformal for all $n, k > 0$.

4. Boundary distortion of wandering disks

Suppose f is a quasiconformal homeomorphism of T^2, and Δ is a simply connected domain in T^2. By the Riemann mapping theorem, Δ is conformally equivalent to the upper half plane $\mathbf{H}$ by a conformal map ϕ. Similarly for $\Delta' = f(\Delta)$ and a conformal map ψ.

We can choose ϕ and ψ so that $\psi \circ f \circ \phi^{-1}$ takes $\{\infty\}$ to $\{\infty\}$. Since it is quasiconformal, it extends as usual to a boundary homeomorphism of $\partial\mathbf{H} = \mathbf{R}$. We call this boundary homeomorphism the *ideal boundary mapping* of f, denoted $f|_{\partial\Delta}$. It is uniquely determined up to pre- and post-composition with Möbius transformations, which leave the quasisymmetry of the map unaffected. We restate

Theorem 1. *Let $f \in \mathrm{QC}(T^2)$ have Denjoy type. Then either*

(i) *$m(\Gamma) > 0$ and the collection $\{f^n|_\Gamma\}$ has unbounded quasiconformal distortion, or*

(ii) *the collection $\{f^n\}$, considered as mappings of the ideal boundaries of the wandering domains, has unbounded quasisymmetric distortion.*

Proof. If $\Gamma = T^2$, then (i) holds by the Corollary of Proposition 2 and our hypothesis that f is not conjugate to a translation.

Suppose $\Gamma \neq T^2$. Assuming that the conclusion of the theorem fails, our technique is to extend $f|_\Gamma$ to a new mapping $\tilde{f}$ in such a way that $\mathrm{dil}(\tilde{f}^n)$ is bounded globally. By the Corollary of Proposition 2, this means that $\tilde{f}$ is conjugate to a translation, contradicting the fact that $\Gamma \neq T^2$ is a minimal set for $\tilde{f}$.

Suppose there exists $K > 1$ such that $\mathrm{qs}(f^n|_{\partial\Delta}) \leq K$ for all $n \in Z$ and all components Δ of $T^2 \setminus \Gamma$. Fix a particular component Δ_0 and let $\{\Delta_i\}_{i=-\infty}^{+\infty}$ be the orbit of Δ_0 and write $f_i = f|_{\Delta_i}$.

Then by choosing appropriate conformal isomorphisms

$$\phi_i\colon \Delta_i \to \mathbf{H},$$

we have $\phi_{i+1} f_i \phi_i^{-1} \in \mathrm{QC}(\mathbf{H})$ and this map induces a boundary mapping $g_i \in \mathrm{QS}(\mathbf{R})$.

Our assumption means that

$$\mathrm{qs}(g_{i+j} \circ \cdots \circ g_{i+1}) \leq K$$

for some $K > 1$ and all $i \in Z$, $j \in Z^+$.

Let G_0 be the standard grid S, and define

$$G_i = g_{i-1} \circ g_{i-2} \circ \cdots \circ g_1 \circ g_0(S)$$

for $i > 0$, and

$$G_i = g_i^{-1} \circ \cdots \circ g_2^{-1} \circ g_1^{-1}(S)$$

for $i < 0$.

Then $g_i(G_i) = G_{i+1}$ for all $i \in Z$. By the extension theorem, for each i there is an extension $\hat{g}_i = \mathscr{E}(g_i, G_i)$ of g_i to $\mathbf{H}$ such that

$$\operatorname{dil}(\hat{g}_{i+j} \circ \cdots \circ \hat{g}_{i+1}) \leq K'(K)$$

for all $i \in Z$, $j \in Z^+$. Pulling back by means of the ϕ_i's, we obtain new maps $\tilde{f}_i\colon \Delta_i \to \Delta_{i+1}$ for which the dilatation of arbitrarily long compositions is bounded by K'.

Now we claim that these maps can be glued together with $f|_\Gamma$ to make a homeomorphism of T^2.

For this we state the

Glueing lemma. *Let $f\colon \mathbf{C} \to \mathbf{C}$ be quasiconformal, Δ a bounded simply connected domain. Suppose $g\colon \mathbf{H} \to \mathbf{H}$ is quasiconformal, and has the same ideal boundary values as $f|_\Delta$; i.e., for some conformal equivalences $\phi\colon \Delta \to \mathbf{H}$ and $\psi\colon f(\Delta) \to \mathbf{H}$, $\psi f \phi^{-1}|_{\mathbf{R}} = g|_{\mathbf{R}}$.*

Then the map $h\colon \mathbf{C} \to \mathbf{C}$, defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in \mathbf{C} \setminus \Delta, \\ \psi^{-1} g \phi & \text{for } x \in \Delta, \end{cases}$$

is a quasiconformal homeomorphism.

Proof. Consider the map

$$\Theta = \begin{cases} \mathrm{id} & \text{off } \Delta, \\ f^{-1}(\psi^{-1} g \phi) & \text{on } \Delta. \end{cases}$$

On $\mathbf{H}$,

$$\phi \Theta \phi^{-1} = \phi f^{-1}(\psi^{-1} g \phi)\phi^{-1} = (\psi f \phi^{-1})^{-1} g$$

and by our hypothesis this is qc and extends to the identity on $\mathbf{R} = \partial \mathbf{H}$.

By Lemma 2 below, this map moves points a bounded distance in the Poincaré metric on $\mathbf{H}$. Hence Θ moves points of Δ a bounded distance in the Poincaré metric on Δ.

However, since the ratio of the Euclidean metric to the Poincaré metric tends to zero at $\partial\Delta$, this means that Θ is continuous at $\partial\Delta$. Therefore it is a homeomorphism. By Bers' lemma, stated below, Θ is quasiconformal. Hence $h = f\Theta$ is quasiconformal. □

Lemma 2. *If $g\colon \mathbf{H} \to \mathbf{H}$ is K-quasiconformal and $g|_{\mathbf{R}} = \mathrm{id}$, then g moves points of $\mathbf{H}$ by a uniformly bounded amount with respect to the Poincaré metric.*

Proof. By conjugating with a Möbius transformation, it is enough to show that the image of the point i must lie in some compact set depending only on K. But this is clear since the set of K-quasiconformal homeomorphisms of $\mathbf{H}$ that fix three points on the boundary form a compact family in the compact-open topology.

Bers' lemma [1]. *Let U be an open set in the plane, g a plane homeomorphism such that $g|_U$ is* qc *and $g = \mathrm{id}$ off U. The g is* qc *(and has no dilatation off U).*

The result of the glueing lemma is that we obtain, by glueing one disk at a time and passing to a limit, a new quasiconformal map $\tilde{f}$ that agrees with f on Γ. Off Γ, there is a uniform bound on the dilatation of arbitrarily long iterates of $\tilde{f}$. On Γ, the dilatation of $\tilde{f}^n$ must agree with that of f^n. □

Corollary of Theorem 1. *Let $f \in \mathrm{QC}(T^2)$ have Denjoy type with $\Gamma \neq T^2$. If the components of $T^2 \setminus \Gamma$ are all K-quasidisks, then the collection $\{f^n\}$, considered as mappings of the ideal boundaries of the wandering domains, has unbounded quasisymmetric distortion.*

Proof. Since every point of Γ is a limit of components of $T^2 \setminus \Gamma$ and f is differentiable a.e., the hypothesis implies that $\mathrm{dil}(f^n, x) \leq K^2$ for all n, and a.e. $x \in \Gamma$. Now apply Theorem 1. □

5. Invariant conformal structures on Γ

If $S \subset T^2$ is an invariant set for f, then by an *f-invariant conformal structure on S* we mean a measurable, essentially bounded choice $c(x)$ of similarity class on $T_x(T^2)$ for each $x \in S$, such that $(f^*c)(x) = c(x)$ for all $x \in S$.

Remark. The proof of Proposition 2 shows that a differentiable mapping f has an invariant conformal structure on S if and only if the family of iterates of f has uniformly bounded dilatation on S.

Given an f-invariant conformal structure on S, we can arbitrarily extend to a measurable, essentially bounded structure on all of T^2. Via the measurable Riemann mapping theorem, this determines a complex structure $\mathbf{C}/L$ with global coordinate ϕ. In this coordinate system, f is pointwise conformal on S—that is, $\phi f \phi^{-1}$ is pointwise conformal on $\phi(S)$ a.e.

Recall that the integral of a C^{r+Z} conformal structure on $\mathbf{C}$ is a C^{r+1+Z} quasiconformal mapping. Hence f preserves a C^{r+Z} conformal structure on S if and only if f is pointwise conformal on S relative to a C^{r+1+Z} complex structure on T^2.

Theorem 2. *Let $f \in \mathrm{Diff}_1(T^2)$ have Denjoy type, and suppose f preserves a C^{1+Z} conformal structure on Γ.*

Then f cannot be C^{2+Z}.

Equivalently, Theorem 2 says that if f is of class C^{2+Z} then f cannot be pointwise conformal on Γ relative to any complex structure on T^2 with the same underlying C^{2+Z} smooth structure.

Corollary 1. *Let $f \in \mathrm{Diff}_{2+Z}(T^2)$ have Denjoy type. Then $T^2 \setminus \Gamma$ cannot be a union of pairwise disjoint circular disks.*

Proof. If $T^2 \setminus \Gamma$ were a union of round circular disks, then each point $p \in \Gamma$ would be a limit of arbitrarily small such disks. Since f preserves the family of such disks, this means that Df_p must be a conformal linear map relative to the C^∞ conformal structure induced by the metric. Now apply Theorem 2. ▫

This argument actually proves more, for which we need the following terminology. Two bounded domains in $\mathbf{R}^2$ are *similar* if one can be taken to the other by a *Euclidean similarity*, i.e., a composition of a dilation, rotation, and translation. Clearly, similarity is an equivalence relation on the collection of all bounded domains in $\mathbf{R}^2$. We call the equivalence classes *shapes.*

A shape is *generic* if the only elements of $\mathrm{SL}(2,\mathbf{R})$ preserving it are the elements of $\mathrm{SO}(2,\mathbf{R})$. A little thought should convince the reader that most shapes are generic, including circular and square shapes. However, other elliptical and rectangular shapes are not generic.

Corollary 2. *Let $f \in \mathrm{Diff}_{2+Z}(T^2)$ have Denjoy type, with $\Gamma \neq T^2$. Then the components of $T^2 \setminus \Gamma$ (i.e., the wandering domains of f) cannot all have the same generic shape.*

Proof. If they did all have the same generic shape, f would be pointwise conformal on Γ as before. ▫

Proof of Theorem 2. Assume the contrary: that $f \in C^{2+Z}$ preserves a C^{1+Z} conformal structure c on Γ. Extend c to a C^{1+Z} structure on all of T^2, and let $\phi\colon T^2 \to \mathbf{C}/L$ be an induced global coordinate. Then $g = \phi f \phi^{-1}$ is pointwise conformal on $\Gamma' \equiv \phi(\Gamma) \subset \mathbf{C}/L$, and $g \in C^{2+Z}$.

Now $\mu_g \in C^{1+Z}$, and conformality on Γ' means $\mu_g = 0$ on Γ'. By the topology of Γ' (*c.f.* Proposition 1), we must have $D\mu_g = 0$ on Γ' also. Hence the same is true of a lift $\tilde{g}\colon \mathbf{C} \to \mathbf{C}$ of g. By Proposition 3 below, $\mathrm{dil}(\tilde{g}^n, x)$ is bounded by a constant independent of $n \in Z$ or $x \in \mathbf{C}$. Therefore $\{g^n\}$ has bounded dilatation on $\mathbf{C}/L$, and by the Corollary of Proposition 2, this means that g, hence f, is quasiconformally conjugate to a translation. This contradicts our hypothesis that f has Denjoy type. ▫

Proposition 3. *Let f be a C^{2+Z} orientation preserving diffeomorphism of $\mathbf{C}$, and U an open subset of $\mathbf{C}$. Let $V = \bigcup_{n\in Z} f^n(U)$. Suppose*

(a) $A \equiv \sum_{n\in Z} \mathrm{area}\big(f^n(U)\big) < +\infty$,
(b) $|\mu_f| \le \delta < 1$ *on* V, *and*

(c) μ_f and $D\mu_f$ are zero on $\cup f^n(\partial U)$.

Then $\operatorname{dil}(f^n|_V)$ *is bounded by a constant depending only on* A, *the Zygmund constant of* $D\mu_f$, *and* δ.

The proof of Proposition 3 requires two lemmas.

Lemma 3. *If* $f\colon [a,b] \to \mathbf{R}$ *is* C*-Zygmund and* $f(a) = f(b) = 0$, *then, for all* $x \in [a,b]$,

$$|f(x)| \le (C/2)|b-a|.$$

Proof. We are supposing that for all $x, h \in \mathbf{R}$ with $x, x+h, x-h \in [a,b]$,

$$|f(x+h) + f(x-h) - 2f(x)| \le C|h|.$$

If I is a subinterval with endpoints y and z and midpoint x, this implies that

$$|f(x)| \le \tfrac{1}{4}C|I| + \bigl(|f(y)| + \tfrac{1}{2}|f(z)|\bigr). \tag{1}$$

Let $L = |b-a|$. For $k = 0, 1, 2, \ldots$, consider the collection $\mathscr{C}_k$ of closed dyadic subintervals of $[a,b]$, i.e. intervals with endpoints

$$a + (p/2^k)(b-a) \qquad \text{and} \qquad a + \bigl((p+1)/2^k\bigr)(b-a),$$

where p is an integer between 0 and $2^k - 1$.

Define $B_0 = \{a, b\}$, and for $k = 1, 2, 3, \ldots$, define B_k to be the finite set of points x in $[a,b]$ such that x is an endpoint of an interval of $\mathscr{C}_k$ but is an endpoint of no interval of $\mathscr{C}_j$ for any $j < k$. (B_k is the set of endpoints that "arise at stage k".)

Claim. For all $x \in B_k$, $|f(x)| \le \frac{1}{2}CL(1 - 2^{-k})$.

We prove the claim below, but first note that this implies $|f(x)| \le \frac{1}{2}CL$ for all x in the dense set $\bigcup_k B_k$. Since f is continuous, this yields the conclusion of Lemma 3.

The claim is proved by simple induction. The case $k = 0$ is simply our hypothesis that $f(a) = f(b) = 0$. Let $k > 0$ and suppose that for $j = 0, 1, 2, \ldots, k-1$,

$$|f(y)| \le \tfrac{1}{2}CL(1 - 2^{-j}) \quad \text{for all } y \in B_j.$$

Take $x \in B_k$. Then x is the midpoint of some dyadic interval of length $L2^{-(k-1)}$, with endpoints, say, x' and x''. By (1) above,

$$|f(x)| \le CL/2^{-(k+1)} + \tfrac{1}{2}\bigl(|f(x')| + |f(x'')|\bigr).$$

By induction, since x' and x'' must appear at some stage before the kth, $|f(x')|$ and $|f(x'')|$ are no more than $\frac{1}{2}CL(1 - 2^{-(k-1)})$. Therefore

$$|f(x)| \le CL/2^{-(k+1)} + \tfrac{1}{2}CL(1 - 2^{-(k-1)}) = \tfrac{1}{2}CL(1 - 2^{-k}). \ \square$$

Definition. A *chord* of an open set $U \subset \mathbf{C}$ is a connected component of $L \cap U$, where L is a line in $\mathbf{C}$.

Lemma 4. *Let $U \subset \mathbf{C}$ be open, and for $y \in U$ denote by $l(y)$ the length of the shortest chord of U through y.*

Then for every $y \in U$, $\big(l(y)\big)^2 \le (4/\pi)\,\mathrm{area}(U)$.

Proof. Fix $y \in U$. By translation if necessary, assume that $y = 0$.

Let $L(\theta)$ denote the chord of U through 0 inclined at angle θ from the positive x-axis, $0 < \theta < \pi$. Let $l_1(\theta)$ and $l_2(\theta)$ denote the lengths of $L(\theta) \cap \{z : \mathrm{Im}(z) > 0\}$ and $L(\theta) \cap \{z : \mathrm{Im}(z) < 0\}$, respectively.

Then

$$\mathrm{area}(U) \ge \tfrac{1}{2} \int_0^{\pi} l_1(\theta)^2 + l_2(\theta)^2 \, d\theta.$$

However, $l_1(\theta) + l_2(\theta) \ge l(y)$, and hence $l_1(\theta)^2 + l_2(\theta)^2 \ge \frac{1}{2} l(y)^2$. Therefore

$$\mathrm{area}(U) \ge \tfrac{1}{2} \int_0^{\pi} (1/2) l(y)^2 \, d\theta = \tfrac{1}{4} \pi l(y)^2. \ \square$$

Proof of Proposition 3 (finite area argument). Write $\nu = \mu_f$. Choose $y \in U$. Let σ be the shortest chord of U through y. By applying Lemma 3 to $D\nu$ along σ and then integrating, we obtain

$$|\nu(y)| \le \tfrac{1}{2} C |\sigma|^2,$$

where C is the Zygmund constant for $D\nu$. By Lemma 4, this quantity is less than or equal to $C\,\mathrm{area}(U)$.

Now let $\{y_n\}$ be an f-orbit in V, with $y_n \in f^n(U)$. By applying the above inequality to $f^n(U)$ and summing, we find that $\Sigma |\nu(y_n)| \le CA$.

Therefore $\prod \big(1 + |\nu(y_n)|\big) < \infty$ and $\prod \big(1 - |\nu(y_n)|\big) > 0$. This means

$$\prod \mathrm{dil}(f, y_n) = \prod \left(\frac{1 + |\nu(y_n)|}{1 - |\nu(y_n)|} \right) < M < \infty,$$

where M depends only on CA and δ. However, the dilatation is submultiplicative along orbits:

$$\mathrm{dil}(f^n, y_0) \le \prod_{i=0}^{n-1} \mathrm{dil}(f, y_i).$$

Hence we have $\mathrm{dil}(f^n, y)$ is controlled by a constant as claimed. $\square$

Open questions. The primary question is: can any $f \in \mathrm{Diff}_{2+Z}(T^2)$ have Denjoy type? Theorem 2 above says any such map cannot preserve a C^{1+Z} conformal structure on Γ. Can this be improved to the statement that any such map cannot preserve a measurable, bounded conformal structure on Γ?

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